A Fermi Model of Black Hole

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Outline

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Problems with Black Hole

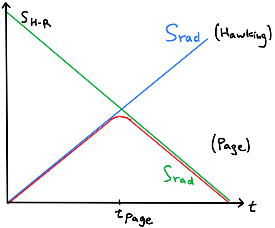
1. Bekenstein-Hawking entropy:

$$S_{\mathrm{B-H}} = \frac{A}{4G\hbar}.$$

   What is the interpretation? entanglement? microstates? Why area dependence?

2. Black hole interior: existence of singularity
   What replace the continuum spacetime description inside the black hole?

3. Information problem: non-unitarity evolution appears in 2 ways.
   - Hawking radiation is a "mixed thermal state", while the initial state can be a pure state.
   - Hawking curve behaviour of S_{rad} violates unitarity.



What should be quantum degrees of freedom of black hole so that one can evolve them quantum mechanically?

Top-Down Approach

- Obviously, all these problems are due to a lack of understanding of the quantum d.o.f. of black hole.
- The best thing to do is to start from a theory X of quantum gravity, construct the classical black hole as a solution of it, then the properties of quantum black hole should follow. $X =$ e.g. string theory, branes, AdS/CFT, matrix model etc

This top-down approach has been quite successful in string.

- Duality in QFT e.g. Montone-Olive S-duality
- Non-commutative geometry can be derived from first principle
- For BH, some of the progress made are:
  1. *Bekerstein-Hawking entropy*: Microstate counting (Strominger-Vafa), fuzzy ball proposal (Mathur), and progress in index computation.
  2. *Page curve:* AdS/CFT Island proposal (Engelhardt, Wall, Penington, Almheiri, Marlof, Maxfield, Mahajan, Maldacena, Zhao, ...)

Yet we still do not know:

- what fundamental degrees of freedom of BH are being counted?
- how are they related to the Hawking radiation?
- how the evolution of them leads to the Page curve and unitary evolution of the black hole?

The building of QM

- Historically, QM was built from bottom-up. Blackbody radiation, discreteness of atomic spectrum etc point clearly to a break down of classical physics
- The postulate of quantized orbits in the Bohr model of atom, together with various other postulates (uncertainty relations, wave-particle duality etc) played important roles in the develpment of QM, and have eventually got built in to the theory of QM.

Bottom up approach to BH

- Current construction of string theory, AdS/CFT, matrix model etc have captured truth about quantum gravity, but not complete.
- One may take BH as a theoretical laboratory and adopt a *bottom-up approach* to learn about what are needed for the fundamental theory.

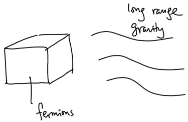
Outline

Model of the interior of BH

- We consider a matrix model of fermionic degrees of freedom coupled with bosonic degrees of freedom as a fundamental model of quantum gravity. Schematically,

$$L = i\psi^\dagger\dot{\psi} + h(X)\psi^\dagger\psi - V(X), \quad \text{e.g. BFSS}$$

  Presumably spacetime/gravity will emerge in some “effective” way.
- We consider a (spherical) box of the fermion d.o.f as follows:



- Outside the black hole, we assume that the usual spacetime and general relativity has emerged as effective description. (see Maldacena for complementary consideration).

- Instead of committing to a specific matrix model, e.g. BFSS matrix model, we will search for the properties of the matrix model so that the box of fermi system resembles a BH.
- The energy eigenstates of the fermi system is discrete. e.g. for the QM $L = \frac{i}{2}\sum_{i=1}^{n} \dot{\psi}^i \psi^i$,

$$\{\psi^i, \psi^j\} = 2\delta^{ij}, \quad i, j = 1, \cdots n.$$

gives the oscillators $\{c^{\dagger i}, c^j\} = \delta^{ij}$,

$$c^i := \frac{1}{\sqrt{2}}(\psi^{2i} + i\psi^{2i-1}), \quad i = 1, \cdots, \frac{n}{2}$$

We can use them to build the Fock space with $|0\rangle$ that obeys $c^i|0\rangle = 0$,

$$|i\rangle = c^{\dagger i}|0\rangle.$$

That's how fermion arises.

For the general fermionic QM,

- Introduce density of states $g(E)$, $g(E)dE$ is the number of energy eigenstates contained within the energy interval $(E, E + dE)$.
- For the general form $g(E) = c_0 V E^\alpha$, total number of states $N_S = \int dE g(E) n(E)$ and total internal energy $U = \int dE g(E) n(E) E$

$$N_S = \frac{c_0 V}{\alpha + 1}\mu^{\alpha+1}, \quad U = \frac{c_0 V}{\alpha + 2}\mu^{\alpha+2}.$$

  μ is the Fermi level.
- "internal energy pressure" $P_U := -(\partial U/\partial V)_N$

$$P_U = -\frac{\alpha + 1}{\alpha + 2} N_S \left(\frac{\partial \mu}{\partial V}\right)_N .$$

  $N = 2N_S$ is the total number of "fermions".

E.g. Neutron star

- For a star made of non-relativistic neutron, $\alpha = 1/2$. μ is determined by the number of neutrons,

$$\mu = \frac{1}{2m}\left(\frac{6\pi^2 N}{V}\right)^{2/3} .$$

System internal energy

$$P_U \sim \frac{N^{5/3}\hbar^2}{mR^5},$$

- For mass M, the gravitational pressure acting on the star can be computed from the metric:

$$P_g = -\frac{GM^2}{2\pi R^4}$$

Equilibrium $P_U + P_g = 0$ gives the mass-radius relation

$$R \sim M^{-1/3}$$

This is characteristic of object made of particles.

Mass-Radius relation

- The fermionic d.o.f. of our system are NOT particles, so we can expect a different expression of μ, and hence a different $R(M)$ relation.
- Let us consider a Fermi energy of the form $\mu = \frac{a}{R}$, then

$$P_U = \frac{U}{4\pi R^3}$$

- Assume equivalence principle $U = M$, then the system is stable against the pressure of gravity if $-\frac{GM^2}{2\pi R^4} + \frac{M}{4\pi R^3} = 0$. This gives

$$R = 2GM,$$

which is precisely the Schwarzschild radius.

Microstate counting

- The requirement of equivalence principle actually demands that $\alpha = 0$:

$$U \sim V\mu^{\alpha+2} \stackrel{!}{=} M \sim R \Longrightarrow \alpha = 0.$$

- From $U = \frac{N_S}{2}\frac{a}{R} = M$, we obtain that

$$N_S = \frac{1}{\pi a} \cdot \frac{\pi R^2}{G,}$$

  the number of energy eigenstates of BH has an area dependence!

Q. Is this the B-H entropy?

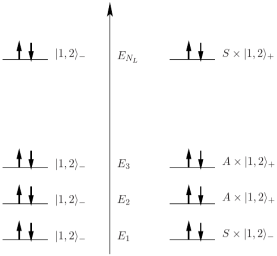
- If there is only fermionic d.o.f., then the ground state is unique and hence entropy is zero!
- However, there is also a bosonic wave function:

$$\Psi = \chi_F(\psi)\chi_B(X)$$

- e.g. if there are two "orbital" wavefunctions associated with each energy eigenstate, then one has the set of wavefunctions for each energy level:

$$\chi_1(1)\chi_1(2)|1,2\rangle_-, \chi_2(1)\chi_2(2)|1,2\rangle_-, \left(\chi_1(1)\chi_2(2)+\chi_1(2)\chi_2(1)\right)|1,2\rangle_-,$$
$$\left(\chi_1(1)\chi_2(2)-\chi_1(2)\chi_2(1)\right)|1,2\rangle_+$$

  where $|1,2\rangle_\pm := \frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2 \pm |-\rangle_1|+\rangle_2\right)$ are the symm/anti-symm spin wave functions.
- Note that these states are entangled Bell states.



- As a result, the ground state wave function is given by

$$\Psi_{\{n_E\}} = \prod_{0<E<\mu} \psi_{E,n_E},$$

for a given specification $\{n_E\}$ of symmetry of the ground state.

- As there is a choice of 4 states ψ_{E,n_E} for each energy level E, the ground state has a degeneracy of

$$\mathcal{G} = 4^{N_L} = 2^{N_S/2}.$$

This give rises to the entropy

$$S = \log \mathcal{G} = N_S/2.$$

This reproduces the Bekenstein-Hawking entropy if

$$a = \frac{1}{2\pi}, \quad c_0 = \frac{3\pi}{G}$$

- Summarizing, our model is defined by the Fermi level

$$\mu = \frac{1}{2\pi R}$$

and the constant density of states

$$g(E) = \frac{3\pi V}{G}.$$

Charged BH

- IN GR, charged BH is described by the Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{R_s}{r} + \frac{R_Q^2}{r^2}\right)dt^2 + \left(1 - \frac{R_s}{r} + \frac{R_Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

where the horizon radius R satisfies the quadratic relation

$$R^2 - RR_S + R_Q^2 = 0,$$

with $R_S = 2GM$ and $R_Q = \sqrt{GQ^2}$.

- We consider our model for the fermionic d.o.f.,

$$g(E) = \frac{3\pi V}{G}, \quad \mu = \frac{1}{2\pi R}.$$

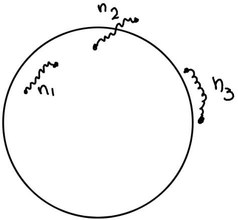
- This gives the internal energy of the Fermi sea $U_F = \frac{R}{2G}$. Now there is also a Coulombic energy $U_Q = \int_R^\infty u\, 4\pi r^2 dr$ due to the presence of an electric field. Here $u = E^2/(8\pi) = Q^2/(8\pi r^4)$ is the electric energy density outside the black hole. We get $U_Q = \frac{Q^2}{2R}$.
- Setting the total internal energy $U = M$,

$$U = \frac{R}{2G} + \frac{Q^2}{2R}$$

  We obtain immediately the $R(M, Q)$ relation!
- Stability of the systema and microstate counting works out similarly.
- Also work for BH with cosmological constant.

Outline

Dynamical Evolution of Entangled Particles



- With d.o.f. for the BH, we can now discuss it's quantum mechanical evolution.
- Let $n_1(\omega, t)d\omega$ be the no of entangled pairs with energies in the regions $(\omega, \omega + d\omega)$ and both inside the horizon.
- $n_2(\omega, t)d\omega$ be the bumber of entangled pairs that has one quanta inside the horizon and the other outside,
- $n_3(\omega, t)d\omega$ be the number of entangled pairs that are located entirely outside the horizon.

- In general tunneling is possible, we can write down simple dynamical eqns,

$$\frac{\partial n_1}{\partial t} = -2\Gamma\, n_1, \tag{1}$$

$$\frac{\partial n_2}{\partial t} = 2\Gamma n_1 - \Gamma n_2, \tag{2}$$

$$\frac{\partial n_3}{\partial t} = \Gamma n_2, \tag{3}$$

- Our model allows clear determination of various entropies of BH in terms of the total number of each types of entangled pairs

$$N_i(t) = \int d\omega n_i(\omega, t), \quad i = 1, 2, 3.$$

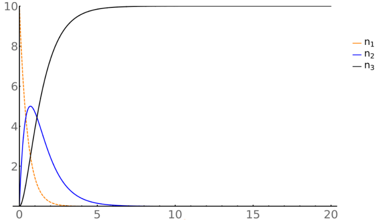
  - N_1 gives the coarse-grained entropy $S_{\mathrm{B-H}}$ of the BH at time t,
  - N_2 gives the entanglement entropy S_{rad} of Hawking radiation.
  - N_3 gives the amount of entanglement information contained in the Hawking radiation.

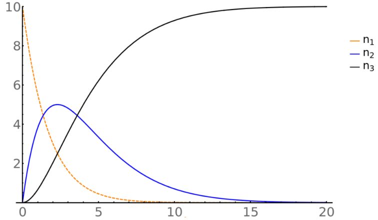
Page Curve

Let us consider a black hole formed at $t = 0$, implying the conditions $n_2 = n_3 = 0$ initially.

1. RHS of (1) is non-positive, N_1 will continue to decrease until it reaches zero at some time t_1.
2. The competition of the tunneling effects in N_2 give rises to the Page curve.
3. N_3 increases monotonically until N_2 reaches zero at $t = t_E$, then it remains a constant.

The above analysis is general. With a semiclassical $M(t)$, plots can be made that confirm the general analysis.





Information Retrival

- It follows immediately from the conservation equation that:

$$\frac{d}{dt}(N_1 + N_2 + N_3) = 0.$$

- At $t = 0$, we have $N_1 = N_{10}$, $N_2 = N_3 = 0$. At $t = t_E$, we have $N_1 = N_2 = 0$ and $N_3 = N_{3f} =$ a constant, it follows immediately that

$$N_{3f} = N_{10},$$

meaning that all the entanglement information originally stored in the Bell pairs are returned to the exterior observer via the Hawking radiation.

Outline

- We adopted a bottom-up apparoch to spacetime and quantum gravity. We proposed a model of black hole interior as a system of fermi d.o.f.
- The system of fermions has the following properties:
  - Energetics of the fermi sea gives a pressure that provide the stability of the system against gravitational collapse, resulting in a specific size of the system $R = R(M, Q)$.
  - Microstates are made of entangled fermion pairs. Taking into possible degenerace of the bosonic wave function, the counting of the microstates give rises to an entropy

$$S = \frac{A}{4G}$$

  - Dynamical evolution of the entangled pairs give rises to Page curve and information retriveal of the system.

These provides strong support that the model indeed has something to do with a quantum black hole.

More things to do

In increasing degree of difficulty:

1. Need to find the matrix QM $L = L(\psi, X)$ whose energy eigenstates has a constant density of states.
2. In our work, the boundary of the box is identified with the horizon. Classical gravity is assumed outside the box. This division is artifical.

   Proper understanding of the fermi liquid outside the black hole will allow us to understand the origin of the tunneling in the QM and the true property of the horizon. This will also lead to observational consequence for BH.

3. In general, the distribution of energy eigenstates will depend on the configuration of X. It is possible that each solution to the matrix model will give a fermi liquid with it's density of states, with spacetime M and metric arises as an effective description.

$$\lambda'\mathrm{s} \to g(E) \to (M, g_{\mu\nu})$$

   How?

4. What about time? IKKT instatonic like matrix model?

Thank you for listening!