

Vafa-Witten Theory: Invariants, Floer Homologies, Higgs Bundles, a Geometric Langlands Correspondence, and Categorification

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Outline of Talk

- Introduction and Motivation
- Summary of Results
- Main Body of the Talk
- Conclusion

Introduction and Motivation

In this talk, we will discuss the **Vafa-Witten (VW) twist** of a 4d $\mathcal{N} = 4$ SYM gauge theory on M_4 .

The motivations for doing so are to:

- Derive a **novel VW invariant** of M_4 , and relate it to Gromov-Witten (GW) invariants via an $\mathcal{N} = (4, 4)$ A -model.
- Derive a **novel Vafa-Witten Atiyah-Floer correspondence**, and thereby a **physical proof and generalization** of the **Abouzaid-Manolescu conjecture** of hypercohomology of perverse sheaves in [1].
- Obtain a **Langlands dual** of the invariants, Floer homologies and hypercohomology stated hitherto, as well as the quantum and classical Geometric Langlands correspondence [2].
- Obtain a physical framework for **higher categorification** of VW theory.

Introduction and Motivation

This talk is based on

- Ong, Zhi-Cong and Tan, Meng-Chwan, *Vafa-Witten Theory: Invariants, Floer Homologies, Higgs Bundles, a Geometric Langlands Correspondence, and Categorification*, arXiv preprint arXiv:2203.17115 (ATMP in press)

Built on earlier insights in

- Bershadsky, Michael, et al, *Topological reduction of 4D SYM to 2D σ -models*, Nuclear Physics B 448.1-2, 166-186 (1995).
- Birmingham, Danny, et al, *Topological field theory*, Physics Reports 209.4-5, 129-340 (1991).
- Gukov, Sergei, *Surface operators and knot homologies*, New Trends in Mathematical Physics, 2009 - Springer.
- Kapustin, Anton and Witten, Edward. *Electric-magnetic duality and the geometric Langlands program*, Communications in Number Theory and Physics Volume 1, Number 1, (2007).

Summary of Results

1. If the scalar curvature of M_4 and the gauge group G are not simultaneously non-negative and locally a product of $SU(2)$'s, the theory localizes on a moduli space of configurations satisfying the VW equations. The invariant is the partition function

$$\mathcal{Z}_{\text{VW}, M_4}(\tau, G) = \sum_k a_k q^{m_k}, \quad q = e^{2\pi i \tau}$$

k denotes the k^{th} sector of the moduli space \mathcal{M}_{VW} of the VW equations. The number a_k is

$$a_k = \int_{\mathcal{M}_{\text{VW}}^k} \Omega^0 \wedge e(T_{\mathcal{M}_{\text{VW}}^k}), \quad \text{where } \Omega^0(\mathcal{M}_{\text{VW}}^k) = (1 + B^4)^{\dim_{\mathbb{C}} \mathcal{M}_{\text{VW}}^k}$$

B is a coordinate on $\mathcal{M}_{\text{VW}}^k(A, B)$, e is the signed Euler class of the tangent bundle $T_{\mathcal{M}_{\text{VW}}^k}$.

Summary of Results

2. Compactify VW theory on $M_4 = \Sigma \times C$ along C , where both Σ and C are closed Riemann surfaces of genus $g = 1$ and $g \geq 2$, respectively. We arrive at an A -model in complex structure I on Σ with $\mathcal{N} = (4, 4)$ supersymmetry and target space $\mathcal{M}_H^G(C)$, the moduli space of Hitchin's equations on C . Topological invariance implies a 4d-2d correspondence

$$\mathcal{Z}_{\text{VW}, M_4}(\tau, G) = \mathcal{Z}_{\text{GW}, \Sigma}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) = \sum_l \tilde{a}_l q^{\tilde{m}_l}$$

where l denotes the l^{th} sector of the moduli space $\mathcal{M}_{\text{maps}}$ of holomorphic maps for *genus one*, the rational number \tilde{a}_l is

$$\tilde{a}_l = \int_{\mathcal{M}_{\text{maps}}^l} e(\mathcal{V})$$

where e is the signed Euler class of the vector bundle \mathcal{V} with fiber $H^0(\Sigma, K \otimes \Phi^* T^* \mathcal{M}_{\text{maps}}^l)$ and canonical bundle K on Σ .

Summary of Results

3. Boundary VW theory on $M_4 = M_3 \times \mathbb{R}^+$, with M_3 a closed three-manifold in the temporal gauge allows us to recast the 4d theory as 1d supersymmetric quantum mechanics (SQM) on the space of all **complexified** connections on M_3 , with potential being the **complex** Chern-Simons functional. The VW partition function is then

$$\mathcal{Z}_{\text{VW}, M_4}(\tau, G) = \sum_k \mathcal{F}_{\text{VW}}^{G, \tau}(\Psi_{M_3}^k) = \sum_k \text{HF}_k^{\text{VW}}(M_3, G, \tau) = \mathcal{Z}_{\text{VW}, M_3}^{\text{Floer}}(\tau, G)$$

4. We then Heegaard split M_3 along the Riemann surface C to obtain an equivalent open A -model with boundaries given by Lagrangian (A, B, A) -branes L_0, L_1 in $\mathcal{M}_{\text{Higgs}}^G(C)$, which leads us to a VW Atiyah-Floer correspondence as

$$\text{HF}_*^{\text{VW}}(M_3, G, \tau) \cong \text{HF}_*^{\text{Lagr}}(\mathcal{M}_{\text{Higgs}}^G(C), L_0, L_1, \tau)$$

Summary of Results

5. This allows us to physically realize the Abouzaid-Manolescu conjecture for the hypercohomology $\mathrm{HP}^*(M_3)$ of a perverse sheaf of vanishing cycles in the moduli space of irreducible flat $SL(2, \mathbb{C})$ -connections on M_3 , which can be generalized to

$$\mathrm{HP}^*(M_3) \cong \mathrm{HF}_*^{\mathrm{inst}}(M_3, G_{\mathbb{C}}, \tau)$$

6. S -duality of $\mathcal{N} = 4$ VW theory implies a Langlands duality of the aforementioned invariants and Floer homologies.

7. Also, when we replace M_3 with $I \times C$ where $C \rightarrow 0$, from S -duality, we have a homological mirror symmetry of the category of A -branes

$$\mathrm{Cat}_{A\text{-branes}}(\tau, \mathcal{M}_{\mathrm{Higgs}}^G(C)) \longleftrightarrow \mathrm{Cat}_{A\text{-branes}}\left(-\frac{1}{n_{\mathrm{g}}\tau}, \mathcal{M}_{\mathrm{Higgs}}^{LG}(C)\right)$$

Summary of Results

8. If $\text{Re}(\tau) = 0$, we obtain a quantum geometric Langlands correspondence

$$\mathcal{D}_{-h^\vee\text{-mod}}^c(q, \text{Bun}_{G_C}) \longleftrightarrow \mathcal{D}_{-Lh^\vee\text{-mod}}^c\left(-\frac{1}{n_{\mathfrak{g}}q}, \text{Bun}_{L_G C}\right)$$

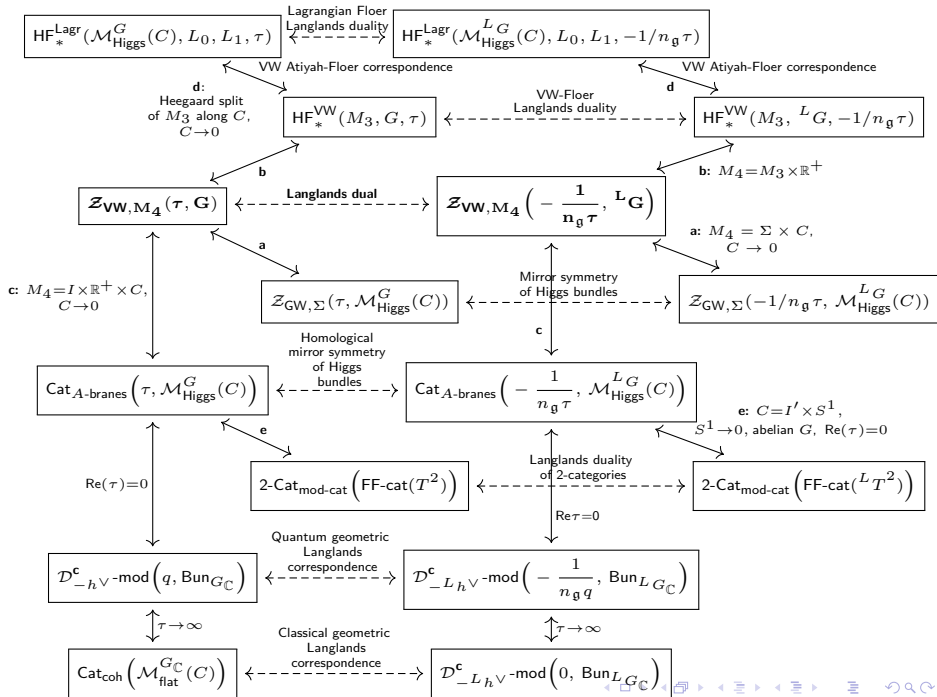
9. In the “classical” $\tau \rightarrow \infty$ limit, this becomes the classical geometric Langlands correspondence

$$\text{Cat}_{\text{coh}}(\mathcal{M}_{\text{flat}}^{G_C}(C)) \longleftrightarrow \mathcal{D}_{-Lh^\vee\text{-mod}}^c(0, \text{Bun}_{L_G C})$$

10. One can also observe that by successively adding boundaries to the underlying manifold as we have done, the VW invariant will be categorified as

$$\mathcal{Z}_{\text{VW}} \xrightarrow{\text{categorification}} \text{HF}_*^{\text{VW}} \xrightarrow{\text{categorification}} \text{Cat}_{A\text{-branes}} \xrightarrow{\text{categorification}} 2\text{-Cat}_{\text{mod-cat}}(\text{FF-cat}(T^2))$$

The last 2-category is that of module categories over the Fukaya- Floer category of T^2 when we further let $C = I' \times S^1$ where $S^1 \rightarrow 0$, and it is assigned to S^1 , just like how $\text{Cat}_{A\text{-branes}}$ is assigned to C , HF_*^{VW} is assigned to M_3 , and \mathcal{Z}_{VW} is assigned to M_4 .



LET'S EXPLAIN HOW WE GOT THESE RESULTS

- VW theory has a **single scalar supercharge** Q , whose BPS equations are obtained by setting the Q -variation of fermions to zero:

$$\begin{aligned} F_{\mu\nu}^+ + \frac{1}{2}[B_{\mu\nu}, C] + \frac{1}{4}[B_{\mu\rho}, B_{\lambda\nu}]g^{\rho\lambda} &= 0, \\ \mathcal{D}_\mu C + \mathcal{D}_\nu B^{\nu\mu} &= 0. \end{aligned} \tag{1}$$

- We then set $C = 0$ to ensure that there are no reducible gauge connections A .
- The 2-form B need not vanish if the scalar curvature of M_4 and the gauge group G are not simultaneously non-negative and locally a product of $SU(2)$'s, and we will assume this to be the case here.

A Vafa-Witten Invariant of M_4

- VW theory is a balanced TQFT (same number of fermion pair zero modes), and the path integral localizes to the moduli space \mathcal{M}_{VW} , whence the only non-vanishing topological invariant is the partition function

$$\mathcal{Z}_{\text{VW}, M_4}(\tau, G) = \sum_k a_k q^{m_k} \quad (2)$$

Here, $q = e^{2\pi i\tau}$, k denotes the k^{th} sector of \mathcal{M}_{VW} , the number a_k is

$$a_k = \int_{\mathcal{M}_{\text{VW}}^k} \Omega^0 \wedge e(T_{\mathcal{M}_{\text{VW}}^k}), \quad \text{where } \Omega^0(\mathcal{M}_{\text{VW}}^k) = (1 + B^4)^{\dim_{\mathbb{C}} \mathcal{M}_{\text{VW}}^k}$$

B is a coordinate on $\mathcal{M}_{\text{VW}}^k(A, B)$, e is the signed Euler class of the tangent bundle $T_{\mathcal{M}_{\text{VW}}^k}$, and m_k is the corresponding VW number

$$m_k = \frac{1}{8\pi^2} \int_{M_4} \text{Tr} \left(F_{(k)} \wedge F_{(k)} + dB_{(k)} \wedge \star DB_{(k)} + B_{(k)} \wedge d(\star DB_{(k)}) \right)$$

$\mathcal{N} = (4, 4)$ A-model, Higgs Bundles and GW Theory

- We consider a block diagonal metric g for $M_4 = \Sigma \times C$, where the genus are $g = 1$ and $g \geq 2$ for Σ and C , respectively.

$$g = \text{diag}(g_\Sigma, \epsilon g_C), \quad (3)$$

where ϵ is a small parameter to deform g_C .

- When $\epsilon \rightarrow 0$, in order for the action to remain well-defined, i.e. finite, we obtain the following conditions along C :

$$F_C - \varphi \wedge \varphi = D\varphi = D^*\varphi = 0. \quad (4)$$

Here, A_C and a section $\varphi \in \Omega^1(C)$ modulo gauge transformations span **Hitchin's moduli space** $\mathcal{M}_H^G(C)$.

- We get a sigma model on Σ with a map $\Phi(X, Y) : \Sigma \rightarrow \mathcal{M}_H^G(C)$, where the bosonic scalars (X, Y) on Σ correspond to (A_C, φ_C) .
- The BPS equations of the sigma model are **holomorphic maps**, obtained from the dimensional reduction of (1):

$$\partial_{\bar{z}} X^i = \partial_{\bar{z}} Y^i = 0. \quad (5)$$

$\mathcal{N} = (4, 4)$ A -model, Higgs Bundles and GW Theory

- This is an A -model, and it can be further shown that the complex structure is I , whence the target space $\mathcal{M}_H^G(C) = \mathcal{M}_{\text{Higgs}}^G(C)$.
- The topological invariant is the partition function, a GW invariant:

$$\mathcal{Z}_{\text{GW}, \Sigma}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) = \sum_l \tilde{a}_l q^{\tilde{m}_l} \quad (6)$$

l denotes the l^{th} sector of $\mathcal{M}_{\text{maps}}$ for *genus one* Σ , the rational number \tilde{a}_l is given by [6]

$$\tilde{a}_l = \int_{\mathcal{M}_{\text{maps}}^l} e(\mathcal{V})$$

$\mathcal{N} = (4, 4)$ A-model, Higgs Bundles and GW Theory

where e is the signed Euler class of the vector bundle \mathcal{V} with fiber $H^0(\Sigma, K \otimes \Phi^* T^* \mathcal{M}_{\text{maps}}^l)$ and canonical bundle K on Σ , and \tilde{m}_l is the corresponding worldsheet instanton number given by

$$\tilde{m}_l = \frac{1}{2\pi} \int_{\Sigma} \Phi_l^*(\omega_I)$$

- Topological invariance implies a **4d-2d correspondence**:

$$\mathcal{Z}_{\text{VW}, M_4}(\tau, G) = \mathcal{Z}_{\text{GW}, \Sigma}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) \quad (7)$$

between the VW invariant of $M_4 = \Sigma \times C$ and the GW invariant of $\mathcal{M}_{\text{Higgs}}^G(C)$.

1d SQM from Boundary Vafa-Witten Theory

- Let $M_4 = M_3 \times \mathbb{R}^+$, where the M_3 boundary is a closed three-manifold, and \mathbb{R}^+ is the 'time' coordinate. Using the temporal gauge $A^0 = 0$, and exploiting the self-duality of the 2-form $B_{\mu\nu}$, VW equations in (1) become

$$\begin{aligned}\dot{A}^i + \frac{1}{2}\epsilon^{ijk}(F_{jk} - [B_j, B_k]) &= 0, \\ \dot{B}^i + \epsilon^{ijk}(\partial_j B_k + [A_j, B_k]) &= 0.\end{aligned}\tag{8}$$

- If we define a complexified connection $\mathcal{A} = A + iB \in \Omega^1(M_3)$, of a $G_{\mathbb{C}}$ -bundle on M_3 , we can rewrite (8) as a **flow equation**:

$$\frac{d\mathcal{A}^i}{dt} + sg_{\mathfrak{A}}^{ij} \frac{\partial V(\mathcal{A})}{\partial \mathcal{A}^j} = 0.\tag{9}$$

- \mathfrak{A} is the space of complexified connections \mathcal{A} with metric $g_{\mathfrak{A}}^{ij}$, s is a tuneable parameter, and

$$V(\mathcal{A}) = -\frac{1}{4\pi^2} \int_{M_3} \text{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right).\tag{10}$$

1d SQM from Boundary Vafa-Witten Theory

- The 4d action of boundary VW theory can be rewritten as a **1d SQM model** with target space \mathfrak{A} :

$$S_{\text{VW}}^{\text{bdry}} = \frac{1}{e^2} \int dt \left| \frac{d\mathcal{A}^i}{dt} + sg_{\mathfrak{A}}^{ij} \frac{\partial V(\mathcal{A})}{\partial \mathcal{A}^j} \right|^2 + \dots \quad (11)$$

- The partition function of boundary VW theory will localize onto the configurations that minimize the above term.
- That is, it will be an **algebraic count of time-invariant critical points of the complex Chern-Simons functional**, corresponding to flat $G_{\mathbb{C}}$ -connections on M_3 .
- The complex Chern-Simons functional $V(\mathcal{A})$ is a Morse functional, whose **critical points generate a Floer complex**.
- VW flow lines between critical points, as described by the gradient flow equation (9), can be interpreted as Floer differentials, whence **the number of outgoing flow lines at each critical point is the degree of the corresponding chain in the complex**.

Vafa-Witten Floer Homology Assigned to M_3

- The partition function of boundary VW theory is originally expressed as

$$\mathcal{Z}_{\text{VW},M_4}(\tau, G) = \int_{\mathcal{M}_{\text{VW}}} \mathcal{F}(\Psi_{M_3}) e^{-S_{\text{VW}}^{\text{bdry}}} = \sum_k \left\langle \mathcal{F}_{\text{VW}}^{G,\tau}(\Psi_{M_3}^k) \right\rangle. \quad (12)$$

- By comparing with the results from the 1d SQM perspective, we have

$$\mathcal{F}_{\text{VW}}^{G,\tau}(\Psi_{M_3}^k) \in \text{HF}_k^{\text{VW}}(M_3, G, \tau), \quad (13)$$

where the HF's are the **VW Floer homology classes**.

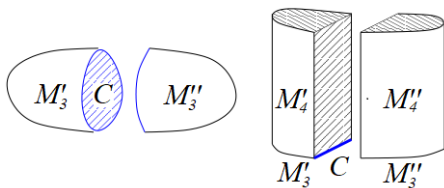
- Thus, boundary VW theory allows us to define a novel VW Floer homology assigned to M_3 as

$$\boxed{\mathcal{Z}_{\text{VW},M_4}(\tau, G) = \sum_k \mathcal{F}_{\text{VW}}^{G,\tau}(\Psi_{M_3}^k) = \sum_k \text{HF}_k^{\text{VW}}(M_3, G, \tau) = \mathcal{Z}_{\text{VW},M_3}^{\text{Floer}}(\tau, G)} \quad (14)$$

where the second and third expressions are understood to be expectation values of operators.

A Vafa-Witten Atiyah-Floer Correspondence

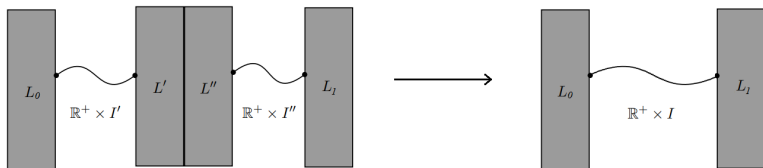
- We perform a Heegaard split of $M_3 = M'_3 \cup_C M''_3$ along C by writing $M_4 = (\mathbb{R}^+ \times I' \times C) \cup_C (\mathbb{R}^+ \times I'' \times C)$:



- When $C \rightarrow 0$, we have an open A -model in complex structure I on $\mathbb{R}^+ \times I'$ and $\mathbb{R}^+ \times I''$, with target space $\mathcal{M}_{\text{Higgs}}^G(C)$.
- These describe open strings with worldsheets $\mathbb{R}^+ \times I'$ and $\mathbb{R}^+ \times I''$ that propagate in $\mathcal{M}_{\text{Higgs}}^G(C)$ and end on A -branes. Specifically, we have a certain (A, B, A) -brane that is an A -brane in $\mathcal{M}_H^G(C)$ in complex structure K , that will correspond to **flat $G_{\mathbb{C}}$ -connections on C which can be extended to M'_3** , as required.

A Vafa-Witten Atiyah-Floer Correspondence

- With two split pieces M_4' and M_4'' , when $C \rightarrow 0$, we have two strings, each ending on pairs of Lagrangian branes (L_0, L') and (L'', L_1) .



- Gluing the open worldsheets together along their common boundary L' and L'' gives a single A -model on $\mathbb{R}^+ \times I$, with a single string extending from L_0 to L_1 , which is equivalent to gluing M_4' and M_4'' along $C \times \mathbb{R}^+$.
- Next, we recast the A -model here as an SQM model, where the target space is $\mathcal{P}(L_0, L_1)$, the **space of smooth trajectories from L_0 to L_1** (arising from the interval I that connects them).

A Vafa-Witten Atiyah-Floer Correspondence

- The 2d BPS equation for this open A -model can be written as a **gradient flow equation** for the 1d SQM model

$$\frac{\partial Z^l}{\partial t} + i \frac{\partial Z^l}{\partial s} = 0. \quad (15)$$

- Critical points correspond to **stationary trajectories in** $\mathcal{P}(L_0, L_1)$, i.e., the intersection points of L_0 and L_1 , which generate the chains of the Lagrangian Floer complex.
- Intersection points belong to Lagrangian Floer homology classes

$$\sum_i (L_0 \cap L_1)_i^n \in \text{HF}_n^{\text{Lagr}}(\mathcal{M}_{\text{Higgs}}^G(C), L_0, L_1). \quad (16)$$

- Floer differentials are the **outgoing flow lines at each** $L_0 \cap L_1$, **which number would be the degree of the corresponding chain in the complex.**

A Vafa-Witten Atiyah-Floer Correspondence

- The partition function of the open A -model is then given by

$$\mathcal{Z}_{A,L}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) = \sum_n \text{HF}_n^{\text{Lagr}}(\mathcal{M}_{\text{Higgs}}^G(C), L_0, L_1, \tau). \quad (17)$$

- Topological invariance gives an **equivalence of partition functions**, which implies

$$\sum_k \text{HF}_k^{\text{VW}}(M_3, G, \tau) = \sum_n \text{HF}_n^{\text{Lagr}}(\mathcal{M}_{\text{Higgs}}^G(C), L_0, L_1, \tau). \quad (18)$$

- It can be shown that we can identify the k and n indices, whence we have

$$\boxed{\text{HF}_*^{\text{VW}}(M_3, G, \tau) \cong \text{HF}_*^{\text{Lagr}}(\mathcal{M}_{\text{Higgs}}^G(C), L_0, L_1, \tau)} \quad (19)$$

This gives a Vafa-Witten Atiyah-Floer Correspondence.

The Abouzaid-Manolescu Conjecture and its Generalization

- (A, B, A) -branes of the open A -model can be interpreted as Lagrangian branes in $\mathcal{M}_H^G(C)$ in complex structure K i.e., $\mathcal{M}_{\text{flat}}^{G_{\mathbb{C}}}(C)$, the moduli space of irreducible flat $G_{\mathbb{C}}$ -connections on C . The **VW Atiyah-Floer correspondence is then the Atiyah-Floer correspondence for $G_{\mathbb{C}}$ -instantons.**

$$\boxed{\text{HF}_*^{\text{inst}}(M_3, G_{\mathbb{C}}, \tau) \cong \text{HF}_*^{\text{Lagr}}(\mathcal{M}_{\text{flat}}^{G_{\mathbb{C}}}(C), L_0, L_1, \tau)} \quad (20)$$

- $\text{HP}^*(M_3)$ is the hypercohomology of the perverse sheaf of vanishing cycles in $X_{\text{irr}}(C)$, the moduli space of irreducible flat $SL(2, \mathbb{C})$ -connections on C .
- There is a one-to-one correspondence between the gradings of $\text{HP}^*(M_3)$ and $\text{HF}_*^{\text{Lagr}}$, and the chains underlying their complexes, whence we can identify

$$\text{HP}^*(M_3) \cong \text{HF}_*^{\text{Lagr}}(X_{\text{irr}}(C), L_0, L_1, \tau), \quad (21)$$

in agreement with [3, Remark 6.15].

The Abouzaid-Manolescu Conjecture and its Generalization

- According to (20) and (21), for $G_{\mathbb{C}} = SL(2, \mathbb{C})$, we have

$$\boxed{HP^*(M_3) \cong HF_*^{\text{inst}}(M_3, SL(2, \mathbb{C}), \tau)} \quad (22)$$

This is just the the Abouzaid-Manolescu conjecture in [1].

- The Atiyah-Floer correspondence for $G_{\mathbb{C}}$ instantons in (20), and a $G_{\mathbb{C}}$ generalization of (21) in [3, Remark 6.15], imply that a generalization of the Abouzaid-Manolescu conjecture is also possible as

$$\boxed{HP^*(M_3, G_{\mathbb{C}}) \cong HF_*^{\text{inst}}(M_3, G_{\mathbb{C}}, \tau)} \quad (23)$$

Langlands Duality of Invariants and Floer Homologies

- $\mathcal{N} = 4$ supersymmetric Yang-Mills theories have an $SL(2, \mathbb{Z})$ (or a subgroup thereof) symmetry on τ relating a theory to its S -dual, giving

$$\mathcal{Z}_{\text{VW}, M_4}(\tau, G) \longleftrightarrow \mathcal{Z}_{\text{VW}, M_4}\left(-\frac{1}{n_{\mathfrak{g}}\tau}, {}^L G\right)$$

$$\mathcal{Z}_{\text{GW}, \Sigma}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) \longleftrightarrow \mathcal{Z}_{\text{GW}, \Sigma}\left(-\frac{1}{n_{\mathfrak{g}}\tau}, \mathcal{M}_{\text{Higgs}}^{{}^L G}(C)\right)$$

$$\text{HF}_*^{\text{VW}}(M_3, G, \tau) \longleftrightarrow \text{HF}_*^{\text{VW}}(M_3, {}^L G, -1/n_{\mathfrak{g}}\tau)$$

$$\text{HF}_*^{\text{Lagr}}(\mathcal{M}_{\text{Higgs}}^G(C), L_0, L_1, \tau) \longleftrightarrow \text{HF}_*^{\text{Lagr}}(\mathcal{M}_{\text{Higgs}}^{{}^L G}(C), L_0, L_1, -1/n_{\mathfrak{g}}\tau)$$

$$\text{HP}^*(M_3, G_{\mathbb{C}}, \tau) \longleftrightarrow \text{HP}^*(M_3, {}^L G_{\mathbb{C}}, -1/n_{\mathfrak{g}}\tau)$$

A Quantum and Classical Geometric Langlands Correspondence

- If we let $M_4 = I \times \mathbb{R}^+ \times C$ with $C \rightarrow 0$, S -duality gives a homological mirror symmetry of the category of A -branes

$$\text{Cat}_{A\text{-branes}}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) \longleftrightarrow \text{Cat}_{A\text{-branes}}\left(-\frac{1}{n_g \tau}, \mathcal{M}_{\text{Higgs}}^{LG}(C)\right) \quad (24)$$

- When $\text{Re}(\tau) = 0$, the category of A -branes can be identified with a category of twisted D -modules on $\text{Bun}_{G_C}(C)$ with parameter $q = \tau$ [4], whence we have

$$\mathcal{D}_{-h^\vee\text{-mod}}^c(q, \text{Bun}_{G_C}) \longleftrightarrow \mathcal{D}_{-Lh^\vee\text{-mod}}^c\left(-\frac{1}{n_g q}, \text{Bun}_{L G_C}\right) \quad (25)$$

a quantum geometric Langlands correspondence [2]. In the “classical limit” of $q \rightarrow \infty$, we have the classical correspondence

$$\text{Cat}_{\text{coh}}(\mathcal{M}_{\text{flat}}^{G_C}(C)) \longleftrightarrow \mathcal{D}_{-Lh^\vee\text{-mod}}^c(0, \text{Bun}_{L G_C}) \quad (26)$$

Categorification of Vafa-Witten Theory

- Categorification is naturally realized in our physical framework:

VW theory on M_4	\rightsquigarrow	number	\mathcal{Z}_{VW}
VW theory on $\mathbb{R}^+ \times M_3$	\rightsquigarrow	vector	HF_*^{VW}
VW theory on $\mathbb{R}^+ \times I \times C$	\rightsquigarrow	1-category	$\text{Cat}_{A\text{-branes}}$
VW theory on $\mathbb{R}^+ \times I \times I' \times S^1$	\rightsquigarrow	2-category	2-Cat
VW theory on $\mathbb{R}^+ \times I \times I' \times [0, 1]$	\rightsquigarrow	3-category	3-Cat .

(27)

- Notice from our discussion hitherto that as we go down the list, the categories get assigned to M_3 , C , ..., and are determined by the category of boundaries of the effective 1d, 2d, ... theory on \mathbb{R}^+ , $\mathbb{R}^+ \times I$, ...
- Therefore, 2-Cat will be the 2-category of 2d boundaries of the 3d theory on $\mathbb{R}^+ \times I \times I'$ given by VW theory compactified on S^1 , that is assigned to S^1 .

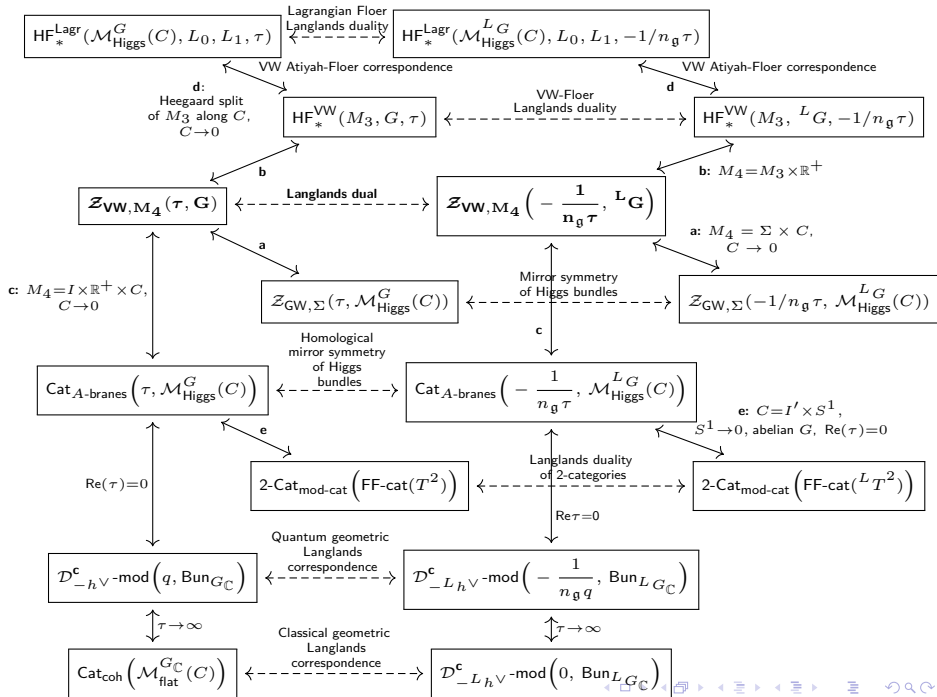
Categorification of Vafa-Witten Theory

- These 2d boundaries can be realized as surface defects.
- For abelian G and $\text{Re}(\tau) = 0$, the category of such surface defects is the 2-category $2\text{-Cat}_{\text{mod-cat}}(\text{FF-cat}(T^2))$ of module categories over the Fukaya-Floer category of T^2 [5].
- S -duality of VW theory then implies that they enjoy a Langlands duality

$$\boxed{2\text{-Cat}_{\text{mod-cat}}(\text{FF-cat}(T^2)) \longleftrightarrow 2\text{-Cat}_{\text{mod-cat}}(\text{FF-cat}({}^L T^2))} \quad (28)$$

where ${}^L T^2$ is the dual torus with the radii of the circles inverted.

- Similarly, the 3-Cat will be the 3-category of 3d boundaries of VW theory on $\mathbb{R}^+ \times I \times I' \times [0, 1]$ compactified on two points, that is assigned to a point.



Conclusion

- We have physically derived a novel VW invariant of M_4 .
- We have a 4d-2d correspondence between VW invariants and GW invariants.
- We have recast boundary VW theory as a 1d SQM model, thereby physically deriving a novel Vafa-Witten Floer homology.
- We went further to physically derive a Vafa-Witten Atiyah-Floer correspondence, which in turn allowed us to physically realize and generalize the Abouzaid-Manolescu conjecture in [1].
- S -duality of VW theory allowed us to obtain Langlands duals of all the invariants and Floer homologies. In certain cases, we could also obtain a quantum and classical geometric Langlands correspondence.
- Our physical framework also allows for a higher categorification of VW theory, whence S -duality again implies a Langlands duality of the relevant higher categories.

THANKS FOR LISTENING!

- [1] Abouzaid, Mohammed, and Ciprian Manolescu, *A sheaf-theoretic model for $SL(2, \mathbb{C})$ Floer homology*, *Journal of the European Mathematical Society* **93** (2020) 3641-3695.
- [2] Frenkel, Edward, *Lectures on the Langlands program and conformal field theory*, *Frontiers in number theory, physics, and geometry II* **152** (2007) 387-533.
- [3] Brav, Christopher, et al. "Symmetries and stabilization for sheaves of vanishing cycles", arXiv preprint arXiv:1211.3259 (2012).
- [4] Kapustin, Anton, *A note on quantum geometric Langlands duality, gauge theory, and quantization of the moduli space of flat connections*, arXiv preprint arXiv:0811.3264 (2008).
- [5] Kapustin, Anton, Kevin Setter, and Ketan Vyas, *Surface operators in four-dimensional topological gauge theory and Langlands duality*, arXiv preprint arXiv:1002.0385 (2010).
- [6] Hori, Kentaro, Katz, Sheldon, Pandharipande, Rahul and Vakil, Ravi and Zaslow, Eric *Mirror symmetry*, *American Mathematical Soc. Vol1*(2003)