Thermal Entropy in Calabi-Yau Quantum Mechanics

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- Entropy has played a significant role in recent developments in our understandings of holography and quantum gravity. Since the seminal proposal of holographic entanglement entropy in Ryu and Takayanagi, 2006, there have been many generalizations and applications.

- In particular, a generalized version of the entropy may follow the Page curve of the black hole entropy during the Hawking radiation process, see Zhenbin Yang, Masamichi Miyaji's talks. These significant progress appear to be getting close toward a consensus on the resolution of the famous black hole information paradox.

- The thermal mixed state in a quantum system can be obtained as the reduced state by tracing over one party in a thermofield double (TFD) state, which is a pure quantum state entangled between two similar systems. The von Neumann entropy of the thermal state is the entanglement entropy of the TFD state.

- The use of TFD states has been instrumental in our understandings of holography with two boundaries, as well as in the study of von Neumann algebra. (See e.g. some of Maldacena, Witten's papers)

- Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ denote the eigenvalues of a Hamiltonian. For a thermal state, the probability at an excited state is $p_i = \frac{1}{Z} e^{-\beta \lambda_i}$, where $\beta \equiv \frac{1}{k_B T}$ is the inverse temperature and $Z = \sum_i e^{-\beta \lambda_i}$ is the partition function. The von Neumann entropy can be written as

$$S = -\sum_{i=1}^{\infty} p_i \log(p_i) = \log(Z) + \frac{\beta}{Z} \sum_i \lambda_i e^{-\beta \lambda_i} = \log(Z) + T \partial_T \log(Z). \quad (1)$$

- For example, as a warm-up exercise, for the simple case of a harmonic oscillator $\widehat{H} = \frac{\widehat{p}^2}{2} + \frac{\omega^2 \widehat{x}^2}{2}$, it is not difficult compute the von Neumann entropy of a thermal state exactly (set $\hbar = 1$)

$$S = \frac{\beta \omega}{e^{\beta \omega} - 1} - \log(1 - e^{-\beta \omega}). \quad (2)$$

- It is well known that one can couple two harmonic oscillators together with a quadratic interaction, then the ground state of the combined system is a TFD state Srednicki:1993.

- We will be interested in the high temperature limit where the probability is more evenly distributed between all excited states, so the entropy should be maximized. For the harmonic oscillator

$$S \sim \log(T), \qquad T \sim \infty. \tag{3}$$

- More generally, we will use the number of states defined by

$$N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j < \lambda\}. \tag{4}$$

- The partition function can be computed by

$$Z = \int_0^\infty e^{-\beta\lambda} dN(\lambda). \tag{5}$$

To compute the asymptotic behavior of the entropy, we only need the asymptotic behavior of $N(\lambda)$. For the harmonic oscillator, the number of states goes like $N(\lambda) \sim \lambda$, so we have $Z \sim T$ in the high temperature limit and the first term in (1) dominates. We recover the result (3).

Calabi-Yau Quantum Mechanics

- We consider quantum systems derived from toric Calabi-Yau geometries, where the Hamiltonians are exponential functions of the position and momentum operators, e.g.

$$\begin{aligned} \widehat{H} &= e^{\widehat{x}} + e^{-\widehat{x}} + e^{\widehat{p}} + e^{-\widehat{p}}, && \mathbb{P}^1 \times \mathbb{P}^1 \text{ model}, \\ \widehat{H} &= e^{\widehat{x}} + e^{\widehat{p}} + e^{-\widehat{x}-\widehat{p}}, && \mathbb{P}^2 \text{ model}. \end{aligned} \tag{6}$$

- The perturbative quantization conditions are given by the Nekrasov-Shatashvili limit of refined topological string theory.

- Furthermore, based on some works on numerical calculations of the quantum spectrum Kallen and Marino, arXiv:1308.6485; MH and Wang, arXiv:1406.6178, the exact quantization conditions including all non-perturbative effects were conjectured in Grassi, Hatsuda, and Marino, arXiv:1410.3382; X. Wang, Zhang, and MH, arXiv:1505.05360, where the two seemingly different proposals turned out to be related by the blowup equations Grassi and J. Gu, arXiv:1609.05914.

- Now known as the TS/ST (Topological String/Spectral Theory) correspondence, it has attracted the attentions of many mathematicians as well, with some promising results toward a proof of the conjecture.

- Some nice mathematical results on the asymptotics of the energy eigenvalues were proven in Laptev, Schimmer, and Takhtajan, 2016, 2019.

- The the number of states grow like

$$N(\lambda) \sim \log^2(\lambda), \qquad \lambda \sim \infty \tag{7}$$

  where we have neglected the coefficient factor which depends on specific models.

- We see that the logarithmic growth is much slower than the linear growth of the harmonic oscillator. As a result, the inverse Hamiltonian $\widehat{H}^{-1}$ is a trace class operator for the Calabi-Yau models, while it is not for the harmonic oscillator, by checking the divergence of $\int^{\infty} \frac{dN(\lambda)}{\lambda}$.

- In the high temperature limit, the partition function is

$$Z \sim \int_1^\infty e^{-\beta\lambda} \frac{\log(\lambda)}{\lambda} d\lambda. \tag{8}$$

This integral can be represented by the Meijer G-function. Using its series expansion as well as some alternative elementary calculations, we find the leading asymptotic behavior

$$Z \sim \log^2(T), \qquad T \sim \infty. \tag{9}$$

- So as in the harmonic oscillator case, the first term in (1) dominates and we have

$$S \sim \log(Z) \sim \log(\log(T)), \qquad T \sim \infty. \tag{10}$$

So the entropy grow much slower than the harmonic oscillator in the high temperature limit.

Another Class of Models

- We consider a different class of models with polynomial potentials
$$\widehat{H} = e^{\widehat{p}} + e^{-\widehat{p}} + W(\widehat{x}), \tag{11}$$
where $W(x) = x^{2N} + \cdots$ is chosen to be an even degree $2N$ polynomial, so that we have a confining potential and an infinite discrete spectrum.

- The mathematical result for the asymptotics of the number of states
$$N(\lambda) \sim \lambda^{\frac{1}{2N}} \log(\lambda), \qquad \lambda \sim \infty. \tag{12}$$
The growth is faster than the Calabi-Yau models in the previous section, but slower than the standard harmonic oscillator.

- Similarly, we find the asymptotic behavior of the partition function $Z \sim T^{\frac{1}{2N}} \log(T)$ and the asymptotic entropy
$$S \sim \log(Z) \sim \log(T), \qquad T \sim \infty. \tag{13}$$
So the scaling behavior is actually the same as the standard harmonic oscillator.

No Finite Bound for Entropy

- We consider whether it is possible to have a finite upper bound for thermal entropy in the infinite temperature limit $T \to \infty$ for some quantum systems.

- We assume that the Hilbert space is infinite dimensional where the energy eigenvalues can be shifted to be all positive.
  Further assume the sum in the partition function is convergent for any finite temperature, so the partition function $Z(T)$ is well defined for any T, i.e. there is no exponential growth of the number of states $N(\lambda)$ as in string theory.

- It is easy to check by taking derivative that both $Z(T)$ and $S(T)$ are monotonically increasing functions of T

$$S(T) = \log(Z) + T\partial_T \log(Z) > \log(Z). \tag{14}$$

  Since the Hilbert space is infinite dimensional, the partition function $Z(T)$ tends to infinity as $T \to \infty$, therefore the entropy also tends to infinity and there is no finite upper bound.

Some Discussions

- Although we focus on simple quantum systems, our study may provide some useful experience for relevant questions in quantum gravity.

- In the context of the influential Swampland Program Vafa:2005, some recent works have studied the species scale, the emergence string proposal and their thermodynamics. In certain limit e.g. near boundary of the moduli space, a tower of infinite number of string states may become light.

- Another way to explore such emergence is the high temperature limit considered here, where the highly excited states become equally probable. Of course, it is well known that string theory has a Hagedorn temperature, inverse proportional to the string length, where the partition function diverges. In those contexts, the high temperature limit should probably mean a temperature approaching the Hagedorn temperature.

- A folklore of quantum gravity is the finiteness of entropy, in contrast to its divergence in generic calculations in quantum field theory.

- Of course, the dimension of Hilbert space is infinite in perturbative string theory but this is not necessarily in conflict with the folklore. In a countable (separable) infinite dimensional Hilbert space, the von Neumann entropy of a mixed state of trace class would be generically still finite except in some very contrived circumstances.

- For example, for a probability distribution that scales as power law $p_n \sim n^{-\alpha}$ among an orthogonal basis of states, the convergence of the sum $\sum_{n=1}^{\infty} p_n$ is equivalent to $\alpha > 1$, in which case the von Neumann entropy $-\sum_{n=1}^{\infty} p_n \log(p_n)$ is also finite.

- We may consider a more contrived probability distribution $p_n \sim \frac{1}{n \log^{\alpha}(n)}$. For $\alpha \leq 1$ the sum $\sum_{n=1}^{\infty} p_n$ is divergent, while for $\alpha > 2$ both sums $\sum_{n=1}^{\infty} p_n$ and $-\sum_{n=1}^{\infty} p_n \log(p_n)$ are convergent. So in this case in a limited range $1 < \alpha \leq 2$ we can have a probability distribution where the entropy is infinite.

- An important source of motivation comes from the finite horizon area of de Sitter space, which appears to be the current state of our universe. (cosmological horizon area $\sim$ entropy, Gibbons and Hawking, 1977.) Some discussions of dark energy.

- Consider a 4d de Sitter space with a cosmological constant $\Lambda > 0$. Some physical quantities (set $\hbar = c = 1$)
  The horizon size $r \sim \Lambda^{-\frac{1}{2}}$,
  The vacuum energy density $\rho_{vac} \sim m_p^2 \Lambda$,
  where $m_p \sim l_p^{-1}$ is the Planck mass.

- Astrophysical observations suggest

$$\frac{\rho_{vac}}{m_p^4} \sim \frac{\Lambda}{m_p^2} \sim \frac{l_p^2}{r^2} \sim 10^{-120} \tag{15}$$

- The famous cosmological constant problem "why it is so small" can be rephrased as:
  Why the entropy of our universe is so big $S \sim \frac{r^2}{l_p^2} \sim 10^{120}$?

- Adding matters in a de Sitter space decreases the horizon area, so the empty de Sitter is a state of maximal entropy.
  E. Witten, arXiv:2308.03663: "... it is natural to think of the Hartle-Hawking no boundary state as a universal state of maximum entropy, ...", "... it is something like the infinite temperature limit of the thermofield double state."

- In the quantum models we studied, the entropy is indeed finite at a finite temperature, but tends to infinity in the infinite temperature limit.

- Although we are not successful in calculating the entropy of our universe, it is useful to learn some lessons from these models and pose the question for future research.

- Is it possible to get a finite maximal entropy in a similar natural limit in some other situations?

- A similar situation appeared in our earlier study of the entropy of Berenstein-Maldacena-Nastase (BMN) strings Huang:2019. In that case, the pp-wave spacetime background is infinitely curved, the strings become effectively infinitely long and tensionless with degenerate spectra, so the Hagedorn temperature is zero.

- Instead, a real non-negative genus counting parameter in the dual BMN double scaling limit becomes the effective string coupling g, playing a similar role of temperature as in the current context.

- Due to the structure of dual free CFT correlators, we are only accessing a countable infinite dimensional subspace of the whole Hilbert space of excited string states.

- It was found that at finite coupling g, the entropy is indeed also finite, while it is naively expected that as $g \to \infty$, the probability would be evenly distributed among the infinite dimensional Hilbert subspace, so the entropy should likely tend to infinity, which by itself does not seem to violate any fundamental principle of quantum gravity.

- Nevertheless, it would be a pleasant surprise if it turns out that the entropy of BMN strings does have a finite upper bound as $g \to \infty$, strongly confirming a folklore of quantum gravity in an implausible fashion. (Previously I proved $S < (2+\epsilon)\log g$.)
  Such a bound may be related to the entropy of our current universe $S \sim 10^{120}$, thus could provide a natural estimate of the cosmological constant.

- An encouraging hint is that for the Calabi-Yau models, which are related to topological string theory, a toy version of quantum gravity, the entropy does grow much slower (10) than the conventional models. It would be interesting to settle this issue in the future.

Thank You