Chaos on Photon Rings

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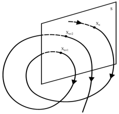
Outline

Introduction I

Chaos exist when a non-linear deterministic dynamical system has non-periodic orbits which are extra sensitive to the initial conditions. Usual ways to recognize chaos:

- Poincare section

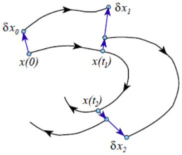


A picture of chaos is given by a scattered Poincare section.

- Leading Lyapunov Exponent

$$\|\delta x(t)\| \approx e^{\lambda t} \|\delta x_0\|$$

It is computed at large time and small $\|\delta x_0\|$ several times during the trajectory.



Positive coefficients signal the existence of chaos.

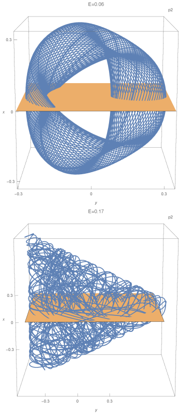
Introduction II

Study of the Hénon-Heiles system's Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(x_1^2 + x_2^2\right) + x_1 x_2^2 - \frac{x_1^3}{3}$$

The non-linear terms in the Hamiltonian is responsible for chaos.

Phase Space | Poincare section



Introduction III: When chaos is present?

Demonstrate the question for the generalized Hénon-Heiles Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(x_1^2 + x_2^2\right) + c x_1 x_2^2 + d\,\frac{x_1^3}{3}\ ,$$

where c and d are constants.

- It can be found analytically applying formalisms of non-integrability that except

$$\frac{c}{d} = 0,\ 1,\ \frac{1}{6}\ .$$

  the Hamiltonian system is non-integrable! (Ito 85)
- Chaos is present for the rest of the values.

Side Comment:

Holographic and Gravitational applications of such methods include: marginally (β-)deformed theories, Factorization of the S-matrix, non-relativistic (anisotropic) theories, D-brane backgrounds, Sasaki-Einstein, confining theories...:

(D.G., Zayas, Zoubos, Tseytlin, Lunin, Basu, Ishii, Murata, Yoshida, Nunez, Thompson, Banerjee, Bhattacharyya, Morales-Ruiz, Ramis,...: 2012-...)

Chaos around Black holes?

- It is natural for geodesics to develop instabilities and chaos due to the non-linear nature of gravitational fields on GR.
- Geodesics naturally display a rich structure and convey important information on the black holes.
- A particularly interesting class of them are the nearly bound null geodesics that comprise the photon ring.
- Their Lyapunov exponents are expected to be measured experimentally for the astrophysical black holes.
- A well established approximate relation between the quasinormal modes and the null geodesics exists.

  (Cardoso, Miranda, Berti, Witek, Zanchin, 0812.1806; ...)
- Holographic principle of asymptotically flat black holes.

  (Hadar, Kapec, Lupsasca, Strominger, 2205.05064)

Bounds/Universality on photon rings?

For spherically symmetric (Schwarzschild) black holes for particle motion in the near horizon regime it has been noticed

$$\lambda \leq \kappa = 2\pi T \ ,$$

(Hashimoto, Tanahashi, 1610.06070) .

Although for generic black holes around the horizon:

$$\lambda^2 \simeq \kappa^2 + \frac{1}{4}\left(g_{tt}^{(2)} G_{rr}^{(1)} - g_{tt}^{(1)} G_{rr}^{(2)}\right)(r - r_h) - G_{rr}^{(1)}\sqrt{-g_{tt}^{(1)}}\, V''(r_h)(r - r_h)^{3/2} + \ldots$$

$G_{rr} := 1/g_{rr}$.

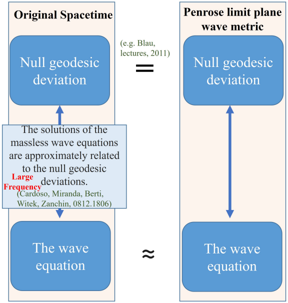
Implementing all geodesic conditions and the null energy conditions are not enough to ensure in general a bound! (D.G., 2112.02081; ...) .

There are many other specific black hole examples. (..., Zhao, Li, Lu 1809.04616; Lei, Ge, Ran 2008.01384; Jeong, Lee, Lee, Lee 2301.12198; ... many other)

Any bounds related to photon ring geodesics?

An alternative approach for the near photon ring physics

- Penrose limit of generic class of space-times on null geodesics of the photon ring to obtain the plane wave metric.
- The metric of the plain wave is associated with the Lyapunov exponent and the quasinormal modes.



- The construction presented is for generic large class of black holes.
- The surface gravity (at least for certain limits) appears naturally on the results.

Null Geodesics in plane waves

The plane wave metric (of a metric g) in Brinkmann coordinates:

$$ds^2 = 2dudv + A_{ij}(u)x^i x^j du^2 + d\vec{x}^2 \ .$$

The massless geodesics satisfy

$$-\dot{u}\dot{v} = \frac{1}{2}A_{ij}x^i x^j \dot{u}^2 + \frac{1}{2}\dot{\vec{x}}^2 \ ,$$

with a conserved conjugate momentum $p_v = \dot{u}$, so that

$$u = p_u \tau \ .$$

The geodesic equations for the transverse coordinates x^i

$$\ddot{x}(\tau)^i = A_{ij}p_v^2 x^j := -\omega_{ij}^2 x^j \ ,$$

are of the form of the non-relativistic harmonic oscillator with a frequency matrix ω_{ij}.

Deviations of families of geodesics

The separation δx between nearby geodesics is given by

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu_{\nu k l}\dot{x}^\nu \dot{x}^k \delta x^l \ , \qquad \frac{D}{D\tau} := \dot{x}^\kappa \nabla_\kappa ,$$

where the transverse deviation equation is reduced to

$$\frac{d^2}{du^2}\delta x^i = A_{ij}\delta x^j \ ,$$

the harmonic oscillator equation.

R_{iubu} is the only non-vanishing component of the Riemann curvature tensor of a plane wave metric and gives the $-A_{ij}$ matrix.

Stable and unstable directions?

The only non-zero component of the Einstein tensor of the plane wave metric:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = R_{uu} = -\delta^{ij} A_{ij}$$

For spacetimes satisfying the vacuum Einstein equations without the presence of null fluxes and matter:

$$TrA = 0 \ \Rightarrow A_{11} = -A_{22}.$$

Immediate implications:

- They exist both stable and unstable directions with related frequency and Lyapunov exponent.

Which Black Holes?

The Hamilton-Jacobi (H-J) equation of geodesics is separable for a stationary axisymmetric spacetime written in (t, θ, ϕ, r) coordinates of the form:

$$g_{\mu\nu}(r,\theta) = \begin{bmatrix} -\frac{C_1 C_3}{C_4^2 - C_3 C_5} & 0 & \frac{C_1 C_4}{C_4^2 - C_3 C_5} & 0 \\ 0 & \frac{C_1}{B_2} & 0 & 0 \\ \frac{C_1 C_4}{C_4^2 - C_3 C_5} & 0 & -\frac{C_1 C_5}{C_4^2 - C_3 C_5} & 0 \\ 0 & 0 & C_4 & \frac{C_1}{A_2} \end{bmatrix} .$$

where $C_i = A_i(r) + B_i(\theta)$. (Papadopoulos, Kokkotas, 2020)

The H-J equation for null geodesics is

$$g^{\mu\nu}\partial_\mu S \partial_\nu S = 0$$

with $p_\mu = \partial_\mu S$. Separability for motion suggests

$$S = -Et + L\phi + S_r(r) + S_\theta(\theta) \ ,$$

We obtain two separable equations depending on r and θ

$$A_2 S_r'^2 + A_5 E^2 + A_3 L^2 - 2A_4 EL = -\mathcal{K} \ , \quad B_2 S_\theta'^2 + B_5 E^2 + B_3 L^2 - 2B_4 EL = \mathcal{K} \ .$$

On the spherical photon ring r_0, we find the algebraic equations

$$b = \left(\frac{A_4'}{A_3'} \pm \frac{\sqrt{{A_4'}^2 - A_3' A_5'}}{A_3'} \right) \Bigg|_{r_0} ,$$

$$k_E = -A_5 + 2\frac{A_4'}{A_3'}\left(A_4 - A_3 \frac{A_4'}{A_3'} \right) + \frac{A_3 A_5'}{A_3'} \pm 2 \frac{\sqrt{{A_4'}^2 - A_3' A_5'}(A_3 A_4' - A_3' A_4)}{A_3'(r)^2} \Bigg|_{r_0} ,$$

where

$$b = \frac{L}{E} , \qquad k_E = \frac{\mathcal{K}}{E^2} .$$

The equations are solvable for this general background.

Penrose Limit: Parallel Frames along null Geodesics

The Killing-Yano antisymmetric tensor is essential for the construction. We can use the Killing tensor

$$Y_a^c Y_c^b = K_a^b$$

and additionally satisfy

$$\nabla_c Y_{ab} + \nabla_a Y_{cb} = 0 \ .$$

In order to solve the last equation let us try to make a more specific choice of the functions $A_i(r), B_i(\theta)$

$$A_1(r) = \Lambda(r)^2 \ , \quad A_2 = \Delta(r) \ , \quad A_3(r) = -\frac{a^2\Phi(r)}{\Delta(r)} \ ,$$

$$A_4(r) = -\frac{a\Phi(r)\left(\Lambda^2(r) + a^2\right)}{\Delta(r)} \ , \quad A_5(r) = \frac{A_4(r)^2}{A_3(r)} \ .$$

and

$$B_1(\theta) = a^2 c_\theta^2 \ , \quad B_2(\theta) = 1 \ , \quad B_3(\theta) = s_\theta^{-2} \ , \quad B_4(\theta) = a \ , \quad B_5(\theta) = \frac{B_4(\theta)^2}{B_3(\theta)} \ .$$

The Killing-Yano tensor for this class of metrics is

$$Y(r,\theta) = s_1 \begin{bmatrix} 0 & ars_\theta & 0 & -ac_\theta \\ -ars_\theta & 0 & r(a^2+r^2)s_\theta & 0 \\ 0 & -r(a^2+r^2)s_\theta & 0 & a^2 c_\theta s_\theta^2 \\ ac_\theta & 0 & -a^2 c_\theta s_\theta^2 & 0 \end{bmatrix}.$$

with

$$\Phi(r) = 1\ , \qquad \Lambda(r) = \pm r + c_1\ , \qquad s_1^2 = 1,$$

while $\Delta(r)$ remains arbitrary.

This class of metrics includes:

- Kerr, Kerr-Newmann, mass deformations of Kerr...
- Schwarzschild, Reissner–Nordström, Kiselev...

Penrose limit

$$ds^2 = 2dudv + A_{ij}x^i x^j du^2 + dx_1^2 + dx_2^2 \ , \qquad x_1 \sim r, \ x_2 \sim \theta \ .$$

where

$$A_{ij} = -R_{\mu\nu\alpha\beta} u^\mu e^{(i)\nu} u^\alpha e^{(j)\beta}\big|_{null \ geodesic} \ .$$

To construct the parallel frames for the null geodesics we have

$$u_\mu = \partial_\mu S \ , \quad u_\mu u^\mu = n_\mu n^\mu = 0 \ , \quad e_\mu^{(i)} e^{\mu(j)} = \delta^{ij} \ ,$$

the parallel propagated tetrad

$$e^{(1)\mu} = \frac{1}{C}\left(u^\alpha h_\alpha{}^\mu - u(u^\alpha \xi_\alpha) u^\mu\right) \ , \qquad e^{(2)\mu} = \frac{1}{K}\left(u^\alpha f_\alpha{}^\mu\right) \ ,$$

$$n^\mu = -\frac{1}{C} e^{(1)\alpha} h_\alpha{}^\mu + \frac{1}{2C^4}\left(C_\beta{}^\gamma C_{\gamma\delta} u^\beta u^\delta + u^2 (\xi_\alpha u^\alpha)^2 C^2\right) u^\mu \ .$$

$$K^2 = K_{\alpha\beta} u^\alpha u^\beta \ , \qquad C^2 = C_{\alpha\beta} u^\alpha u^\beta \ , \qquad C_{\alpha\beta} = h_{\alpha\gamma} h_\beta{}^\gamma \ .$$

and

$$h = \star Y = -a\left(a^2 + r^2\right) c_\theta s_\theta d\theta \wedge d\phi - r dt \wedge dr - a^2 c_\theta s_\theta dt \wedge d\theta + a r s_\theta^2 d\phi \wedge dr \ .$$

(Fransen, 2301.06999; Kubiznak, Frolov, Krtous, Connell, 0811.0012,...) .

Penrose Limit: Generic Results

For the equatorial geodesics we get the simplified expressions for A_{ij}

$$
\begin{aligned}
A_{11} &= 4\frac{2\Delta(r_0\Delta'' - \Delta') - r_0\Delta'^2}{r_0^3\Delta'^2} \ , \\
A_{22} &= 2\frac{4\Delta(2r_0 - \Delta') + r_0\Delta'^2}{r_0^3\Delta'^2} \ , \\
A_{21} &= A_{12} = 0 \ .
\end{aligned}
$$

evaluated on the photon ring r_0.

- A_{ij} is always diagonal.
- In general $A_{11} \neq A_{22}$!
  No Einstein equations has been used so far.

Massless Scalar Wave Equation

The Massless Scalar Wave Equation is

$$\nabla_\mu \nabla^\mu \Phi = 0 \ , \qquad \Phi = e^{ip_v v + ip_u u}\phi_1(x_1)\phi_2(x_2)$$

The wave equation separates to

$$\frac{1}{2p_v^2}\phi_i''(x_i) + A_{ii}\frac{x_i^2}{2}\phi_i(x_i) = \left(\frac{p_u}{2p_v} \pm c\right)\phi_i(x_i)$$

When $TrA = 0 \Rightarrow A_{11} = -A_{22}$ implies one harmonic oscillator(stable direction) and one mirror inverted oscillator(unstable direction)

Surface Gravity

Surface gravity: a measure of the gravitation force on a static observer at the horizon surface as seen by an observer at infinity.

$$\kappa \sim V\alpha \ , \quad V : redshift\ factor , \quad \alpha : acceleration.$$

It is equal to

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu K_\nu)(\nabla^\mu K^\nu)\Big|_{r_h} \ .$$

For the black holes under study

$$\kappa = \frac{\Delta'(r_h)}{2(r_h^2 + a^2)} \ .$$

Where for spherically symmetric static black holes we have

$$\kappa = -\frac{g'_{tt}}{2\sqrt{-g_{tt} g_{rr}}} \ ,$$

and for a Schwarzschild black hole simplifies to $\kappa = (4m)^{-1} = 2\pi T$.

Applicability of the formalism on certain Black holes I

Schwarzschild black hole:

$$\Delta(r) = r^2 - 2mr\ , \qquad a = 0\ ,$$

The radius of the photon ring is

$$r_0 = 3m\ , \qquad b^2 = 27m^2\ .$$

The Penrose limit plane wave in Brinkmann coordinates

$$A_{11} = -A_{22} = \frac{1}{3m^2} = \frac{16}{3}\kappa^2\ , \qquad A_{12} = A_{21} = 0\ .$$

(Fransen, 2023)

Which implies

$$\lambda = \frac{4\kappa}{3\sqrt{3}} < \kappa\ , \qquad \omega = \lambda\ .$$

Applicability of the formalism on certain Black holes II

Kerr Black hole

$$\Delta(r) = r^2 - 2mr + a^2$$

The Penrose Limit reads

$$A_{11} = -A_{22} = \frac{3}{r_0^2}$$

In the limit of large mass the Penrose limit becomes

$$A_{11} \simeq \frac{16}{3}\kappa^2_{Kerr} + C_0(a,m) + \ldots = \frac{16}{3}\kappa^2_{Schw} + C_1(a,m) + \ldots$$

A_{11} approaches the surface gravity from above

$$\lambda \simeq \kappa_{Kerr} + C_2(a,m) \pm \ldots \geq \kappa_{Kerr}$$

Applicability of the formalism on certain Black holes III

Kerr-Newman black hole

$$\Delta(r) = r^2 - 2mr + a^2 + q^2$$

The Penrose limit metric elements

$$A_{11} = \frac{3mr_0 - 4q^2}{r_0^2\,(q^2 - mr_0)}\ , \qquad A_{22} = \frac{2q^2 - 3mr_0}{r_0^2\,(q^2 - mr_0)}\ ,$$

with $A_{12} = A_{21} = 0$, and $A_{11} \neq A_{22}$ always unless $q = 0$.
For very massive black holes we obtain

$$A_{11} \simeq \frac{16}{3}\kappa_{RN}^2 + C_1(a,m)\ , \qquad A_{22} \simeq -\frac{16}{3}\kappa_{RN}^2 - C_2(a,m)\ .$$

Conclusions

✓ Penrose limit is interesting alternative and "universal" formalism to study the near photon ring regime physics in relation to black hole properties.

✓ We have applied the framework on a large class of space-times, obtaining readily applicable formulas for the Lyapunov coefficients.

- Ideal to study Universalities and existence of bounds.

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Thank you!