Quasinormal modes of C-metric from superconformal field theories

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Based on work with Yang Lei (Soochow U.), Hongfei Shu (Zhengzhou U.), Kilar Zhang (Shanghai U.) arXiv:2308.16677 and work in progress with Hao Zhao; Pujun Liu.

Elliptic Vertex of M-strings

(Nick) R.D. Zhu*, collaboration with J.-E. Bourgine†, O.Foda♠, and Y. Matsuo*

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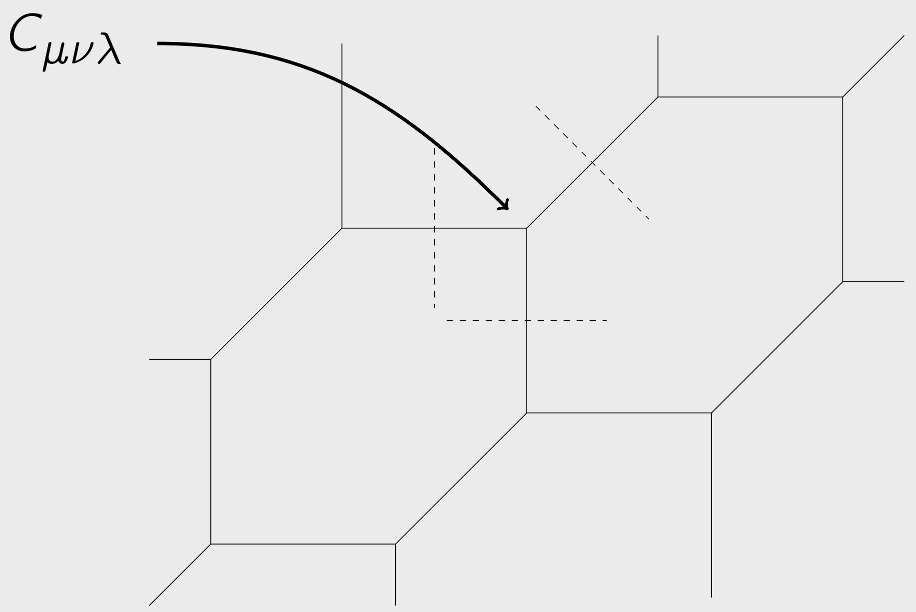
Poster presentation @ 2017 EAJW KEK

Introduction

5d $\mathcal{N} = 1$ quiver gauge theories realized by (p, q)-brane web can be dualized to M-theory compactified on some toric Calabi-Yau manifold. Its instanton partition function can thus be computed from the topological string A-model targeting on the same Calabi-Yau. It is known that the partition function for toric Calabi-Yau can be computed in a Feynman-diagram style with the topological vertex $C_{\mu\nu\lambda}$ assigned to each vertex of the corresponding toric diagram. e.g.



6d $\mathcal{N} = (1, 0)$ on S^1 can be realized in a similar way, by compactifying the brane web along the NS5-brane direction on (5, 6)-plane[1]. The partition function of this 6d theory on $\Omega_{q,t} \times T^2$ can also be computed

Elliptic vertex in elliptic DIM

The elliptic (AFS) vertex can be written down in a very similar way. A lot of preparing works have already done by [3]. It turns out to be an intertwiner of representations of the elliptic DIM introduced also by Saito in [3]. This algebra must be an elliptic deformation of $\mathcal{W}_{1+\infty}[\mu]$. This embedding thus helps us understand the 2d algebraic structure of the elliptic AGT relation. For a numerical factor of the form $1 - x$, we can simply replace it with the θ-function,

$$1 - x \to \theta_p(x).$$

The vertex operator used to define the AFS vertex,

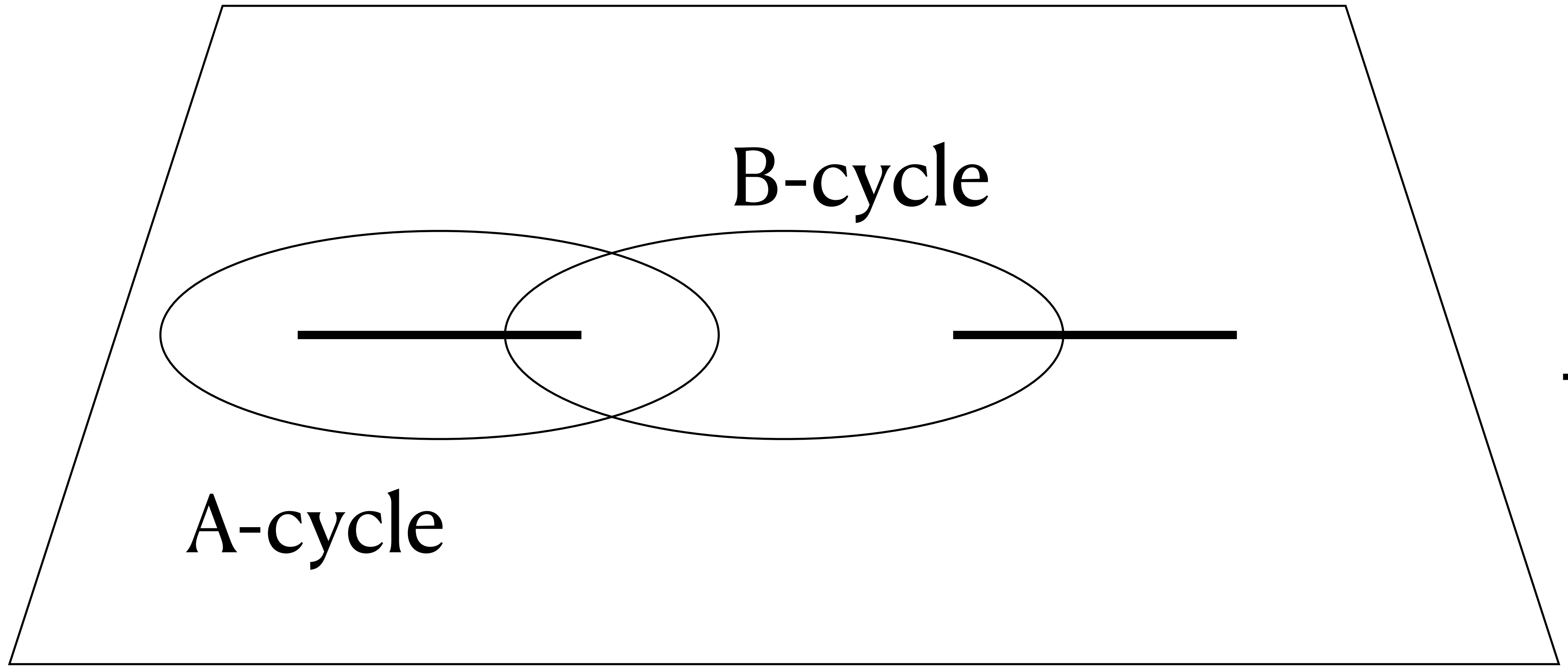
$$V_\pm(z) = \exp\left(\pm \sum_{n>0} \frac{1}{n}\frac{z^{\mp n}}{1 - q^{\mp n}} a_{\pm n}\right), \tag{1}$$

is replaced by

A recent interesting movement: To study the perturbations around BHs with sophisticated methods developed in exact calculations of SUSY gauge theory.

A proposal by [Aminov, Grassi, Hatsuda (2020)]

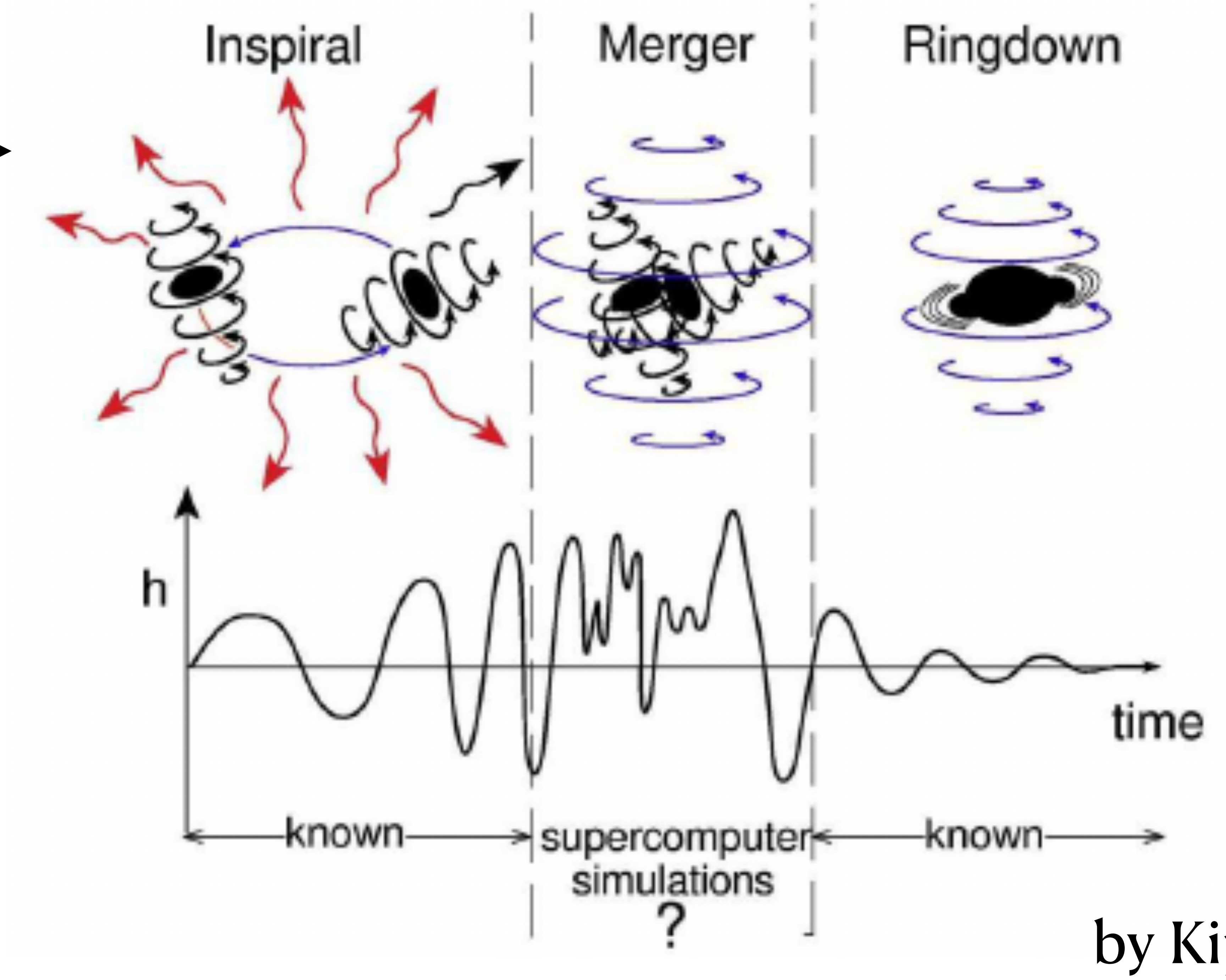
Quantized Seiberg-Witten curve



Tells us how to compute non-perturbative corrections (instantons) to (super) gauge theories in an exact way.

Quasinormal modes

captures the ringdown feature during the gravitational wave releasing



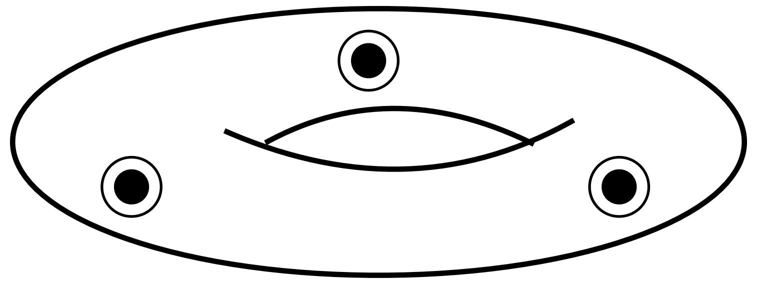
by Kip Thorne

Features of 4d $\mathcal{N} = 2$ supersymmetric gauge theories

- Class-S construction in the string theory context: [Gaiotto (2009)]

6d N=(2,0) theory (low energy theory on M5 branes)

compactify on Σ



Punctured Riemann surface with punctures regular/irregular

4d N=2 theory

- Seiberg-Witten curve (relation with classical integrable models)

The punctured Riemann surface gives a complex curve that governs the low-energy behavior of the gauge theory.

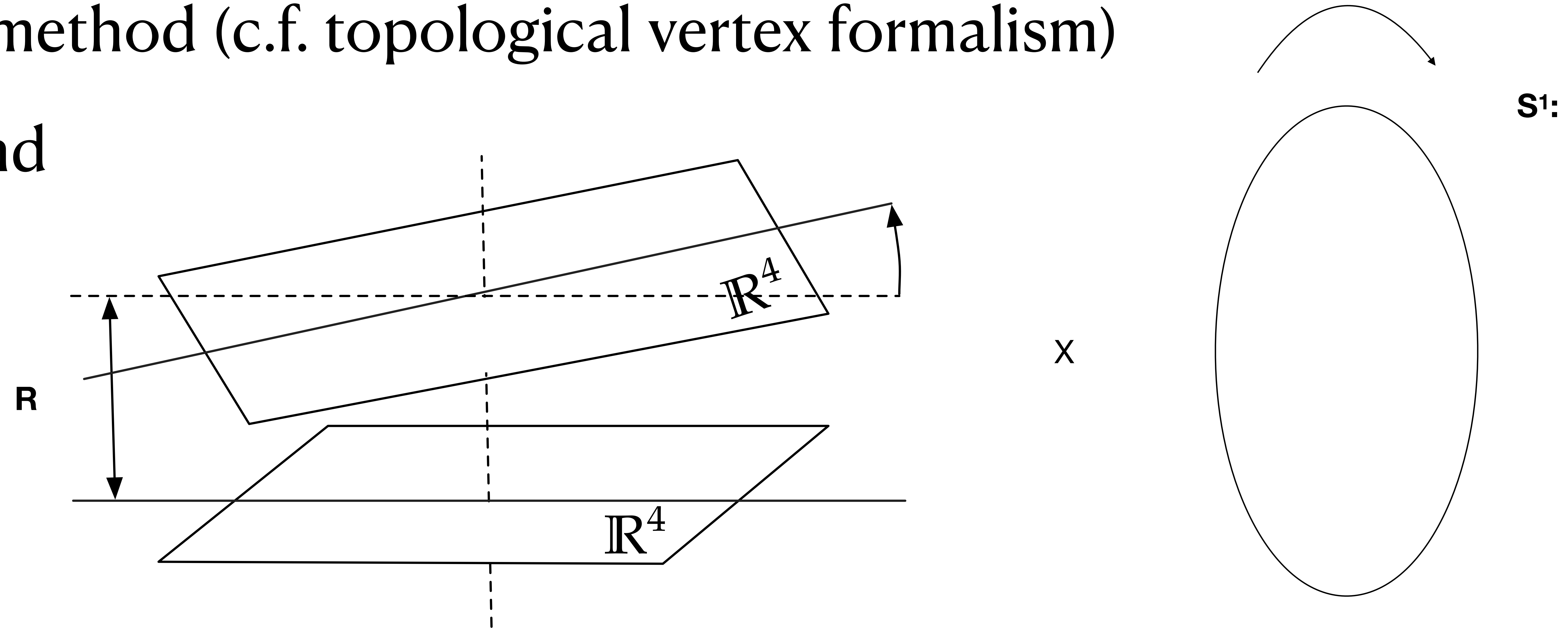
[Seiberg, Witten (1994)]

Now known as the Seiberg-Witten curve.

$$\left(\frac{P_1(x)}{y} + \bar{\Lambda} P_2(x) y\right) = x^2 - u,$$

- Non-perturbative effects can be computed exactly (analytically) with the localization method (c.f. topological vertex formalism)

Ω-background



R: radius of S¹

Identification: $\mathbb{C}^2 \ni (z_1, z_2) \sim (q_1 z_1, q_2 z_2)$ $\quad q_1 = e^{R\epsilon_1} = t, \;\; q_2 = e^{R\epsilon_2} = q^{-1}.$

$$Z = \int \mathcal{D}\phi \; e^{iS} = Z_{classical} \times Z_{pert.} \times Z_{non-pert}.$$

classical action — one-loop factor — non-perturbative corrections

Nekrasov-Shatashvili limit

[Nekrasov, Shatashvili (2009)]

$$\epsilon_1 = \hbar, \quad \epsilon_2 \to 0.$$

The Seiberg-Witten curve gets quantized

$$[x, \log y] = \hbar \qquad\qquad x = \hbar y \partial_y$$

and it becomes a second-order ODE.

given in terms of (defect) partition function

$$\left(\hbar^2 \partial_z^2 + Q_{\mathrm{SW}}(z)\right) \psi(z) = 0,$$

e.g. Σ = sphere with four regular punctures $\to$ 4d SU(2) gauge theory with $N_f = 4$

$$Q_{\mathrm{SW}}(z) = \frac{\frac{1}{4} - a_0^2}{z^2} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_t^2}{(z-t)^2} - \frac{\frac{1}{2} - a_0^2 - a_1^2 - a_t^2 + a_\infty^2 + u}{z(z-1)} + \frac{u}{z(z-t)},$$

(Exactly the singularity structure in class S construction)

$\approx$ Bethe/Gauge correspondence

Such linear ODEs are classified by the singularities

- Perturbations around a black-hole geometry

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + \dots$$

The perturbation obeys the equation of the form

$$\frac{1}{f(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{f(r)}\frac{\mathrm{d}}{\mathrm{d}r}\phi(r)\right) + \left[\omega^2 - V(r)\right]\phi(r) = 0.$$

The equation can potentially have singularities at the zeros of $f(r) = 0$ and $0, \infty$.

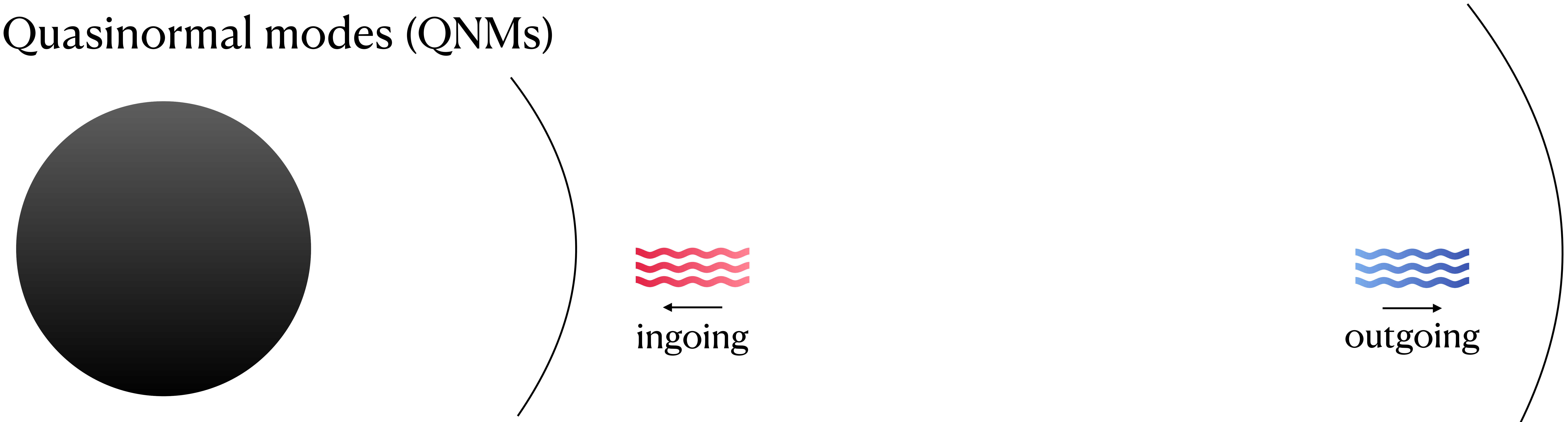
e.g. in the C-metric, dS black hole (s=1,2), four regular singularities.

C-metric:

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)(1 - \alpha^2 r^2)$$

Singularities at $\pm\frac{1}{\alpha},\ r_\pm = M \pm \sqrt{M^2 - Q^2}$.

Quasinormal modes (QNMs)



Event horizon
(One singularity in the ODE)

Infinity
or cosmological horizon
(Another singularity in the ODE)

Energy dissipation modes of the perturbations around the blackhole geometry.

$$\omega = \omega' - i\omega''$$

$$\psi(t) = \psi(0)e^{-\omega'' t} \cos \omega' t \, .$$

Quantization condition

B-period quantization = QNM boundary condition [Aminov, Grassi, Hatsuda (2020)]

$$\Pi_B^{(3)}\left(-\ell(\ell+1)+8M^2\omega^2-\frac{1}{4},\mathbf{m},-16M\mathrm{i}\omega,1\right)=2\,\pi\left(n+\frac{1}{2}\right),\quad n=0,1,\ldots,$$

$$\mathbf{m}=\{s-2\mathrm{i}M\omega,-s-2\mathrm{i}M\omega,-2\mathrm{i}M\omega\}.$$

where

$$\Pi_B^{(N_f)}(E,\mathbf{m},\Lambda_{N_f},\hbar)=\partial_a\mathcal{F}^{(N_f)}(a,\mathbf{m},\Lambda_{N_f},\hbar)\Big|_{a=a(E,\mathbf{m},\Lambda_{N_f},\hbar)}.$$

$$\partial_a\mathcal{F}^{(N_f)}(a;\mathbf{m};\Lambda_{N_f},\hbar)=-\,2\,a\,(4-N_f)\log\left[\frac{\Lambda_{N_f}2^{-\frac{1}{(2-N_f/2)}}}{\hbar}\right]-\pi\hbar-2\,\mathrm{i}\,\hbar\log\left[\frac{\Gamma\left(1+\frac{2\mathrm{i}a}{\hbar}\right)}{\Gamma\left(1-\frac{2\mathrm{i}a}{\hbar}\right)}\right]$$
$$-\,\mathrm{i}\,\hbar\sum_{j=1}^{N_f}\log\left[\frac{\Gamma\left(\frac{1}{2}+\frac{m_j-\mathrm{i}a}{\hbar}\right)}{\Gamma\left(\frac{1}{2}+\frac{m_j+\mathrm{i}a}{\hbar}\right)}\right]+\frac{\partial\mathcal{F}_{\text{inst}}^{(N_f)}(a;\mathbf{m};\Lambda_{N_f},\hbar)}{\partial a}.$$

← instanton part

Computational accuracy for Schwarzschild blackhole

with my laptop @5-instanton (+low-order Pade)

Numerical results w/ QNMspectral package [Jansen (2017)]

For scalar QNM, n=0, l=0

0.222016 – 0.209263 i vs Num $0.22090988 - 0.20979143i$

For scalar QNM, n=0, l=1

0.496038 – 0.186285 i vs Num $0.496527 - 0.184975i$

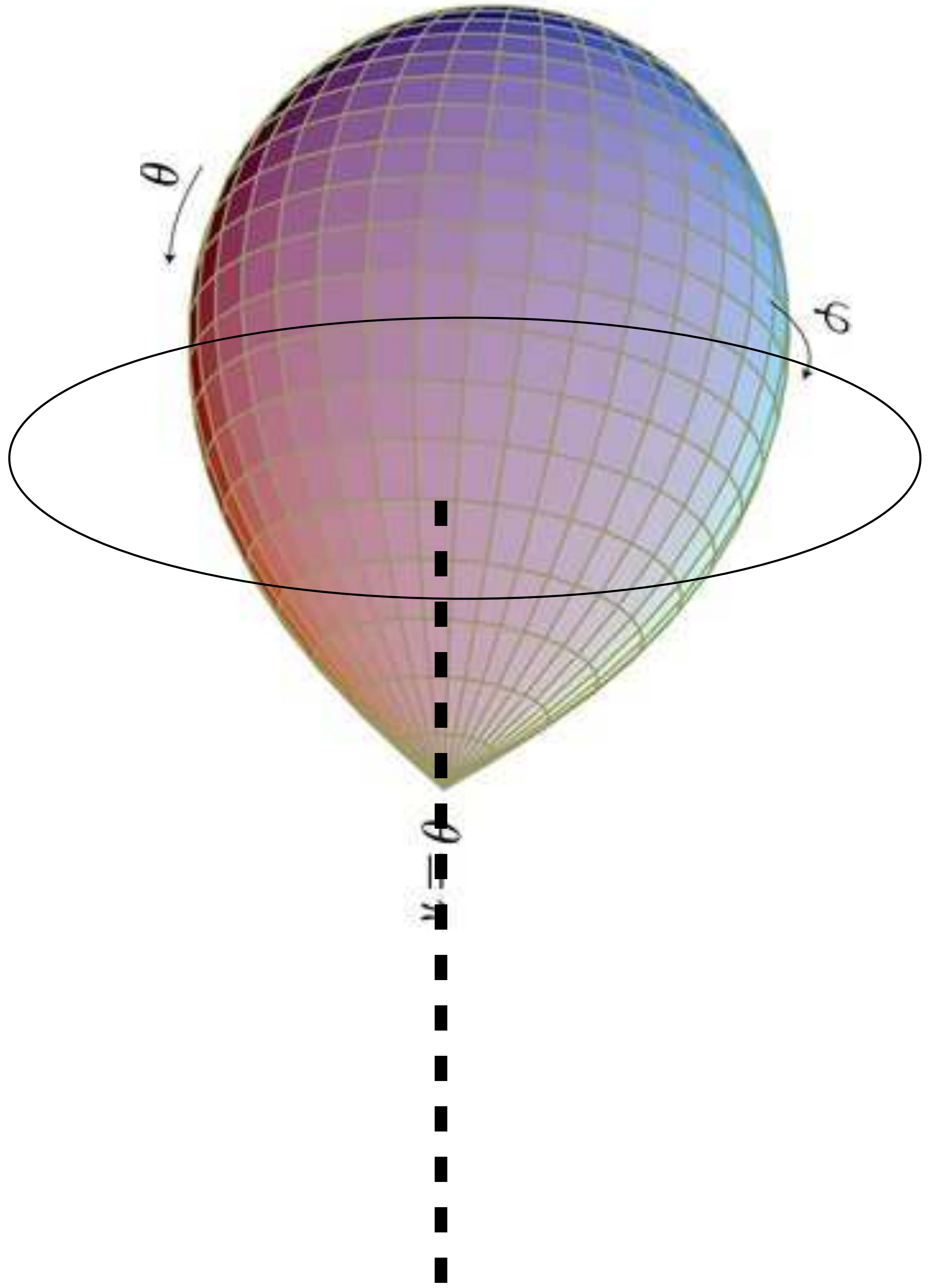
For scalar QNM, n=0, l=2

0.741044 – 0.0208661 i vs Num $0.7473 - 0.1779i$

For larger n and l, the convergence is getting worse.

Quasinormal modes

c-metric



Cosmic string

[Griffiths, Krtous, Podolsky (2006)]

$$\mathrm{d}s^2 = \frac{1}{(1-\alpha r\cos\theta)^2}\left(-f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + \frac{r^2\mathrm{d}\theta^2}{P(\theta)} + P(\theta)r^2\sin^2\theta\mathrm{d}\varphi^2\right),$$

where

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)(1-\alpha^2 r^2),$$
$$P(\theta) = 1 - 2\alpha M\cos\theta + \alpha^2 Q^2\cos^2\theta.$$

Radial part of Klein-Gordon equation: four regular singularities

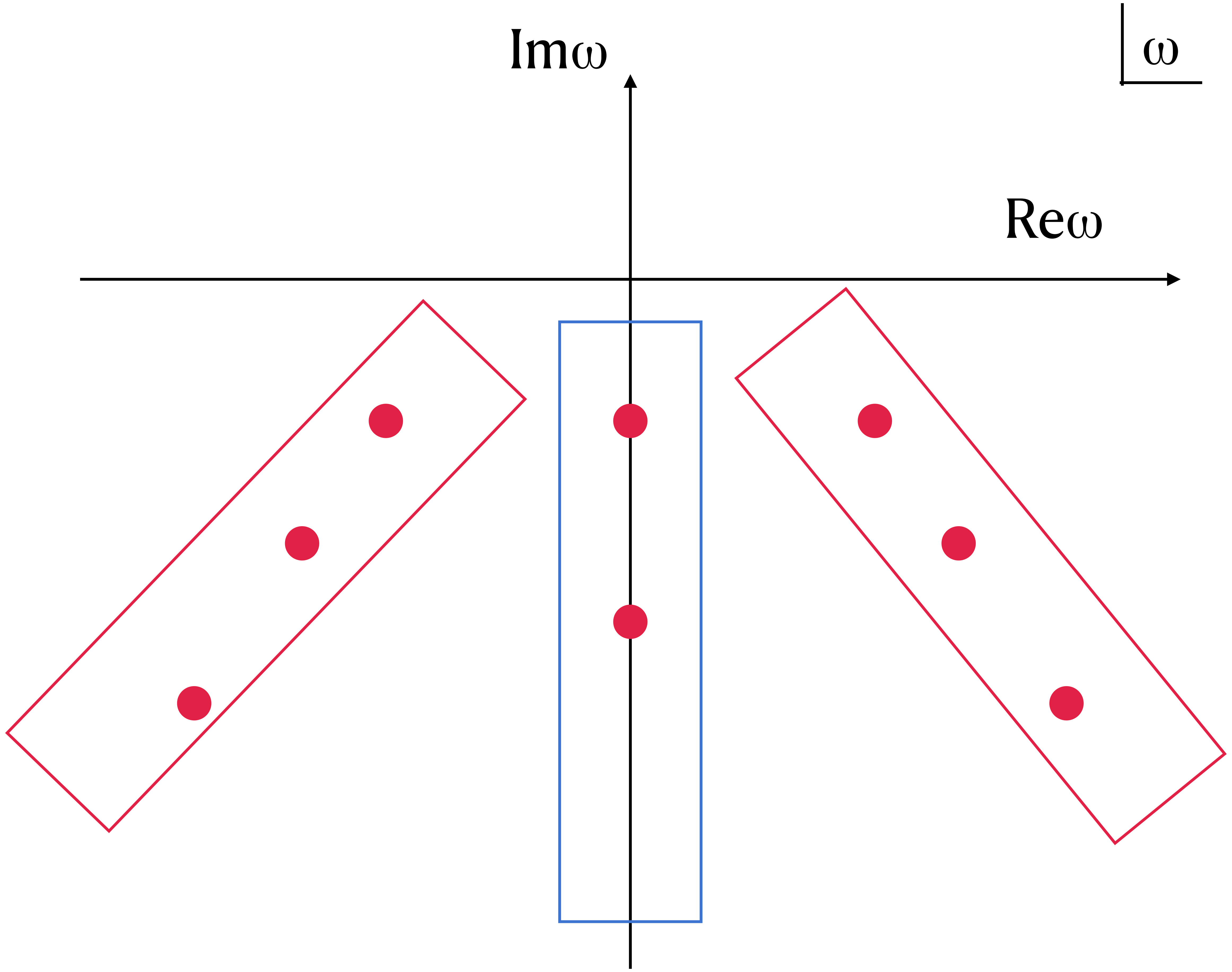
←4d SU(2) gauge theory with $N_f = 4$

Angular part of Klein-Gordon equation: five regular singularities

←4d SU(2) quiver gauge theory (superconformal)

Quasinormal modes in c-metric

[Destounis, Fontana, Mena (2020)]



Several families of QNMs exist

1) photon sphere modes

2) accelerating modes

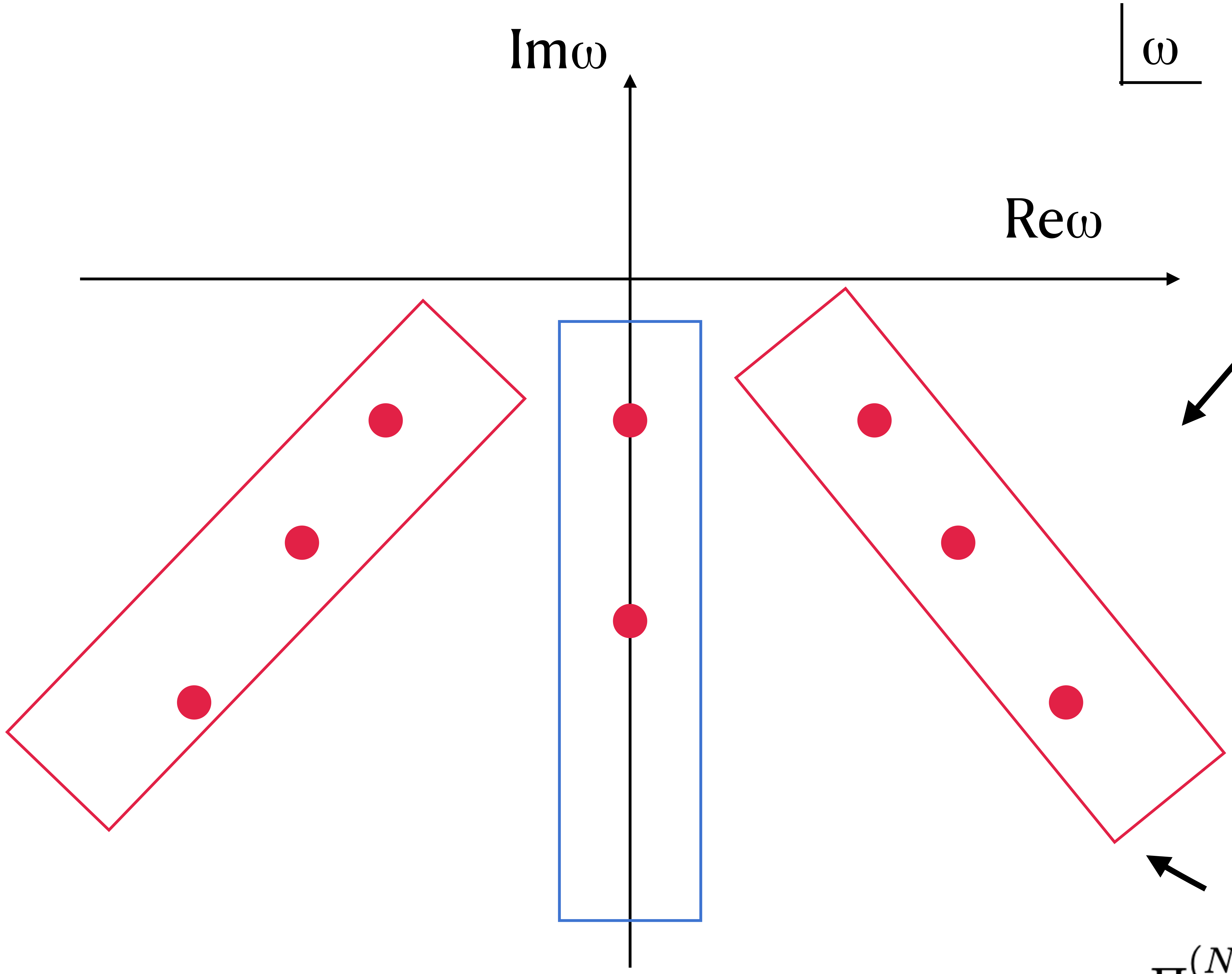
3) near extremal modes

Only become very important in the case of large BH charge Q

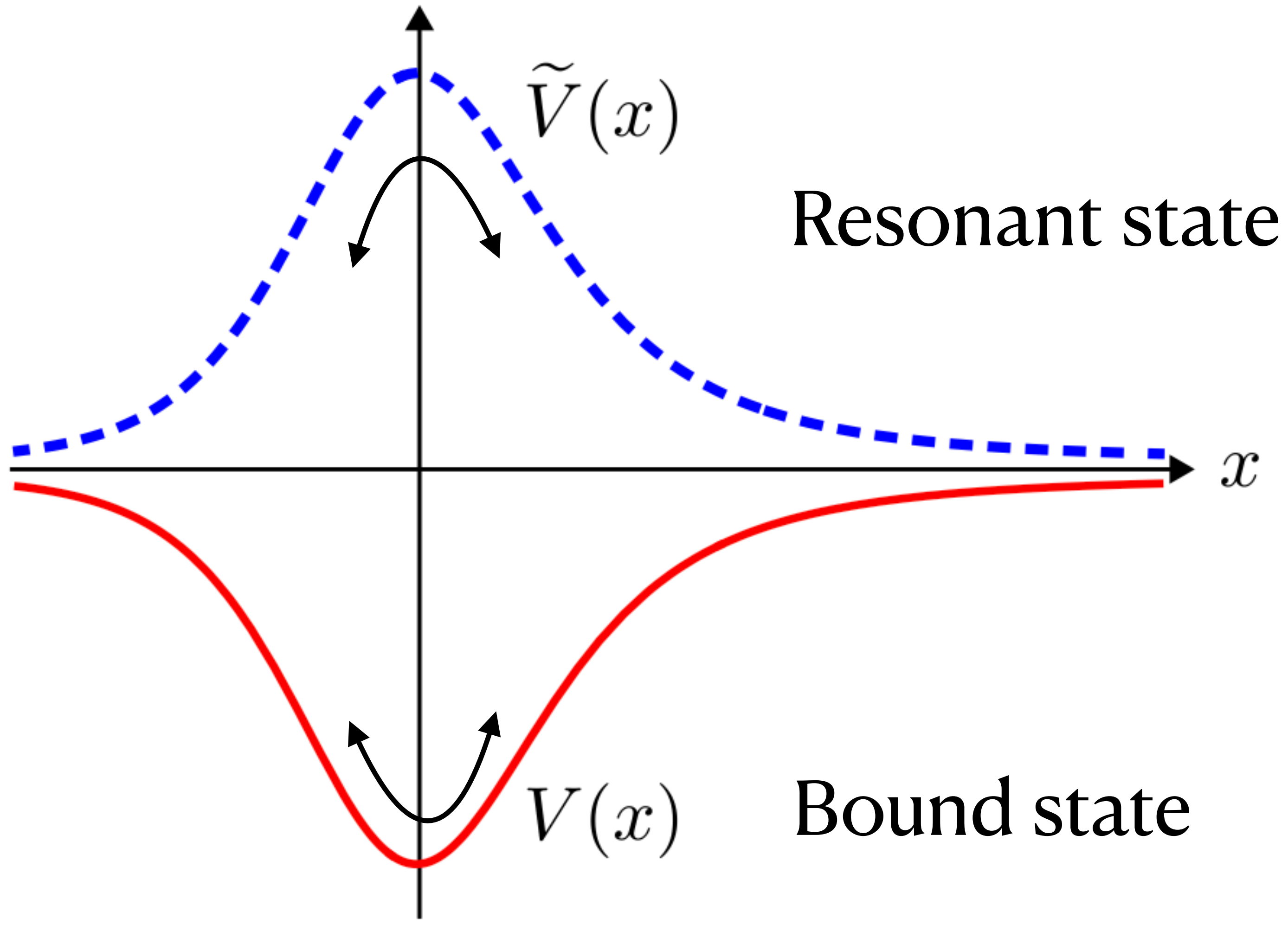
How to reproduce several families from one quantization condition?

Direct computation

[Lei, Shu, Zhang, RZ (2023)]



Bender-Wu approach/uniform WKB



Harmonic oscillator approximation

Instanton approach

$$\Pi_B^{(N_f)}(E, \mathbf{m}, \Lambda_{N_f}, \hbar) = \partial_a \mathcal{F}^{(N_f)}(a, \mathbf{m}, \Lambda_{N_f}, \hbar)\Big|_{a=a(E,\mathbf{m},\Lambda_{N_f},\hbar)}.$$

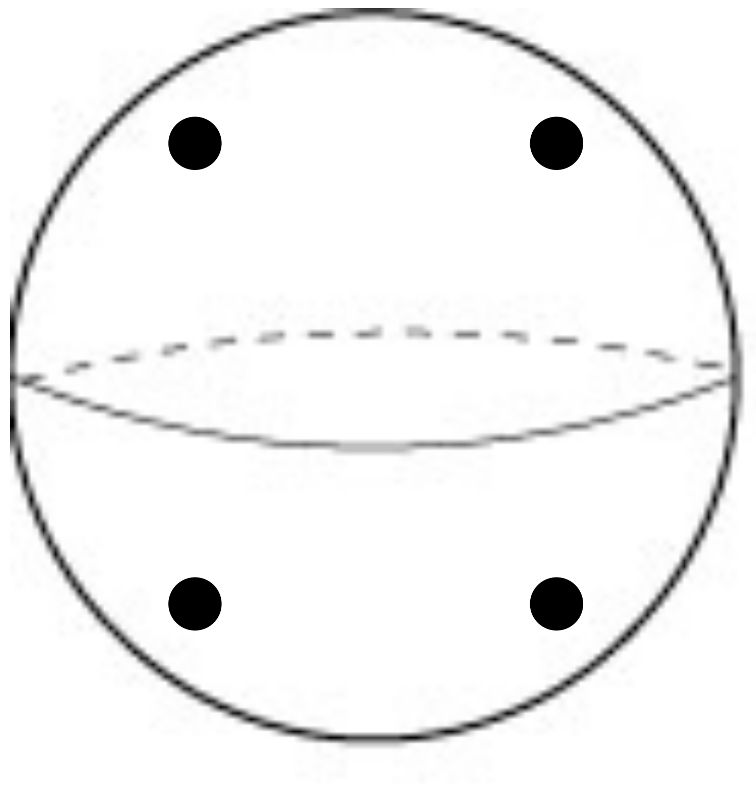
Message: correct “partially”, but tricky and needs to be improved!

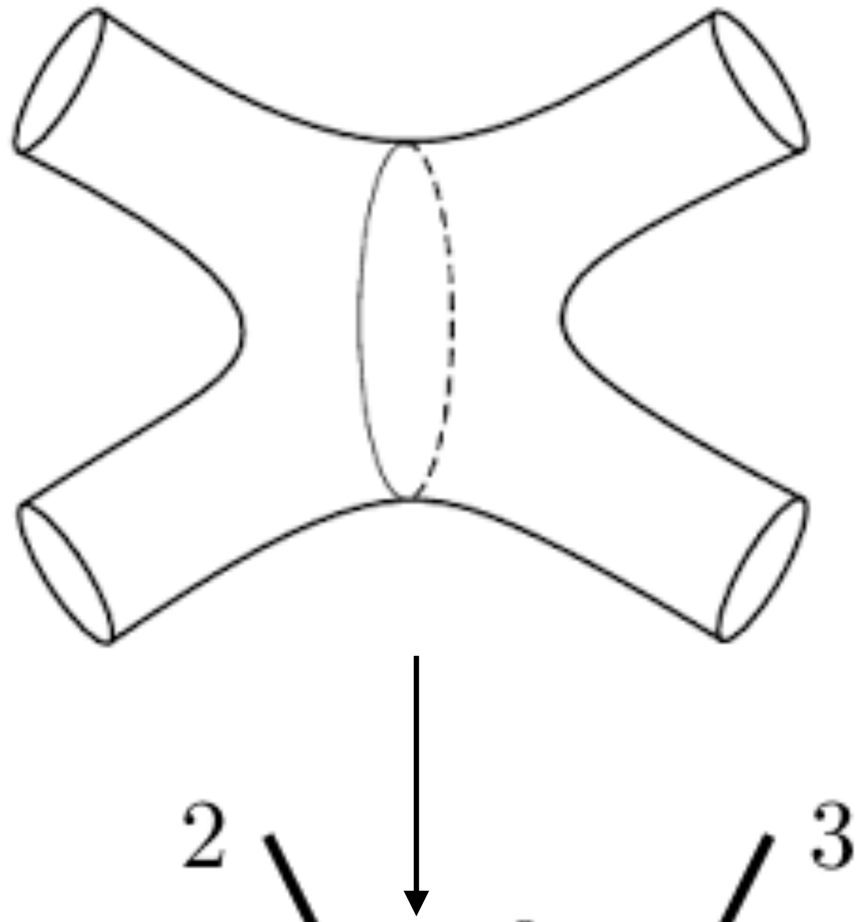
Alday-Gaiotto-Tachikawa relation

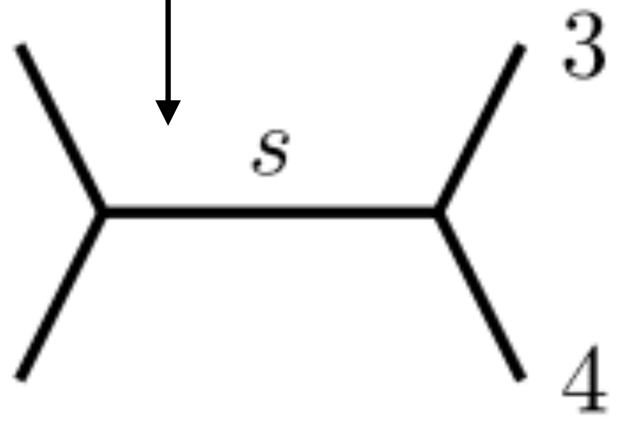
[Alday, Gaiotto,Tachikawa (2009)]

[Willard (2009)]

4d N=2 SU(2) supersymmetric gauge theory | 2d CFT

- $\Sigma =$  = (sphere with 4 punctures)



$$\sum_{s \in S} C_{12s} C_{s34}$$ 

4d N=2 Nf=4 SU(2) theory | puncture → primary operator

Nekrasov partition function = 4-pt correlation function

(In the NS limit $b \to 0$, the BPZ satisfied by the conformal block reduces to the Heun equation)

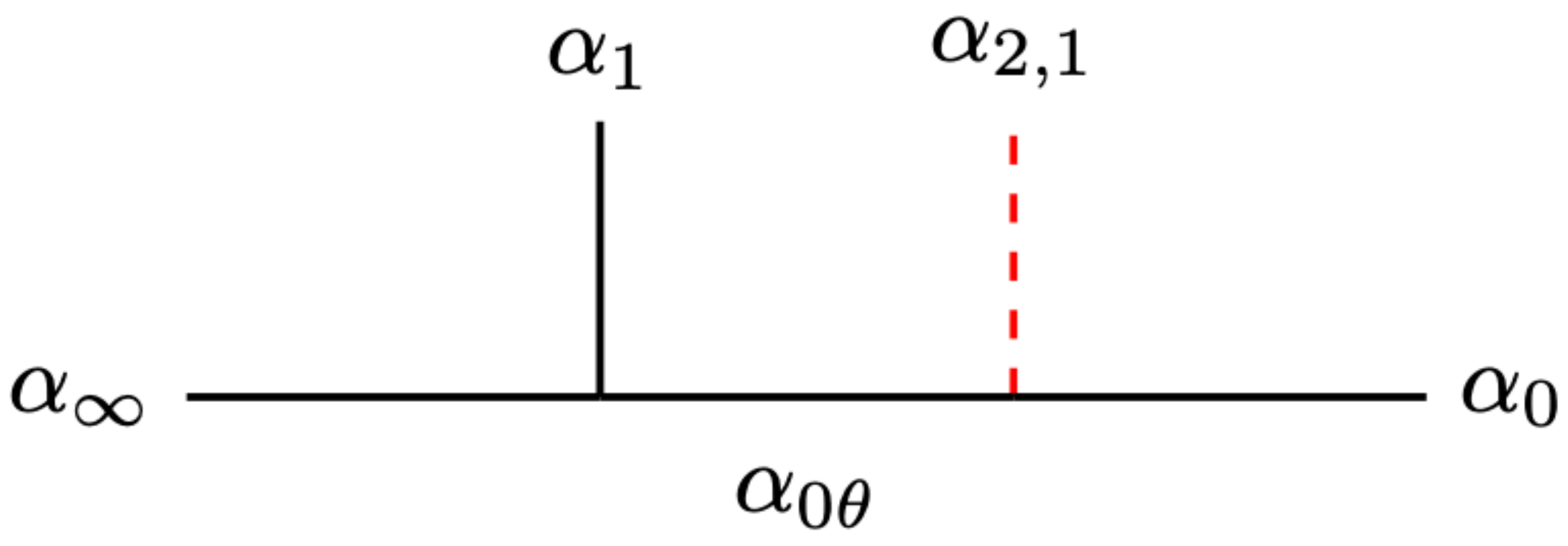
Connection formula from 2d CFT

Example: Hypergeometric function

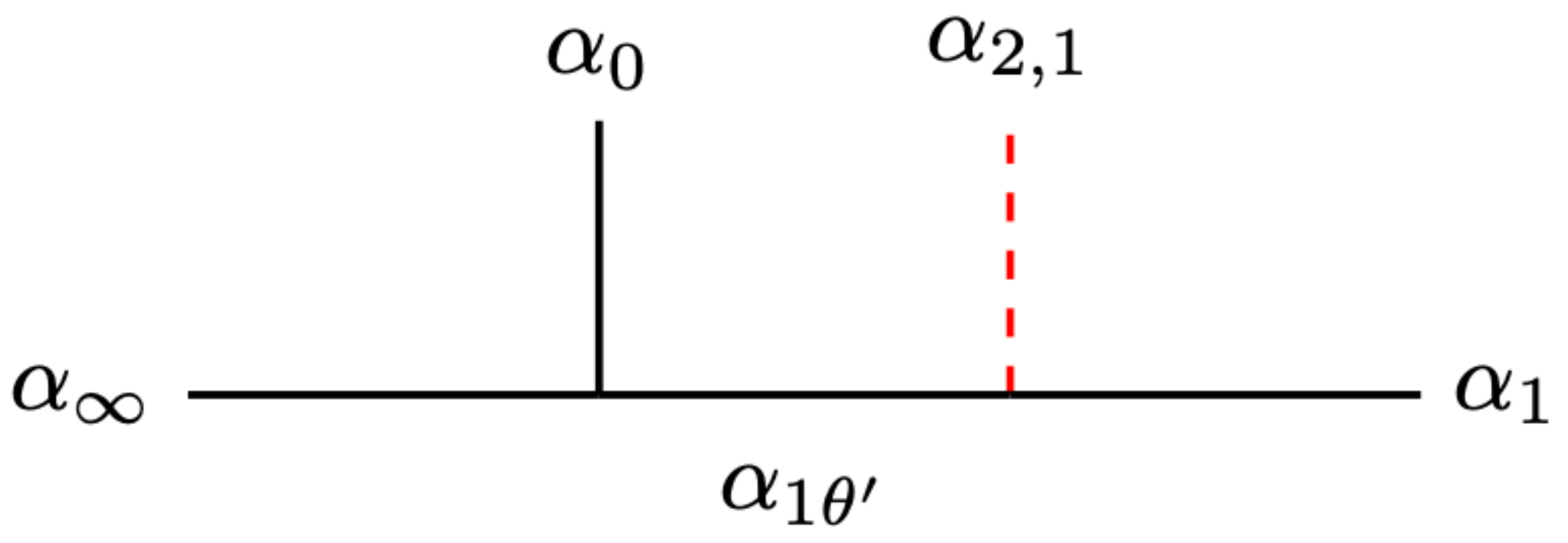
$$
{}_2F_1(a,b,c;y) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b,1+a+b-c;1-y) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-y)^{c-a-b} {}_2F_1(c-a,c-b,1+c-a-b;1-y).
$$

Solutions around y~0 re-expanded in terms of solutions around y~1.

Translated to CFT language,



$$= \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}$$

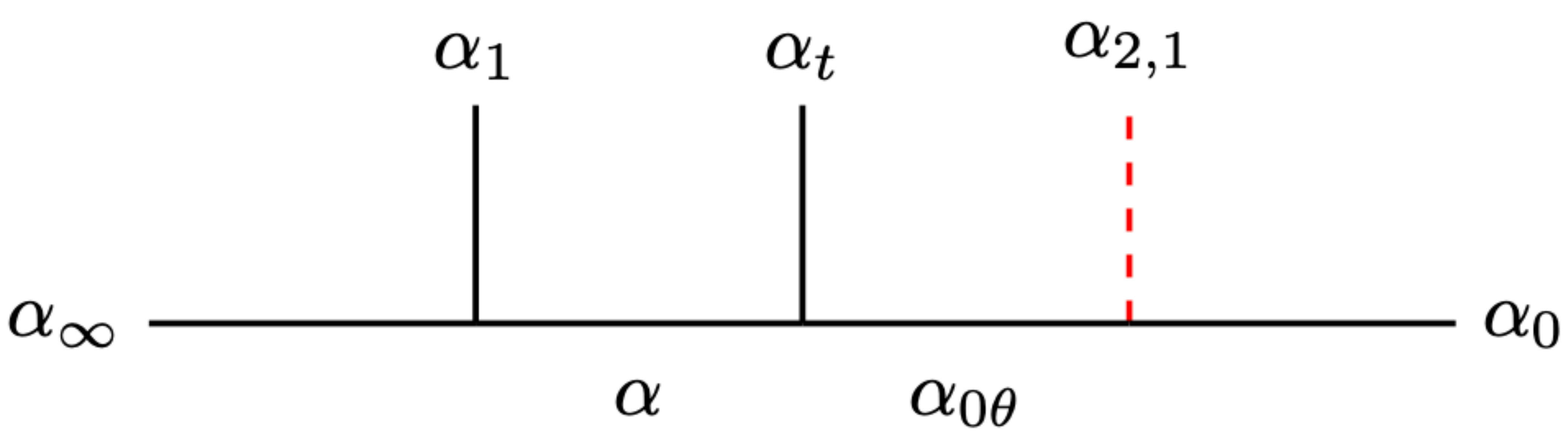


.

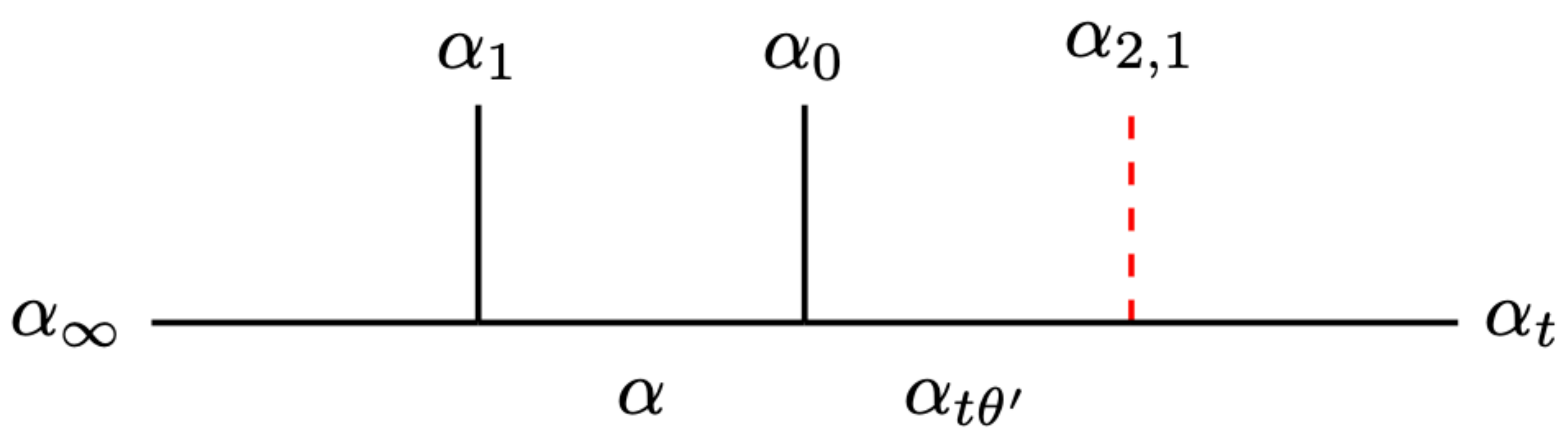
$$
\mathcal{M}_{\theta\theta'}(b\alpha_0, b\alpha_1; b\alpha_\infty) = \frac{\Gamma(-2\theta' b\alpha_1)\Gamma(1+2\theta b\alpha_0)}{\Gamma\left(\frac{1}{2}+\theta b\alpha_0 - \theta' b\alpha_1 + b\alpha_\infty\right)\Gamma\left(\frac{1}{2}+\theta b\alpha_0 - \theta' b\alpha_1 - b\alpha_\infty\right)}
$$

This can be easily generalized to higher-pt conformal blocks

c.f. [Bonelli, Iossa, Lichtig, Tanzini (2022)]



$$= \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}$$



.

More precisely

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t \\ \end{matrix}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix};t,\frac{z}{t}\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0,b\alpha;b\alpha_t)\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_{2,1} \\ \end{matrix}\,\alpha_{\theta'}\,\begin{matrix}\alpha_t \\ \alpha_0\end{matrix};z,\frac{t}{z}\right).$$

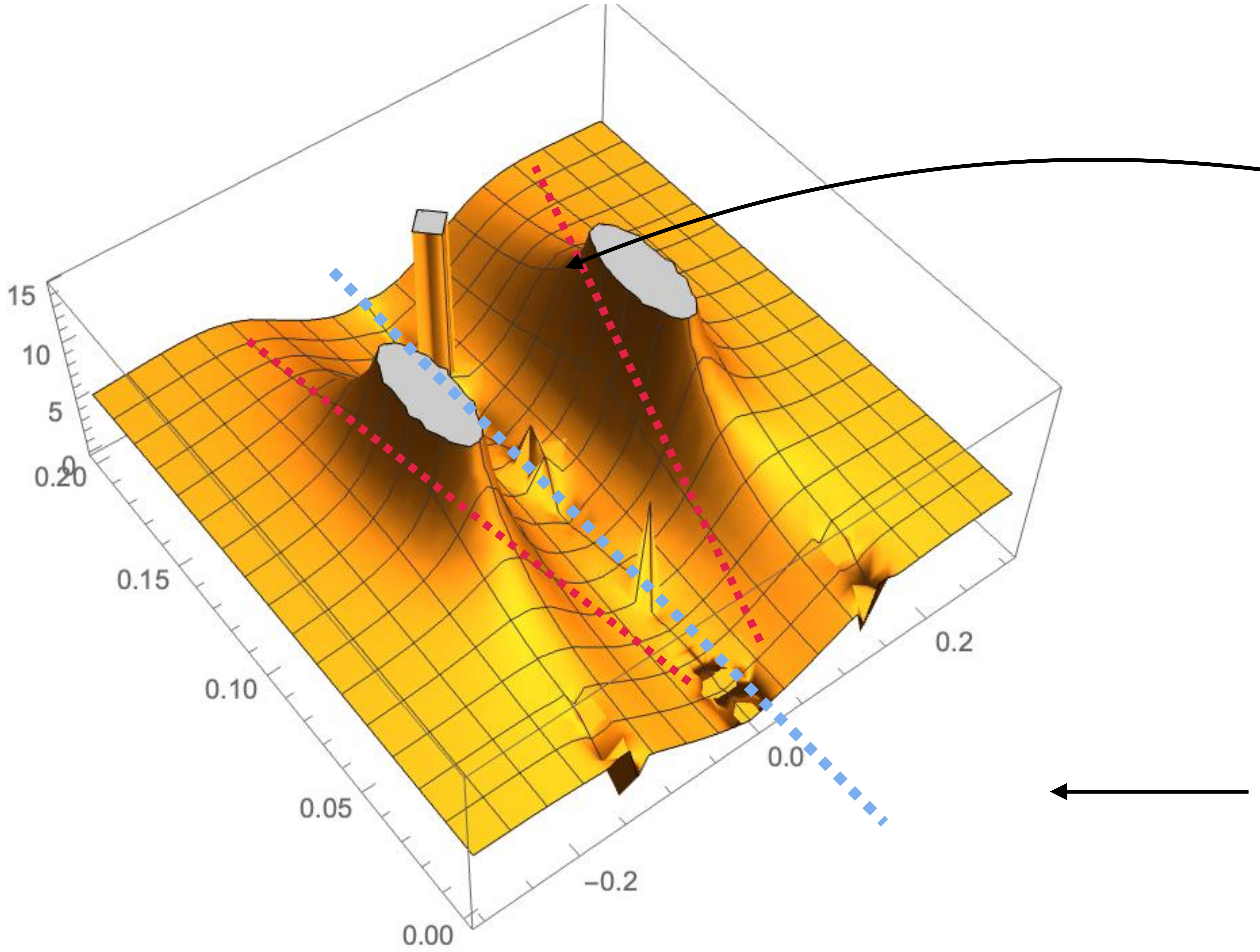
Some other connection formula:

$$\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}\,a\,\begin{matrix}a_t \\ \end{matrix}\,a_{0\theta}\,\begin{matrix}a_{2,1} \\ a_0\end{matrix};t,\frac{z}{t}\right) =$$

$$= \sum_{\theta'}\left(\sum_{\sigma}\mathcal{M}_{\theta\sigma}(a_0,a;a_t)\mathcal{M}_{(-\sigma)\theta'}(a,a_\infty;a_1)t^{-\sigma a}e^{-\frac{\sigma}{2}\partial_a F}\right)t^{-\frac{1}{2}}z\mathcal{F}\left(\begin{matrix}a_t \\ a_0\end{matrix}\,a\,\begin{matrix}a_1 \\ \end{matrix}\,a_{\infty\theta'}\,\begin{matrix}a_{2,1} \\ a_\infty\end{matrix};t,\frac{1}{z}\right).$$

Correct formula

$$\sum_{\sigma} \frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2a_0)\Gamma(2a_\infty)t^{-\sigma a}e^{-\frac{\sigma}{2}\partial_a F}}{\Gamma(\frac{1}{2}+a_0-\sigma a+a_t)\Gamma(\frac{1}{2}+a_0-\sigma a-a_t)\Gamma(\frac{1}{2}-\sigma a+a_\infty+a_1)\Gamma(\frac{1}{2}-\sigma a+a_\infty-a_1)} = 0.$$



Photon sphere modes

accelerating modes

[Lei, Shu, Zhang, RZ (2023)]

Numerical tests

[Lei, Shu, Zhang, RZ (2023)]

QNMspectral

	$\alpha M = 0.05$ $Q/M = 0.3$
$\ell = 0$ $m_0 = 0$	$\lambda = 0.3317$ $\omega_{\mathrm{PS}_1} = 0.111 - 0.104i$ $\omega_{\alpha_1} = -0.0506i$ $\omega_{\alpha_2} = -0.103i$

$\alpha M = 0.3$ $Q/M = 0.999$
$\lambda = 0.303261$
$\omega_{\mathrm{PS}_1} = 0.11168 - 0.08142i$
$\omega_{\mathrm{NE}_1} = -0.0412i$
$\omega_{\mathrm{NE}_2} = -0.084i$

Connection formula @3-instanton (without Pade)

	$\alpha M = 0.05$ $Q/M = 0.3$
t	0.174209

→ 0. 111416-0. 10266i

→ -0. 0505002i

→ -0. 10341i

$\alpha M = 0.3$ $Q/M = 0.999$
0.0572592

→ 0. 111516-0. 0839528i

→ -0. 0412224i

→ -0. 0837135i

Analytic results

[Lei, Shu, Zhang, RZ (2023)]

Leading order at $\alpha \to 0$, matching with that guessed from numerical appraoch

- Accelerating modes

$$\omega_{\alpha_n} = -i\alpha\left(\frac{1}{2} + n + \sqrt{\lambda - \frac{1}{12}}\right) + \cdots$$

- Near-extremal modes

$$\omega_{\mathrm{NE}_n} = -\frac{i\sqrt{M-Q}\,(1-\alpha^2Q^2)\left(1+2n+\sqrt{\frac{4\lambda}{1-\alpha^2Q^2} - \frac{1}{3}}\right)}{\sqrt{2}Q^{3/2}} + \cdots$$

$$= -\frac{i\sqrt{1-Q/M}\,(1-\alpha^2M^2)\left(1+2n+\sqrt{\frac{4\lambda}{1-\alpha^2M^2} - \frac{1}{3}}\right)}{\sqrt{2}M} + \cdots$$

$$\sum_\sigma \frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2a_0)\Gamma(2a_\infty)t^{-\sigma a}e^{-\frac{\sigma}{2}\partial_a F}}{\Gamma(\frac{1}{2}+a_0-\sigma a+a_t)\Gamma(\frac{1}{2}+a_0-\sigma a-a_t)\Gamma(\frac{1}{2}-\sigma a+a_\infty+a_1)\Gamma(\frac{1}{2}-\sigma a+a_\infty-a_1)} = 0.$$

NE Accelerating

What was the problem in Aminov-Grassi-Hatsuda's original proposal?

Rewriting the connection formula:

$$\exp\left(i\partial_a \mathcal{F}^{N_f=4}\right) = -1.$$

Taking log of the connection formula, we recover the B-cycle quantization condition,

$$\Pi_B = \partial_a \mathcal{F}^{N_f=4} = \pi(2n+1)$$

- Losing zeroes from the perturbative part

$$\sum_\sigma \frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2a_0)\Gamma(2a_\infty)t^{-\sigma a}e^{-\frac{\sigma}{2}\partial_a F}}{\Gamma(\frac{1}{2}+a_0-\sigma a+a_t)\Gamma(\frac{1}{2}+a_0-\sigma a-a_t)\Gamma(\frac{1}{2}-\sigma a+a_\infty+a_1)\Gamma(\frac{1}{2}-\sigma a+a_\infty-a_1)} = 0.$$

diverging

Open problem: how to recover it in the language of exact WKB/resurgence theory?

dS blackhole: similar structure

Numerical results in dS BHs

	3-instanton	Numerical data	t
$\Lambda M^2 = 0.06$ $l = 1$	$\omega_{\mathrm{PS}_1} = 0.175119 - 0.0663035i$ $\omega_{\mathrm{PS}_2} = 0.156801 - 0.198668i$ $\omega_{\mathrm{dS}_1} = -0.29994i$	$0.170891 - 0.063809i$ $0.158918 - 0.193747i$ $-0.288i$	0.476876
$\Lambda M^2 = 0.08$ $l = 1$	$\omega_{\mathrm{PS}_1} = 0.13476 - 0.0505651i$ $\omega_{\mathrm{PS}_2} = 0.128529 - 0.15182i$	$0.133914 - 0.050199i$ $0.128470 - 0.151125i$	0.367459
$\Lambda M^2 = 0.1$ $l = 1$	$\omega_{\mathrm{PS}_1} = 0.0803948 - 0.0302815i$ $\omega_{\mathrm{PS}_2} = 0.0792948 - 0.0908679i$ $\omega_{\mathrm{PS}_3} = 0.0770229 - 0.15155i$ $\omega_{\mathrm{dS}_1} = -0.440043i$	$0.0803459 - 0.0302718i$ $0.0792632 - 0.0908461i$ $0.077009 - 0.151528i$ $-0.44i$	0.223449

[Lei, Shu, Zhang, RZ (2023)]

Summary

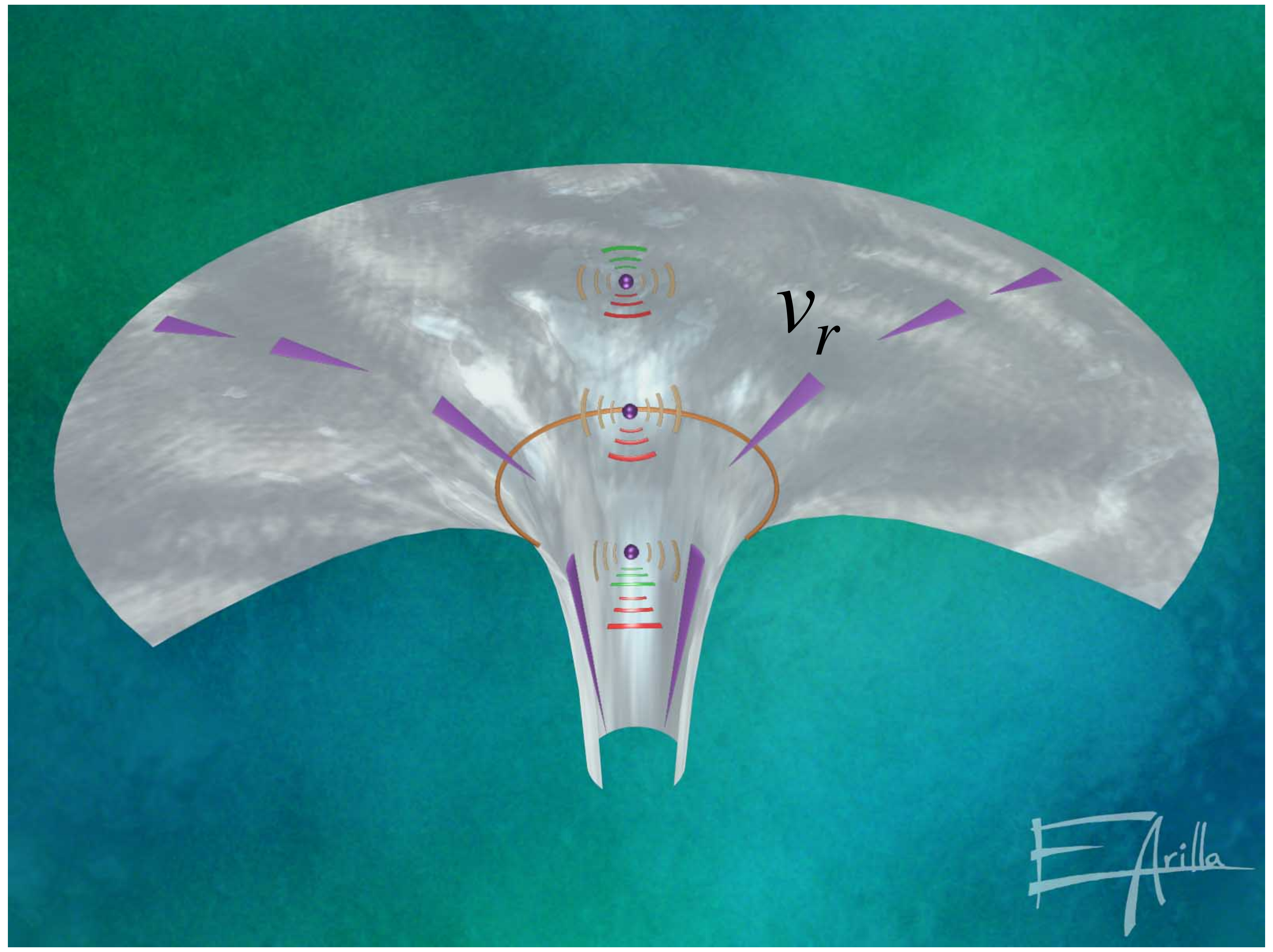
- Instanton partition function + quantized Seiberg-Witten curve
  + connection formula of conformal blocks
  combined together solve the Heun-type equation.
  (Semi-analytical approach, e.g. Taylor expansion over some parameter)

- All QNMs are found by the connection formula.

There are many many black-hole-like geometries!

- Sonic/acoustic black hole (to be compared with experiments?)

$$f = 1 - \left(\frac{v_r^2}{c_s^2}\right)$$



by Enrique Arilla

But often encounters irregular singularity and the BH is dual to Argyres-Douglas-like theories (without Lagrangian description).

How to take the colliding limit of singularities in a controlled way?

→ Connection formula + numerical experiments?

- Lifshitz brane to Lifshitz geometry

Four regular singularities to 1 regular+1 rank-4 irregular

[WIP with Yang Lei, Hao Zhao]

Backups

More to be done:

Angular part

[Lei, Shu, Zhang, RZ (2023)]

[WIP with Pujun Liu]

$$\frac{d^2\chi(\theta)}{dz^2} - (m^2 - V_\theta)\chi(\theta) = 0,$$

where

$$\mathrm{d}z = \frac{\mathrm{d}\theta}{P(\theta)\sin\theta},$$

$$V_\theta = P(\theta)\left(\lambda\sin^2\theta - \frac{P(\theta)\sin^2\theta}{3} + \frac{\sin\theta\cos\theta P'(\theta)}{2} + \frac{\sin^2\theta P''(\theta)}{6}\right).$$

Equation with 5 regular singularities

$$Q_{SW}(z) = \frac{\alpha_5}{z^2} + \frac{\beta_5}{(z-t)^2} + \frac{\gamma_5}{(z-q)^2} + \frac{\delta_5}{(z-1)^2} + \frac{\eta_5}{z(z-1)} + \frac{\kappa_5}{z(z-t)} + \frac{\mu_5}{z(z-q)}.$$

can be solved with SU(2)xSU(2) (quiver gauge theory) partition function or 5-pt conformal block.

Results get worse at large t:

$$\alpha M = 0.5$$
$$Q/M = 0.8$$
$$\lambda = 0.194511$$
$$\omega_{\mathrm{PS}_1} = 0.03945 - 0.04122i$$
$$\omega_{\mathrm{PS}_2} = 0.03876 - 0.1237i$$

	3-instanton	5-instanton
$t = 0.83333$		
	0.0338046−0.033456i	0.0347785−0.0364286i
	0.0572973−0.0986153i	0.0514567−0.107756i

$t = 0.167777$

3-instanton

0.0394654−0.0412316i
0.0387606−0.123697i

Dictionary 2

$$z = \frac{1+r_-\alpha}{1+r_+\alpha}\frac{r-r_+}{r-r_-},$$

with

$$t = \frac{(1-\alpha r_+)(1+\alpha r_-)}{(1+\alpha r_+)(1-\alpha r_-)},$$

$$r = r_+ \leftrightarrow z = 0, \quad r = \frac{1}{\alpha} \leftrightarrow z = t.$$

[Lei, Shu, Zhang, RZ (2023)]

dS blackhole: similar structure

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2),$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{L^2}, \quad \Lambda = \frac{3}{L^2}.$$

For spin $s = 1,2$, $Q = 0$ only four singularities as solutions to

$$x^4 - \tilde{L}^2x^2 + \tilde{L}^2x - Q'^2\tilde{L}^2 = 0,$$

where

$$Q' = \frac{Q}{2M}, \qquad \tilde{L} := \frac{L}{2M}.$$

More than one type of modes exist: PS modes, dS modes, near-extremal modes.

In other cases, more complicated singularity structure! 5 or more singularities!

[WIP with Pujun Liu]