The renormalization structure of Gross-Neveu Model

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Gross-Neveu model

- Gross-Neveu model is a 2-d renormalizable theory with 4-fermion interactions [Gross, Neveu 1974] :

$$L = \bar{\Psi}_i(i\not{\partial} - m)\Psi^i - \frac{g}{2}(\bar{\Psi}_i\Psi^i)^2 \ . \tag{1}$$

- Similar to QCD, the Gross-Neveu model is asymptotic free. Fermion mass can be generated by dynamic symmetry breaking, and the massless theory is unstable.
- Connected to higher spin fields in AdS [Giombi 2016].
- In the large N limit, Gross-Neveu model has a UV completion in 4-d, which is Gross-Neveu-Yukawa model [Fei, Giombi, Klebanov 2014] :

$$L_{\mathrm{GNY}} = \bar{\Psi}_i(i\not{\partial} - m)\Psi^i - \frac{1}{2}(\partial\sigma)^2 - g\sigma\bar{\Psi}_i\Psi^i - \frac{h}{4!}\sigma^4 \ . \tag{2}$$

- In the $N \to 0$ limit, Gross-Neveu model describes the physics of random bond Ising model[Dotsenko 1981].

Evanescent operators

- Besides $(\bar{\Psi}_i \Psi^i)^2$, there are also other 4-fermion operators

$$L = \bar{\Psi}_i (i \not{\partial} - m) \Psi^i - \frac{1}{4} \sum_{n=0}^{\infty} g_n O_n, \quad O_n = \frac{1}{n!} (\bar{\Psi}_i \gamma^{\mu_1 \cdots \mu_n} \Psi^i)(\bar{\Psi}_i \gamma_{\mu_1 \cdots \mu_n} \Psi^i) \ . \tag{3}$$

- O_n are evanescent operators when $n \geq 3$, but their contribution cannot be neglected in dimensional regularization.
- In fact, the original Gross-Neveu model is not renormalizable in dimensional regularization. A divergence proportional to O_3 first appear at 3-loop [Vasil'ev 1997]:

$$\langle \Psi \Psi \bar{\Psi} \bar{\Psi} \rangle^{(3)} \sim \frac{(4 - 3\zeta_3) g^4}{\epsilon} O_3 + \cdots \tag{4}$$

- At 4-loop, a divergence proportional to O_4 appears [Gracey 2016].

O_n divergence appear at n-loop ?

- n-loop 4-fermion correlation function has the following structure :

$$\langle\Psi\Psi\bar{\Psi}\bar{\Psi}\rangle^{(n)} = \frac{(\bar{u}_3\, \not{l}_1 \cdots \not{l}_n u_1)(\bar{u}_4\, \not{l}_1 \cdots \not{l}_n u_2)}{D_1 \cdots D_{2n}} + \cdots \tag{5}$$

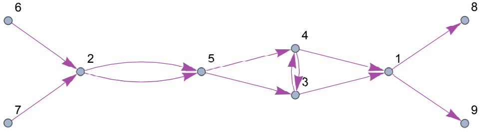

Figure 1: A 4 loop Feynman diagram contributing to 4 fermion amplitude.

- O_n appears if we anti-symmetrize the fermion chain, and perform a PV reduction :

$$\begin{aligned}
&(\bar{u}_3[\not{l}_1 \cdots \not{l}_n]u_1)(\bar{u}_4[\not{l}_1 \cdots \not{l}_n]u_2) \\
=&\frac{1}{(n!)^2}\delta^{\mu_1\cdots\mu_n}_{l_1\cdots l_n}\delta^{\nu_1\cdots\nu_n}_{l_1\cdots l_n}(\bar{u}_3\gamma^{\mu_1\cdots\mu_n}u_1)(\bar{u}_4\gamma^{\nu_1\cdots\nu_n}u_2) \\
\rightarrow&\frac{(-1)^n}{(-d)_n}G(l_1,\cdots,l_n)(\bar{u}_3\gamma^{\mu_1\cdots\mu_n}u_1)(\bar{u}_4\gamma_{\mu_1\cdots\mu_n}u_2) \\
\rightarrow&\frac{(-1)^n}{(-d)_n}G(l_1,\cdots,l_n)O_n
\end{aligned} \tag{6}$$

Not all O_n divergence appear

- Although integrals proportional to O_n appear at n-loop, these integral may not be divergent. For example, O_1 and O_2 do not appear up to 4-loop [Gracey 2016].
- O_5 does not appear at 5 loop [Huang, QJ, Li, to appear].
- There seems to be a pattern:

$$\begin{aligned} L =&\bar{\Psi}_i(i\not{\partial} - m)\Psi^i - \frac{g}{4}O_0 - \frac{g^4 Z_3}{4}O_3 - \frac{g^5 Z_4}{4}O_4 \\ &- \frac{g^8 Z_7}{4}O_7 - \frac{g^9 Z_8}{4}O_8 - \frac{g^{12} Z_{11}}{4}O_{11} - \frac{g^{13} Z_{12}}{4}O_{12} + \cdots \end{aligned} \tag{7}$$

- Why some operators appear, while others do not? Is this pattern governed by a symmetry?

Hidden symmetry

- By introducing a series of auxiliary tensor fields $\sigma_{\mu_1 \cdots \mu_n}$, the Lagragian can be rewritten as,

$$L = \bar{\Psi}_i (i \not{\partial} - m) \Psi^i + \frac{1}{2n!} \sum_{n=0}^{\infty} \sigma_{\mu_1 \cdots \mu_n} \left[\sigma^{\mu_1 \cdots \mu_n} - \sqrt{2 g_n} \bar{\Psi}_i \gamma^{\mu_1 \cdots \mu_n} \Psi^i \right] \tag{8}$$

- The new Feynman rule only contain a $\sigma \bar{\Psi} \Psi$ 3-vertex :

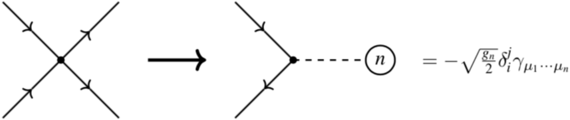

Figure 2: New Feynman rules with auxiliary fields.

Fermion chain

- Each $n\Psi - n\bar{\Psi}$ Feynman diagram contains n fermion chains and several fermion loops. These fermion chains and fermion loops are connected by auxiliary fields.

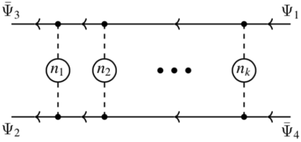

Figure 3: A Feynman diagram with 4 external fermions.

- The fermion chain connecting $\Psi_1 - \bar{\Psi}_3$ produces the integral

$$F_1 = \frac{\bar{u}_3 \Gamma^{\mu_1}_{(n_1)} \not{l}_1 \Gamma^{\mu_2}_{(n_2)} \not{l}_2 \cdots \Gamma^{\mu_k}_{(n_k)} \not{l}_k \Gamma^{\mu_{k+1}}_{(n_{k+1})} u_1}{l_1^2 l_2^2 \cdots l_k^2} \tag{9}$$

in which $\Gamma^{\mu_i}_{(n_i)} \equiv \gamma^{\mu_{i1} \cdots \mu_{in_i}}$ is from the $\sigma\bar{\Psi}\Psi$ 3-vertex.

Fermion chain

- After integral reduction, and anti-symmetrization of gamma matrices,

$$F_1 \to \sum_m c_m \bar{u}_3 \gamma_{\nu_1 \cdots \nu_m} u_1 \tag{10}$$

- Since gamma matrices contract in pairs, the number of gamma matrices stays even (odd) if the original integral contains even (odd) of gamma matrices.
- This means m satisfies

$$(-1)^{k-m+\sum_{i=1}^{k+1} n_i} = 1 \tag{11}$$

The reversed fermion chain

- Let us interchange the position of Ψ_1 and $\bar{\Psi}_3$ in the diagram, and reverse the direction of fermion chain connecting them. The other parts of the Feynman diagram remain unchanged:

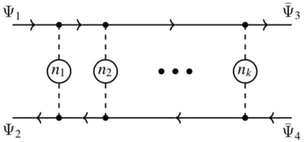

Figure 4: The reversed fermion chain.

$$F_2 = \frac{\bar{u}_3 \Gamma^{\mu_{k+1}}_{(n_{k+1})} (-\not{l}_k) \Gamma^{\mu_k}_{(n_k)} \cdots (-\not{l}_2) \Gamma^{\mu_2}_{(n_2)} (-\not{l}_1) \Gamma^{\mu_1}_{(n_1)} u_1}{l_1^2 l_2^2 \cdots l_k^2} \tag{12}$$

- We define an operator $\mathcal{R}$, which reverse the order of gamma matrix products,

$$F_2 = (-1)^{k+\sum_{i=1}^{k+1} \frac{n_i(n_i-1)}{2}} \frac{\bar{u}_3 \mathcal{R}\left(\Gamma^{\mu_1}_{(n_1)} \not{l}_1 \Gamma^{\mu_2}_{(n_2)} \not{l}_2 \cdots \Gamma^{\mu_k}_{(n_k)} \not{l}_k \Gamma^{\mu_{k+1}}_{(n_{k+1})}\right) u_1}{l_1^2 l_2^2 \cdots l_k^2} \tag{13}$$

The reversed fermion chain

- The operator $\mathcal{R}$ commutes with gamma matrix algebra:

$$\begin{aligned} F_2 \to & (-1)^{k+\sum_{i=1}^{k+1} \frac{n_i(n_i-1)}{2}} \sum_m c_m \bar{u}_3 \mathcal{R} \gamma_{\nu_1 \cdots \nu_m} u_1 \\ = & \sum_m (-1)^{k+\frac{m(m-1)}{2}+\sum_{i=1}^{k+1} \frac{n_i(n_i-1)}{2}} c_m \bar{u}_3 \gamma_{\nu_1 \cdots \nu_m} u_1 \\ = & \sum_m (-1)^{\frac{m(m+1)}{2}+\sum_{i=1}^{k+1} \frac{n_i(n_i+1)}{2}} c_m \bar{u}_3 \gamma_{\nu_1 \cdots \nu_m} u_1 \end{aligned} \tag{14}$$

- The total contribution of two fermion chains reads

$$F_1 + F_2 = \sum_m \left[1 + (-1)^{\frac{m(m+1)}{2}+\sum_{i=1}^{k+1} \frac{n_i(n_i+1)}{2}}\right] c_m \bar{u}_3 \gamma_{\nu_1 \cdots \nu_m} u_1 \tag{15}$$

- The coefficient of $\bar{u}_3 \gamma_{\nu_1 \cdots \nu_m} u_1$ is non-zero only if

$$(-1)^{\frac{m(m+1)}{2}} = (-1)^{\sum_{i=1}^{k+1} \frac{n_i(n_i+1)}{2}} \tag{16}$$

Charge conjugation

- When we reverse the fermion chain, the factor $(-1)^{\frac{n(n+1)}{2}}$ is produced by $\bar{\Psi}_i\gamma^{\mu_1\cdots\mu_n}\Psi^i$.
- The same factor can be produced by charge conjugation:

$$\mathcal{C}\bar{\Psi}_i\gamma^{\mu_1\cdots\mu_n}\Psi^i = (-1)^{\frac{n(n+1)}{2}}\bar{\Psi}_i\gamma^{\mu_1\cdots\mu_n}\Psi^i \tag{17}$$

- $\mathcal{C}$-even : $n = 0, 3, 4, 7, 8, \cdots$.
- $\mathcal{C}$-odd : $n = 1, 2, 5, 6, 9, 10, \cdots$.
- A $\mathcal{C}$-odd (even) fermion chain only produces $\mathcal{C}$-odd (even) divergent terms:

$$(-1)^{\frac{m(m+1)}{2}} = (-1)^{\sum_{i=1}^{k+1}\frac{n_i(n_i+1)}{2}} \tag{18}$$

- $\mathcal{C}$-odd fermion loops vanish :

$$F_1 + F_2 \propto 1 + (-1)^{\sum_i \frac{n_i(n_i+1)}{2}} \tag{19}$$

$\mathcal{C}$-even Gross-Neveu model

- Suppose a Feynman diagram only contains $\mathcal{C}$-even vertices, then the UV divergence is also $\mathcal{C}$-even.
- Therefore, if we only add $\mathcal{C}$-even vertices to the Lagrangian, the Gross-Neveu model is still renormalizable:

$$\begin{aligned} L =& \bar{\Psi}_i(i\,\partial\!\!\!/ - m)\Psi^i - \frac{g}{4}O_0 - \frac{g^4 Z_3}{4}O_3 - \frac{g^5 Z_4}{4}O_4 \\ &- \frac{g^8 Z_7}{4}O_7 - \frac{g^9 Z_8}{4}O_8 - \frac{g^{12} Z_{11}}{4}O_{11} - \frac{g^{13} Z_{12}}{4}O_{12} + \cdots \end{aligned} \tag{20}$$

- This explains why we did not find O_1, O_2 and O_5 divergences.
- New divergent terms proportional to O_7 may appear at 7-loop.

Towards 7-loop computations

- Almost all techniques which are useful at computing lower loop Feynman integrals, including IBP reduction, PV reduction, differential equation, UV subtraction, are not efficient at 7-loop. Simple tasks like generating Feynman diagrams become very challenging at 7-loop.
- Each $\sigma \in S_{n+1}$ corresponds to a n-loop 4-fermion Feynman diagram:

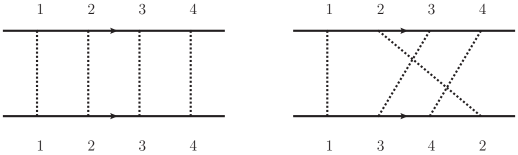

Figure 5: The Feynman diagrams corresponding to [1234] and [1342].

- The integral corresponding to σ will be denoted by $I_\sigma N_\sigma$. I_σ are the denominators, and N_σ are numerators produced by fermion chains.

Towards 7-loop computations

- If we set all external momenta to 0, the numerator N_σ becomes

$$N_\sigma = \text{sign}(\sigma)(\bar{u}_2[\not{l}_1 \cdots \not{l}_n]u_1)(\bar{u}_4[\not{l}_1 \cdots \not{l}_n]u_3) \tag{21}$$

  All Feynman diagrams have the same numerators!

- After tensor reduction

$$N_\sigma = \frac{(-1)^n}{(-d)_n}\text{sign}(\sigma)G(l_1, \cdots, l_n)O_n \tag{22}$$

- Using dimensional shift [Tarasov 1996], n-loop vacuum integrals with a $G(l_1, \cdots, l_n)$ factor can be related to $d+2$ dimension integrals without this factor:

$$\sum_{\sigma \in S_{n+1}} I_\sigma N_\sigma \Big|_{p_i=0} = (-\frac{1}{2})^n O_n \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) I_\sigma \Big|_{p_i=0, d=4} \tag{23}$$

Towards 7-loop computations

- Now let us retrieve the external momenta on both sides of the equation:

$$\sum_{\sigma\in S_{n+1}} I_\sigma N_\sigma \sim (-\frac{1}{2})^n O_n \sum_{\sigma\in S_{n+1}} \text{sign}(\sigma) I_\sigma \Big|_{d=4} \tag{24}$$

- While the local divergence is unaltered by this "retrieve p_i" operation, the total UV divergence will be changed. But we have proved that in this case, the total UV divergences of two sides are still the same!
- Now $I_\sigma\Big|_{d=4}$ are 4-d integrals without numerators. They are actually 4-point integrals in the ϕ^4 theory, and can be computed using graphical functions [Schnetz 2021]:

$$= \frac{1}{18\epsilon}\Big(36 + 9z + 9\bar{z} + 4z\bar{z} - (18 + 9z + 9\bar{z} + 6z\bar{z})\ln z\Big)$$
$$+ \frac{1}{108}\Big(864 - 38z\bar{z} + (-216 + 54z\bar{z})\ln z - (54 + 27z + 27\bar{z} + 18z\bar{z})\ln^2 z\Big) \tag{25}$$
$$+ \mathcal{O}(\epsilon)$$

Towards 7-loop computations

- The graphical functions was used to compute the 7-loop beta functions in ϕ^4 theory [Schnetz 2022].
- We have correctly reproduced 3 and 4 loop renormalization factors in Gross-Neveu model using graphical functions. 7 loop computations should finish in 1 or 2 days!
- It would be the first time to obtain a beyong-5-loop beta function in a model with non-zero spin.
- Thank you for your attention!