# Progress in the numerical studies of the type IIB matrix model

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East Asia Joint Workshop on Fields and Strings 2023
Xian, November 13th, 2023

# Type IIB matrix model

Ishibashi Kawai, Kitazawa, AT (1996)

A conjectured nonperturbative formulation of superstring theory

$$S = -N\mathrm{Tr}\left(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right)$$

$N \times N$ Hermitian matrices

$A_\mu$ : 10D Lorentz vector $(\mu = 0, 1, \ldots, 9)$

$\psi$ : 10D Majorana-Weyl spinor

SO(9,1) symmetry

The action takes the form of the dimensional reduction of 10D N=1 SYM.

Space-time does not exist a priori,
but emerges from the degrees of freedom of matrices.

→ Dimensionality of space-time

# Crucial properties: 10D N=2 SUSY

$$Q^{(1)} \begin{cases} \delta^{(1)} A_\mu = i\bar{\epsilon}_1 \Gamma_\mu \psi \\ \delta^{(1)} \psi = \frac{i}{2} \Gamma^{\mu\nu} [A_\mu, A_\nu] \epsilon_1 \end{cases} \qquad Q^{(2)} \begin{cases} \delta^{(2)} A_\mu = 0 \\ \delta^{(2)} \psi = \epsilon_2 1_N \end{cases} \qquad P_\mu \begin{cases} \delta_{\mathrm{T}} A_\mu = c_\mu 1_N \\ \delta_{\mathrm{T}} \psi = 0 \end{cases}$$

dimensional reduction of
10D N=1 SUSY

$$\begin{cases} \tilde{Q}^{(1)} = Q^{(1)} + Q^{(2)} \\ \tilde{Q}^{(2)} = i(Q^{(1)} - Q^{(2)}) \end{cases} \Longrightarrow \quad [\bar{\epsilon}_1 \tilde{Q}^{(i)}, \bar{\epsilon}_2 \tilde{Q}^{(j)}] = -2\delta^{ij} \bar{\epsilon}_1 \Gamma^\mu \epsilon_2 P_\mu$$

10D N=2 SUSY if $P_\mu$ is identified with momentum, which generates shift of $A_\mu$

**The space-time is represented as the eigenvalue distribution of $A_\mu$ .**

The fact that the model has maximal SUSY suggests strongly that the model includes gravity.

# Crucial properties: connection to the world sheet action

Green-Schwarz action of Schild-type for type IIB superstring with $\kappa$ symmetry fixed

$$S_{\mathrm{S}} = \int d\tau d\sigma \sqrt{-g}\left[\frac{1}{4}\{X_\mu, X_\nu\}\{X^\mu, X^\nu\} - \frac{i}{2}\bar{\Psi}\Gamma^\mu\{X_\mu, \Psi\}\right]$$

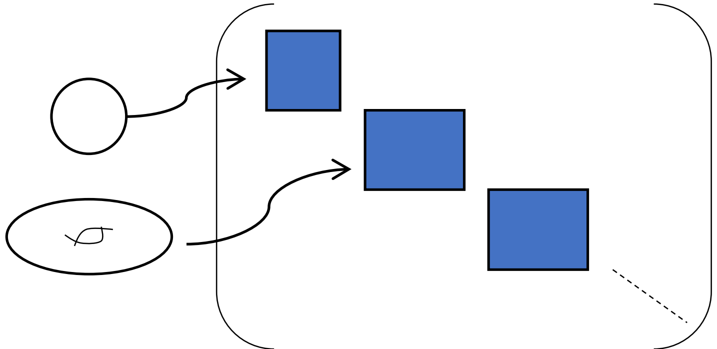
$$\{X, Y\} = \frac{1}{\sqrt{-g}}\left(\frac{\partial X}{\partial \tau}\frac{\partial Y}{\partial \sigma} - \frac{\partial X}{\partial \sigma}\frac{\partial Y}{\partial \tau}\right)$$

matrix regularization

$$\begin{cases} X_\mu(\tau, \sigma) \to A_\mu \\ \Psi(\tau, \sigma) \to \psi \\ \{\ ,\ \} \to \frac{1}{i}[\ ,\ ] \\ \int d\tau d\sigma \to \mathrm{Tr} \end{cases}$$

type IIB matrix model

$$S = -N\mathrm{Tr}\left(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right)$$




multi strings

2nd quantized

# Crucial properties (cont'd)

- Long distance behavior of interaction between D-branes is reproduced.

- Light-cone string field theory for type IIB superstring is reproduced from SD equations for Wilson loops under reasonable assumptions.

Fukuma-Kawai-Kitazawa-AT (1997)

# Plan of the present talk

1. Introduction
2. Lorentzian vs Euclidean
3. Introducing IR regulator into the Lorentzian model
4. How to investigate the model
5. Results of numerical simulations
6. Summary and outlook

# Lorentzian vs Euclidean

# Partition function of Lorentzian model

$$S = -N\mathrm{Tr}\left(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right)$$

$$Z = \int dA d\psi\ e^{iS} = \int dA\ \mathrm{Pf}\mathcal{M}(A)\ e^{iS_b}$$

pure phase factor ⟹ sign problem

$\mathrm{Pf}\mathcal{M}(A)$: polynomial in $A_\mu$ , real

$e^{iS}$: connection to worldsheet theory

$$S_b = -\frac{N}{4}\mathrm{Tr}\left([A^\mu, A^\nu][A_\mu, A_\nu]\right)$$

The model is not well-defined as it is.

We need to regularize the model.

# Euclidean model

$$Z = \int d\tilde{A}\ \mathrm{Pf}\mathcal{M}_{\mathrm{E}}(\tilde{A})\ e^{-\tilde{S}_{\mathrm{b}}}$$

connection to worldsheet theory

$$\tilde{S}_{\mathrm{b}} = \frac{N}{4}\sum_{\mu,\nu=0}^{9} \mathrm{Tr}(-[\tilde{A}_\mu, \tilde{A}_\nu]^2)$$ : positive semi-definite

SO(10) symmetry

$\Gamma^{10} = i\Gamma^0$ in $\mathrm{Pf}\mathcal{M}_{\mathrm{E}}(\tilde{A})$ : complex ⟹ sign problem

The Euclidean model is well-defined without cutoff.

Krauth, Nicolai, Staudacher ('98) Austing, Wheater ('01)

Numerical simulations showed SSB of SO(10) to SO(3) due to less fluctuations of the complex phase of Pfaffian for lower dimensions

Nishimura, Vernizzi (2000) Anagnostopoulos, Azuma, Ito, Nishimura, Okubo, Papadoudis (2020)

3D space emerges, but time does not emerge ⟹ study the Lorentzian model

# Deformation of integration contour

We try to define the Lorentzian model by deformation of integration contour

$$Z = \int dA \,\mathrm{Pf}\mathcal{M}(A)\, e^{iS_b} = \int dA \,\mathrm{Pf}\mathcal{M}(A)\, e^{-\tilde{S}_b}$$

Euclidean model

$$\tilde{S}_\mathrm{b} = -i\frac{N}{4}\left(-2\sum_{i=1}^{9}\mathrm{Tr}(-[A_0,A_i]^2) + \sum_{i,j=1}^{9}\mathrm{Tr}(-[A_i,A_j]^2)\right) \Leftrightarrow \tilde{S}_\mathrm{b} = \frac{N}{4}\sum_{\mu,\nu=0}^{9}\mathrm{Tr}(-[\tilde{A}_\mu,\tilde{A}_\nu]^2)$$

We deform the integration contour $\quad 0 \le u \le 1$

$$A_0 = e^{-i\frac{i}{2}\pi u}\, e^{i\frac{1}{8}\pi u}\,\tilde{A}_0 \qquad A_i = e^{i\frac{1}{8}\pi u}\,\tilde{A}_i \qquad \tilde{A}_0,\ \tilde{A}_i \ \text{: Hermitian matrices}$$

$$= e^{-i\frac{3}{8}\pi u}\,\tilde{A}_0$$

Wick rotation in target space

Wick rotation in worldsheet

$u = 0$ Lorentzian model

$u = 1$ Euclidean model

# Contour deformed theory is well-defined

Y. Asano, private communication ('19)

$$\tilde{S}_b = \frac{N}{4}\left(2e^{\frac{i\pi}{2}(1-u)}\sum_{i=1}^{9}\mathrm{Tr}(-[\tilde{A}_0,\tilde{A}_i]^2) + e^{\frac{i\pi}{2}(u-1)}\sum_{i,j=1}^{9}\mathrm{Tr}(-[\tilde{A}_i,\tilde{A}_j]^2)\right)$$

$\mathrm{Tr}(-[\tilde{A}_\mu,\tilde{A}_\nu]^2)$ (no sum)

: positive semi-definite

real part is positive for $0 < u \leq 1$

➡ Contour deformed theory is well-defined

According to Cauchy's theorem, $\langle\mathcal{O}(e^{-i\frac{3\pi}{8}u}\tilde{A}_0, e^{i\frac{\pi}{8}u}\tilde{A}_i)\rangle_u$ is independent of $u$

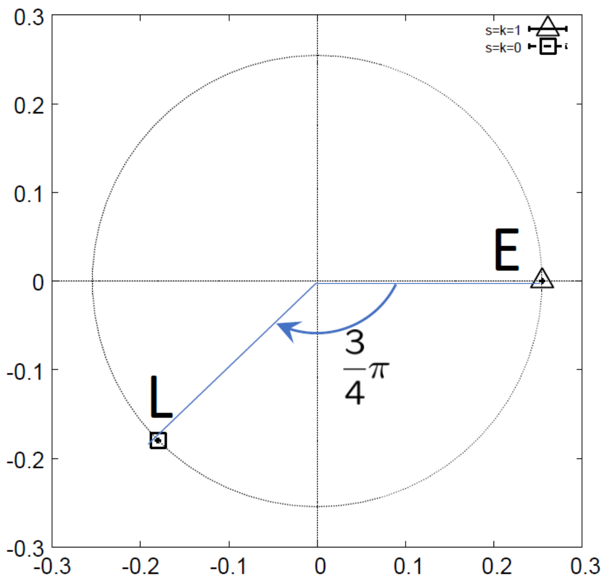
If we define the Lorentzian model by taking the $u \to +0$ limit

$$\langle\mathcal{O}(A_0, A_i)\rangle_L = \langle\mathcal{O}(e^{-i\frac{3}{8}\pi}\tilde{A}_0, e^{i\frac{1}{4}\pi}\tilde{A}_0)\rangle_E$$

# Conformation of the equivalence by simulation

10D bosonic model

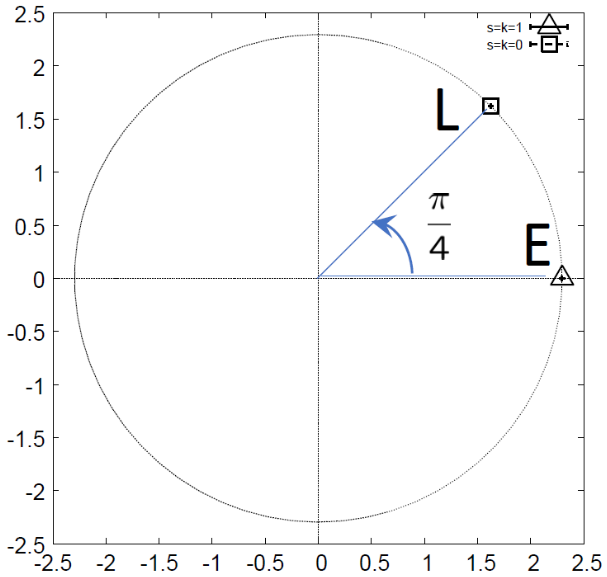
$$\left\langle \frac{1}{N}\mathrm{Tr}(A_0^2)\right\rangle_L = e^{-i\frac{3}{4}\pi}\left\langle \frac{1}{N}\mathrm{Tr}(\tilde{A}_0^2)\right\rangle_E$$




$$A_0 = e^{-i\frac{3}{8}\pi}\tilde{A}_0$$

$$A_i = e^{i\frac{1}{8}\pi}\tilde{A}_i$$

$$\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i^2)\right\rangle_L = e^{i\frac{1}{4}\pi}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(\tilde{A}_i^2)\right\rangle_E$$




The emergent space-time is complex and Euclidean.

Can we define the Lorentzian model in a different manner?

# Introducing IR regulator into the Lorentzian model

# Introducing a Lorentz invariant mass term

As an IR regulator, we introduce a Lorentz invariant mass term

$$S_m = -\frac{1}{2}N\gamma \mathrm{Tr}(A_\mu A^\mu) = \frac{1}{2}N\gamma(e^{i\epsilon}\mathrm{Tr}(A_0^2) - e^{-i\epsilon}\sum_{i=1}^{9}\mathrm{Tr}(A_i^2)) \qquad \epsilon \to 0 \qquad \gamma > 0$$

$$Z = \int dA \ \mathrm{Pf}\mathcal{M}(A) \ e^{-(\tilde{S}_b + \tilde{S}_m)}$$

$$\tilde{S}_m = -i\frac{1}{2}N\gamma(\mathrm{Tr}(A_0^2) - \sum_{i=1}^{9}\mathrm{Tr}(A_i^2)) = \frac{1}{2}N\gamma(e^{-i\frac{1}{2}\pi(1+\frac{3}{2}u)}\mathrm{Tr}(\tilde{A}_0^2) + e^{i\frac{1}{2}\pi(1+\frac{1}{2}u)}\sum_{i=1}^{9}\mathrm{Tr}(\tilde{A}_i^2))$$

$$\begin{cases} A_0 = e^{-i\frac{3}{8}\pi u}\tilde{A}_0 \\ A_i = e^{i\frac{1}{8}\pi u}\tilde{A}_i \end{cases}$$

real part is negative for $0 < u \leq 1$

We cannot deform the integration contour as before. We define the model by more general deformation of contour (Picard-Lefshetz theory).

We define the model by taking the $\gamma \to 0$ limit after taking the $N \to \infty$ limit.

Equivalence to the Euclidean model can be violated.

# Classical solutions

Hatakeyama, Matsumoto, Nishimura, AT, Yosprakob (2020)

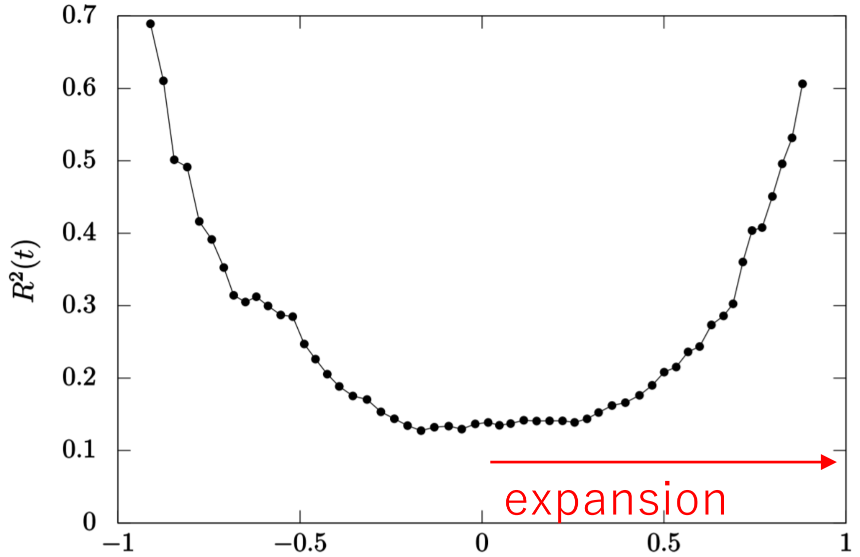
$$Z = \int dA\ e^{i(A^4+\gamma A^2)} \sim \int d\tilde{A}\ e^{i\gamma^2(\tilde{A}^4+\tilde{A}^2)} \qquad A_\mu = \sqrt{\gamma}\tilde{A}_\mu \qquad \gamma^2 \leftrightarrow \frac{1}{\hbar}$$

Classical solutions dominate at large $\gamma$ .

Classical EOM $\quad [A^\nu,[A_\nu,A_\mu]] = \gamma A_\mu$

$A_\mu = 0$ is always a solution (trivial saddle).

Typical solutions exhibit expanding behavior for $\gamma > 0$ (non-trivial saddles).




But space-time dimensionality is not fixed at the classical level.

Those solutions are hermitian so that they reside on the original contour before deformation (relevant saddles in the Picard-Lefshetz theory, which contribute to summation over saddles). We expect that non-trivial saddles are more dominant than the trivial one due to large entropy when N is large.

# How to investigate the model

# Complex Langevin Method

Parisi (1983), Klauder (1984)

We use the complex Langevin method to overcome the sign problem.

$A_\mu$ : Hermitian matrix $A_\mu \in su(N)$ ➡ $A_\mu$ : general complex matrix $A_\mu \in gl(N,\mathbb{C})$

Complex Langevin equation

$$\frac{d(A_\mu)_{ab}}{dt_L} = -\frac{\partial S_{\rm eff}}{\partial (A_\mu)_{ba}} + (\eta_\mu(t_L))_{ab}$$

$$S_{\rm eff} = (\tilde{S}_{\rm b} + \tilde{S}_{\rm m}) - \log {\rm Pf}\mathcal{M}(A)$$

$t_L$ : Langevin time ~ discretized in practice

$\eta_\mu$ : Gaussian noise

zero eigenvalue of ${\rm Pf}\mathcal{M}(A)$

➡ singular drift problem

We can calculate the expectation value of holomorphic observables by sampling them around sufficiently large $t_L$ .

# Avoiding the singular drift problem

To avoid the singular drift problem, we add a mass term to the fermionic action.

$$S_f = -\frac{N}{2}\mathrm{Tr}\left(\bar{\psi}\Gamma^{\mu}[A_{\mu},\psi] + im_f\bar{\psi}\Gamma^7\Gamma^{8\dagger}\Gamma^9\psi\right)$$

$m_f = \infty$ : bosonic

$m_f = 0$ : SUSY

The effect of fermions is weakened for finite $m_f$

$m_f$ should be as small as possible

# Controlling the quantum fluctuation of bosonic matrices

$$S_m = \frac{1}{2} N\gamma \mathrm{Tr} \left( \mathrm{Tr}(A_0)^2 - \sum_{i=1}^{d} \mathrm{Tr}(A_i)^2 - \mu \sum_{i=d+1}^{9} \mathrm{Tr}(A_i)^2 \right)$$

$d$ , $\mu$ : parameters that can control the quantum fluctuations of bosonic matrices

For large $\mu$, the bosonic degrees of freedom reduces effectively to $(d+1)$ -dimensional one.

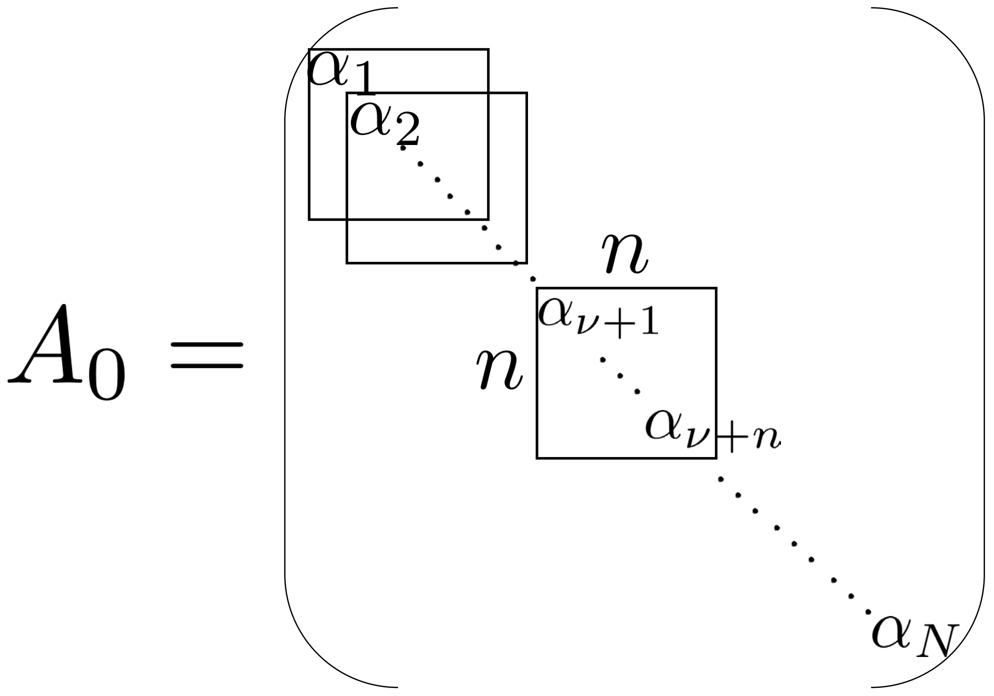
By choosing $d$ and $\mu$ appropriately, we can expect to realize a situation which is close to one where SUSY is respected.

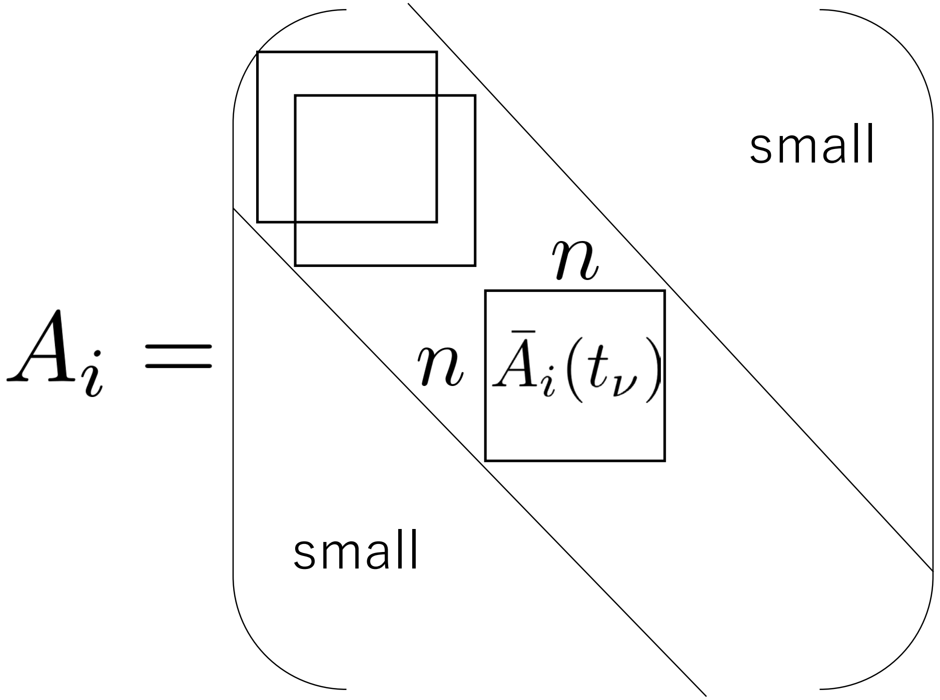
Eventually, we want to take $N \to \infty,\ m_f \to 0,\ \mu \to 1,\ \gamma \to 0$ ➡ target theory

# Extracting the time evolution

Kim-Nishimura-AT (2011)

We take the gauge in which $A_0$ is diagonal.




The state of the universe at time $t_\nu$

$\alpha_1 < \alpha_2 < \ldots < \alpha_N$

definition of time $t_\nu = \frac{1}{n}\sum_{i=1}^{n} \alpha_{\nu+i}$

$A_i$ has band-diagonal structure, which is nontrivial dynamical property.

locality of time is guaranteed. ~ 'emergence of time evolution'

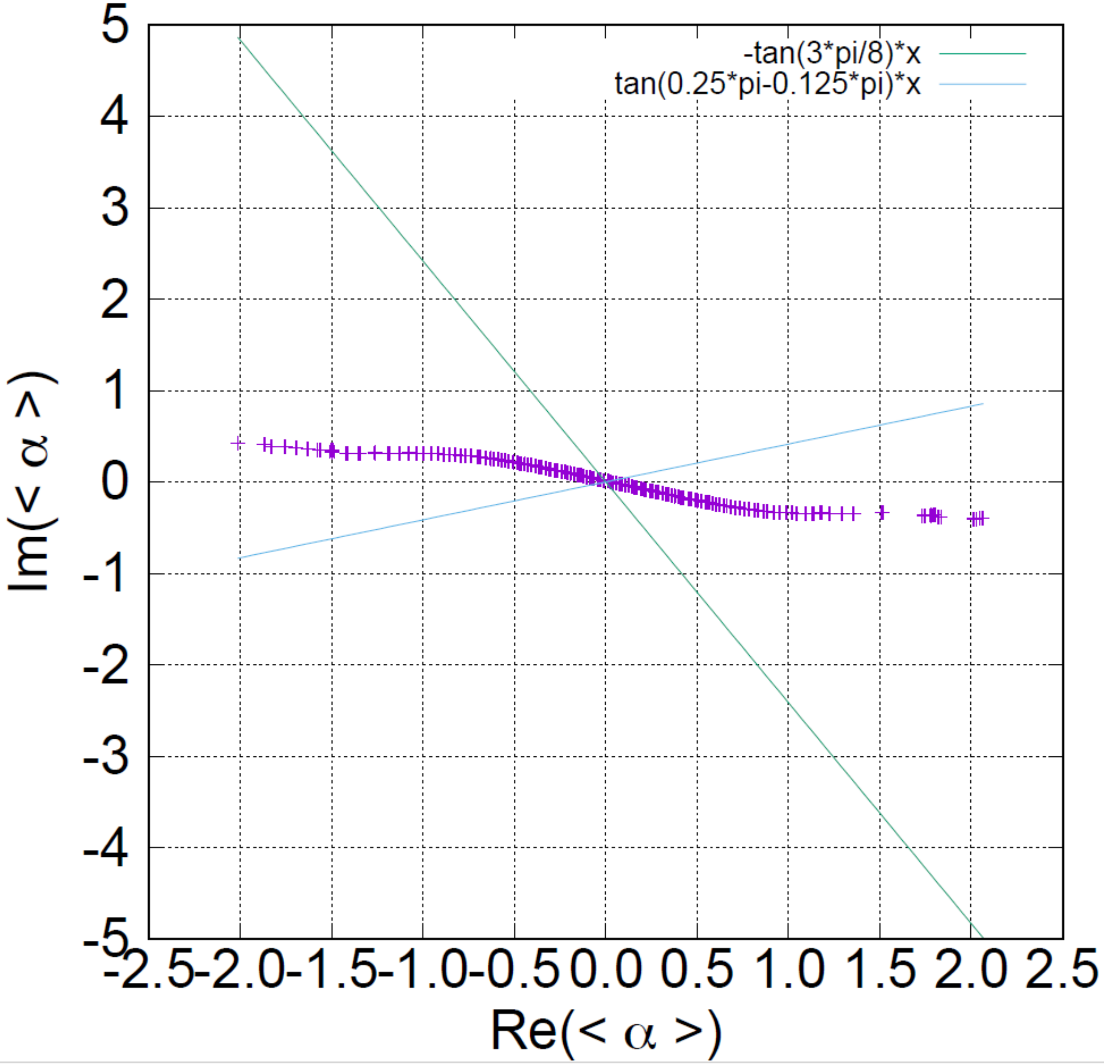
# Results of numerical simulations

$$N = 128,\ \gamma = 4,\ d = 5,\ \mu = 10,\ m_f = 3.5$$

# Emergence of real time

Nontrivial because of the complex weight $e^{iS_b+iS_m}$

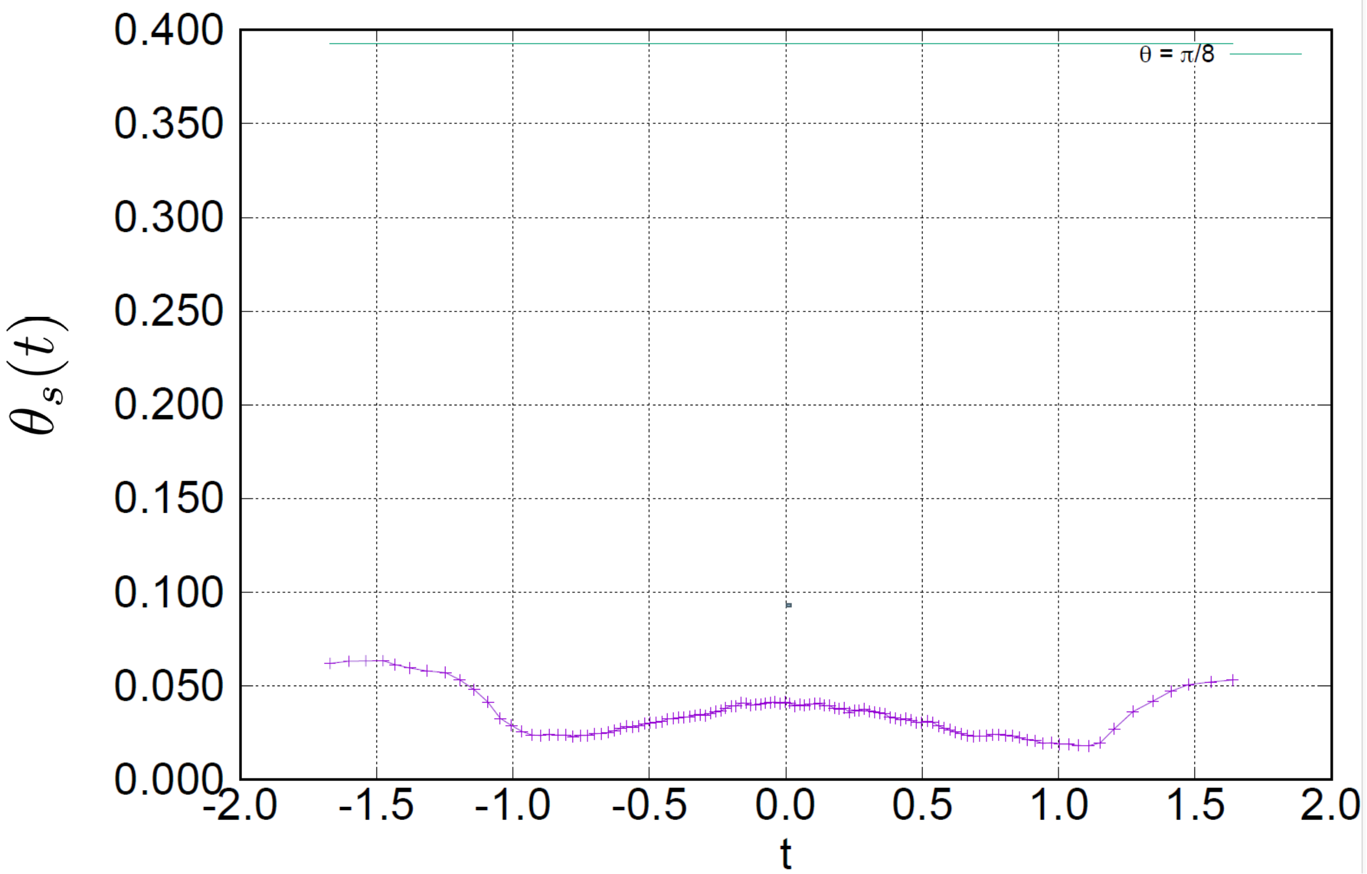
$$A_0 = \begin{pmatrix} \alpha_1 & & & & & & \\ & \alpha_2 & & & & & \\ & & \ddots & & & & \\ & & & \alpha_{\nu+1} & & & \\ & & & & \ddots & & \\ & & & & & \alpha_{\nu+n} & \\ & & & & & & \ddots \\ & & & & & & & \alpha_N \end{pmatrix}$$



# Emergence of real space

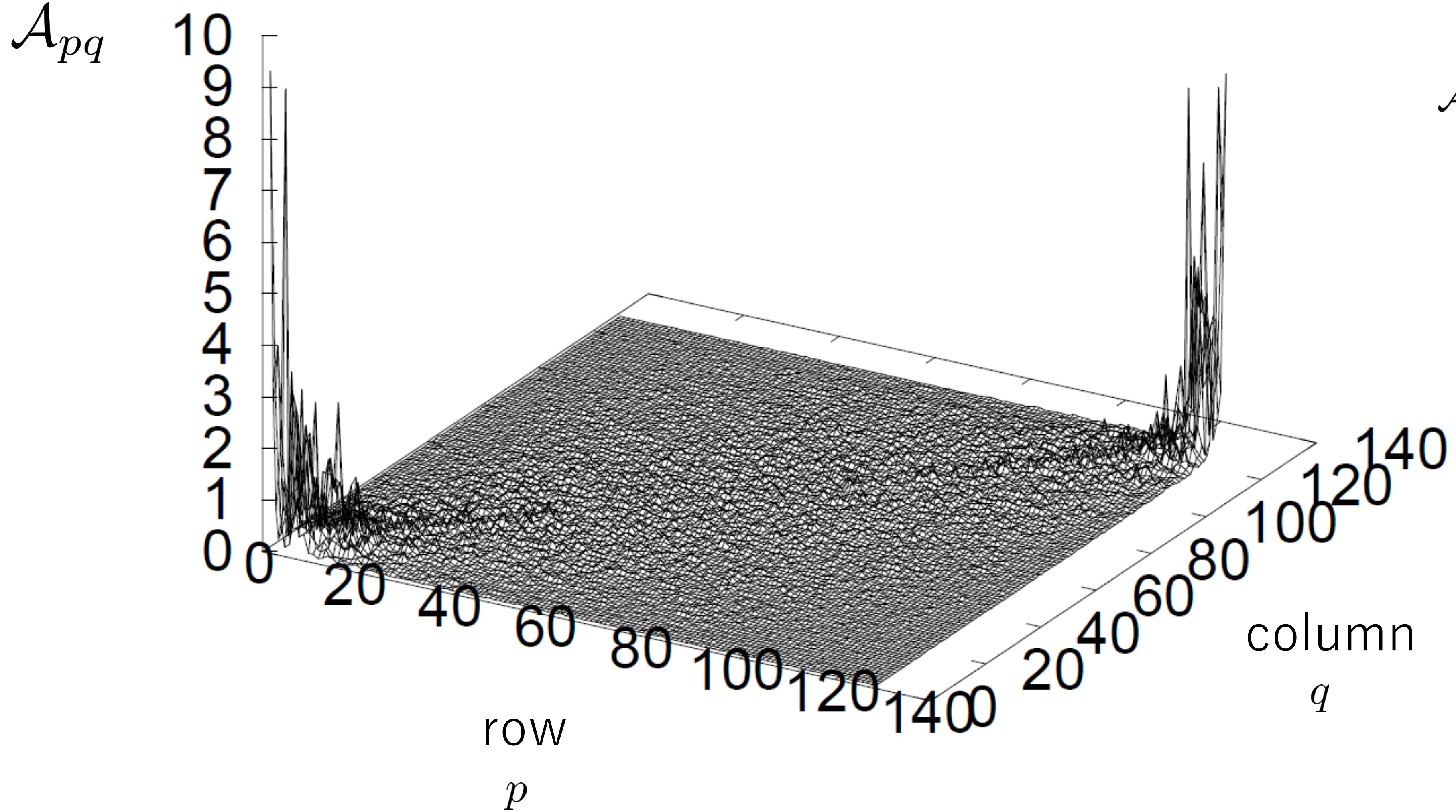
Nontrivial because of the complex weight $e^{iS_b+iS_m}$

$$\mathrm{tr}(\bar{A}_i(t))^2 = e^{2i\theta_s(t)}|\mathrm{tr}(\bar{A}_i(t))^2|$$



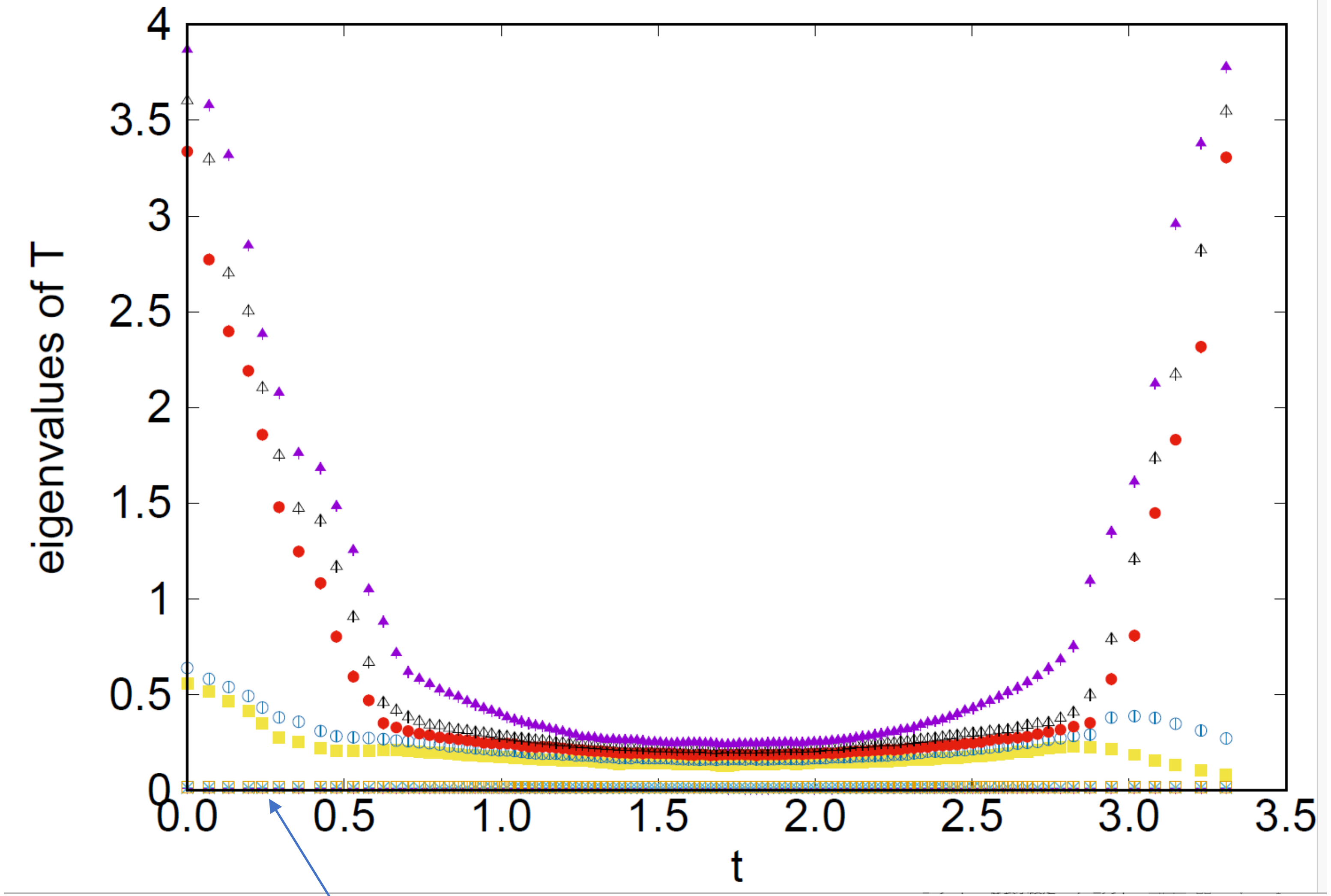
# Band diagonal structure

Emergence of time evolution

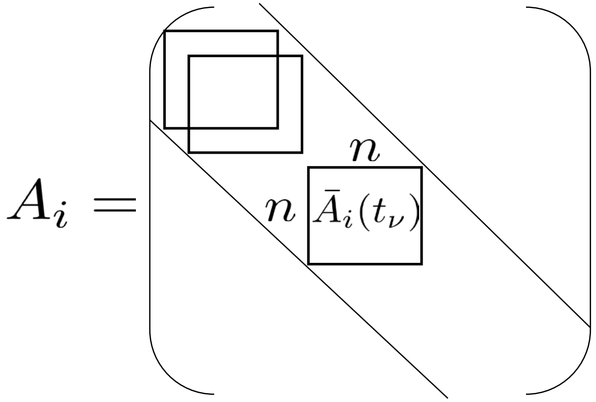



$$\mathcal{A}_{pq} = \frac{1}{9}\sum_{i=1}^{9} |(A_i)_{pq}|^2$$

# Emergence of (3+1)-dimensional space-time




four eigenvalues ← $d=5,\ \mu=10$




$$T_{ij}(t) = \frac{1}{n}\mathrm{tr}(\bar{A}_i(t)\bar{A}_j(t))$$

analog of moment of inertia tensor

Emergence of (3+1)-dimensional expanding space-time

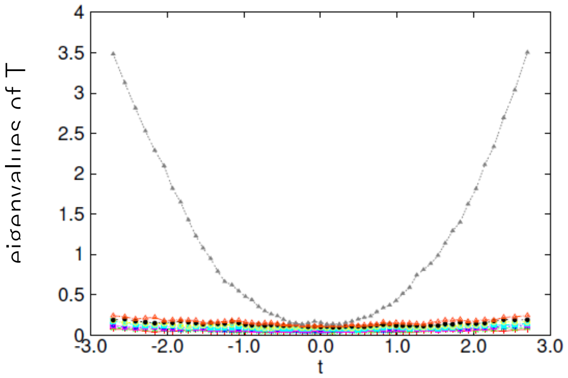
# Discussion on mechanism for emergence of (3+1)-dimensional expanding space-time

1. Effects of the bosons favor low dimensionality of space

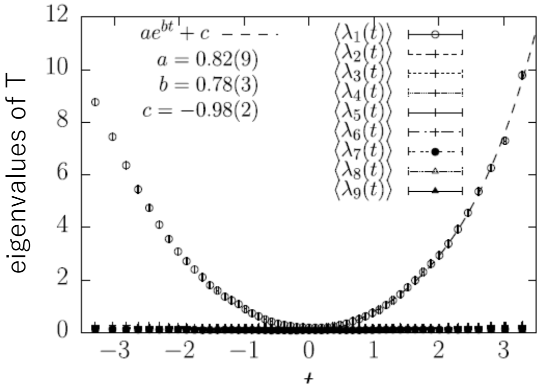
$$S_b \sim \frac{1}{2}\mathrm{Tr}([A_0, A_i]^2) - \frac{1}{4}\mathrm{Tr}([A_i, A_j]^2)$$

$$N = 64,\ \gamma = 2.6,\ (d = 9),\ \mu = 1$$

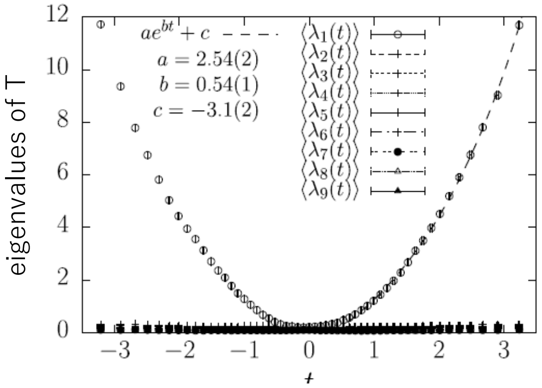
bosonic model




$m_{\mathrm{f}} = 10$




$m_{\mathrm{f}} = 5$




(1+1)-dimensional expanding space-time emerges

# Discussion on mechanism for emergence of (3+1)-dimensional expanding space-time (cont'd)

2. $\mathrm{Pf}\mathcal{M}(A_0, A_1, \cdots, A_9) = 0$ if there are only two nonzero $A_\mu$ at $m_f = 0$

Krauth, Nicolai, Staudacher (1998)

It is expected that configurations with 3D expanding space are enhanced by combination of these two effects in a situation which is close to one where SUSY is respected

# Summary and outlook

- The Lorentzian model is equivalent to the Euclidean model if we define it by the deformation of the integration contour. The emergent space-time is complex and Euclidean.
- We introduce a Lorentz invariant mass term, which invalidates the contour deformation. It plays a role of IR regulator for expanding universe.
- We define the model by more general contour deformation (the Picard-Lefshetz theory). The model defined in this manner is different from the Euclidean model.
- We observe (3+1)-dimensional expanding space-time in some cases with appropriate $N, \ \gamma, \ d, \ \mu, \ m_f$
- Do we observe (3+1)-dimensional expanding space-time when we take the limit in which $N \to \infty, \ m_f \to 0, \ \mu \to 1, \ \gamma \to 0$ to obtain the target theory?