Scattering Amplitudes @ intersection of QFT, Strings & Maths

何 颂

Institute of Theoretical Physics
Chinese Academy of Sciences

partly based on works with N. Arkani-Hamed, T. Lam JHEP 02 (2021) 069, SIGMA (2022) …
N. Arkani-Hamed, T. Lam, G. Salvatori, H. Thomas (2019) …
N. Arkani-Hamed, Y. Bai, G.Yan JHEP 1805 (2018) 096
& F. Cachazo, E. Y. Yuan
PRL 113 (2014) PRD90 (2014) JHEP 1407 (2014) JHEP 1501 (2015)
JHEP 1507 (2015) PRD92 (2015) JHEP 1608 (2016) …

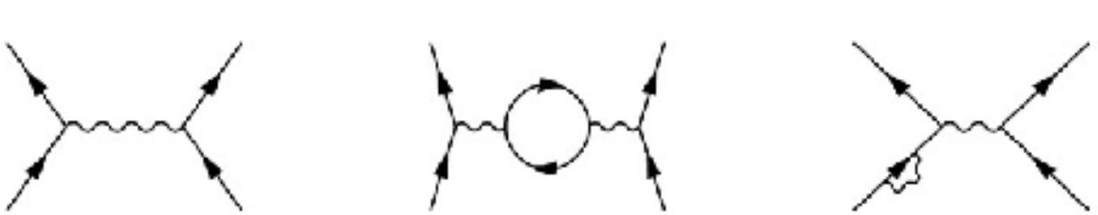
Peng Huanwu Center for Fundamental Theory
USTC
March 8, 2022

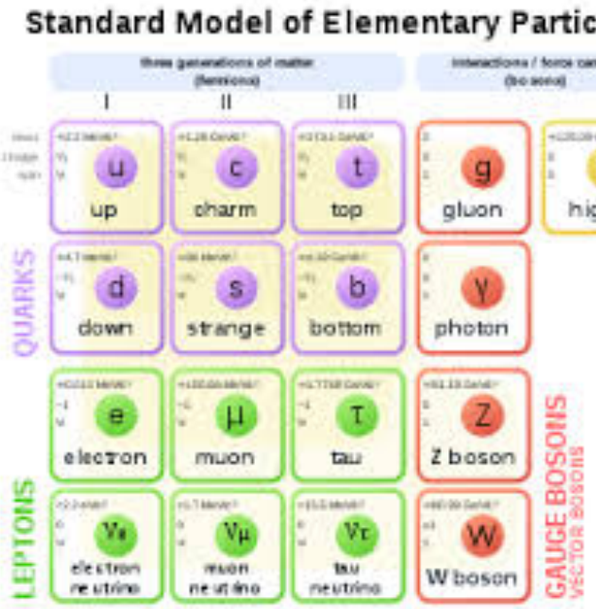
Quantum Field Theory (QFT)

Most successful theoretical framework to describe Nature:
particle physics, condensed matter, cosmology, strings

inevitable & universal: consequence of QM & relativity!
fundamental interactions unified @ high energy

simple picture in perturbation theory: Feynman diagrams

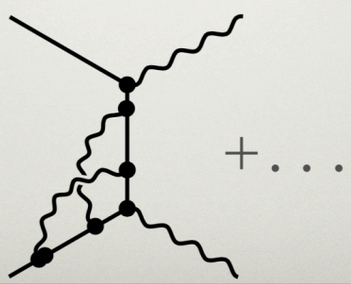




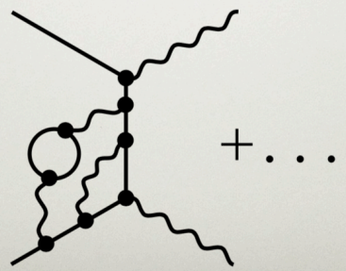
incredible accuracy!
e.g. g-factor of electron
magnetic dipole moment

$g_e^{\text{theory}} = 2 + \frac{\alpha}{\pi} + \ldots$ [1947]
$g_e^{\text{expt}} = 2.0023\ldots$ [1947]

$g_e^{\text{theory}} = 2.0023193\ldots$ [1957]
$g_e^{\text{expt}} = 2.00231931\ldots$ [1972]



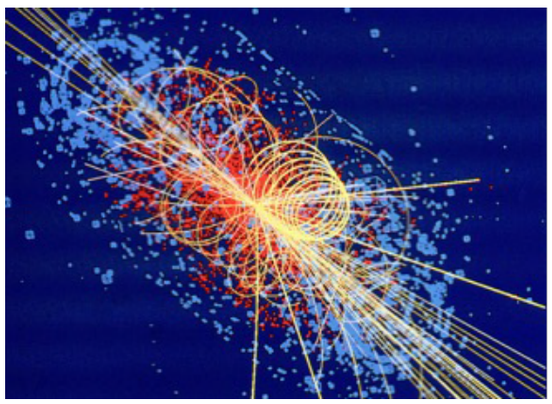
$g_e^{\text{theory}} = 2.0023193044\ldots$ [1990]
$g_e^{\text{expt}} = 2.00231931\ldots$ [1972]



$g_e^{\text{theory}} = 2.00231930435801\ldots$ [2012]
$g_e^{\text{expt}} = 2.002319304361\ldots$ [2011]

S-matrix in QFT

- Colliders at high energies need amplitudes of many gluons/quarks



gg → gg … g

- Fundamental level our understanding of QFT & gravity incomplete:
  strong coupling, dualities, hidden symmetries, quantum gravity & cosmology…

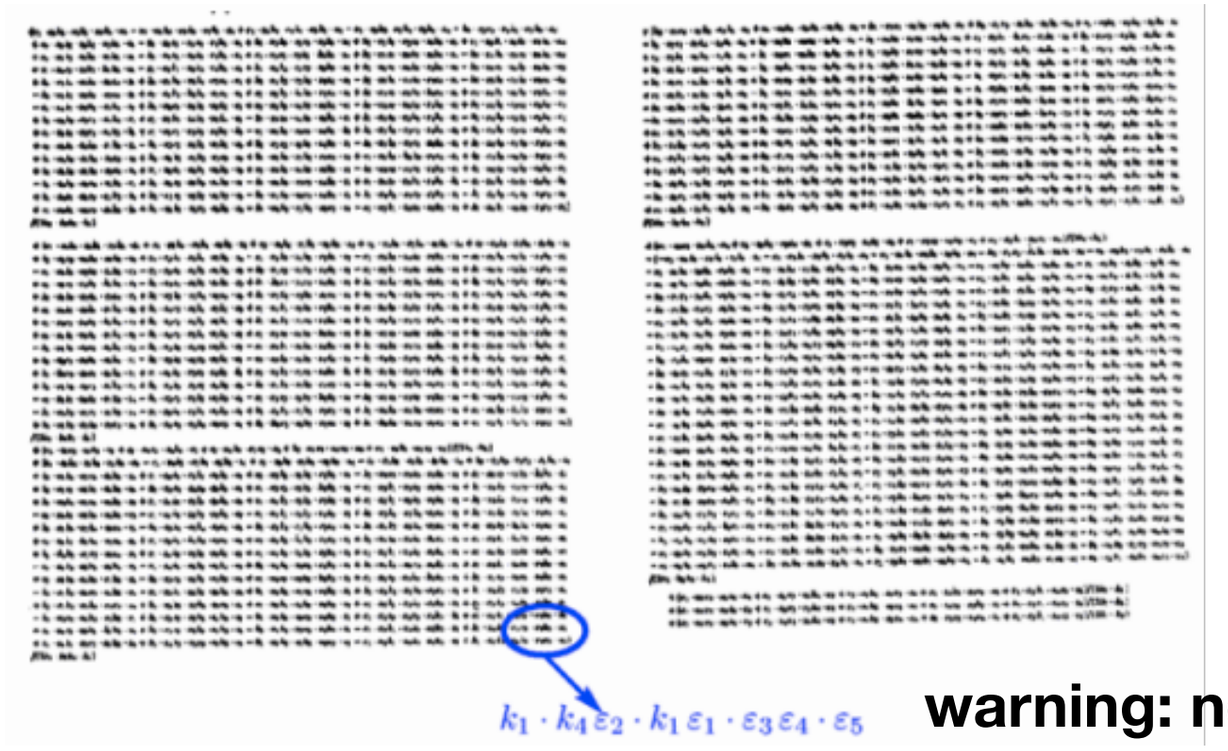
simplicity, new structures & relations seen in perturbative scattering amplitudes!

- Goal: new tools, ideas & theories for QFT+gravity from studying S-matrix

Impossible computations?

Feynman diagrams manifest locality & unitarity, but usually no manifest symmetry

Challenging for more legs/loops: many diagrams, lots of terms, huge redundancy



Process	N_{FG}
$gg \to 2g$	4
$gg \to 3g$	25
$gg \to 4g$	220
$gg \to 5g$	2485
$gg \to 6g$	34300
$gg \to 7g$	559405
$gg \to 8g$	10525900
$gg \to 9g$	224449225
$gg \to 10g$	5348843500

warning: not with your bare hands!

Gluons: 2 states $h = \pm$, but manifest locality requires 4 states (huge redundancies)
Much worse for graviton scattering: redundancies from diff invariance

A prior no reason to expect any simplicity or structures in the S-matrix

Parke-Taylor formula

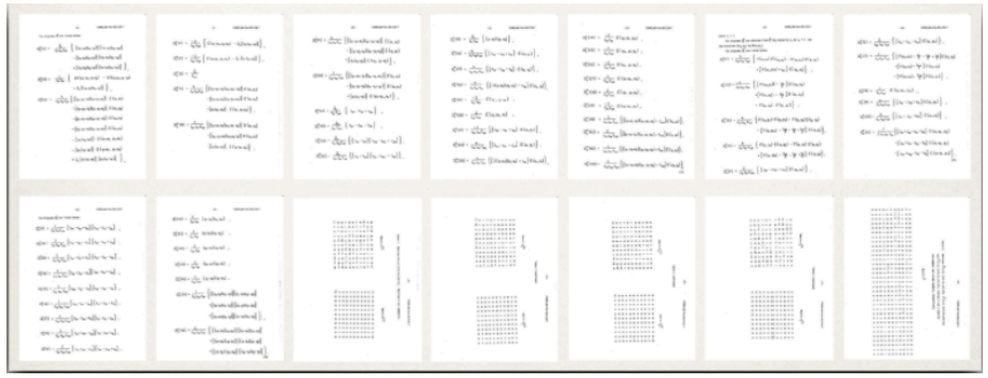
1985: heroic calculation of tree amp gg → gggg (results ~10 pages)

GLUONIC TWO GOES TO FOUR

Stephen J. Parke and T.R. Taylor
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510
U.S.A.

ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.



Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

MHV: Maximally helicity violating
(all out-going) amps for all + or one - vanish!

Spinor-helicity variables

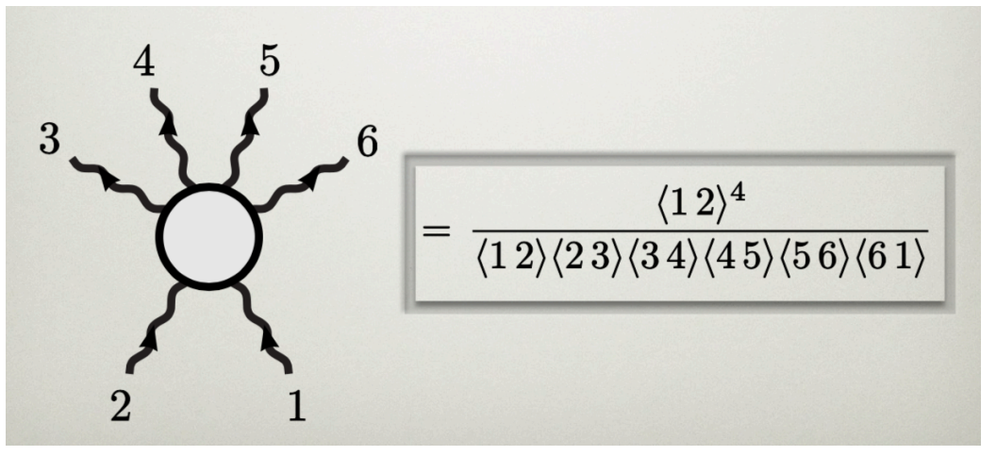
$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \tilde{\lambda}_{\dot{a}}$$

$$\langle 12 \rangle = \epsilon_{ab} \lambda_a^{(1)} \lambda_b^{(2)}$$

$$[12] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_{\dot{a}}^{(1)} \tilde{\lambda}_{\dot{b}}^{(2)}$$

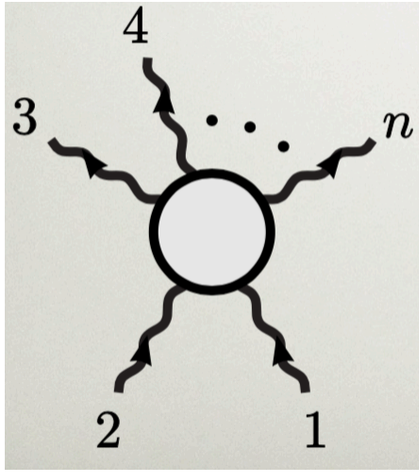
(Mangano, Parke, Xu 1987)

6 months later they realized



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle}$$

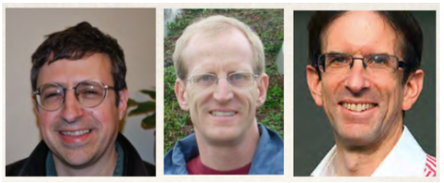
conjecture for n-pt MHV amp [Parke,Taylor]



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,\cdots\,\langle n\,1\rangle}$$

A very selective history

- 1986 - 2000: **spinor-helicity + generalized unitarity**



→ tree & one-loop gluon amps in QCD & N=4 SYM… powerful generalized unitarity method: cuts of loops = products of tree amp

- Twistor strings (2003) … **BCFW recursion**: all trees in QCD

new unitarity methods → one-loop QCD & more

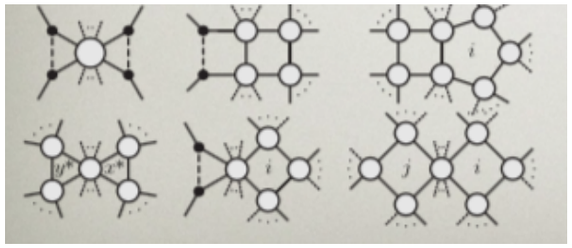
→NLO revolution -> NNLO, loop integrands, integrals & polylogs, …

New math structures (2009-): Grassmannian for all-loop integrands in N=4 SYM (hydrogen atom of QFT) + bootstrap, integrability, AdS/CFT…

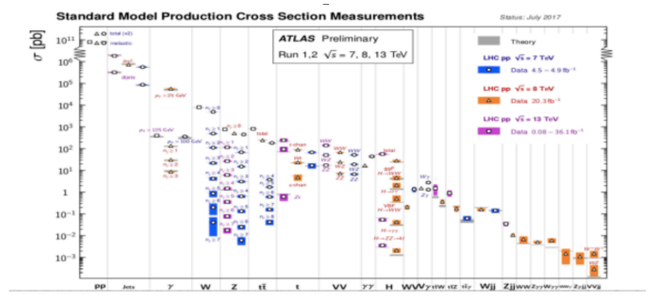
double copy (gauge theories, gravity & strings)—> CHY formulation etc.
geometric pictures for QFT & strings —> amplituhedron, associahedron, etc.

(numerous topics & names omitted here…)

Who do we connect to?



Multi-loop integrands, integrals,…



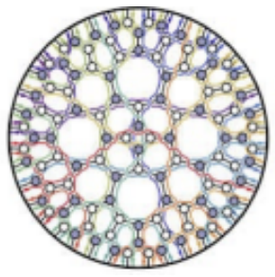
Formal QFT

Mathematics

Amplitudes

Collider Phenomenology

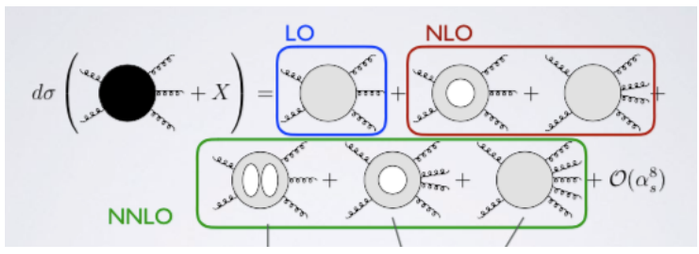
precision frontier



Geometries, combinatorics,..

String Theory

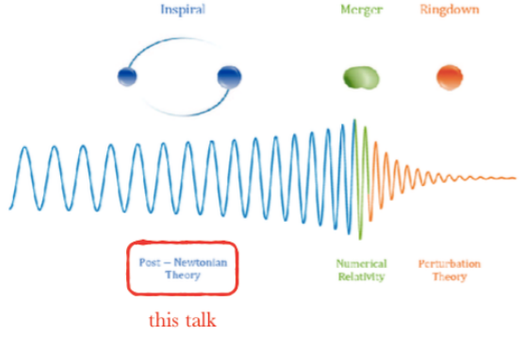
Gravity



$$\int_{\mathcal{M}_{0;4}} + \int_{\mathcal{M}_{1;4}} + \int_{\mathcal{M}_{2;4}} + \int_{\mathcal{M}_{3;4}} \cdots$$

$$\mathcal{A}^{g\text{-loop}}_{\text{string}}(1,2,\ldots,n) \sim \int_{\mathcal{M}_{g,n}} \left\langle \left(\prod_j \text{PCO}(w_j)\right) V_1(z_1)\, V_2(z_2) \ldots V_n(z_n) \right\rangle_g$$

CFT, string perturbation,…



gravity amps
-> post-Newtonian

S-matrices for (super)-gravity

GR as Effective Field Theory (EFT): computing perturbative amps for (super)-gravity

Crucial new insight for quantum gravity: UV behavior, hidden symmetries/structures?

Important for classical gravity e.g. post-newtonian for GW (potential, angle…)

However, Feynman diagrams seem hopeless

(infinite # of vertices, even 3-pt has >100 terms!)

3 loops  $\sim 10^{20}$ TERMS No surprise it has never been calculated via Feynman diagrams.

4 loops  $\sim 10^{26}$ TERMS

5 loops  $\sim 10^{31}$ TERMS More terms than atoms in your brain!

On-shell: all gravity trees can be computed & directly related to tree in Yang-Mills (“QCD meets gravity”)

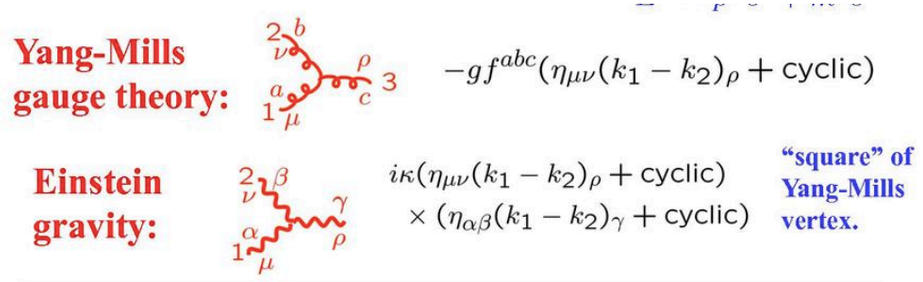
Generalized unitarity —> recycle trees into loops

N=8 supergravity amps computed to 5 loops, with surprisingly good UV behavior!

Gravity=(Gauge Theory)^2

1985: Kawai, Lewellen, Tye (KLT): **"closed string amp=open-string amp^2"**

Field-theory limit:



2008: Bern, Carrasco, Johansson (BCJ): **double-copy construction**

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$$n_s + n_t + n_u = 0$$

$$\mathcal{A}_4^{\text{tree}}\Big|_{c_i \to n_i} \equiv \mathcal{M}_4^{\text{tree}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

extended to classical solutions, curved background etc.-> hidden symmetry & structure of classical gravity!

If you have a set of duality satisfying numerators.
To get:

gauge theory → gravity theory

simply take

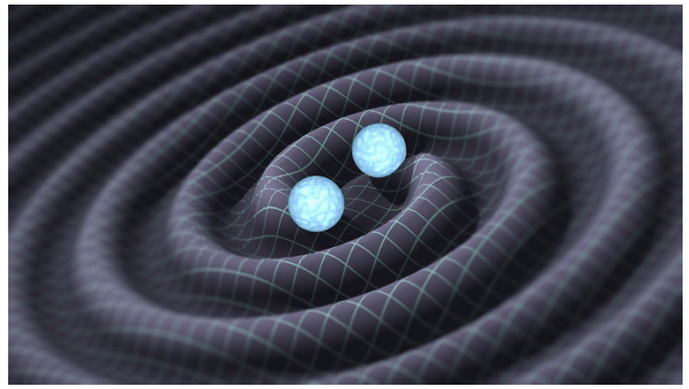
color factor → kinematic numerator

black hole ⟷ **point electric charges**

Schwarzschild ~ (Coulomb)²

Gravitational waves

How to help calculations needed for LIGO (inspiral)?

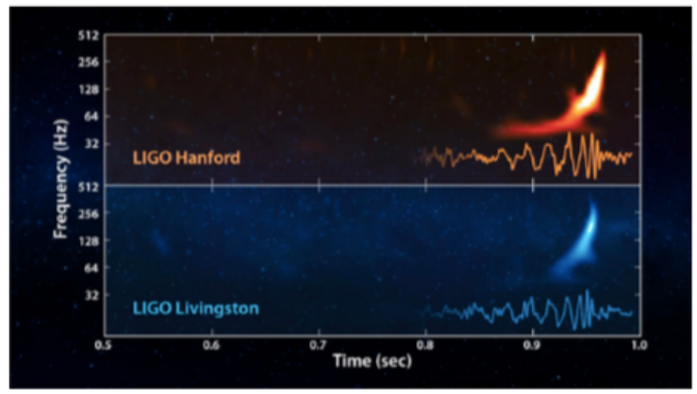


Classical limits from quantum scattering amplitudes

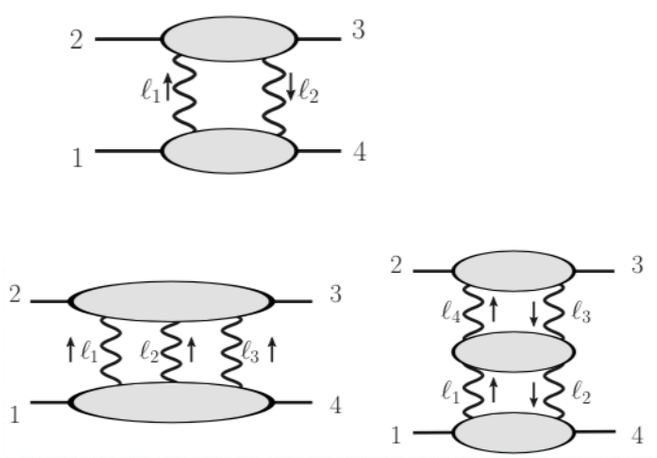
New tools e.g. double-copy simplifies GW calculations

Post-Newtonian/Minkowski from (EFT) amplitudes

[Goldberger, Rothstein, Porto,…]
[Bern, Cheung, Roiban, Shen, Solon, Zeng; …]
[…]



		0PN	1PN	2PN	3PN	4PN	5PN	…
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	…
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	…
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	…
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	…
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	…
…						…	…	



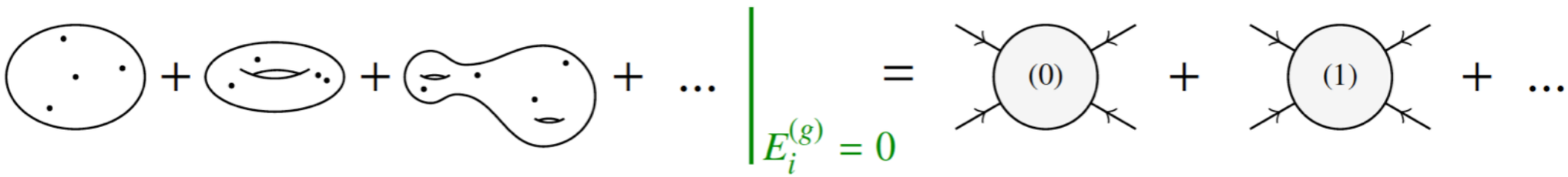
New formulation of QFT

- Twistor string theory [Witten 2003]: worldsheet model for N=4 SYM tree amps failed at loops, but led to BCFW, CSW & many new developments!

- How universal is Witten's twistor string? no SUSY? any spacetime dim? more general theories: (pure) Yang-Mils, gravity, effective field theories? loop level?

- CHY formulation: scattering of massless particles in any dim [Cachazo, SH, Yuan 2013]

  - *compact formulas* for amps of gluons, gravitons, scalars, (fermions?!) etc.
  - *manifest* gauge (diff) invariance, soft theorems, double-copy & new relations, etc.
  - *worldsheet picture:* ambitwsitor strings etc. [Mason, Skinner; Adamo et al; Berkovits; Siegel…]



Scattering equations & CHY formulas

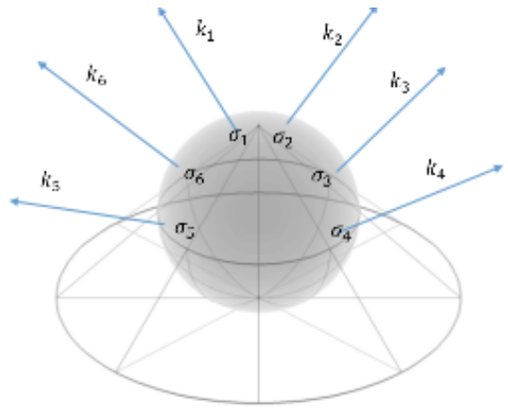
$$E_a := \sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \qquad a = 1, 2, \ldots, n$$ [CHY 2013]

- saddle-point equations of “Koba-Nielson” factor in string theory [Gross, Mende;…]

- moduli space of n-punctured Riemann sphere knows locality (& unitarity) of tree amps

$SL(2,\mathbb{C})$ symmetry:

n-3 variables, n-3 equations



$$M_n = \int \underbrace{\frac{d^n\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k,\epsilon,\sigma\}) = \sum_{\{\sigma\}\in \mathrm{solns.}} \frac{\mathcal{I}(\{k,\epsilon,\sigma\})}{J(\{\sigma\})}$$

- New picture: scattering of massless particles via worldsheet correlators

- Feynman diagrams, Lagrangians, even spacetime itself become emergent

Gluons & Gravitons: gauge (diff.) invariance [CHY]

- Two basic building blocks: color & kinematics (polarization)

$$PT(\alpha) := \frac{1}{\sigma_{\alpha(1),\alpha(2)}\sigma_{\alpha(2),\alpha(3)}\cdots\sigma_{\alpha(n),\alpha(1)}}; \qquad \mathrm{Pf}'\Psi \sim \langle V^{(0)}(\sigma_1)\ldots V^{(-1)}(\sigma_i)\ldots V^{(-1)}(\sigma_j)\ldots V^{(0)}(\sigma_n)\rangle$$

- All tree amps in bi-adjoint scalar, Yang-Mills and Einstein Gravity!

$$\mathcal{M}_n^{\phi^3} = \int d\mu_n\, \mathcal{C}_n\, \mathcal{C}'_n \qquad \mathcal{C}_n = \sum_{\pi} \mathrm{PT}(\pi)\mathrm{Tr}(T^{\pi(1)}\cdots T^{\pi(n)})$$

$$\mathcal{M}_n^{\mathrm{YM}} = \int d\mu_n\, \mathcal{C}_n\, \mathrm{Pf}'\Psi(\epsilon) \qquad \mathcal{L}_{\phi^3} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{3!} f^{I\,J\,K} f^{I'\,J'\,K'}\ \phi^{I\,I'}\phi^{J\,J'}\phi^{K\,K'}$$

$$\mathcal{M}_n^{\mathrm{GR}} = \int d\mu_n\, \mathrm{Pf}'\Psi(\epsilon)\, \mathrm{Pf}'\Psi(\tilde{\epsilon})$$

hidden simplicity of gluons/gravitons in any dimension!

$$\mathrm{Pf}'\Psi := \frac{\mathrm{Pf}|\Psi|^{i,j}_{i,j}}{\sigma_{i,j}}$$

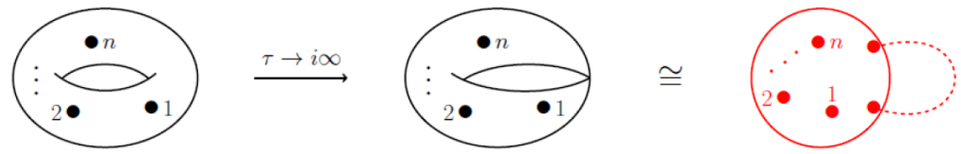
$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix},$$

$$A_{a,b} := \begin{cases} \frac{k_a\cdot k_b}{\sigma_{a,b}} & a\neq b \\ 0 & a=b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a\cdot\epsilon_b}{\sigma_{a,b}} & a\neq b \\ 0 & a=b \end{cases},$$

$$C_{a,b} := \begin{cases} \frac{\epsilon_a\cdot k_b}{\sigma_{a,b}} & a\neq b \\ -\sum_{c\neq a} C_{a,c} & a=b \end{cases}$$

Defining feature: Pfaffian is gauge invariant by SE -> gauge & diff. invariance!

Loops & (ambi-twistor) strings



- Ambitwistor strings (2d chiral CFT) [Mason,Skinner]: derive CHY formulas from CFT correlators

- Higher genus too difficult! -> loop amps from nodal Riemann sphere [Geyer, Mason, Monteiro,Tourkine,…]

- possible to obtain higher-genus string correlators from ambitwistor/CHY integrands [Geyer, Monteiro;…]

- another method: 1-loop CHY from forward limit of trees in higher dim, $\ell^2 \neq 0$ [SH, Yuan; CHY]

$$M^{(1)} = \int \frac{d^d \ell}{\ell^2} \lim_{k_\pm \to \pm \ell} \int \prod_{i=2}^{n} \delta \left(\frac{\ell \cdot k_i}{\sigma_i} + \sum_{j=1, j\neq i}^{n} \frac{k_i \cdot k_j}{\sigma_{ij}} \right) \hat{I}(\ell)$$

- 1-loop KLT formula for gauge theories + gravity, etc., manifest double copy [SH Schlotterer, 17 PRL] equivalence of two methods: both from superstring amps [SH Schlotterer, Y. Zhang 18] higher loops?

- New relations: QFT amps <-> string amps, also for bosonic/heterotic strings [SH, F. Teng Y. Zhang 19 PRL]

Goldstone particles from Adler zero

EFTs for Goldstone particles (symmetry breaking) e.g. pions, DBI, Galileon etc. [CHY 14] [Cheung et al 14]

What is special about them? Amplitudes vanish in soft limit: enhanced Adler zero!

From CHY: a new ingredient with Adler zero $(\det' A_n)|_{p_i \sim \mathcal{O}(\tau)} = \mathcal{O}(\tau^2)$.

- $M_n = \int d\mu_n \, (\mathrm{Pf}' A)^2 \, \mathrm{PT}$, adjoint scalars with two derivative coupling?

  U(N) NLSM (the chiral Lagrangian) $\mathcal{L} = \mathrm{Tr}(\partial_\mu U^+ \partial^\mu U)$

- $M_n = \int d\mu_n \, (\mathrm{Pf}' A)^2 \, \mathrm{Pf}' \Psi$, higher-derivative-coupled photons?

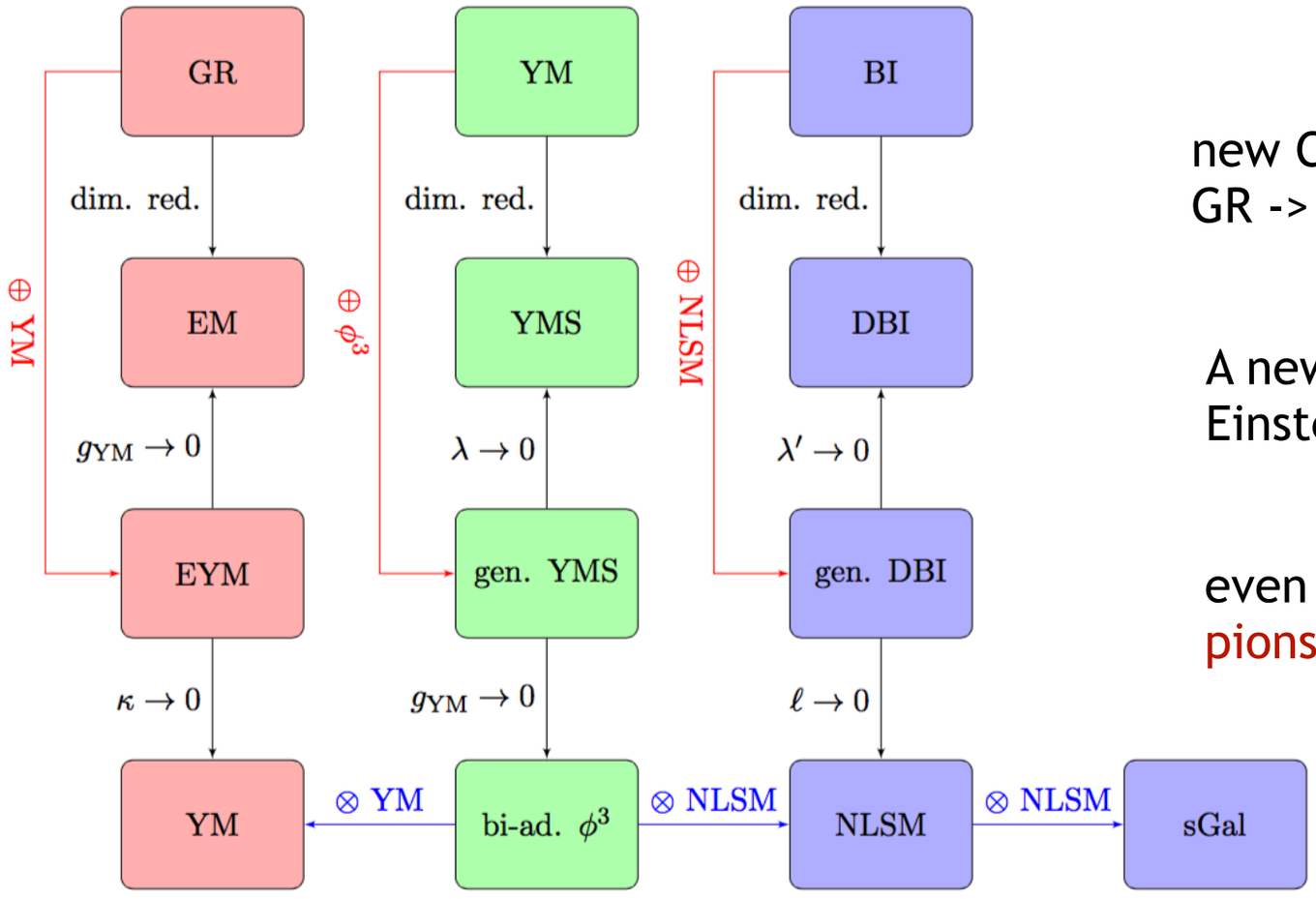
  Born-Infeld theory (BI) & DBI by dim reduction $\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} - \ell F_{\mu\nu} - \ell^2 \partial_\mu \phi \partial_\nu \phi)}$

- a special Galileon (single scalar with many derivatives) $M_n^{\mathrm{sGal}} = \int d\mu_n \, (\mathrm{Pf}' A)^4$

	Gauge Theories		
I	GR (s=2)	YM (s=1)	BI (s=1)
II	YM (s=1)	ϕ^3 (s=0)	NLSM (s=0)

	Effective Field Theories			
I	sGal (τ^3)	NLSM (τ^1)	BI (τ^1)	DBI (τ^2)
II	NLSM (τ^1)	ϕ^3 (τ^{-1})	YM (τ^{-1})	YMs (τ^0)

A landscape of massless theories



new CHY from old ones by e.g. dim reduction
GR -> Einstein-Maxwell, YM -> YM-scalar

A new operation as direct sum of two particles -> Einstein-Yang-Mills, Yang-Mills + bi-adjoint scalars

even more interesting relations [CHY 14][Cheung et al]:
pions from special dimension reduction of gluons!

These amplitudes are strongly constrained (even uniquely determined) by symmetries:
gauge invariance & Adler zero; deeply connected to each other!

Double-copy as direct product

- Double copy $\text{“}GR \sim YM \otimes YM\text{”}$ or more precisely $GR = YM^2/\phi^3$
- CHY: KLT kernel is the inverse of bi-adjoint scalar amps: S=m^-1
- Direct product of amplitudes in two theories: discover new double-copies

Double copies from CHY

$$A \equiv L \otimes R = \int d\mu_n \; I_L \; I_R$$

$$A_L(\alpha) = \int d\mu_n \; I_L \; PT(\alpha)$$

$$A_R(\beta) = \int d\mu_n \; I_R \; PT(\beta)$$

$$m(\alpha|\beta) = \int d\mu_n \, PT(\alpha) \, PT(\beta)$$

$$A = \sum_{\alpha,\beta \in S_{n-3}} A_L(\alpha) m^{-1}(\alpha|\beta) A_R(\beta)$$

L⊗R	L	R
GR	YM	YM
BI	YM	NLSM
DBI	YMS	NLSM
sGal	NLSM	NLSM

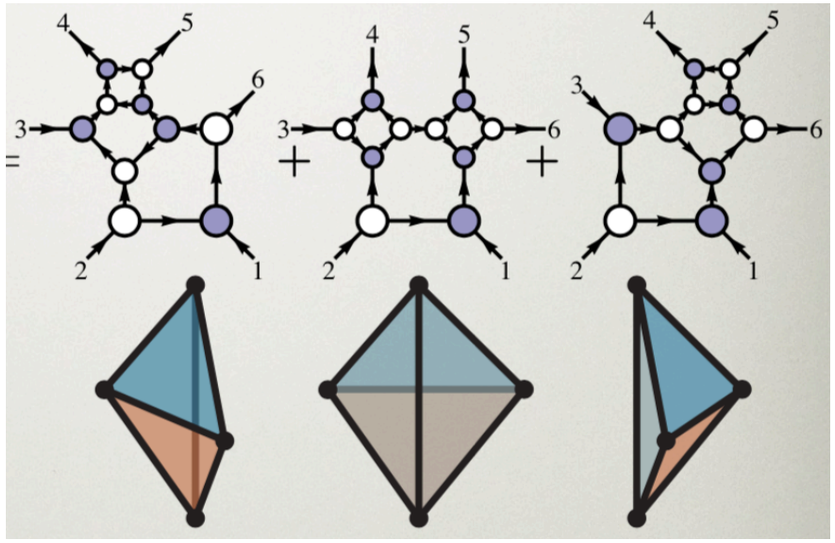
- None of these manifest from Lagrangian/Feynman diagrams: deeper reason?

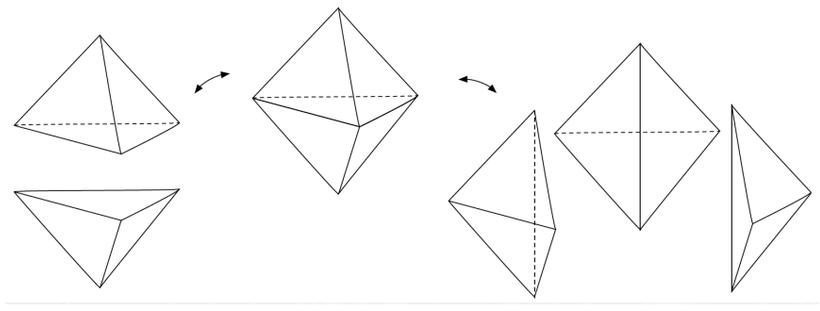
Amplituhedron

N=4 Super-Yang-Mills: hydrogen atom of QFT

deep connections with geometries, reflecting the infinite-dim symmetry -> integrability

Amplitudes are volume of some “polytopes”!
emergence of QM & spacetime from geometry!





New maths: positive geometry (real)
with a unique canonical form (complex):
only logarithmic singularities
@ boundaries (residues recursively defined)

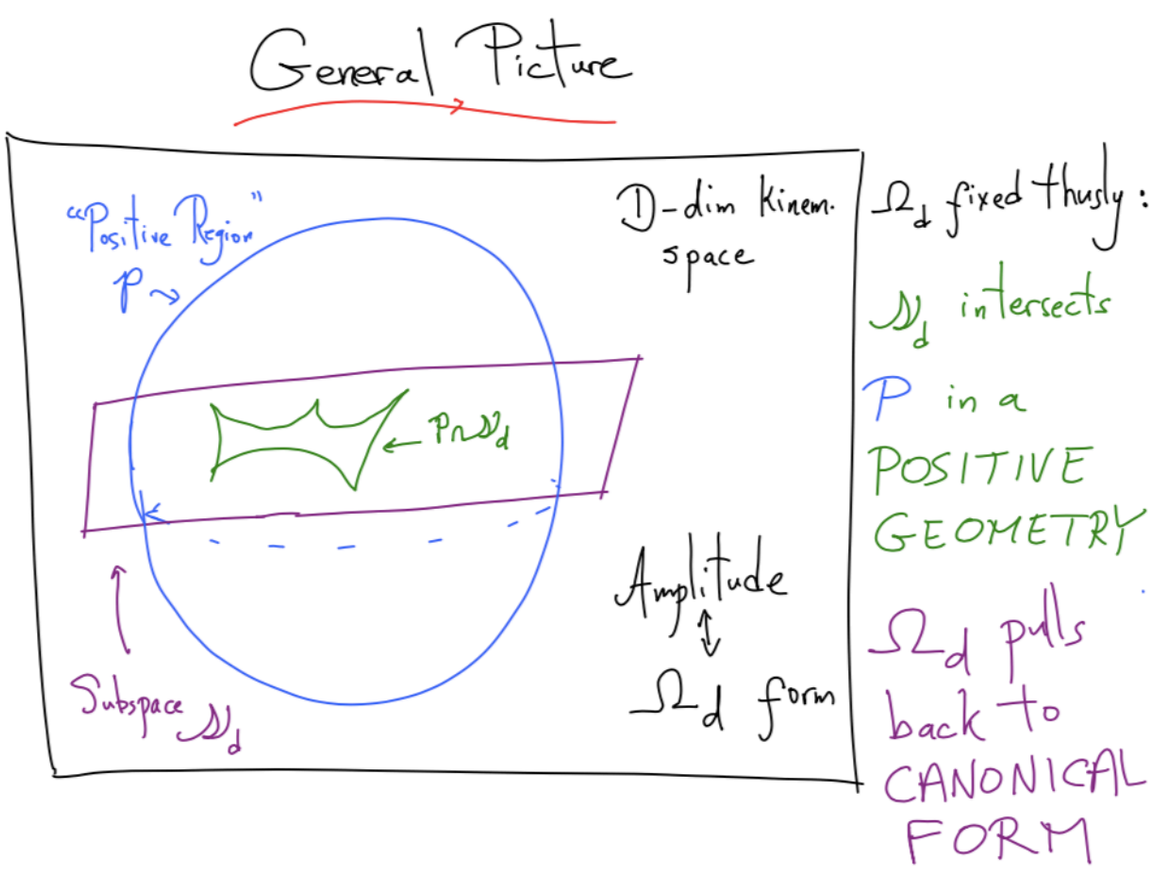
$$Y = C \cdot Z$$ external data: mom-twistors

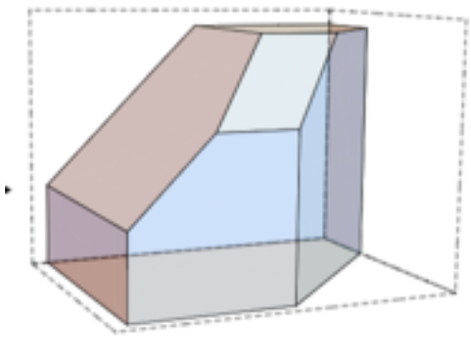
common gen. of polytopes & Grassmannians

differential forms in kinematic space for any helicity amplitudes in any gauge theories [SH, C. Zhang, 18]

-> momentum amplituhedron for SYM and ABJM!
[Damagard, Ferro, Lukowski, 19 ; SH, C. Kuo, Y. Zhang, 21]

Amplitudes as differential forms





Generalize amplituhedron to general theories in any dimensions (even phi^3)!

Bi-adjoint scalar: Amp (form)="volume" of associahedron in kinematic space

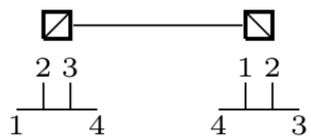
Geometrize color & its duality to kinematics, forms for gluon/pion amps etc.

Locality & unitarity emerges purely from geometries @ infinity of spacetime!

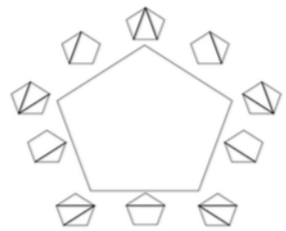
Kinematic associahedron

[Arkani-Hamed, Bai, SH, Yan, 2018]

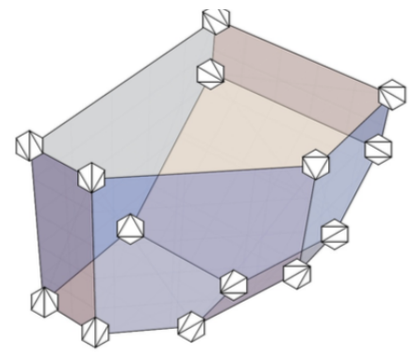
Associahedron of dim. (n-3): faces 1:1 corresp. with triangulation of n-gon



Positive region Δ_n: all planar variables $s_{i,i+1,\cdots,j} \geq 0$ (top-dimension)



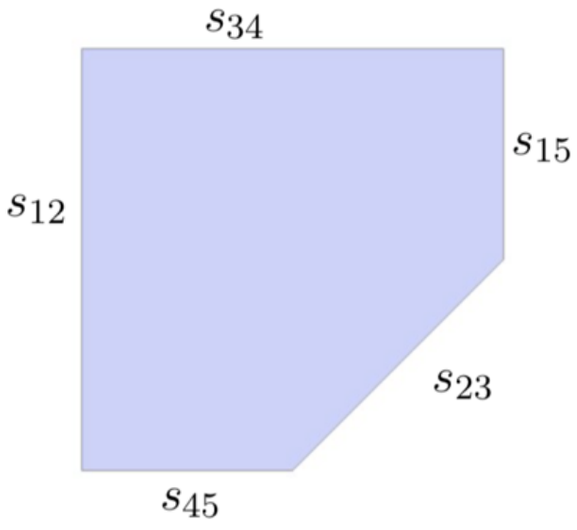
Subspace H_n: $-s_{i\,j} = c_{i,j}$ as *positive constants*, for all non-adjacent pairs $1 \leq i, j < n$; we have $\frac{(n-2)(n-3)}{2}$ conditions $\Longrightarrow$ $\dim H_n = n-3$

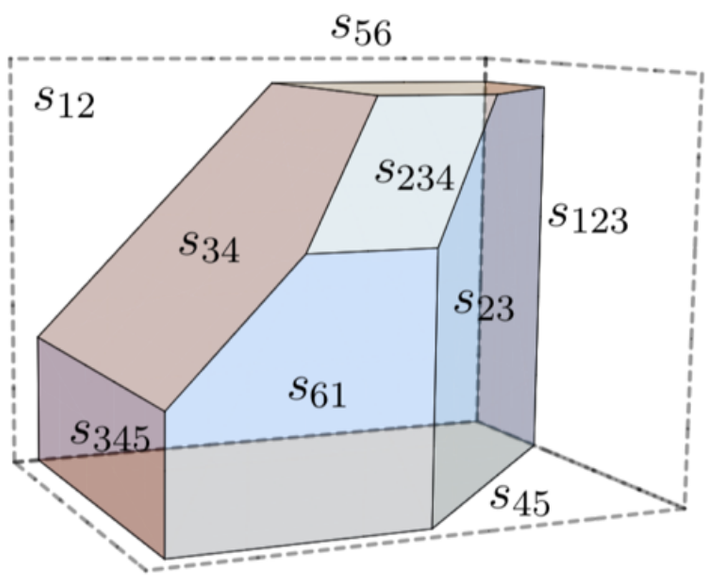


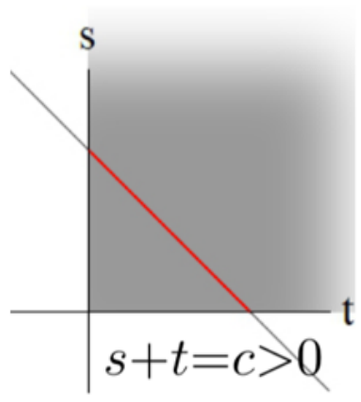
Kinematic Associahedron is their intersection! $\mathcal{A}_n := \Delta_n \cap H_n$

$e.g.\ \mathcal{A}_4 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$

encode singularities of any (colored) massless amplitudes at tree level: gluons, pions, etc.







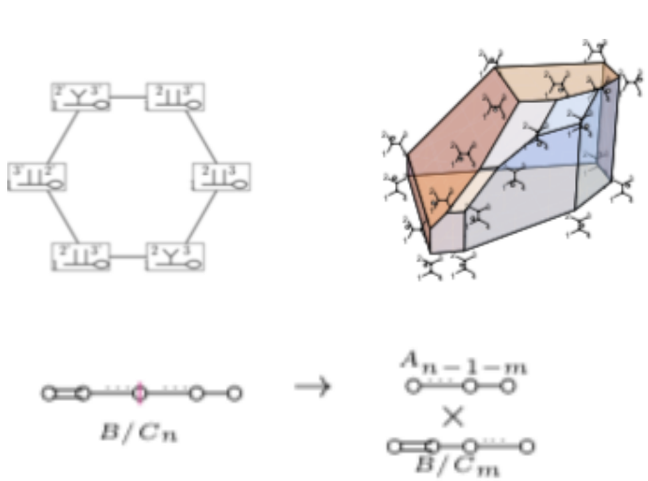
Amplitudes as "volume" [Arkani-Hamed, SH, Salvatori, Thomas, 2019]

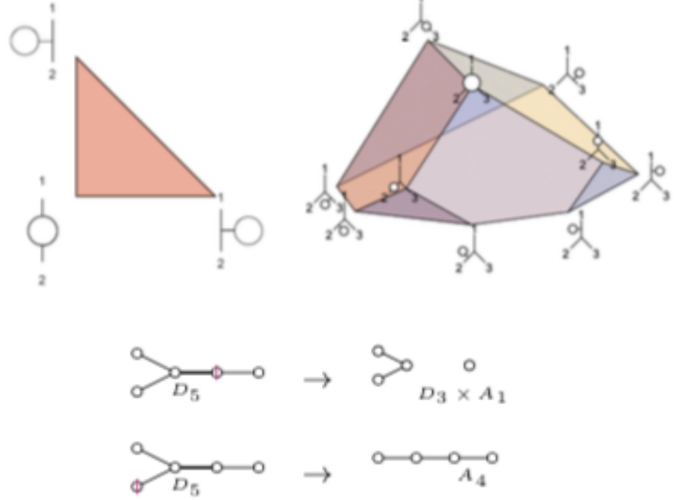
Canonical form of $\mathcal{A}_n$ = Pullback of Ω_n to $H_n \propto$ planar ϕ^3 amplitude!

$$e.g. \qquad \Omega(\mathcal{A}_4) = \Omega_4^{(1)}|_{H_4} = (\frac{ds}{s} - \frac{dt}{t})|_{-u=c>0} = (\frac{1}{s} + \frac{1}{t})\, ds$$

$$\Omega(\mathcal{A}_5) = \Omega_5^{(2)}|_{H_5} = \left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right) ds_{12} \wedge ds_{34}$$

- Associahedron is the (tree) "amplituhedron" for scalar: amps="volume"
- Feynman-diagram expansion=special triangulation -> many new representations





- Extend to "cluster polytope" of finite types: B/C: cyclohedron <—> tadpoles; type D <—> one-loop planar phi^3 (all with "factorizing" boundaries)
- Hidden symmetry (invisible in FD's) manifest by geometry (analog in N=4 SYM)

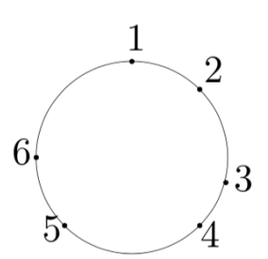
Generalized string amps [Arkani-Hamed, SH, Lam, Thomas, 2019]

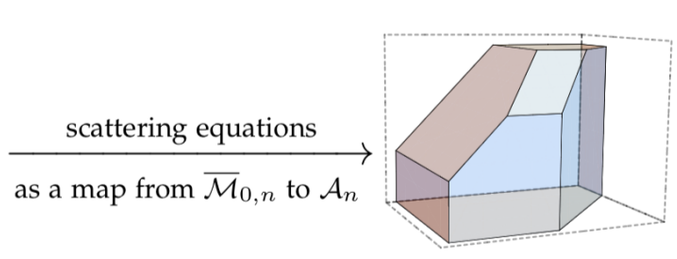
A well-known associahedron: minimal blow-up of the open-string worldsheet $\mathcal{M}^+_{0,n} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_n\}/\mathrm{SL}(2,\mathbb{R})$ [Deligne, Mumford]

a geometric origin of scattering eqs & CHY

The *canonical form* of $\overline{\mathcal{M}}^+_{0,n}$ is the "Parke-Taylor" form

$$\omega_n^{\mathrm{WS}} := \frac{1}{\mathrm{vol}\,[\mathrm{SL}(2)]} \prod_{a=1}^{n} \frac{d\sigma_a}{\sigma_a - \sigma_{a+1}} := \mathrm{PT}(1,2,\cdots,n)\, d\mu_n$$

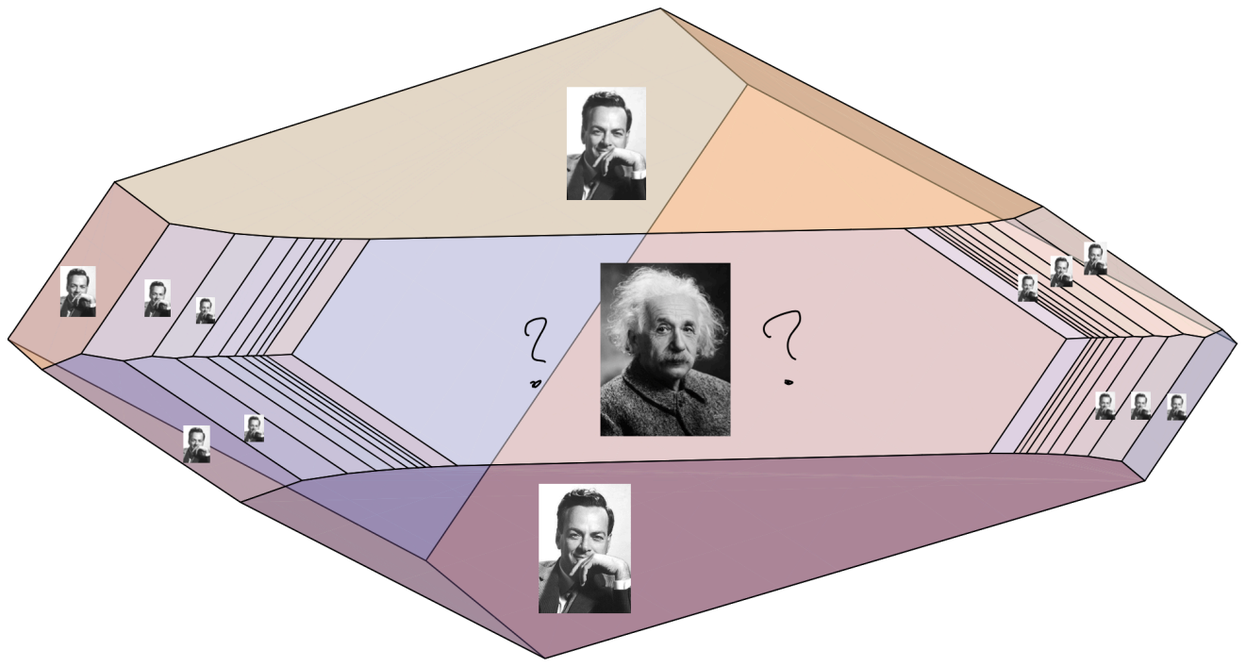




- Generalize $\mathscr{M}_{0,n}$(worldsheet associahedron) to other types: binary geometries
- Natural "string integrals" for all finite types: α'-deform. of loop ϕ^3 amps
- Field-theory (particles) $\alpha' \to 0$ = CHY formula with $\alpha' \to \infty$ (saddle points)
- Higher-genus surfaces vs. higher-loop amps?

Particles & strings from geometries

- Surfacehedra [Arkani-Hamed et al]: curves on surface w. any genus (all loops!)
- Infinite polytopes: truncations <-> (infinite) cluster algebras & quivers
- Canonical forms -> all-loop non-planar tr(phi^3) integrand



- Now also natural string-like integrals for surfacehedra (infinite product!)
- Appearance of “gravity” (like closed-string) from positivity (open-string)
- Goal: strings (& particles) without string (worldsheet) <— new geometries

Stringy canonical forms [Arkani-Hamed, SH, Lam, 2019, 2020…]

vast generalizations of (open-)string amplitudes $\mathbf{I}_n(\{s\}) = (\alpha')^{n-3} \int_{\mathscr{M}^+_{0,n}} \frac{d^{n-3}z}{z_{1,2}\cdots z_{n,1}} \prod_{a<b} (z_{a,b})^{\alpha' s_{a,b}}$

positive parametrization, e.g. $z_3 = 1 + x_2\,, \quad z_4 = 1 + x_2 + x_3, \quad \ldots, \quad z_{n-1} = 1 + x_2 + \cdots + x_{n-2} \implies$

integral w. positive polynomials: $\mathscr{I}_{\{p\}}(\mathbf{X}, \{c\}) = \alpha'^d \int_{\mathbb{R}^d_{>0}} \prod_i^d \frac{dx_i}{x_i}\, x_i^{\alpha' X_i} \prod_I p_I(\mathbf{x})^{-\alpha' c_I}$

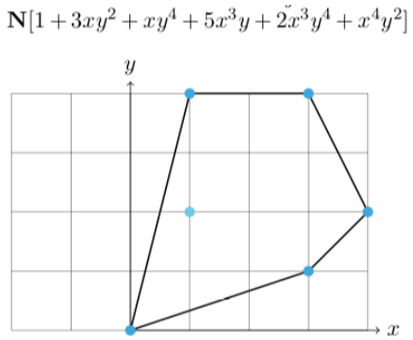
long history ("Euler-Mellin"…) $\rightarrow$ $d^d\mathbf{X}\mathscr{I}\,(\mathbf{X})$ a diff. form exhibits geometries & remarkable properties

- leading order $(\alpha' \to 0)$, residues on "massless poles" (+ convergence): controlled by polytope P
- "stringy" properties: meromorphic with"massive poles", exponential UV softness, "channel duality"
- scattering (saddle-point) equations & twisted (co-)homology
- dual u variables, tropical compactifications, (complex) closed-stringy integrals …

Newton polytopes

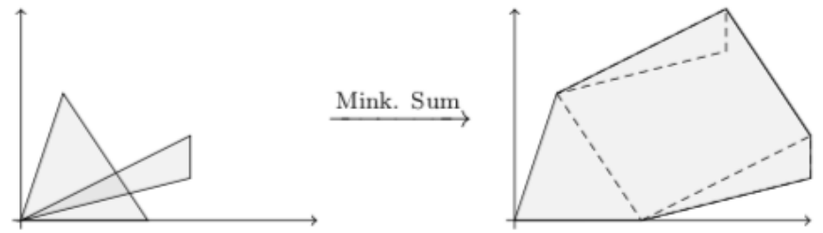
single polynomial: $\mathcal{J}_p(\mathbf{X}, c) = \alpha'^d \int_0^\infty \prod_{i=1}^d \frac{dx_i}{x_i}\, x_i^{\alpha' X_i}\, p(\mathbf{x})^{-\alpha' c}$

converges iff $\mathbf{X}$ is inside Newton polytope $c\,\mathbf{N}(p)$, $\lim_{\alpha'\to 0} \mathcal{J}_p(\mathbf{X}, c) = \underline{\Omega}(c\ \mathbf{N}[p(\mathbf{x})]; \mathbf{X})$



(also see [Nilsson, Passare; Berkesch et al])

trivial to gen. to $\mathcal{J}_{\{p\}}(\mathbf{X}, \{c\}) = \alpha'^d \int_{\mathbb{R}^d_{>0}} \prod_{i=1}^d \frac{dx_i}{x_i}\, x_i^{\alpha' X_i} \prod_{I=1}^m p_I(\mathbf{x})^{-\alpha' c_I}$

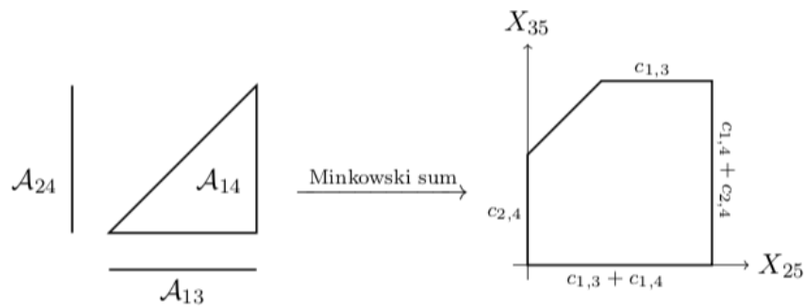


converges iff $\mathbf{X}$ is inside Minkowski sum $\mathscr{P} := \sum_I c_I \mathbf{N}[p_I(\mathbf{x})]$

$\lim_{\alpha'\to 0} d^d\mathbf{X}\mathcal{J}_{\{p\}}(\mathbf{X}, \{c\}) = \Omega(\mathscr{P}; \mathbf{X})$ α'-deformations of canonical form of any (rational) polytope

e.g. ABHY = Minkowski sum of simplices, one for each $c_{i,j} > 0$

$$\mathbf{I}_n = \mathcal{J}_{\mathscr{A}_{n-3}} = \int \prod_{i=2}^{n-2} \frac{dy_i}{y_i} y_i^{\alpha' X_{i,n}} \prod_{1\le i<j-1<n} p_{ij}(\mathbf{y})^{-\alpha' c_{i,j}}$$



re-discover open-string integrals & moduli space from ABHY!

Applications of stringy integrals

(1). stringy integral for gen. associahedra $\mathscr{P}(\Phi)$ of finite type: $\mathcal{I}_\Phi(\{S\}) = (\alpha')^d \int_{\mathbf{A}(\Phi)_+/T} (\omega/T) \prod_{\gamma\in\Gamma} x_\gamma^{\alpha' S_\gamma}$

leading order → ABHY cluster polytopes: $\lim_{\alpha'\to 0} d^d\mathbf{X}\mathcal{I}_\Phi(\mathbf{X},\{c\}) = \Omega(\mathscr{P}(\Phi^\vee,c);\mathbf{X})$

ABCD: α'-extension of ϕ^3 amps w. factorization at finite α' e.g. $\mathcal{I}_{D_n} \to \mathcal{I}_{A_m}\times\mathcal{I}_{D_{n-m-1}}$, $\mathcal{I}_{A_{n-3}}\times\mathcal{I}_{A_1}\times\mathcal{I}_{A_1}$, $\mathcal{I}_{A_{n-1}}$

(2). stringy integral over $G_+(k,n)/T$: $\mathcal{I}_{k,n}(\{S\}) := (\alpha')^d \int_{\mathrm{G}_+(k,n)/T} (\omega_{k,n}/T)\prod_I \Delta_I^{\alpha' s_I}$ $\quad d := (k-1)(n-k-1)$

$D = \binom{n}{k} - n$ higher-k gen. of string integrals ($\mathscr{M}^+_{0,n} \sim \mathrm{G}_+(2,n)/T$): k=4 -> symbol alphabet in N=4 SYM

$\mathscr{P}_{k,n} = \sum_I S_I N[\Delta_I]$ [Arkani-Hamed, Lam, Spradlin 20] ↔ tropical $G_+(k,n)$ [Speyer, Williams; Cachazo et al; Drummond et al]

(3). Feynman integrals (parametric form) -> A-hypergeometric functions, GKZ system… [work in progress]

leading UV/IR behavior given by Symanzik polytopes (tropical geometry) [Arkani-Hamed, Hillman, Mizera 22]

Summary & outlook

Scattering Amplitudes: one of the most exciting frontiers of hep-th
rich structures/applications to formal QFT, gravity, strings, math etc.

New Picture: general massless S-matrix via punctured Riemann spheres;
higher-genus for loops. A (weak-weak) QFT/String duality for S-matrix?

New Relations: *gluons*, *pions*, *gravitons* … double copy for quantum gravity
Double copy beyond amps: *classical solutions*, *gravity waves*, ……

New Maths: geometries in kinematic space & amps as differential forms
"*theory at infinity*": *geometry/combinatorics → Lorentz inv. + unitarity*

"Marble statues in the Forest beyond Quantum Mechanics & Spacetime"
What will we see next?

Thank You!

Acknowledgement: pictures from N. Arkani-Hamed, Z. Bern, Y. Geyer, J. Trnka, G. Salvatori