Symmetry TFT for Subsystem Symmetry

Qiang Jia

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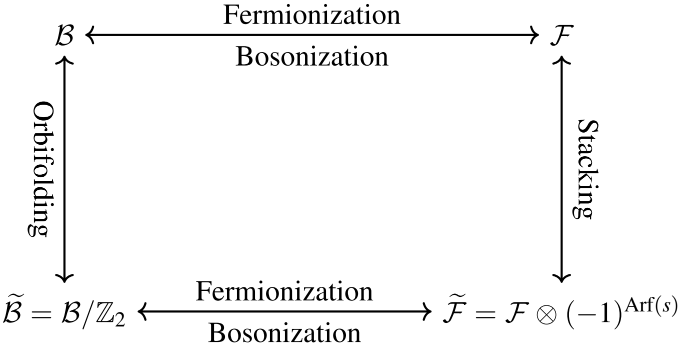
with Weiguang Cao (IPMU)

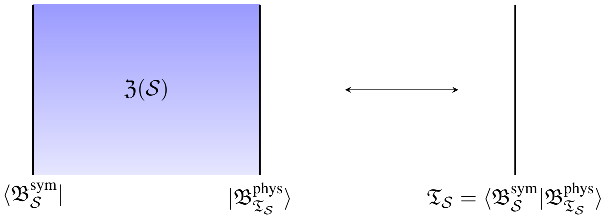
Symmetries and Dualities

- Symmetries often serve as guiding principles for theoretical explorations
- Given a (generalized) symmetry, there exist various symmetry operations such as gauging and stacking invertible phases onto a given system
- e.x. 2D theory with $\mathbb{Z}_2$ Symmetry has the following dualities



Symmetry TFT*

- A d-dim theory $\mathfrak{T}_S$ on $\mathcal{M}_d$ with finite symmetry S can be expanded as
  - A (d+1)-dim TFT $\mathfrak{T}_S$ on $[0,1] \times M_d$
  - A topological boundary $\mathfrak{B}_S^{\text{sym}}$. It encodes the symmetry S.
  - A dynamical boundary $\mathfrak{B}_{\mathfrak{T}_S}^{\text{phys}}$. It depends on the details of $\mathfrak{T}_S$
- Dualities in $\mathfrak{T}_S$ are interpreted as changing topological boundary $\mathfrak{B}_S^{\text{sym}}$ while fixing dynamical boundary $\mathfrak{B}_{\mathfrak{T}_S}^{\text{phys}}$.



*For a recent review, see Lakshya Bhardwaj, Sakura Schäfer-Nameki. 2023

Example: $\mathbb{Z}_2$ symmetry in 2D

- The SymTFT $\mathfrak{Z}_{\mathbb{Z}_2}$ is 3D BF theory with level 2

$$S_{\mathfrak{Z}_{\mathbb{Z}_2}} = \frac{2}{2\pi} \int_{[0,1]\times M_2} \hat{A} \wedge dA,$$

- For any 1-cycle, one can construct electric/magnetic line operators (anyons)

$$U[\Gamma] = \exp\left[i \oint_\Gamma A\right], \quad \hat{U}[\Gamma] = \exp\left[i \oint_\Gamma \hat{A}\right]$$

with $U[\Gamma]^2 = 1$, $\hat{U}[\Gamma]^2 = 1$ and Γ is a 1-cycle.

- They generate two copies of $\mathbb{Z}_2$ 1-form symmetries with a mixed 't Hooft anomaly.

Topological boundary state $\mathfrak{Z}_{\mathbb{Z}_2}$

- Assume $M_2 = T^2$ and $\gamma \in H^1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. The Wilson loops satisfy the following algebra when restricted on T^2,

$$U_\gamma \hat{U}_{\gamma'} = (-1)^{\int \gamma \wedge \gamma'} \hat{U}_{\gamma'} U_\gamma$$

  where $U_\gamma \equiv U[\Gamma_\gamma]$ with Γ_γ the Poincare dual of γ.

- The quantum algebra induce a Hilbert space. There exists three sets of maximally commuting set of operators

Operators	$\lvert \mathfrak{B}^{\text{sym}}_{\mathbb{Z}_2} \rangle$	$\mathbb{Z}_2$ generator
U_γ	$U_\gamma \lvert a \rangle = (-1)^{\int \gamma \wedge a} \lvert a \rangle$	$\hat{U}_\gamma \lvert a \rangle = \lvert a + \gamma \rangle$
$\hat{U}_\gamma$	$\hat{U}_\gamma \lvert \hat{a} \rangle = (-1)^{\int \gamma \wedge \hat{a}} \lvert \tilde{a} \rangle$	$U_\gamma \lvert \hat{a} \rangle = \lvert \hat{a} + \gamma \rangle$
$U_{F,\gamma} \equiv U_\gamma \hat{U}_\gamma$	$U_{F,\gamma} \lvert s \rangle = (-1)^{\text{Arf}(s+\gamma)} \lvert s \rangle$	$U_\gamma \lvert s \rangle = \lvert s + \gamma \rangle$

where $\lvert a \rangle = \lvert a_1, a_2 \rangle, \lvert \hat{a} \rangle = \lvert \hat{a}_1, \hat{a}_2 \rangle, \lvert s \rangle = \lvert s_1, s_2 \rangle$.

2D Dualities

- Topological boundary states $|\mathfrak{B}^{\mathrm{sym}}_{\mathbb{Z}_2}\rangle$: $|a\rangle$, $|\hat{a}\rangle$, $|s\rangle$ with,

$$\begin{cases} |\hat{a}\rangle = \frac{1}{2}\sum_{a\in H^1(T^2,\mathbb{Z}_2)}(-1)^{\int a\wedge\hat{a}}|a\rangle \\ |s\rangle = \frac{1}{2}\sum_{a\in H^1(T^2,\mathbb{Z}_2)}(-1)^{\mathrm{Arf}(s+a)}|a\rangle \end{cases}$$

- Given any 2D bosonic theory $\mathfrak{T}_{\mathbb{Z}_2}$ with $\mathbb{Z}_2$ symmetry, the dynamical boundary state $|\mathfrak{B}^{\mathrm{phys}}_{\mathfrak{T}_{\mathbb{Z}_2}}\rangle$ is constructed as,

$$|\mathfrak{B}^{\mathrm{phys}}_{\mathfrak{T}_{\mathbb{Z}_2}}\rangle = \frac{1}{2}\sum_{a_1,a_2\in\mathbb{Z}_2} Z[a_1,a_2]|a_1,a_2\rangle$$

- Changing topological boundary state implements the duality,
  - Original theory : $Z_{\mathfrak{T}_{\mathbb{Z}_2}}[a] = \langle a|\mathfrak{B}^{\mathrm{phys}}_{\mathfrak{T}_{\mathbb{Z}_2}}\rangle$
  - KW-transformation : $Z_{\hat{\mathfrak{T}}_{\mathbb{Z}_2}}[\hat{a}] = \langle \hat{a}|\mathfrak{B}^{\mathrm{phys}}_{\mathfrak{T}_{\mathbb{Z}_2}}\rangle$
  - JW-transformation : $Z_{\mathfrak{T}_{\mathbb{Z}_2,F}}[s] = \langle s|\mathfrak{B}^{\mathrm{phys}}_{\mathfrak{T}_{\mathbb{Z}_2}}\rangle$

Motivation of this work

- Symmetry TFT depends only on the Symmetry group G
- Possible generalization of the discrete group G

Symmetry	Codimension	Invertibility	Topologicalness
Ordinary	=1	Yes	Yes
Higher-form	>1	Yes	Yes
Non-invertible	=1	No	Yes
Subsystem	=1	Yes	Restricted

- Some examples of the corresponding SymTFT:
  - $\mathbb{Z}_N$ k-form symmetry : BF-theory $\frac{N}{2\pi}\int A_{k+1} \wedge dB_{D-k-2}$
  - Non-Invertible : Turaev-Viro theory†
  - ...
- However, the SymTFT for Subsystem symmetry is still absent and we wish to fill this gap

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†See Justin Kaidi, Kantaro Ohmori, Yunqin Zheng. 2022 for a physical construction.

Subsystem Symmetry

- Subsystem symmetries are related to fracton model
- Let's consider the (2+1)-lattice as a concrete model. The spatial lattice is $L_x \times L_y$ periodic lattice, and on each site there is a spin-1/2 state $|s\rangle_{i,j}$. Denote the Pauli matrices on each site as $X_{i,j}, Y_{i,j}, Z_{i,j}$ such that,

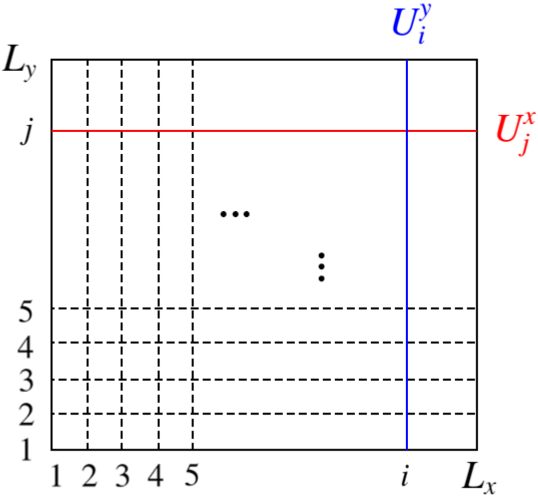
$$X_{i,j}|s\rangle_{i,j} = |-s\rangle_{i,j}, \quad Z_{i,j}|s\rangle_{i,j} = s|s\rangle_{i,j}.$$

- The generators of subsystem $\mathbb{Z}_2$ global symmetry are line operators acting on each row and column

$$U_j^x = \prod_{i=1}^{L_x} X_{i,j}, \quad U_i^y = \prod_{j=1}^{L_y} X_{i,j}, \quad (U_j^x)^2 = (U_i^y)^2 = 1.$$

- Only $L_x + L_y - 1$ operators are independent,

$$\prod_{j=1}^{L_y} U_j^x \prod_{i=1}^{L_x} U_i^y = 1.$$



U_i^y
L_y
j
U_j^x
5
4
3
2
1
1 2 3 4 5
i
L_x

Twist sectors

- If we put the subsystem $\mathbb{Z}_2$ line defects along the time direction, they change the boundary condition of each column and row,

$$|s_{i+L_x,j}\rangle = |(-1)^{t^x_j} s_{i,j}\rangle, \quad |s_{i,j+L_y}\rangle = |(-1)^{t^y_i} s_{i,j}\rangle$$

and also

$$|s_{i+L_x,i+L_y}\rangle = |(-1)^{t^{xy}+t^x_j+t^y_i} s_{i,j}\rangle,$$

with

$$t^y_{i+L_x} = t^y_i + t^{xy}, \quad t^x_{j+L_y} = t^x_j + t^{xy}.$$

- The Hamiltonian with subsystem $\mathbb{Z}_2$ symmetry depends only on the combinations $w^x_{j+\frac{1}{2}}, w^y_{i+\frac{1}{2}}$,

$$w^x_{j+\frac{1}{2}} \equiv t^x_j + t^x_{j+1}, \quad w^y_{i+\frac{1}{2}} \equiv t^y_i + t^y_{i+1}, \quad \sum_{j=1}^{L_y} w^y_{i+\frac{1}{2}} = \sum_{i=1}^{L_x} w^x_{j+\frac{1}{2}} = t^{xy}.$$

- There are also $L_x + L_y - 1$ of them.

SymTFT for (2+1)D subsystem symmetry

We want to explore the Symmetry TFT for subsystem symmetry, there are several questions we need to answer.

- What is the corresponding (3+1)D topological field theory?
- What are the gauge invariant operators
- What are the possible topological boundary states?
- How they implement the dualities

Symmetry:	Ordinary 2D $\mathbb{Z}_2$	(2+1)D subsystem
SymTFT:	3D level-2 BF theory	?
Operators:	Wilson loop $U(\Gamma)$ and $\hat{U}[\Gamma]$	?
Top. Boundary	Eigenstates of U or $\hat{U}$ or $U\hat{U}$	?
KW duality	$U \leftrightarrow \hat{U}$	?
JW duality	$U \leftrightarrow U\hat{U}$	?

SymTFT for (2+1)D subsystem symmetry

- We find the candidate for subsystem SymTFT of our interest is the $(3+1)$d 2-foliated BF theory with level 2[‡],

$$S_{\text{2-foliated}} = \frac{2}{2\pi}\int b\wedge dc + \sum_{k=1,2} dB^k\wedge C^k\wedge dx^k + \sum_{k=1,2} b\wedge C^k\wedge dx^k$$

and it is equivalent to the exotic tensor theory,

$$S_{\text{exotic}} = \frac{N}{2\pi}\int \Big[A^\tau(\partial_z\hat{A}^{xy} - \partial_x\partial_y\hat{A}^z) - A^z(\partial_\tau\hat{A}^{xy} - \partial_x\partial_y\hat{A}^\tau) - A^{xy}(\partial_\tau\hat{A}^z - \partial_z\hat{A}^\tau)\Big]$$

- In the exotic theory, there exists a naive $SL(2,\mathbb{Z}_N)$ symmetry

$$S: \quad A\to\hat{A}, \quad \hat{A}\to -A,$$
$$T: \quad A\to A, \quad \hat{A}\to\hat{A}+A.$$

where $A = (A_{xy}, A_z, A_\tau)$ and $\hat{A} = (\hat{A}_{xy}, \hat{A}_z, \hat{A}_\tau)$.

[‡]Kantaro Ohmori, Shutaro Shimamura. 2022

Operators and Algebras

- Assume the boundary $M_3 = T^2 \times S^1$ parametrized by (x, y, z), where z is the "time" direction. The gauge invariant operators restricting to M_3 are line/strip operators,

$$W(x,y) = \exp\left(i \oint dz A^z\right),$$
$$W(x_1,x_2) = \exp\left(i \int_{x_1}^{x_2} dx \oint dy A^{xy}\right),$$
$$W(y_1,y_2) = \exp\left(i \int_{y_1}^{y_2} dy \oint dx A^{xy}\right),$$

  for A and $\hat{W}(x,y), \hat{W}(x_1,x_2), \hat{W}(y_1,y_2)$ for $\hat{A}$. All $W, \hat{W}$ satisfy $W^2 = \hat{W}^2 = 1$.
- The quantum algebras are

$$W(x_1,x_2)\hat{W}(x,y) = -\hat{W}(x,y)W(x_1,x_2), \quad \text{if } x_1 < x < x_2,$$
$$W(y_1,y_2)\hat{W}(x,y) = -\hat{W}(x,y)W(y_1,y_2), \quad \text{if } y_1 < y < y_2,$$

- Using EOM, we can decompose $W(x,y)$ (or $\hat{W}(x,y)$) as,

$$W(x,y) = W_{z,y}(x)W_{z,x}(y)$$

Electric topological boundary

- Eigenstates of W-operators: $|\mathbf{w}\rangle = |w_{z,x;j}, w_{z,y;i}, w_{x;j+\frac{1}{2}}, w_{y;i+\frac{1}{2}}\rangle$, where and the electric operators W are diagonalized as

$$\begin{cases} W_{z,x}(y_j)|\mathbf{w}\rangle = (-1)^{w_{z,x;j}}|\mathbf{w}\rangle \\ W_{z,y}(x_i)|\mathbf{w}\rangle = (-1)^{w_{z,y;i}}|\mathbf{w}\rangle \\ W(y_j, y_{j+1})|\mathbf{w}\rangle = (-1)^{w_{x;j+\frac{1}{2}}}|\mathbf{w}\rangle \\ W(x_i, x_{i+1})|\mathbf{w}\rangle = (-1)^{w_{y;i+\frac{1}{2}}}|\mathbf{w}\rangle \end{cases}$$

- On the other hand, the magnetic operators $\hat{W}$ conjugate to electric operators W will shift the eigenvalues when acting on the state $|\mathbf{w}\rangle$

$$\begin{cases} \hat{W}(y_{j'-\frac{1}{2}}, y_{j'+\frac{1}{2}})|\mathbf{w}\rangle = |w_{z,x;j} + \delta_{j,j'}, w_{z,y;i}, w_{x;j+\frac{1}{2}}, w_{y;i+\frac{1}{2}}\rangle \\ \hat{W}(x_{i'-\frac{1}{2}}, x_{i'+\frac{1}{2}})|\mathbf{w}\rangle = |w_{z,x;j}, w_{z,y;i} + \delta_{i,i'}, w_{x;j+\frac{1}{2}}, w_{y;i+\frac{1}{2}}\rangle \\ \hat{W}_{z,x}(y_{j'+\frac{1}{2}})|\mathbf{w}\rangle = |w_{z,x;j}, w_{z,y;i}, w_{x;j+\frac{1}{2}} + \delta_{j,j'}, w_{y;i+\frac{1}{2}}\rangle \\ \hat{W}_{z,y}(x_{i'+\frac{1}{2}})|\mathbf{w}\rangle = |w_{z,x;j}, w_{z,y;i}, w_{x;j+\frac{1}{2}}, w_{y;i+\frac{1}{2}} + \delta_{i,i'}\rangle \end{cases}$$

- $\hat{W}$ operators generate the subsystem symmetry

Magnetic topological boundary

- Eigenstates of $\hat{W}$-operators: $|\hat{\mathbf{w}}\rangle = |\hat{w}_{z,x;j+\frac{1}{2}}, \hat{w}_{z,y;i+\frac{1}{2}}, \hat{w}_{x;j}, \hat{w}_{y;i}\rangle$ where $\hat{W}$ operators are diagonalized as,

$$\begin{cases} \hat{W}_{z,x}(y_{j+\frac{1}{2}})|\hat{\mathbf{w}}\rangle = (-1)^{\hat{w}_{z,x;j+\frac{1}{2}}}|\hat{\mathbf{w}}\rangle \\ \hat{W}_{z,y}(x_{i+\frac{1}{2}})|\hat{\mathbf{w}}\rangle = (-1)^{\hat{w}_{z,y;i+\frac{1}{2}}}|\hat{\mathbf{w}}\rangle \\ \hat{W}(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})|\hat{\mathbf{w}}\rangle = (-1)^{\hat{w}_{x;j}}|\hat{\mathbf{w}}\rangle \\ \hat{W}(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})|\hat{\mathbf{w}}\rangle = (-1)^{\hat{w}_{y;i}}|\hat{\mathbf{w}}\rangle \end{cases}$$

- The electric operators W will serve as symmetry generator instead,

$$\begin{cases} W(y_{j'}, y_{j'+1})|\hat{\mathbf{w}}\rangle = |\hat{w}_{z,x;j+\frac{1}{2}} + \delta_{j,j'}, \hat{w}_{z,y;i+\frac{1}{2}}, \hat{w}_{x;j}, \hat{w}_{y;i}\rangle \\ W(x_{i'}, x_{i'+1})|\hat{\mathbf{w}}\rangle = |\hat{w}_{z,x;j+\frac{1}{2}}, \hat{w}_{z,y;i+\frac{1}{2}} + \delta_{i,i'}, \hat{w}_{x;j}, \hat{w}_{y;i}\rangle \\ W_{z,x}(y_{j'})|\hat{\mathbf{w}}\rangle = |\hat{w}_{z,x;j+\frac{1}{2}}, \hat{w}_{z,y;i+\frac{1}{2}}, \hat{w}_{x;j} + \delta_{j,j'}, \hat{w}_{y;i}\rangle \\ W_{z,y}(x_{i'})|\hat{\mathbf{w}}\rangle = |\hat{w}_{z,x;j+\frac{1}{2}}, \hat{w}_{z,y;i+\frac{1}{2}}, \hat{w}_{x;j}, \hat{w}_{y;i} + \delta_{i,i'}\rangle \end{cases}$$

KW-duality[§]

- Given any (2+1)D theory $\mathfrak{T}_{\text{sub}}$ with subsystem symmetry, the dynamical boundary state is constructed as,

$$|\mathfrak{B}^{\text{phys}}_{\mathfrak{T}_{\text{sub}}}\rangle = \sum_{\mathbf{w}} Z_{\mathfrak{T}_{\text{sub}}}[\mathbf{w}]|\mathbf{w}\rangle,$$

- The original theory is $Z_{\mathfrak{T}_{\text{sub}}}[\mathbf{w}] = \langle \mathbf{w}|\mathfrak{B}^{\text{phys}}_{\mathfrak{T}_{\text{sub}}}\rangle$
- The KW-dual is given by $Z_{\hat{\mathfrak{T}}_{\text{sub}}}[\hat{\mathbf{w}}] = \langle \hat{\mathbf{w}}|\mathfrak{B}^{\text{phys}}_{\mathfrak{T}_{\text{sub}}}\rangle$

$$\begin{aligned} Z_{\hat{\mathfrak{T}}_{\text{sub}}}[\hat{\mathbf{w}}] =& \frac{1}{2^{(L_x+L_y-1)}} \sum_{\mathbf{w}\in M_v} (-1)^{\sum_i (\hat{w}_{z,y;i+\frac{1}{2}} w_{y;i+\frac{1}{2}} + \hat{w}_{y;i} w_{z,y;i})} \\ & \times (-1)^{\sum_j (\hat{w}_{z,x;j+\frac{1}{2}} w_{x;j+\frac{1}{2}} + \hat{w}_{x;j} w_{z,x;j})} Z_{\mathfrak{T}_{\text{sub}}}[\mathbf{w}] \end{aligned}$$

[§] Weiguang Cao, Linhao Li, Masahito Yamazaki, Yunqin Zheng, 2023.

JW transformation[¶]

- The fermionic state $|\mathbf{s}\rangle = |s_{z,x;j}, s_{z,y;i}, s_{x;j+\frac{1}{2}}, s_{y;i+\frac{1}{2}}\rangle$ is diagonalized by,

$$\begin{cases} \hat{W}_{z,x}(y_{j-\frac{1}{2}}) W_{z,x}(y_j) \hat{W}_{z,x}(y_{j+\frac{1}{2}}) |\mathbf{s}\rangle = (-1)^{s_{z,x;j}} |\mathbf{s}\rangle \\ W_{z,y}(x_i)|\mathbf{s}\rangle = (-1)^{s_{z,y;i}} |\mathbf{s}\rangle \\ \hat{W}(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) W(y_j, y_{j+1}) \hat{W}(y_{j+\frac{1}{2}}, y_{j+\frac{3}{2}}) |\mathbf{s}\rangle = (-1)^{s_{x;j+\frac{1}{2}}} |\mathbf{s}\rangle \\ W(x_i, x_{i+1})|\mathbf{s}\rangle = (-1)^{s_{y;i+\frac{1}{2}}} |\mathbf{s}\rangle \end{cases}$$

- The fermionic subsystem $\mathbb{Z}_2$ parity symmetry is generated by magnetic operators $\hat{W}$

$$\begin{cases} \hat{W}(y_{j'-\frac{1}{2}}, y_{j'+\frac{1}{2}})|\mathbf{s}\rangle = |s_{z,x;j} + \delta_{j,j'}, s_{z,y;i}, s_{x;j+\frac{1}{2}}, s_{y;i+\frac{1}{2}}\rangle \\ \hat{W}(x_{i'-\frac{1}{2}}, x_{i'+\frac{1}{2}})|\mathbf{s}\rangle = |s_{z,x;j}, s_{z,y;i} + \delta_{i,i'}, s_{x;j+\frac{1}{2}}, s_{y;i+\frac{1}{2}}\rangle \\ \hat{W}_{z,x}(y_{j'+\frac{1}{2}})|\mathbf{s}\rangle = |s_{z,x;j}, s_{z,y;i}, s_{x;j+\frac{1}{2}} + \delta_{j,j'}, s_{y;i+\frac{1}{2}}\rangle \\ \hat{W}_{z,y}(x_{i'+\frac{1}{2}})|\mathbf{s}\rangle = |s_{z,x;j}, s_{z,y;i}, s_{x;j+\frac{1}{2}}, s_{y;i+\frac{1}{2}} + \delta_{i,i'}\rangle \end{cases}$$

[¶] Weiguang Cao, Masahito Yamazaki, Yunqin Zheng. 2022

- There exists another fermionic state $|\mathbf{s}'\rangle = |s'_{z,x;j}, s'_{z,y;i}, s'_{x;j+\frac{1}{2}}, s'_{y;i+\frac{1}{2}}\rangle$ which diagonalizes the line operators,

$$W_{z,x}(y_j), \quad \hat{W}_{z,y}(x_{i-\frac{1}{2}})W_{z,y}(x_i)\hat{W}_{z,y}(x_{i+\frac{1}{2}}),$$

and strip operators,

$$W(y_j, y_{j+1}), \quad \hat{W}(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})W(x_i, x_{i+1})\hat{W}(x_{i+\frac{1}{2}}, x_{i+\frac{3}{2}}),$$

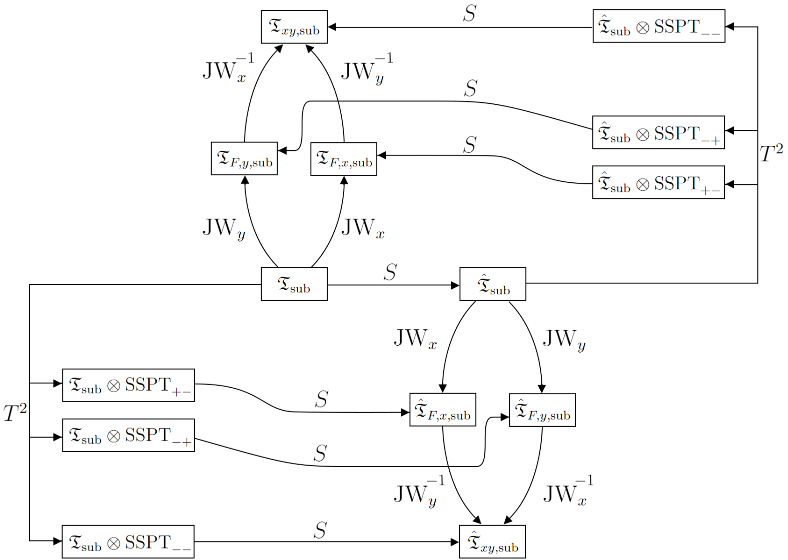
where $W_{z,y}(x_i)$, $W(x_i, x_{i+1})$ are sandwiched by a pair of $\hat{W}$ operators instead.

- Choosing different topological boundary states give two fermionic theories $\mathfrak{T}_{F,x,\text{sub}}$ and $\mathfrak{T}_{F,y,\text{sub}}$ whose partition functions are,

$$Z_{\mathfrak{T}_{F,x,\text{sub}}}[\mathbf{s}] = \langle \mathbf{s}|\mathfrak{B}^{\text{phys}}_{\mathfrak{T}_{\text{sub}}}\rangle, \quad Z_{\mathfrak{T}_{F,y,\text{sub}}}[\mathbf{s}'] = \langle \mathbf{s}'|\mathfrak{B}^{\text{phys}}_{\mathfrak{T}_{\text{sub}}}\rangle$$

They are related to the bosonic theory $\mathfrak{T}_{\text{sub}}$ by performing the subsystem JW transformations along x and y directions respectively.

Dualities in Subsystem Symmetry



Conclusion

- We propose that the SymTFT of (2+1)D subsystem symmetry is the 2-foliated BF theory
- We construct the bosonic/fermionic topological boundary states
- By studying the SymTFT, we obtain a duality web connecting different theories.

Future work

- Extend the study of subsystem SymTFT to other models, like $\mathbb{Z}_N$ subsystem symmetry.
- Study models with subsystem symmetry in higher dimensions, for example, the (3+1)D X-cube model.
- $\dots$

Thank you!