An Observable in Classical Pure AdS_3 Gravity: the Twist along a Geodesic

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based on an on-going work with Xiao-shuai Wang

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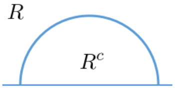
- In this work, we study a diffeomorphism invariant observable in the classical pure AdS_3, namely the twist along a geodesic
- The motivation to study the twist is that the twist only supports on the geodesic so may be a candidate element in the center
- We study the properties of the twist in the Hamiltonian formalism, and we get the following two results:
  - First, we get the system's evolution generated by the twist, which, under a proper gauge choice, exhibits a relative shift along the geodesic
  - Second, we show that the twist commutes with the length of the same geodesic, which supports our proposal that the twist is a candidate element in the center

Motivation: the Hilbert space of gravity

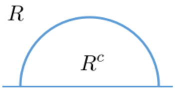
- We first introduce the motivation
- The construction of the Hilbert space of gravity is a long-standing problem, but is far from fully solved even in perturbative level
- One difficulty in solving this problem is from the gravitational constraints, because of which, the Hilbert space of the system is not tensor factorizable

- For example, in the right figure, we divide the whole system into two subsystems R and R^c; while, the Hilbert space of the whole system $\mathcal{H}$ cannot be written as a tensor product of the Hilbert spaces of the individual subsystems as $\mathcal{H} \neq \mathcal{H}_R \otimes \mathcal{H}_{R^c}$



The Hilbert space structure for a system with constraints

- We believe that the Hilbert space of gravity should be very similar as the ones of the gauge theories Casini, Huerta, Rosabal, 1312.1183
    - Here, we first introduce the notion of the **center**, which is formed by a set of commutative operators supporting on the interface
    - Based on the center, the Hilbert space can be decomposed into a set of sub-Hilbert spaces, each of which is tensor factorizable
    - In particular, we represent the Hilbert space as $\mathcal{H} = \oplus_a \mathcal{H}_{R,a} \otimes \mathcal{H}_{R^c,a}$



The importance of the center

- Clearly, the center plays an important role in the previous Hilbert space decomposition
- However, in gravity, our knowledge for the center is very limited
- The only known element in the center is the HRT-area

The HRT-area as an element in the center

- The HRT-area A appears in the RT formula and the JLMS formula as
  - RT formula: $S_{EE} = \langle \frac{A}{4G} \rangle + S_{EE,g}$

    Ryu, Takayanagi, hep-th/0603001

    Ryu, Takayanagi, hep-th/0605073

    Hubeny, Rangamaini, Takayanagi, 0705.0016
  - JLMS formula: $K = \frac{A}{4G} + K_g$

    Jafferis, Lewkowycz, Maldacena, Suh, 1512.06431
- It is in the JLMS formula that the HRT-area A is treated as an operator in gravity and furthermore viewed as an element in the center

  Jafferis, Lewkowycz, Maldacena, Suh, 1512.06431

  Harlow, 1607.03901

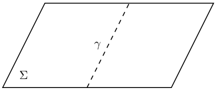
The kink transformation generated by the HRT-area

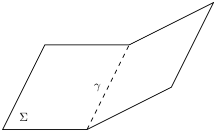
- Further studies have also been performed on the HRT-area in the aspect of viewing it as an operator of gravity
- One interesting result is that the system's evolution generated by the HRT-area A exhibits a kink transformation along the HRT-surface as illustrated in the figures below

Bousso, Chandrasekaran, Shahbazi-Moghaddam, 1906.05299

Bousso, Chandrasekaran, Rath, Shahbazi-Moghaddam, 2007.00230

Kaplan, Marolf, 2203.04270





Other elements in the center

- Besides the HRT-area, we expect there should be some other operators in the center. And these operators should also support on the HRT-surface
- Moreover, assuming that we can indeed construct some operators in the center, we would like to study their properties in the following aspects:
  - First, inspired from the story of the HRT-area, we would like to study the system's evolutions generated by these operators
  - Second, to verify that these operators are really elements in the center, we would like to compute the commutators both between these operators themselves and between these operators with the HRT-area

The classical pure AdS_3 gravity

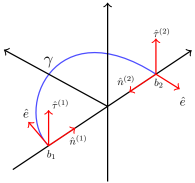
- As a preliminary attempt, we focus on a simple model in this work: the classical pure AdS_3 gravity
- In this model,
  - extremal surfaces reduce to geodesics
  - operators' commutators reduce to observables' brackets
- In particular, we only consider one observable that supports on a geodesic: **the twist along the geodesic**
- And we would like to study its properties, including its generated evolution and the brackets with it

The definition twist along a geodesic

- We now provide the definition of the twist along a geodesic

  Castro, Detournay, Iqbal, Perlmutter, 1405.2792

- Here, we need to introduce two normal frames to the geodesic, denoted by $(\hat{\tau}^{(1)}, \hat{n}^{(1)})$, $(\hat{\tau}^{(2)}, \hat{n}^{(2)})$, which are constructed in the following two steps:
  - First, at each geodesic endpoint, we construct a normal frame by inheriting the frame of the asymptotic boundary
  - Second, by a parallel transport, we extend the definition of the two normal frames to the whole geodesic



- We point out that the two normal frames are different by a relative boost
  $\hat{\tau}^{(2)} = \cosh\zeta\hat{\tau}^{(1)} - \sinh\zeta\hat{n}^{(1)}$
  $\hat{n}^{(2)} = -\sinh\zeta\hat{\tau}^{(1)} + \cosh\zeta\hat{n}^{(1)}$
  where the rapidity ζ is a constant along the geodesic
- We then define this constant rapidity ζ as the twist along the geodesic

The system's evolution generated by the twist

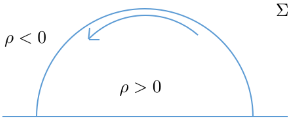
- Having provided the definition of the twist ζ, we now study its properties
- We start from studying the system's evolution generated by the twist ζ
- By saying "the system's evolution generated by the twist ζ", we mean that we view the twist ζ as the Hamiltonian and use it to evolve the system
- Moreover, since the classical system is completely captured by the set of initial data (σ_{ab}, K_{ab}) on a given Cauchy surface, we can represent the system's evolution as the evolution of the set of initial data by the following Poisson equation
  $\Delta\sigma_{ab}(t_0, x) = \lambda\{\sigma_{ab}(t_0, x), \zeta\} + o(\lambda)$
  $\Delta K_{ab}(t_0, x) = \lambda\{K_{ab}(t_0, x), \zeta\} + o(\lambda)$
  Here, λ is an infinitesimal evolution parameter. And we see that the key point is to compute the brackets of the initial data with the twist ζ

The computation of the brackets

- By definition, the twist ζ is a functional of the metric, which in turn can be viewed as a functional of the set of initial data (σ_{ab}, K_{ab})
- Here, we directly provide the expression of the variation of the twist ζ with respect to the variation of the set of initial data as $\delta\zeta = \int ds\big(- e^a n^b \delta K_{ab} + \frac{1}{2} e^m n^n K_{mn} n^a n^b \delta\sigma_{ab}\big)$
- By applying the previous expression to the chain rule $\{\cdot, \zeta\} = \int d^2x \{\cdot, \sigma_{ab}(t_0, x)\} \frac{\delta\zeta}{\delta\sigma_{ab}(t_0,x)} + \{\cdot, K_{ab}(t_0, x)\} \frac{\delta\zeta}{\delta K_{ab}(t_0,x)}$ and also by taking use of the Poisson bracket
  $\{\sigma_{ab}(t, x), \sigma_{cd}(t, y)\} = 0$
  $\{\sigma_{ab}(t, x), K_{cd}(t, y)\} = 8\pi G(\sigma_{ac}\sigma_{bd} + \sigma_{ad}\sigma_{bd} - 2\sigma_{ab}\sigma_{cd}) \frac{1}{\sqrt{\sigma}} \delta^2(x - y)$
  $\{K_{ab}(t, x), K_{cd}(t, y)\} = 8\pi G(\sigma_{ab}K_{cd} - K_{ab}\sigma_{cd}) \frac{1}{\sqrt{\sigma}} \delta^2(x - y)$,
  we can compute the following two brackets
  $\{\sigma_{ab}, \zeta\} = -8\pi G\lambda\delta(\rho)(e_a n_b + n_a e_b)$
  $\{K_{ab}, \zeta\} = -8\pi G K_{mn} e^m n^n \delta(\rho) n_a n_b$

The relative shift along the geodesic

- And we then get the system's evolution generated by the twist ζ as
  $\Delta\sigma_{ab} = -8\pi G\lambda\delta(\rho)(e_a n_b + n_a e_b) + o(\lambda)$
  $\Delta K_{ab} = -8\pi G K_{mn} e^m n^n \delta(\rho) n_a n_b + o(\lambda)$
  which is precisely what we want
- From the first equation, we know that the system's evolution is precisely **a relative shift along the geodesic** for the two sides of the geodesic as the figure illustrated below

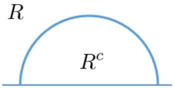


The bracket between the length and the twist

- So far we have studied the system's evolution generated by the twist ζ, and we now switch to the second topic to compute the brackets with the twist ζ
- In particular, we only compute the bracket between the geodesic's length and the twist $\{A, \zeta\}$
- Remember that our original motivation to study the twist ζ is that it may be a candidate element in the center
- But being an element in the center requires the twist ζ at least to be commutative with the HRT-area (which is the geodesic length A in this setup)

- The computation of the bracket $\{A, \zeta\}$ is similar as the previous one: we view both the geodesic length A and the twist ζ as functionals of the set of initial data and compute their bracket by applying the chain rule

- In the first sight, we may expect a divergent result in the bracket $\{A, \zeta\}$, since both the geodesic length A and the twist ζ are defined on the geodesic
$\delta A = \int ds \frac{1}{2} e^a e^b \delta\sigma_{ab}$
$\delta\zeta = \int ds \big(- e^a n^b \delta K_{ab} + \frac{1}{2} e^m n^n K_{mn} n^a n^b \delta\sigma_{ab} \big)$



- But with a careful computation, we show that the bracket indeed vanishes as
  $\{A, \eta\} = \int d^2x 0 \cdot \delta(\rho)^2 = 0$
  Here, the $\delta(\rho)^2$ reflects the fact that both of the two observables are defined on the geodesic. And we believe that the 0 beats the $\delta(\rho)^2$ and leads to a zero in the integral
- Accepting this computation, we then show that the twist ζ indeed commutes with the geodesic's length so can be a candidate element in the center

Summary

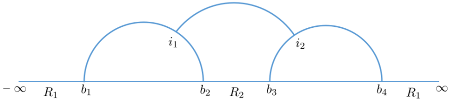
- So far, we have introduced all of our results
- We now provide a short summary:
  - First, we get the system's evolution generated by the twist, which exhibits a relative shift along the geodesic
  - Second, we show that the twist commutes with the length of the same geodesic, which supports our proposal that the twist is a candidate element in the center

The relation to the original motivation

- Remember that our original motivation is to construct the Hilbert space of gravity
- While, the study of this work and also the previous study on the HRT-area should be viewed as some exploratory attempts for this problem
- Generally speaking, the approach that we want to take here is to first study the properties of the operators and then use these properties to construct the Hilbert space
- And to pursue further along this approach, we need to construct more operators and study their properties

Operators defined on a geodesic network

- Partially inspired from the tensor network model of holography, we would like to focus on the operators defined on a geodesic network, for example the one illustrated in the figure below



- Given a geodesic network like this, we can define diffeomorphism invariant operators by measuring the length and the twist along each geodesic segment
- As what we did in this work, we can furthermore study the properties of these operators, including their generated evolutions and their brackets

The Hilbert space of gravity

- Holding these operators, we may try to study the Hilbert space of gravity in some level
  - Among these operators, we may have a set of operators which commute with each other. We view these operators as the center and use them to take Hilbert space decomposition
  - We may also have sets of operators, where the operators in each set form a subalgebra and are commutative with the operators in the center or in other sets. We use the representations of these subalgebras to construct the Hilbert space
- The final goal of this approach is to generalize the discussion to a sufficient fine-grained geodesic network, which may be viewed as a lattice like gravity
- However, we have to admit that this proposal is too vague

Other open questions

- Besides the previous vague proposal, there are also many other interesting open questions
  - For example, we can try to generalize the discussion to systems with matter fields or in higher dimension
  - For example, we can also try to generalize the discussion to systems with finite cutoff which is suggested to correspond to the $T\bar{T}$ deformed CFT_2
  - For example, we may try to study the field theory dual of the twist

Thanks

Thanks for your attention!