Generalized conformal geometry, CFT conservation laws and holography

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Introduction

- How is the bulk dynamics of gravity emergent from CFT?
- The Hamiltonian of AdS$_{d+1}$ gravity is the sum of the constraint from the diffeomorphism along the evolution direction

$$\mathbb{R}_{(d)} + \ell^{-2}[d(d-1) - 2\mathbb{k}^{[\mu_1}{}_{[\mu_1}\mathbb{k}^{\mu_2]}{}_{\mu_2]}] = 0$$

and the constraints perpendicular to the evolution direction

$$\nabla_{[\mu}\mathbb{k}^{\mu}{}_{\nu]} = 0$$

where $\mathbb{k}_{\mu\nu}$ is the extrinsic curvature of the "equal time" hypersurface.

Introduction

- It is natural to identify these constraints with the Weyl and diffeomorphism conservation laws in the dual CFT

$$T^{\mu}{}_{\mu} = \mathcal{A}_{\text{Weyl}} \,, \qquad\qquad \nabla_{\mu} T^{\mu}{}_{\nu} = 0$$

- What is the explicit relation between $T^{\mu}{}_{\nu}$ and $\mathbb{k}^{\mu}{}_{\nu}$??? for example

$$T^{\mu}{}_{\nu} \sim \mathbb{k}^{\mu}{}_{\nu} - \delta^{\mu}_{\nu} \mathbb{k}^{\rho}{}_{\rho} \; ?$$

- A puzzle:
  The constraint for radial evolution is quadratic in $\mathbb{k}$, while the CFT Weyl conservation law is linear in $T^{\mu}{}_{\nu}$.
- To solve such a problem, various approaches have been suggested, e.g. the $T\bar{T}$ deformation.

Introduction

- In our approach, the $SO(2, d)$ symmetry plays the essential role.
- A lesson from AdS/CFT:
  Global symmetry in the boundary field theory
  $\Rightarrow$ Gauge symmetry in the bulk gravity theory
- For a CFT at the d-dimensional boundary, we have the conformal symmetry $SO(2, d)$ as the global symmetry.
- Thus we would expect the local $SO(2, d)$ gauge symmetry in the bulk theory.
- In our past works, the $SO(2, d)$ gauge theory formula of AdS gravity is established for general d.
- Correspondingly, we will also establish the $SO(2, d)$ gauge field formalism of CFT background field.
- In this framework, the bulk Hamiltonian constraints can be naturally reproduced by CFT conservation laws.

Einstein Gravity

- Written in terms of vielbein, the Einstein-Hilbert action is

$$S[e] = \int \epsilon_{a_1 \cdots a_D} \left[\Theta^{a_1 a_2} + \frac{(D-2)}{D\,\ell^2} e^{a_1} \wedge e^{a_2} \right] \wedge e^{a_3} \wedge \cdots \wedge e^{a_D}$$

  where the spin connection $\omega^a{}_b$ is decided by the torsion free condition

$$\mathrm{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$$

  and the curvature is decided by the spin connection

$$\Theta^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

- The Einstein equation is

$$\left(\Theta^{[a_1 a_2} + \ell^{-2} e^{[a_1} \wedge e^{a_2} \right) \wedge e^{a_3} \wedge \cdots \wedge e^{a_{D-1}]} = 0$$

Palatini's 1-st Order Formula

- Palatini: Treating the spin connection $\omega^a{}_b$ as independent variables

$$S[e,\omega] = \int \epsilon_{a_1\cdots a_D}\left[\Theta^{a_1a_2} + \frac{(D-2)}{D\,\ell^2}e^{a_1}\wedge e^{a_2}\right]\wedge e^{a_3}\wedge\cdots\wedge e^{a_D}$$

- The EOM's are

$$\left(\Theta^{[a_1a_2} + \ell^{-2}e^{[a_1}\wedge e^{a_2}\right)\wedge e^{a_3}\wedge\cdots\wedge e^{a_{D-1}]} = 0$$

$$\mathrm{D}e^{[a_1}\wedge e^{a_2}\wedge\cdots\wedge e^{a_{D-2}]} = 0$$

- If the vielbein e^a is not degenerate, the torsion free condition will be automatically implied by the 2nd EOM
- The 1-st order formalism is equivalent to the original second order formalism at least in the classical level.

Uplift to $SO(2,d)$ via ruler

- Basic idea of $SO(1,d) \to SO(2,d)$:

$$A^a{}_b \sim \omega^a{}_b\,, \qquad A^a{}_\bullet \sim \ell^{-1} e^a$$

- In the covariant treatment, we impose a ruler field $Y^{\hat{\alpha}}$ in the vector representation of $SO(2,d)$ to point out the way of $SO(2,d) \to SO(1,d)$
- Ruler field satisfies the constraints $Y^{\hat{\alpha}} Y_{\hat{\alpha}} = -\ell^2$. Thus no new DOF is introduced.
- In the Einstein gauge $Y^a = 0\,, \qquad Y^\bullet = \ell,$

$$\begin{aligned} \mathrm{D}Y^a &= e^a\,, & \mathrm{D}Y^\bullet &= 0\,,\\ \mathrm{D}\mathrm{D}Y^a &= F^a{}_\bullet Y^\bullet = \mathrm{D}e^a\,, & \mathrm{D}\mathrm{D}Y^\bullet &= F^\bullet{}_b Y^b = 0\,. \end{aligned}$$

- The Palatini EOM's can be unified $SO(2,d)$ covariantly as

$$F^{[\hat{\alpha}_1\hat{\alpha}_2} \wedge \mathrm{D}Y^{\hat{\alpha}_3} \wedge \cdots \wedge \mathrm{D}Y^{\hat{\alpha}_d]} = 0\,.$$

SO(2, d) action

- We can realize the previous uplift in terms of a gauge invariant action constructed by $SO(2,d)$ covariant fields

$$\int \epsilon_{a_1\cdots a_D}\left[\Theta^{a_1a_2}+\frac{(D-2)}{D\,\ell^2}e^{a_1}\wedge e^{a_2}\right]\wedge e^{a_3}\wedge\cdots\wedge e^{a_D}$$
$$\sim\int \epsilon_{\hat\alpha_1\cdots\hat\alpha_{D+1}}\left[F^{\hat\alpha_1\hat\alpha_2}-\frac{2}{D\,\ell^2}\mathrm{D}Y^{\hat\alpha_1}\wedge \mathrm{D}Y^{\hat\alpha_2}\right]\wedge \mathrm{D}Y^{\hat\alpha_3}\wedge\cdots\wedge \mathrm{D}Y^{\hat\alpha_D}\,Y^{\hat\alpha_{D+1}}$$

- In the $D=3$ C-S formalism, the Y field is implicitly imposed when one decides the vielbein from the gauge field $e^a=A^a_L+A^a_R$.
- The metric is induced by the $Y^{\hat\alpha}$ field

$$ds^2=\mathrm{D}_MY^{\hat\alpha}\,\mathrm{D}_NY_{\hat\alpha}\,dx^Mdx^N\,.$$

$SO(2,d)$ action

- The bulk covariant action of AdS gravity

$$
\begin{aligned}
S^{\text{cov}}_{(d+1)} &= \int_M \mathcal{L}^{\text{cov}}_{(d+1)} \\
&= \frac{1}{2\kappa^2 \ell\,(d-1)!} \int_M \epsilon_{\hat{\alpha}\hat{\beta}\hat{\alpha}_1\cdots\hat{\alpha}_d} \left[F^{\hat{\alpha}\hat{\beta}} - \frac{2}{(d+1)\ell^2} \mathrm{D}Y^{\hat{\alpha}} \wedge \mathrm{D}Y^{\hat{\beta}}\right] \wedge \mathrm{D}Y^{\hat{\alpha}_1} \wedge \cdots \wedge \mathrm{D}Y^{\hat{\alpha}_{d-1}}\, Y^{\hat{\alpha}_d}
\end{aligned}
$$

- The EOM's are

$$F^{[\hat{\alpha}_1\hat{\alpha}_2} \wedge \mathrm{D}Y^{\hat{\alpha}_3} \wedge \cdots \wedge \mathrm{D}Y^{\hat{\alpha}_{D-1}]} = 0\,,$$

  as well as the EOM from δY

$$\epsilon_{\hat{\alpha}\hat{\beta}\hat{\alpha}_1\cdots\hat{\alpha}_d} Y^{\hat{\beta}} \mathrm{D}(F^{\hat{\alpha}_1\hat{\alpha}_2} \wedge \mathrm{D}Y^{\hat{\alpha}_2} \wedge \cdots \wedge \mathrm{D}Y^{\hat{\alpha}_d}) = 0\,.$$

- Providing the 1st EOM, the 2nd EOM will be automatically satisfied. In fact, it vanishes only when the torsion free condition is satisfied.
- Thus the new EOM will not introduce any further constraints, and the above system is equivalent to the original Einstein gravity classically.

The bulk $SO(2,d)$ structure

- Starting from Y and A, we can use $\{Y, \mathrm{D}_M Y\}$ as an intrinsic basis for $SO(2,d)$ vector space since

$$Y^{\hat{\alpha}} Y_{\hat{\alpha}} = -\ell^2 \,, \qquad \mathrm{D}_M Y^{\hat{\alpha}} \mathrm{D}_N Y_{\hat{\alpha}} = g_{MN} \,, \qquad Y^{\hat{\alpha}} \mathrm{D}_N Y_{\hat{\alpha}} = 0 \,.$$

- How to express the derivatives of the basis it terms of the base itself $\mathrm{D}_{M_2} \mathrm{D}_{M_1} Y^{\hat{\alpha}} = (\cdots) Y^{\hat{\alpha}} + (\cdots) \mathrm{D}_N Y^{\hat{\alpha}}$?
- The answer implies the metric+torsion structure:

$$\mathrm{D}_{M_2} \mathrm{D}_{M_1} Y^{\hat{\alpha}} = \ell^{-2} g_{M_1 M_2} Y^{\hat{\alpha}} + \Gamma^N{}_{M_1 M_2} \mathrm{D}_N Y^{\hat{\alpha}} \,,$$

where

$$\begin{aligned} \Gamma^M{}_{NP} &= \hat{\Gamma}^M{}_{NP} + t^M{}_{NP} \,, \\ \hat{\Gamma}_{M M_1 M_2} &= \frac{1}{2} \left(\partial_{M_2} g_{N M_1} + \partial_{M_1} g_{N M_2} - \partial_N g_{M_1 M_2} \right) \,, \\ t_{N M_1 M_2} &= \frac{1}{2} \Big[(F_{N M_1})^{\hat{\beta}_1 \hat{\beta}_2} \mathrm{D}_{M_2} Y_{\hat{\beta}_1} + (F_{N M_2})^{\hat{\beta}_1 \hat{\beta}_2} \mathrm{D}_{M_1} Y_{\hat{\beta}_1} - (F_{M_1 M_2})^{\hat{\beta}_1 \hat{\beta}_2} \mathrm{D}_N Y_{\hat{\beta}_1} \Big] Y_{\hat{\beta}_2} \,. \end{aligned}$$

The bulk $SO(2,d)$ structure

- Decomposed on the intrinsic basis, the field strength is

$$\begin{aligned} F_{M_1M_2} &= \left(\frac{1}{2}R^{N_1N_2}{}_{M_1M_2} + \ell^{-2}\delta^{N_1}_{[M_1}\delta^{N_2}_{M_2]}\right)\tau_{N_1N_2} - 2\ell^{-1}t^{N}{}_{[M_1M_2]}\tau_N \\ &= \left(\frac{1}{2}\hat{R}^{N_1N_2}{}_{M_1M_2} + \hat{\nabla}_{[M_1}t^{N_1N_2}{}_{M_2]} + t^{N_1}{}_{N[M_1}t^{NN_2}{}_{M_2]} + \ell^{-2}\delta^{N_1}_{[M_1}\delta^{N_2}_{M_2]}\right)\tau_{N_1N_2} \\ &\quad -2\ell^{-1}t^{N}{}_{[M_1M_2]}\tau_N\,. \end{aligned}$$

- Thus the $SO(2,d)$ covariantized torsion free condition is

$$(F_{M_1M_2})^{\hat{\alpha}}{}_{\hat{\beta}}Y^{\hat{\beta}} = 0 \quad \Leftrightarrow \quad [\mathrm{D}_{M_1},\mathrm{D}_{M_2}]Y^{\hat{\alpha}} = 0\,.$$

- The Bianchi identity ⇔ Usual Bianchi identities for GR

$$\begin{aligned} 0 &= \mathrm{D}_{[M_3}F_{M_1M_2]} \\ &= \left(\hat{\nabla}_{[M_3}\hat{R}^{N_1N_2}{}_{M_1M_2]} - \hat{R}^{N}{}_{[M_2M_3M_1]}t^{N_1N_2}{}_{N}\right)\mathrm{D}_{N_1}Y^{[\hat{\alpha}}\mathrm{D}_{N_2}Y^{\hat{\beta}]} \\ &\quad -2\ell^{-2}\hat{R}^{N}{}_{[M_3M_1M_2]}Y^{[\hat{\alpha}}\mathrm{D}_NY^{\hat{\beta}]} \end{aligned}$$

- Substituting into the action, it recovers the Einstein-Hilbert action directly without any particular gauge choice.

Projecting bulk $SO(2,d)$ structure on Σ

- The dual CFT with a local energy scale $z = \zeta(x)$ is defined on the hypersurface $\Sigma = \{(x,z)|z = \zeta(x)\}$.
- The ruler and gauge field on Σ is obtained by the pull back

$$\mathbb{Y}^{\hat{\alpha}}(x;\Sigma) = Y^{\hat{\alpha}}(x,\zeta(x))\,, \qquad \mathbb{A}(x;\Sigma) = A(x,\zeta(x))\,.$$

- It satisfies

$$\mathbb{Y}^{\hat{\alpha}}\mathbb{Y}_{\hat{\alpha}} = -\ell^2\,, \qquad \mathbb{D}_\mu\mathbb{Y}^{\hat{\alpha}}\mathbb{D}_\nu\mathbb{Y}_{\hat{\alpha}} = \mathfrak{g}_{\mu\nu}\,, \qquad \mathbb{Y}^{\hat{\alpha}}\mathbb{D}_\mu\mathbb{Y}_{\hat{\alpha}} = 0\,.$$

where $\mathfrak{g}_{\mu\nu}$ is exactly the induced metric on Σ

Projecting bulk $SO(2,d)$ structure on Σ

- To establish the full basis, we still need $\mathbb{N}^{\hat\alpha}$ which satisfies

$$\mathbb{N}^{\hat\alpha}\mathbb{N}_{\hat\alpha} = \ell^2\,, \quad \mathbb{N}^{\hat\alpha}\mathbb{Y}_{\hat\alpha} = 0\,, \quad \mathbb{N}^{\hat\alpha}\mathbb{D}_\mu\mathbb{Y}_{\hat\alpha} = 0\,.$$

  It can be expressed manifestly as

$$\mathbb{N}_{\hat\alpha} = \frac{1}{d!}\epsilon^{\mu_1\cdots\mu_d}\epsilon_{\hat\alpha\hat\alpha_1\cdots\hat\alpha_{d+1}}\mathbb{D}_{\mu_1}\mathbb{Y}^{\hat\alpha_1}\cdots\mathbb{D}_{\mu_d}\mathbb{Y}^{\hat\alpha_d}\mathbb{Y}^{\hat\alpha_{d+1}}\,.$$

- One additional gauge invariant quantity is

$$\mathbb{D}_\mu\mathbb{N}^{\hat\alpha}\,\mathbb{D}_\nu\mathbb{Y}_{\hat\alpha} = \mathbb{k}_{\mu\nu}$$

  which is just the extrinsic curvature of Σ.

Projecting bulk $SO(2,d)$ structure on Σ

- Taking derivatives of the basis gives

$$\begin{aligned}\mathbb{D}_\mu \mathbb{N}^{\hat\alpha} &= \mathbb{k}_\mu{}^\nu \mathbb{D}_\nu \mathbb{Y}^{\hat\alpha}\,,\\ \mathbb{D}_\mu \mathbb{D}_\nu \mathbb{Y}^{\hat\alpha} &= \ell^{-2}\mathbb{g}_{\mu\nu}\mathbb{Y}^{\hat\alpha} + \Gamma^\rho{}_{\nu\mu}\mathbb{D}_\rho \mathbb{Y}^{\hat\alpha} - \ell^{-2}\mathbb{k}_{\mu\nu}\mathbb{N}^{\hat\alpha}\,,\end{aligned}$$

where

$$\begin{aligned}\mathbb{\Gamma}^\mu{}_{\nu\rho} &= \hat{\mathbb{\Gamma}}^\mu{}_{\nu\rho} + \mathbb{t}^\mu{}_{\nu\rho}\,,\\ \hat{\mathbb{\Gamma}}^\mu{}_{\mu_1\mu_2} &= \tfrac{1}{2}\mathbb{g}^{\mu\nu}\left(\partial_{\mu_2}\mathbb{g}_{\nu\mu_1} + \partial_{\mu_1}\mathbb{g}_{\nu\mu_2} - \partial_\nu \mathbb{g}_{\mu_1\mu_2}\right)\,,\\ \mathbb{t}^\mu{}_{\mu_1\mu_2} &= \frac{1}{2}\mathbb{g}^{\mu\nu}\Big[(\mathbb{F}_{\nu\mu_1})^{\hat\beta_1\hat\beta_2}\mathbb{D}_{\mu_2}\mathbb{Y}_{\hat\beta_1} + (\mathbb{F}_{\nu\mu_2})^{\hat\beta_1\hat\beta_2}\mathbb{D}_{\mu_1}\mathbb{Y}_{\hat\beta_1} - (\mathbb{F}_{\mu_1\mu_2})^{\hat\beta_1\hat\beta_2}\mathbb{D}_\nu \mathbb{Y}_{\hat\beta_1}\Big]\mathbb{Y}_{\hat\beta_2}\,.\end{aligned}$$

- The field strength is decomposed as

$$\begin{aligned}(\mathbb{F}_{\mu_1\mu_2})^{\hat\alpha\hat\beta} &= \Big[\mathbb{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} + 2\ell^{-2}\left(\delta^{\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2]} - \mathbb{k}_{[\mu_1}{}^{\nu_1}\mathbb{k}_{\mu_2]}{}^{\nu_2}\right)\Big]\mathbb{D}_{\nu_1}\mathbb{Y}^{[\hat\alpha}\mathbb{D}_{\nu_2}\mathbb{Y}^{\hat\beta]}\\ &\quad -4\ell^{-4}\mathbb{k}_{[\mu_1\mu_2]}\mathbb{Y}^{[\hat\alpha}\mathbb{N}^{\hat\beta]} + 4\ell^{-2}\mathbb{t}^\nu{}_{[\mu_2\mu_1]}\mathbb{Y}^{[\hat\alpha}\mathbb{D}_\nu \mathbb{Y}^{\hat\beta]}\\ &\quad -4\ell^{-2}\left(\mathbb{\nabla}_{[\mu_1}\mathbb{k}_{\mu_2]}{}^\nu + \mathbb{t}^{\nu_1}{}_{[\mu_2\mu_1]}\mathbb{k}_{\nu_1}{}^\nu\right)\mathbb{N}^{[\hat\alpha}\mathbb{D}_\nu \mathbb{Y}^{\hat\beta]}\end{aligned}$$

Projecting bulk $SO(2,d)$ structure on Σ

- The pull back of bulk Einstein equation is

$$\begin{aligned} 0 &= \epsilon_{\hat{\alpha}\hat{\beta}\hat{\alpha}_1\cdots\hat{\alpha}_d}\varepsilon^{\mu_1\cdots\mu_d}(\mathbb{F}_{\mu_1\mu_2})^{\hat{\alpha}_1\hat{\alpha}_2}\mathbb{D}_{\mu_3}\mathbb{Y}^{\hat{\alpha}_3}\cdots\mathbb{D}_{\mu_d}\mathbb{Y}^{\hat{\alpha}_d} \\ &= \epsilon_{\hat{\alpha}\hat{\beta}\hat{\alpha}_1\cdots\hat{\alpha}_d}\varepsilon^{\mu_1\cdots\mu_d} \\ &\quad \Big[\Big(\mathbb{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} + 2\ell^{-2}\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} - 2\ell^{-2}\mathbb{k}_{\mu_1}{}^{\nu_1}\mathbb{k}_{\mu_2}{}^{\nu_2}\Big)\mathbb{D}_{\nu_1}\mathbb{Y}^{\hat{\alpha}_1}\mathbb{D}_{\nu_2}\mathbb{Y}^{\hat{\alpha}_2}\mathbb{D}_{\mu_3}\mathbb{Y}^{\hat{\alpha}_3}\cdots\mathbb{D}_{\mu_d}\mathbb{Y}^{\hat{\alpha}_d} \\ &\qquad - 2\ell^{-4}\mathbb{k}_{[\mu_1\mu_2]}\mathbb{Y}^{\hat{\alpha}_1}\mathbb{N}^{\hat{\alpha}_2}\mathbb{D}_{\mu_3}\mathbb{Y}^{\hat{\alpha}_3}\cdots\mathbb{D}_{\mu_d}\mathbb{Y}^{\hat{\alpha}_d} \\ &\qquad + 2\ell^{-2}\mathbb{t}^{\nu}{}_{[\mu_2\mu_1]}\mathbb{Y}^{\hat{\alpha}_1}\mathbb{D}_{\nu}\mathbb{Y}^{\hat{\alpha}_2}\mathbb{D}_{\mu_3}\mathbb{Y}^{\hat{\alpha}_3}\cdots\mathbb{D}_{\mu_d}\mathbb{Y}^{\hat{\alpha}_d} \\ &\qquad - 2\ell^{-2}\left(\mathbb{\nabla}_{[\mu_1}\mathbb{k}_{\mu_2]}{}^{\nu} + \mathbb{t}^{\nu_1}{}_{[\mu_2\mu_1]}\mathbb{k}_{\nu_1}{}^{\nu}\right)\mathbb{N}^{\hat{\alpha}_1}\mathbb{D}_{\nu}\mathbb{Y}^{\hat{\alpha}_2}\mathbb{D}_{\mu_3}\mathbb{Y}^{\hat{\alpha}_3}\cdots\mathbb{D}_{\mu_d}\mathbb{Y}^{\hat{\alpha}_d}\Big] \end{aligned}$$

- In the bulk torsion free case

$$\mathbb{k}_{[\mu\nu]} = 0\,, \qquad \mathbb{t}^{\mu}{}_{\nu\rho} = 0\,,$$

  the rest constraints are just the Hamiltonian constraints for bulk diffeomorphism

$$\begin{aligned} \mathbb{R} + \ell^{-2}[d(d-1) - 2\mathbb{k}^{[\mu_1}{}_{[\mu_1}\mathbb{k}^{\mu_2]}{}_{\mu_2]}] &= 0 \\ \mathbb{\nabla}_{[\mu}\mathbb{k}^{\mu}{}_{\nu]} &= 0 \end{aligned}$$

- How to recover them in CFT?

CFT $SO(2,d)$ structure

- The CFT $SO(2,d)$ structure is based on the null ruler $\boldsymbol{X}^{\hat\alpha}$

  $$\boldsymbol{X}^{\hat\alpha}\boldsymbol{X}_{\hat\alpha}=0$$

  and the corresponding $SO(2,d)$ gauge field $\boldsymbol{A}_\mu$.
- Their Weyl transformations are simply

  $$\delta_{\mathrm{W}}\boldsymbol{X}^{\hat\alpha}=\omega\boldsymbol{X}^{\hat\alpha}\,,\qquad \delta_{\mathrm{W}}\boldsymbol{A}_\mu=0$$

- The corresponding covariant derivative is $\boldsymbol{D}_\mu$. We have

  $$\boldsymbol{D}_\mu\boldsymbol{X}^{\hat\alpha}\boldsymbol{D}_\nu\boldsymbol{X}_{\hat\alpha}=\boldsymbol{g}_{\mu\nu}\,,\qquad \boldsymbol{X}^{\hat\alpha}\boldsymbol{D}_\nu\boldsymbol{X}_{\hat\alpha}=0$$

  where $\boldsymbol{g}_{\mu\nu}$ is the CFT metric.
- One can deduce the Weyl transformation of the CFT metric

  $$\delta_{\mathrm{W}}\boldsymbol{g}_{\mu\nu}=2\omega\boldsymbol{g}_{\mu\nu}$$

CFT $SO(2,d)$ structure

- To establish the CFT intrinsic basis, we also need to introduce $Z^{\hat{\alpha}}$

$$Z^{\hat{\alpha}} X_{\hat{\alpha}} = 1\,, \qquad Z^{\hat{\alpha}} D_\nu X_{\hat{\alpha}} = 0\,, \qquad Z^{\hat{\alpha}} Z_{\hat{\alpha}} = 0\,.$$

- One additional gauge invariant quantity is

$$f_{\mu\nu} = D_\mu Z^{\hat{\alpha}}\, D_\nu X_{\hat{\alpha}}$$

- Taking derivatives on the basis gives

$$\begin{aligned} D_\mu Z^{\hat{\alpha}} &= f_\mu{}^\nu D_\nu X^{\hat{\alpha}}\,, \\ D_{\mu_1} D_{\mu_2} X^{\hat{\alpha}} &= -g_{\mu_1\mu_2} Z^{\hat{\alpha}} + \Gamma^\nu{}_{\mu_2\mu_1} D_\nu X^{\hat{\alpha}} - f_{\mu_1\mu_2} X^{\hat{\alpha}}\,, \end{aligned}$$

where

$$\begin{aligned} \Gamma^\mu{}_{\nu\rho} &= \hat{\Gamma}^\mu{}_{\nu\rho} + t^\mu{}_{\nu\rho}\,, \\ \hat{\Gamma}^\mu{}_{\mu_1\mu_2} &= \tfrac{1}{2} g^{\mu\nu} \left(\partial_{\mu_2} g_{\nu\mu_1} + \partial_{\mu_1} g_{\nu\mu_2} - \partial_\nu g_{\mu_1\mu_2}\right)\,, \\ t^\mu{}_{\mu_1\mu_2} &= \frac{1}{2} g^{\mu\nu} \left[(F_{\nu\mu_1})^{\hat{\beta}_1\hat{\beta}_2} D_{\mu_2} X_{\hat{\beta}_1} + (F_{\nu\mu_2})^{\hat{\beta}_1\hat{\beta}_2} D_{\mu_1} X_{\hat{\beta}_1} - (F_{\mu_1\mu_2})^{\hat{\beta}_1\hat{\beta}_2} D_\nu X_{\hat{\beta}_1}\right] X_{\hat{\beta}_2}\,. \end{aligned}$$

CFT $SO(2,d)$ structure

- The field strength is decomposed as

$$\begin{aligned}(\boldsymbol{F}_{\mu_1\mu_2})^{\hat{\alpha}\hat{\beta}} \;=\; & \left[\boldsymbol{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} - 4\delta^{\nu_1}_{[\mu_1}\boldsymbol{f}_{\mu_2]}{}^{\nu_2}\right]\boldsymbol{D}_{\nu_1}\boldsymbol{X}^{[\hat{\alpha}}\boldsymbol{D}_{\nu_2}\boldsymbol{X}^{\hat{\beta}]} - 4\boldsymbol{f}_{[\mu_1\mu_2]}\boldsymbol{X}^{[\hat{\alpha}}\boldsymbol{Z}^{\hat{\beta}]} \\ & -4\boldsymbol{t}^{\nu}{}_{[\mu_1\mu_2]}\boldsymbol{D}_{\nu}\boldsymbol{X}^{[\hat{\alpha}}\boldsymbol{Z}^{\hat{\beta}]} - 4\left(\nabla_{[\mu_1}\boldsymbol{f}_{\mu_2]}{}^{\nu} + \boldsymbol{t}^{\nu_1}{}_{[\mu_2\mu_1]}\boldsymbol{f}_{\nu_1}{}^{\nu}\right)\boldsymbol{X}^{[\hat{\alpha}}\boldsymbol{D}_{\nu}\boldsymbol{X}^{\hat{\beta}]}\,.\end{aligned}$$

- The Weyl transformation of various gauge invariant quantities are

$$\begin{aligned}\delta_{\rm W}\boldsymbol{g}_{\mu\nu} \;&=\; 2\omega\boldsymbol{g}_{\mu\nu}\,, \\ \delta_{\rm W}\boldsymbol{f}_{\mu\nu} \;&=\; -\nabla_{\mu}\nabla_{\nu}\omega\,, \\ \delta_{\rm W}\hat{\boldsymbol{\Gamma}}^{\nu}{}_{\mu_1\mu_2} \;&=\; 2\delta^{\nu}_{(\mu_2}\partial_{\mu_1)}\omega - \boldsymbol{g}_{\mu_1\mu_2}\boldsymbol{g}^{\nu\rho}\partial_{\rho}\omega\,, \\ \delta_{\rm W}\boldsymbol{t}^{\nu}{}_{\mu_1\mu_2} \;&=\; 0\,, \\ \delta_{\rm W}\boldsymbol{R}^{\rho}{}_{\sigma\mu\nu} \;&=\; -4\boldsymbol{g}^{\rho\lambda}\partial_{[\sigma}\omega\,\boldsymbol{t}_{\lambda][\mu\nu]} - 2\boldsymbol{g}^{\rho\lambda}\left(\boldsymbol{g}_{\lambda[\mu}\nabla_{\nu]}\nabla_{\sigma}\omega - \boldsymbol{g}_{\sigma[\mu}\nabla_{\nu]}\nabla_{\lambda}\omega\right)\,.\end{aligned}$$

Conformal geometry

- What is the meaning of the additional CFT background field $f_{\mu\nu}$?
- In conformal geometry, there is an important quantity–the Schouten tensor

$$S_{\mu\nu} = \frac{1}{d-2}\left(\hat{R}_{\mu\nu} - \frac{1}{2(d-1)}\hat{R}\, g_{\mu\nu}\right) .$$

- Taking derivative on it gives rise to the Cotton tensor

$$C_{\rho\mu\nu} = 2(d-2)\hat{\nabla}_{[\rho} S_{\mu]\nu} = 2\hat{\nabla}_{[\rho}\left(\hat{R}_{\mu]\nu} - \frac{1}{2(d-1)}\hat{R}\, g_{\mu]\nu}\right) .$$

- The Weyl tensor is just

$$\begin{aligned} W^{\nu_1\nu_2}{}_{\mu_1\mu_2} &= \hat{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} - \frac{4}{d-2}\delta^{[\nu_1}_{[\mu_1}\hat{R}^{\nu_2]}_{\mu_2]} + \frac{2}{(d-2)(d-1)}\hat{R}\,\delta^{[\nu_1}_{[\mu_1}\delta^{\nu_2]}_{\mu_2]} \\ &= \hat{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} - 4\delta^{[\nu_1}_{[\mu_1} S_{\mu_2]}{}^{\nu_2]} . \end{aligned}$$

Generalized conformal geometry

- In the CFT torsion free case

$$f_{[\mu\nu]} = 0\,, \qquad t^{\mu}{}_{\nu\rho} = 0\,.$$

the Weyl transformation of the Schouten tensor is exactly same as $f_{\mu\nu}$

$$\delta_{\mathrm{W}} S_{\mu\nu} = -\nabla_{\mu}\nabla_{\nu}\omega\,.$$

Thus $f_{\mu\nu}$ is naturally the generalized Schouten tensor.

- The $SO(2,d)$ field strength unifies the generalized Weyl tensor, the generalized Cotton tensor and the torsion

$$\begin{aligned}(F_{\mu_1\mu_2})^{\hat{\alpha}\hat{\beta}} = {} & \left[R^{\nu_1\nu_2}{}_{\mu_1\mu_2} - 4\delta^{\nu_1}_{[\mu_1} f_{\mu_2]}{}^{\nu_2}\right] D_{\nu_1}X^{[\hat{\alpha}} D_{\nu_2}X^{\hat{\beta}]} - 4f_{[\mu_1\mu_2]}X^{[\hat{\alpha}}Z^{\hat{\beta}]} \\ & -4t^{\nu}{}_{[\mu_1\mu_2]}D_{\nu}X^{[\hat{\alpha}}Z^{\hat{\beta}]} - 4\left(\nabla_{[\mu_1}f_{\mu_2]}{}^{\nu} + t^{\nu_1}{}_{[\mu_2\mu_1]}f_{\nu_1}{}^{\nu}\right)X^{[\hat{\alpha}}D_{\nu}X^{\hat{\beta}]}\,.\end{aligned}$$

- For $d > 2$, the condition $F = 0$ automatically implies that

$$f_{\mu\nu} = S_{\mu\nu}\,,$$

and further the corresponding geometry is conformal flat.

Bulk$_\Sigma$-CFT relation

- The bulk $SO(2,d)$ data on Σ can be constructed out of the CFT $SO(2,d)$ data as following

$$\begin{aligned} \mathbb{Y}^{\hat{\alpha}} &= \zeta^{-1} \boldsymbol{X}^{\hat{\alpha}} - \frac{\ell^2}{2} \zeta \boldsymbol{Z}^{\hat{\alpha}} \,, \\ (\mathbb{A}_\mu)^{\hat{\alpha}}{}_{\hat{\beta}} &= (\boldsymbol{A}_\mu)^{\hat{\alpha}}{}_{\hat{\beta}} \,, \end{aligned}$$

- The scale ζ appears manifestly in $\mathbb{Y}$ such that the bulk$_\Sigma$ quantities are all Weyl invariant.
- For simplicity, we shall just show the $\zeta = 1$ results later on

$$\mathbb{Y}^{\hat{\alpha}} = \boldsymbol{X}^{\hat{\alpha}} - \frac{\ell^2}{2} \boldsymbol{Z}^{\hat{\alpha}} \,, \qquad (\mathbb{A}_\mu)^{\hat{\alpha}}{}_{\hat{\beta}} = (\boldsymbol{A}_\mu)^{\hat{\alpha}}{}_{\hat{\beta}} \,.$$

- Thus

$$\mathbb{D}_\mu \mathbb{Y}^{\hat{\alpha}} = \boldsymbol{m}_\mu{}^\nu \boldsymbol{D}_\nu \boldsymbol{X}^{\hat{\alpha}} \,, \qquad \boldsymbol{m}_\mu{}^\nu = \delta_\mu{}^\nu - \frac{\ell^2}{2} \boldsymbol{f}_\mu{}^\nu \,.$$

Bulk$_\Sigma$-CFT relation

- Inversely

$$X^{\hat{\alpha}} = \frac{1}{2}(\mathbb{Y}^{\hat{\alpha}} + \mathbb{N}^{\hat{\alpha}})\,, \qquad (A_\mu)^{\hat{\alpha}}{}_{\hat{\beta}} = (\mathbb{A}_\mu)^{\hat{\alpha}}{}_{\hat{\beta}}\,,$$
$$D_\mu X^{\hat{\alpha}} = M_\mu{}^\nu \mathbb{D}_\nu \mathbb{Y}^{\hat{\alpha}}\,, \qquad M_\mu{}^\nu = \tfrac{1}{2}(\delta_\mu^\nu + \mathbb{k}_\mu{}^\nu)\,.$$

- Thus the CFT metric is

$$g_{\mu\nu} = M_\mu{}^{\mu_1} M_\nu{}^{\nu_1} \mathbb{g}_{\mu_1\nu_1}\,.$$

- An interesting corollary:
  For any hypersurface $\Sigma = \{(x, z)|z = \zeta(x)\}$ in pure AdS, the CFT metric $g_{\mu\nu}$ is always conformal flat since $F = 0$.

CFT conservation laws

- The bulk torsion free condition

  $$\mathbb{k}_{[\mu\nu]} = 0\,, \qquad \mathbb{t}^{\mu}{}_{\nu\rho} = 0\,,$$

  is equivalent to

  $$\begin{aligned} &\boldsymbol{f}_{[\mu\nu]} = 0\,, \\ &\boldsymbol{t}_{\nu\mu_2\mu_1} = \boldsymbol{M}_{\nu}{}^{\nu_1}\boldsymbol{M}_{\mu_2}{}^{\mu_3}\boldsymbol{m}_{\mu_1\sigma}\hat{\boldsymbol{\nabla}}_{[\mu_3}\boldsymbol{m}_{\nu_1]}{}^{\sigma} + \boldsymbol{M}_{\nu}{}^{\sigma}\hat{\boldsymbol{\nabla}}_{[\mu_1}\boldsymbol{m}_{\sigma]\mu_2} - \boldsymbol{M}_{\mu_2}{}^{\sigma}\hat{\boldsymbol{\nabla}}_{[\mu_1}\boldsymbol{m}_{\sigma]\nu}\,. \end{aligned}$$

- Now, a generic CFT $S_{\mathrm{CFT}}[\phi;\boldsymbol{g},\boldsymbol{f}]$ is couple to both the metric $\boldsymbol{g}_{\mu\nu}$ and the generalized Schouten tensor $\boldsymbol{f}_{\mu\nu}$.
- The corresponding currents are

  $$\boldsymbol{T}^{\mu\nu} = \frac{1}{\sqrt{\boldsymbol{g}}}\left\langle \frac{\delta S_{\mathrm{CFT}}}{\delta \boldsymbol{g}_{\mu\nu}} \right\rangle\,, \qquad \boldsymbol{J}^{\mu\nu} = \frac{1}{\sqrt{\boldsymbol{g}}}\left\langle \frac{\delta S_{\mathrm{CFT}}}{\delta \boldsymbol{f}_{\mu\nu}} \right\rangle\,.$$

- The presence of $\boldsymbol{f}_{\mu\nu}$ will modify the diffeomorphism and Weyl conservation laws

  $$\begin{gathered} 2\hat{\boldsymbol{\nabla}}_{\nu}(\boldsymbol{T}^{\nu}{}_{\mu} + \boldsymbol{J}^{\rho\nu}\boldsymbol{f}_{\rho\mu}) - \boldsymbol{J}^{\rho\nu}\hat{\boldsymbol{\nabla}}_{\mu}\boldsymbol{f}_{\rho\nu} = 0\,, \\ 2\boldsymbol{g}_{\mu\nu}\boldsymbol{T}^{\mu\nu} - \hat{\boldsymbol{\nabla}}_{\nu}\hat{\boldsymbol{\nabla}}_{\mu}\boldsymbol{J}^{\mu\nu} - \hat{\boldsymbol{\nabla}}_{\rho}(\boldsymbol{t}^{\rho}{}_{\nu\mu}\boldsymbol{J}^{\mu\nu}) = \mathcal{A}_{\mathrm{Weyl}}\,. \end{gathered}$$

Bulk Hamiltonian constraints

- The pull back of bulk Einstein equation on Σ becomes

$$
\begin{aligned}
0 &= \epsilon_{\hat{\alpha}\hat{\beta}\hat{\alpha}_1\cdots\hat{\alpha}_d}\epsilon^{\mu_1\cdots\mu_d}(\mathbb{F}_{\mu_1\mu_2})^{\hat{\alpha}_1\hat{\alpha}_2}\mathbb{D}_{\mu_3}\mathbb{Y}^{\hat{\alpha}_3}\cdots\mathbb{D}_{\mu_d}\mathbb{Y}^{\hat{\alpha}_d} \\
&= \epsilon_{\hat{\alpha}\hat{\beta}\hat{\alpha}_1\cdots\hat{\alpha}_d}\epsilon^{\mu_1\cdots\mu_d}\boldsymbol{m}_{\mu_3}{}^{\nu_3}\boldsymbol{D}_{\nu_3}\boldsymbol{X}^{\hat{\alpha}}\cdots\boldsymbol{m}_{\mu_d}{}^{\nu}\boldsymbol{D}_{\nu_d}\boldsymbol{X}^{\hat{\alpha}_d} \\
&\quad \left[\left(\boldsymbol{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} - 4\delta^{\nu_1}_{[\mu_1}\boldsymbol{f}_{\mu_2]}{}^{\nu_2}\right)\boldsymbol{D}_{\nu_1}\boldsymbol{X}^{[\hat{\alpha}}\boldsymbol{D}_{\nu_2}\boldsymbol{X}^{\hat{\beta}]}\right. \\
&\qquad \left. -4\ell^{-2}\boldsymbol{t}^{\nu}{}_{[\mu_1\mu_2]}\boldsymbol{D}_{\nu}\boldsymbol{X}^{[\hat{\alpha}}(\boldsymbol{X}^{\hat{\beta}]} + \frac{\ell^2}{2}\boldsymbol{Z}^{\hat{\beta}]})\right] \\
\Leftrightarrow &\quad \left\{\begin{array}{c} \hat{\nabla}_{\nu}(\boldsymbol{m}\boldsymbol{M}_{\mu}{}^{\nu}) = 0 \\ \boldsymbol{M}_{\nu_1}{}^{\mu_1}\boldsymbol{M}_{\nu_2}{}^{\mu_2}(\boldsymbol{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} - 4\delta^{\nu_1}_{[\mu_1}\boldsymbol{f}_{\mu_2]}{}^{\nu_2}) = 0 \end{array}\right.
\end{aligned}
$$

where $\boldsymbol{m}$ is the determinate of $\boldsymbol{m}_{\mu}{}^{\nu}$.

- For the vacuum state $|\Omega(\boldsymbol{g},\boldsymbol{f})\rangle$, the currents $\boldsymbol{T}^{\mu\nu}$ and $\boldsymbol{J}^{\mu\nu}$ must be functions of the background fields $\boldsymbol{g}_{\mu\nu}$ and $\boldsymbol{f}_{\mu\nu}$.
- Can we find the explicit formula for $\boldsymbol{T}^{\mu\nu}(\boldsymbol{g},\boldsymbol{f})$ and $\boldsymbol{J}^{\mu\nu}(\boldsymbol{g},\boldsymbol{f})$ such that the above Hamiltonian constraints are recovered from the conservation laws?

Recovering bulk Hamiltonian constraints

- Very recently, we find the following formula up to a overall constant

$$
\begin{aligned}
J^{\mu\nu} &= 2mM^{\nu[\mu}M_{\rho}{}^{\rho]}\,, \\
T^{\mu\nu} &= m\left[\ell^{-2}(dM^{\nu\mu} - M_{\rho}{}^{\rho}g^{\mu\nu}) - 2M^{\mu[\rho}M_{\sigma}{}^{\sigma]}f_{\rho}{}^{\nu}\right].
\end{aligned}
$$

For $f_{\mu\nu} = 0$, it comes back to

$$
J^{\mu\nu} = (d-1)g^{\nu\mu}\,, \qquad T^{\mu\nu} = 0\,.
$$

- Substituting them into the CFT diffeomorphism conservation law, it recovers the bulk constraints for the diffeomorphism on Σ

$$
\begin{aligned}
0 &= 2\hat{\nabla}_{\nu}(T^{\nu}{}_{\mu} + J^{\rho\nu}f_{\rho\mu}) - J^{\rho\nu}\hat{\nabla}_{\mu}f_{\rho\nu} \\
&= 2\ell^{-2}\hat{\nabla}_{\nu}[m(dM_{\mu}{}^{\nu} - M_{\rho}{}^{\rho}\delta^{\mu}_{\nu})] + m(M^{\nu\rho_1}M_{\rho_1}{}^{\rho} - M_{\rho_1}{}^{\rho_1}M^{\nu\rho})\hat{\nabla}_{\mu}f_{\rho\nu} \\
&= 2d\ell^{-2}\hat{\nabla}_{\nu}(mM_{\mu}{}^{\nu})\,.
\end{aligned}
$$

Recovering bulk Hamiltonian constraints

- The LHS of the anomalous CFT Weyl conservation law gives

  $$\begin{aligned}&2\boldsymbol{g}_{\mu\nu}\boldsymbol{T}^{\mu\nu} - \hat{\nabla}_\nu\hat{\nabla}_\mu \boldsymbol{J}^{\mu\nu} - \hat{\nabla}_\rho(\boldsymbol{t}^\rho{}_{\nu\mu}\boldsymbol{J}^{\mu\nu})\\ &= 2\boldsymbol{m}\boldsymbol{M}_{\mu_1}{}^{\nu_1}\boldsymbol{M}_{\mu_2}{}^{\nu_2}\delta^{[\mu_1}_{[\nu_1}\boldsymbol{f}_{\nu_2]}{}^{\mu_2]} + \hat{\nabla}_\nu[\boldsymbol{M}^{\rho\nu}\hat{\nabla}_\mu(\boldsymbol{m}\boldsymbol{M}_\nu{}^\mu)]\,.\end{aligned}$$

  The 2nd term vanishes due to the CFT diffeomorphism conservation law.

- Comparing with the bulk Radial constraint

  $$\boldsymbol{M}_{\nu_1}{}^{\mu_1}\boldsymbol{M}_{\nu_2}{}^{\mu_2}(\boldsymbol{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} - 4\delta^{\nu_1}_{[\mu_1}\boldsymbol{f}_{\mu_2]}{}^{\nu_2}) = 0$$

  we get

  $$\sqrt{\boldsymbol{g}}\mathcal{A}_{\text{Weyl}} = \frac{1}{2}\sqrt{\boldsymbol{g}}\boldsymbol{m}\boldsymbol{M}_{\nu_1}{}^{\mu_1}\boldsymbol{M}_{\nu_2}{}^{\mu_2}\boldsymbol{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} = \frac{1}{2}\sqrt{\mathbb{g}}\mathbb{R}$$

Recovering bulk Hamiltonian constraints

- Since $\mathbb{g}_{\mu\nu}$ itself contains the scale ζ manifestly and is Weyl invariant, the anomaly

  $$\sqrt{\boldsymbol{g}}\mathcal{A}_{\mathrm{Weyl}} = \frac{1}{2}\sqrt{\boldsymbol{g}}\boldsymbol{m}\boldsymbol{M}_{\nu_1}{}^{\mu_1}\boldsymbol{M}_{\nu_2}{}^{\mu_2}\boldsymbol{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} = \frac{1}{2}\sqrt{\mathbb{g}}\mathbb{R}$$

  is not contradict with the Wess-Zumino consistent condition.
- We can perform the ζ expansion for $\mathcal{A}_{\mathrm{Weyl}}$, and the 0-th order term gives rise to the usual Weyl anomaly.
- For $d = 4$ and $\boldsymbol{f}_{\mu\nu} = \boldsymbol{S}_{\mu\nu}$, the 0-th order is

  $$\mathcal{A}^{(0)}_{\mathrm{Weyl}} \propto 2\hat{\boldsymbol{R}}^{\mu\nu}\hat{\boldsymbol{R}}_{\mu\nu} - \frac{2}{3}\hat{\boldsymbol{R}}^2 = (\mathrm{Weyl})^2 - \mathrm{Euler}$$

  which is consistent with the known results in $N = 4$ SYM.

CZ equation & Radial evolution

- Transfer to the bulk notations, we have

$$\begin{aligned}
\delta S_{\mathrm{CFT}} &= \int \mathrm{d}^d x \sqrt{\boldsymbol{g}}(\boldsymbol{T}^{\mu\nu}\delta \boldsymbol{g}_{\mu\nu} + \boldsymbol{J}^{\mu\nu}\delta \boldsymbol{f}_{\mu\nu}) \\
&= \ell^{-2}\int \mathrm{d}^d x \left[\frac{d}{2}\sqrt{\mathbb{g}}\big(2\mathbb{g}^{\nu\mu}\delta\mathbb{k}_{\mu\nu} - \mathbb{k}^{\nu_1\mu_1}\delta\mathbb{g}_{\mu_1\nu_1}\big) - \delta(\sqrt{\mathbb{g}}\mathbb{k}_\rho{}^\rho)\right] \\
&= \ell^{-2}\int \mathrm{d}^d x \sqrt{\mathbb{g}}\Big\{\big[(d-1)\mathbb{D}^\mu\mathbb{Y}_{\hat\alpha}\mathbb{N}_{\hat\beta} - \big(\mathbb{g}^{\mu\nu}\mathbb{k}_\rho{}^\rho - \mathbb{k}^{\nu\mu}\big)\mathbb{D}_\nu\mathbb{Y}_{\hat\alpha}\mathbb{Y}_{\hat\beta}\big]\,\delta(\mathbb{A}_\mu)^{\hat\alpha\hat\beta} \\
&\qquad + \Big[\ell^{-2}\Big(d(d-1) - \mathbb{k}_\mu{}^\mu\mathbb{k}_\rho{}^\rho + \mathbb{k}^{\nu\mu}\mathbb{k}_{\mu\nu}\Big)\mathbb{N}_{\hat\alpha} \\
&\qquad\quad + \mathbb{\nabla}_\mu\left(\mathbb{g}^{\mu\nu}\mathbb{k}_\rho{}^\rho - \mathbb{k}^{\nu\mu}\right)\mathbb{D}_\nu\mathbb{Y}_{\hat\alpha}\Big]\delta\mathbb{Y}^{\hat\alpha}\Big\}
\end{aligned}$$

- By adding the Gibbons Hawking term, we get

$$\delta(S_{\mathrm{CFT}} + S_{\mathrm{GH}}) = \frac{d}{2\ell^2}\int \mathrm{d}^d x\sqrt{\mathbb{g}}\big(2\mathbb{g}^{\nu\mu}\delta\mathbb{k}_{\mu\nu} - \mathbb{k}^{\nu\mu}\delta\mathbb{g}_{\mu\nu}\big)$$

$$= \frac{d}{\ell^2}\int \mathrm{d}^d x\sqrt{\mathbb{g}}\left(\mathbb{D}^\mu\mathbb{Y}_{\hat\alpha}\mathbb{N}_{\hat\beta}\delta(\mathbb{A}_\mu)^{\hat\alpha\hat\beta} + \frac{d}{\ell^2}\mathbb{N}_{\hat\alpha}\delta\mathbb{Y}^{\hat\alpha}\right) = \int \mathrm{d}^d x\left((\Pi^\mu)_{\hat\alpha\hat\beta}\delta(\mathbb{A}_\mu)^{\hat\alpha\hat\beta} + \Pi_{\hat\alpha}\delta\mathbb{Y}^{\hat\alpha}\right)$$

where the Π's are coincide with the canonical momentums obtained in the bulk Hamiltonian formalism.

CZ equation & Radial evolution

- Generically, when the radial constraint is not satisfied, the “hard” deviation of the scale invariance is

$$\begin{aligned}\mathcal{B} &= \sqrt{\boldsymbol{g}}\,\boldsymbol{m}\boldsymbol{M}_{\nu_1}{}^{\mu_1}\boldsymbol{M}_{\nu_2}{}^{\mu_2}(\boldsymbol{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} - 4\delta^{\nu_1}_{[\mu_1}\boldsymbol{f}_{\mu_2]}{}^{\nu_2}) \\ &= \sqrt{\mathbb{g}}\{\mathbb{R} + \ell^{-2}[d(d-1) - 2\mathbb{k}^{[\mu_1}{}_{[\mu_1}\mathbb{k}^{\mu_2]}{}_{\mu_2]}]\}\,.\end{aligned}$$

- From the RG-flow point of view, we can absorb this deviation of scale invariance by addding extra scale dependence of $\mathbb{A}_\mu$. That is, the Callan-Symanzik equation

$$(\Pi^\mu)_{\hat\alpha\hat\beta}\partial_z(\mathbb{A}_\mu)^{\hat\alpha\hat\beta} = \mathcal{B}\,.$$

- This is equivalent to the radial evolution part of the bulk Einstein equation

$$F^{[\hat\alpha_1\hat\alpha_2}{}_{[z\mu_2} \wedge \mathrm{D}_{\mu_2}Y^{\hat\alpha_3} \wedge \cdots \wedge \mathrm{D}_{\mu_d]}Y^{\hat\alpha_d]} = 0$$

  in the $A_z = 0$ gauge.

Summary

- By introducing $SO(2,d)$ ruler and gauge fields, we established
  - $SO(2,d)$ gauge theory description of AdS gravity
  - Generalized conformal geometry for CFT
- Based on the modified diffeomorphism and Weyl conservation laws, the bulk Hamiltonian constraints are recovered by the proper formula of the currents.

Prospects

- CFT metric @ the BH horizon and interior
- Minimal surface
- Explore the CFT on generalized conformal geometry in explicit models
- Flat holography and $ISO(1, d)$ Carrollian CFT