# Topological Vertex for SO(N) Gauge Theories and Beyond

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> **based on arXiv:2012.13303 (JHEP04(2021)292) in collaboration with Hirotaka Hayashi (Tokai Univ.) and work in progress with Satoshi Nawata (Fudan Univ.)**

**What we study:**

**5d** *N***=1 supersymmetric gauge theories (on C2q,t** x **S1)** $\mathsf{gauge}$  group:  $\mathsf{U(N)} \rightarrow \mathsf{SO(N)} \rightarrow \mathsf{G_2} \rightarrow \dots$ 

**Why?**

- **• Exactly solvable (with localization method)**
- **• non-perturbative physics**
- **• dualities (S-duality, fiber-base duality, AGT (dual with CFTs) )**
- **• quantum integrability**
- $5d N=1 \rightarrow 4d N=2$ 
	- **…**

### **Current situation:**

**…**

**many interesting properties discovered in the case of U(N)**

- **• 4d/2d duality (with chiral algebra or non-unitary CFT)**
- **• modular tensor category**
- **• quantum integrability (~Calogero-Sutherland system)**

**but not so many results known in SO(N) or Sp(N) theories**

**because of technical difficulties**

**• in the calculation of Nekrasov partition function, instanton counting is very difficult.**

**what we try to solve**

**• in the study of chiral algebra, BCD-type Macdonald symmetric polynomial is difficult.**

**A sketch of what we did:**



**Plan of Talk:**

- **1. Review 1: SU(N) gauge theories and localization**
- **2. Review 2: brane construction and topological string**
- **3. Review 3: AGT duality and DIM algebra**
- **4. proposal: topological vertex formalism for SO(N) theory**
- **5. consistency checks**
- **6. Kim-Yagi's prescription and physical interpretation**

**5d N=1 gauge theory on C2q,t** x **S1 Its partition function can be found via localization method. Ω-background with two deformation parameters** *q*<sup>1</sup> = *eR*✏<sup>1</sup> = *t, q*<sup>2</sup> = *eR*✏<sup>2</sup> = *q*1*.* (32) < *q*<sup>1</sup> = *e<sup>R</sup>*✏<sup>1</sup> = *t, q*<sup>2</sup> = *e<sup>R</sup>*✏<sup>2</sup> = *q*<sup>1</sup>*.* (32) *R***: radius of S1**

$$
Z = Z_{cl} Z_{one\ loop} Z_{instanton}.
$$
\nperturbative non-perturbative part

perturbative part is completely determined by the root system<br>
of the matrix of all the contract of all the setting of the root system **of the gauge group** *G***.**

$$
Z_{one\ loop} = Z_{Cartan}^G Z_{root}^G,
$$

**expression of the perturbative part:** *z*<br>*L* the perturbative part Cartan*Z<sup>G</sup> µ,λ* <sup>−</sup>*Q*<sup>2</sup> <sup>−</sup>*Q*<sup>2</sup>

$$
Z_{\text{root}}^G = P.E. \left( \left( \frac{q}{(1-q)(1-t)} + \frac{t}{(1-q)(1-t)} \right) \sum_{\alpha \in \Delta_+} e^{-\alpha \cdot a} \right),
$$

$$
Z_{\text{Cartan}}^G = P.E. \left( \frac{\text{rank}(G)}{2} \left( \frac{q}{(1-q)(1-t)} + \frac{t}{(1-q)(1-t)} \right) \right).
$$

where  $\Delta_+$  is the set of all positive roots, *Z* = *ZclZone loopZinstanton.* (163)

$$
P.E. (f(x_1, x_2, \ldots, x_n)) := \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} f(x_1^k, x_2^k, \ldots, x_n^k) \right).
$$

instanton part of the pure SU(*N*) gauge theory with the zero CS level is given by **For obtaining (2.17) we use the Cauchy identity (B.34) to sum over the Voucher the Young diagram over the Voucher the Young diagram over the Young diagram over the Young diagram over the Young diagram over the Young diagr** *Zone loop* = *Z<sup>G</sup>*

> $\mathsf{on}\ \mathsf{counting}$ unτι<br>− **instanton counting**

*k*=0 *|W*(SU(*k*))*| i*=1 *λλ* (1*, q*)=(−1)*|λ<sup>|</sup> sλ*(*q*−*ρ*)*sλ<sup>t</sup>* (*q*−*ρ*)*.* (2.19) *|W*(*G*)*|* = **of the mathematica parameters using the since the diagram (2.10).** Since the diagram (2.10). Since the diagram (2.10) ⎪⎨ *n a* =  $\frac{1}{2}$  =  $\frac{1}{2}$  =  $\frac{1}{2}$  =  $\frac{1}{2}$ *ng dieory. D(p−4) brancs on Dp branc.*  $\mathsf{in} \mathcal{A} \mathsf{d} \quad *F = F.$  (codimension 4 object) in string theory: D(p-4) branes on Dp brane. **in 5d: particle-like**

*.* (2.18)

**instanton counting is not easy** Coulomb branch moduli in the Cartan subalgebra. *q, t* are related to the Ω-deformation parameters *ϵ*1*, ϵ*<sup>2</sup> by *q* = *e*−*ϵ*<sup>1</sup> *, t* = *eϵ*<sup>2</sup> . The unrefined case corresponds to *q* = *t*. *a n example counting is not easy corresponds to <i>a* **correspond counting is not easy conditions to**  $\alpha$ instanton part of the pure SU(*N*) gauge theory with the zero CS level is given by S١

**ADHM construction → Nekrasov partition function** For U(N) or SU(N) theory: ADHIVI CONSTRUCTION  $\rightarrow$  Nekrasov partition function On the other hand, the partition function of the instanton part is more involved. The ADHM construction  $\rightarrow$  Nekrasov partition function n → Nekrasov partition funct *|W*(SU(*k*))*|*  $\overline{\mathbf{a}}$ *k*  $\mathbf{A}$   $\mathbf{B}$   $\mathbf{B}$  *Lea* instituted the Nekronic <sup>q</sup>*<sup>k</sup>* <sup>1</sup>  $\alpha$  partition  $\blacksquare$  Partition function 2*πi*

$$
Z_{\text{loc, inst}}^{\text{SU}(N)} = \sum_{k=0}^{\infty} \mathfrak{q}^k \frac{1}{|W(\text{SU}(k))|} \oint \left(\prod_{i=1}^k \frac{\mathrm{d}\phi_i}{2\pi i}\right) Z_k^{\text{SU}(N)},
$$

**where** and

$$
|W(G)| = \begin{cases} n! & G = \text{SU}(n) \\ 2^{n-1+\delta}n! & G = O(2n+\delta) \\ 2^n n! & G = \text{Sp}(n) \end{cases}
$$

$$
Z_k^{\text{SU}(N)} = \frac{[2\epsilon_+]^k}{[\epsilon_{1,2}]^k} \prod_{i=1}^k \prod_{j=1}^N [\phi_i - a_j \pm \epsilon_+]^{-1} \prod_{\substack{i,j=1 \\ i  

$$
\phi_{ij} = \phi_i - \phi_j, \qquad [x] := 2 \sinh \frac{x}{2} = e^{\frac{x}{2}} - e^{-\frac{x}{2}}, \quad \epsilon_{\pm} = \frac{\epsilon_1 \pm \epsilon_2}{2}.
$$
$$

[*ϵ*1*,*2] *k* anch r *a***<sub>i</sub>: Coulomb branch parameters** a<sub>i</sub>: Coulomb branch parameters **Jeffrey-Kirwan (JK) residue in U(N) case** 

**poles are labeled by a set of N partitions (Young diagram)**

 $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$ 



**pole**

$$
\phi_i = a_j + (r-1)\epsilon_1 + (s-1)\epsilon_2. \quad \Longleftrightarrow \quad \text{box in Young diagram}
$$

here. Then the Frobenius coordinates are given by (*α*1*, <sup>α</sup>*2*, <sup>α</sup>*3*|β*1*, <sup>β</sup>*2*, <sup>β</sup>*3) = ! <sup>9</sup> **\* Frobenius basis of Young diagram**

$$
\lambda=(\alpha_1,\alpha_2,\ldots|\beta_1,\beta_2,\ldots).
$$

# **analytic expression**

**the partition function can be written in terms of Nekrasov factors** A.1 Nekrasov Factor tition function can be written in terms of Nekrasov factors

 $[Nekrasov (2002)]$ *<sup>i</sup>* = *a<sup>j</sup>* + (*r* 1)✏<sup>1</sup> + (*s* 1)✏2*.* (167) *i*  $\blacksquare$  *i* 

$$
N_{\lambda\nu}(Q;q_1,q_2) := \prod_{(i,j)\in\lambda} \left(1 - Qq_2^{-\nu_i+j-1}q_1^{\lambda_j^t-i}\right) \prod_{(i,j)\in\nu} \left(1 - Qq_2^{\lambda_i-j}q_1^{-\nu_j^t+i-1}\right),
$$

**for example, pure U(N) gauge theories can be expressed as**  *N*

$$
Z = \sum_{\{\lambda_1,\lambda_2,\ldots,\lambda_N\}} \prod_{i=1}^N \left( \mathfrak{q}^{|\lambda_i|} N_{\lambda_i\lambda_i}^{-1}(1) \right) \prod_{i < j} N_{\lambda_i\lambda_j}^{-1}(Q_i/Q_j) N_{\lambda_j\lambda_i}^{-1}(Q_j/Q_i).
$$

 $\sim N_{\lambda}$ <sub>*A</sub>* $(m_f)$ </sub> **matter:**  $\sim N_{\lambda\emptyset}(m_f)$ ⇠ *N*;(*m<sup>f</sup>* ) (169)

#### The convention used here is that (*i, j*) 2 is the box in the *i*-th row and *j*-th column of the Young diagram . In some cases, it is more convenient to rescale *Q* to *Qq*1*q*2, and rewrite (A.1.3) to **Chern-Simons term:**

$$
\sim \prod_{(i,j)\in \lambda} q^i t^{-j}.
$$

#### **c.f. SO(N) instanton partition function** integrand of the integral for the *k*-instanton partition function of the pure SO(2*N* +*δ*) gauge  $c.f.$  SO(N) insta **(N) instanton partition function**

$$
Z_k^{\text{SO}(2N+\delta)}
$$
  
=  $(-1)^k \frac{[2\epsilon_+]^k}{[\epsilon_{1,2}]^k} \prod_{i  
where  

$$
\mathcal{S}(\phi) := \frac{[\phi \pm \epsilon_-]}{[\phi \pm \epsilon_+]}.
$$

$$
Z_{\text{loc, inst}}^{\text{SO}(2N+\delta)} = \sum_{k=0}^{\infty} \mathfrak{q}^k \frac{1}{[W(\text{Sp}(k))]}\oint \left(\prod_{i=1}^k \frac{d\phi_i}{2\pi i}\right) Z_k^{\text{SO}(2N+\delta)}.
$$$ 

*z z z i*nst = 2  $\boldsymbol{\kappa}$  a more complicate 2*πi <sup>k</sup> .* (B.12)  $T$  indentified  $\sigma$  anneard not just a more complicated integrand. much more difficult! not just a more complicated integrand.

We do not know how to label the JK poles... **Γ We do not know how to label the JK poles…**

Sp(N) theories are even more difficult!... Yet we want to work on them! *δ* → 0 at the end. As argued in [78, 80], the poles may be labeled by colored Young

# **brane construction in string theory** [Aharony, Hanany, Kol (1997)]



We draw a web diagram on this plane. (balance of tension  $\Rightarrow$ various kind of (p,q) 5-branes stretching along the vector (p,q)) branes.



**In U(N) theory**

**instanton solutions labeled by N Young diagrams**



**D5**

**Our results suggest that in the unrefined limit of SO(N) theories, the above picture still holds. D1**



# **string duality with topological string theory**

**topological string (A-model) on toric Calabi-Yau**



**[Leung, Vafa (1997)]**

**[Aganagic, Klemm, Marino, Vafa (2003)]**

**= (p,q) brane web**

**on S1**

**The topological string is a convenient way to compute the index (partition function on S1) or the instanton partition function of 5d N=1 gauge theories.**

# **Topological String?**

**\*The concrete definition etc. are not useful in this talk.**

**• It is a topologically twisted N=(2,2) sigma model.**

**[Witten, (1988)] [Vafa, (1991)] …**

**• Due to different ways of topological twist, we have A- and B-models. They are connected through the mirror symmetry.**

**[Candelas et al., (1985), Dixon, (1987), Lerche et al., (1989) ….]**

**• There are certainly the open and closed version of the string theory, and there is a open/closed duality. The open theory is deeply related to Chern-Simons theories.**

**[Witten, (1992)] …**

**In the Calabi-Yau language, the web diagram corresponds to**  In the Galabi-Yau language, the web diagram corresponds to<br>the toric diagram, in which each line denotes degenerate locus of the torus fiber (of toric Calabi-Yau). logical strings of A-model, the toric Calabi-Yau's. The geometry is always *T*<sup>2</sup> ⇥ R fibered over R<sup>3</sup> *x* language, the web diagram corresponds to  $\frac{1}{2}$ In the Calabi-Yau language, the web diagram corresponds to the toric diagram, in which each line denotes degenerate locus direction of Calabi Vau Lenguege the web diegrem corresponde to



Simplest examples: **a** Simplest examples:  $\overline{a}$   $\overline{b}$   $\overline{c}$   $\overline{d}$   $\overline{d}$   $\overline{c}$   $\overline{d}$   $\overline{d}$  By choosing *r*↵ = *|y*1*|*



# **The A-model partition function can be computed with the topological vertex.**<br>
[Aganagic, Klemm, Marino, Vafa, (2003)]



It can be expressed in terms of (skew) Schur functions. where the proportional prefactor only depends on . By using the symmetric property, *C*;*,*;*,* =

$$
C_{\mu,\nu,\lambda} = q^{-\frac{\kappa(\mu)}{2}} s_{\lambda}(q^{-\rho}) \sum_{\eta} s_{\mu^t/\eta}(q^{-\lambda-\rho}) s_{\nu/\eta}(q^{-\lambda^t-\rho}).
$$

# ⇠ <sup>Y</sup> It is similar to the Feynman diagram to compute the **partition function of topological vertex,**

$$
Z_{top} = \sum_{\lambda,\mu,\nu,\sigma,\tau,\dots} (-Q_1)^{|\lambda|} (-Q_2)^{|\lambda|} \dots C_{\mu\nu\lambda} C_{\mu\sigma\tau} \dots
$$

*Qi***: Kahler parameters**  $\epsilon$ *t* parameters

**It reproduces part of the full Nekrasov partition function of the corresponding gauge theory.**

$$
Z_{top} = Z_{root}^G Z_{instanton}.
$$

**\*Remark:** 

**following from the pole cancellation (or blow-up equation), one can determine the classical piece and the Cartan part of the full partition function. [Grassi, Hatsuda, Marino (2014)]**

# The original topological string is dual to the "self-dual" **point, with two omega-background parameters** 1<br>11 - 11 *F*

 $\epsilon_1 + \epsilon_2 = 0.$ 

We call it an unrefined setup, and we mainly focus on this special limit in this talk. ÷, **limit in this talk.** 

 $\overline{a}$ **The refined version corresponding to a general omega-background was soon proposed.** was soon proposea.<br>[Awata, Kanno, (2005)] [Iqbal, Kozcaz, Vafa, (2007)] *<sup>q</sup>* ! <sup>1</sup>*, t* <sup>=</sup> *<sup>q</sup>* (30)

$$
C_{\mu,\nu,\lambda}(t,q) = q^{\frac{||\mu^t||^2}{2}} t^{-\frac{||\mu||^2}{2}} P_\lambda(t^{-\rho},q,t) \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta|+|\mu|-|\nu|}{2}} s_{\mu^t/\eta}(q^{-\lambda} t^{-\rho}) s_{\nu/\eta}(t^{-\lambda^t} q^{-\rho}).
$$

$$
q_1 = e^{R\epsilon_1} = t
$$
,  $q_2 = e^{R\epsilon_2} = q^{-1}$ .

It is again expressed in terms of (skew) Schur function.

**\*The refined topological string has no world sheet description, and is based on the melting crystal model picture. There is a special leg, usually named the preferred direction of**  *t* **the vertex.** *q*

Let us have a look at the details of Schur functions.

expressed as an expressed as an expressed as a fermion correlation correlation correlations of the correlation o It can be expressed as an expectation value of a vertex operator.

$$
s_{\lambda/\mu}(\vec{x}) = \langle \mu | V_+(\vec{x}) | \lambda \rangle = \langle \lambda | V_-(\vec{x}) | \mu \rangle ,
$$

**where**  $\mathcal{L}_{\text{max}}$ 

$$
V_{\pm}(\vec{x}) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i} x_i^n J_{\pm n}\right),
$$

$$
|\lambda\rangle = (-1)^{\beta_1+\beta_2+\cdots+\beta_s+\frac{s}{2}}\psi_{-\beta_1}^*\psi_{-\beta_2}^*\ldots\psi_{-\beta_s}^*\psi_{-\alpha_s}\psi_{-\alpha_{(s-1)}}\ldots\psi_{-\alpha_1}|\text{vac}\rangle,
$$

$$
\{\psi_n, \psi_m\} = \{\psi_n^*, \psi_m^*\} = 0, \quad \{\psi_n, \psi_m^*\} = \delta_{n+m,0}, \quad J_n := \sum_{j \in \mathbb{Z}+1/2} \psi_{-j} \psi_{j+n}^*,
$$

$$
[J_n, \psi_k] = \psi_{n+k}, \quad [J_n, \psi_k^*] = -\psi_{n+k}^*, \quad [J_n, J_m] = n\delta_{n+m,0}.
$$

**Then we have**

$$
C_{\mu,\nu,\lambda}(t,q) \propto \sum_{\eta} s_{\mu^t/\eta}(t^{-\lambda}q^{-\rho+\{1/2\}}) s_{\nu/\eta}(q^{-\lambda^t}t^{-\rho-\{1/2\}})
$$
  
=  $\langle \mu^t | V_-(t^{-\lambda}q^{-\rho+\{1/2\}}) V_+(q^{-\lambda^t}t^{-\rho-\{1/2\}}) | \nu \rangle$ .

#### **Cauchy identities of Schur functions revisited**  $\alpha$  is the number of section of the diagonal line in  $\alpha$  and  $\beta$  and  $\beta$  satisfies  $\alpha$ *ψ<sup>α</sup> |*0⟩ = *ψ*<sup>∗</sup> *<sup>β</sup> |*0⟩ = 0 for any *α >* 0, *β >* 0.  $\mathbf{C}$ Cauchy identities of Schur functions revisited Cauchy identities of Schur functions revisited *{ψn, ψm}* = *{ψ*<sup>∗</sup> Cauchy identities of Schur functions revisited<br>Cauchy identities of Schur functions revisited *P*22 *⊎*−β22 *D*22 *W*<br>*P*22 *W* <sup>−</sup>*βsψ*−*αsψ*−*α*(*s*−1) *... <sup>ψ</sup>*−*α*<sup>1</sup> *<sup>|</sup>*0⟩*,* (B.23)

There are two Cauchy identities known for the skew Schur functions. In the skew Schur functions. In the skew Schur functions. In the skew Schur functions are the skew Schur functions. In the skew Schur functions. In the sk **e.g.**  $\theta$ *.g.*  $\theta$  $\bullet$ *.* $\bullet$ *,* $\bullet$ **,**  $\bullet$ **,** 

$$
\sum_{\lambda} s_{\lambda/\mu}(x) s_{\lambda/\nu}(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \sum_{\eta} s_{\nu/\eta}(x) s_{\mu/\eta}(y),
$$

*s*  $\alpha$  **be derived as**  $\alpha$ 

mnlete be **ι**  $\int$ *(x)*  $\int$  (*x*)*sn* (*x*)*sn* (*x*)*sn* (*x*)*sn* (*x*)*sn* (*x*)*sn* (*x*)*s* (*x*) can be derived as *g* are two complete basis (of Fock space)  $\frac{1}{2}$ *Complete basis (of Fock space)* 

$$
\sum_{\lambda} s_{\lambda/\mu}(x) s_{\lambda/\nu}(y) = \sum_{\lambda} \langle \mu | V_{+}(\vec{x}) | \lambda \rangle \langle \lambda | V_{-}(\vec{y}) | \nu \rangle = \langle \mu | V_{+}(\vec{x}) V_{-}(\vec{y}) | \nu \rangle
$$
  
\n
$$
= \prod_{i,j} (1 - x_{i}y_{j})^{-1} \langle \mu | V_{-}(\vec{y}) V_{+}(\vec{x}) | \nu \rangle
$$
  
\n
$$
= \prod_{i,j} (1 - x_{i}y_{j})^{-1} \sum_{\eta} \langle \mu | V_{-}(\vec{y}) | \eta \rangle \langle \eta | V_{+}(\vec{x}) | \nu \rangle
$$
  
\n
$$
= \prod_{i,j} (1 - x_{i}y_{j})^{-1} \sum_{\eta} s_{\nu/\eta}(x) s_{\mu/\eta}(y),
$$
  
\n... ...

with the commutation relation \$ *<sup>s</sup>λ/µ<sup>t</sup>* (*x*)*sλt/ν*(*y*) = % (1 + *xiy<sup>j</sup>* ) -Ca<br>*sνt/η*(*x*)*sµ/η<sup>t</sup>* (*y*)*.* (B.25) **with the commutation relation (Baker-Campbell-Hausdorff formula)**

$$
V_{+}(\vec{x})V_{-}(\vec{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j} V_{-}(\vec{y})V_{+}(\vec{x})
$$

In the topological vertex formalism, these Cauchy identities lead to the appearance of Nekrasov factors in the partition function. *<sup>x</sup>*)*<sup>V</sup>* <sup>−</sup><sup>1</sup> <sup>−</sup> (−*⃗* (1 + *<sup>x</sup>iy<sup>j</sup>* )*<sup>V</sup>* <sup>−</sup><sup>1</sup> <sup>−</sup> (−*⃗*

a tunical quemotion in the computation. **a typical summation in the computation:**

$$
\sum_{\lambda} Q^{|\lambda|} s_{\lambda/\mu} (q^{-\rho-\sigma}) s_{\lambda/\nu} (q^{-\rho-\tau})
$$
\n
$$
= P.E. \left( \frac{q}{(1-q)^2} Q \right) N_{\sigma^t \tau}^{-1}(Q, q) \sum_{\eta} Q^{|\mu|+|\nu|-|\eta|} s_{\nu/\eta} (q^{-\rho-\sigma}) s_{\mu/\eta} (q^{-\rho-\tau})
$$

one-loop factor Nekrasov factor

emark: all summations o<br>Angle direction Remark: all summations over Young diagrams<br>in non-horizontal directions can be taken in this **Remark: all summations over Young diagrams**<br>in non-borizontal directions can be taken in thic **way.**

where the unrefined  $N_{\rm eff}$  factor is defined by the unrefined by th *Nekrasov's formula for U(N) theories,*  $\frac{1}{2}$ **Then we can see that the exact matching with** 

**i.e. eqn. ( \* ).**



#### **Rewriting the vertex operators Rewriting the vertex operators of refinement calculation of refined to reference in the refinement of refineme**

**[Awata, Feigin, Shiraishi (2011)]**

$$
C_{\mu,\nu,\lambda}=\bra{\nu}\Phi\ket{\mu}\otimes\ket{\lambda}
$$

**where** where as intertwiners of representation in the DIM algebra.

$$
\Phi^{(n)}[u,v]: (1,n)_u \otimes (0,1)_v \to (1,n+1)_{-uv},
$$

is the intertwiner in the Ding-Iohara-Miki algebra (quantum *g* (ˆ<sup>1</sup>*z/w*)  $i$  toroidal algebra of affine gl<sub>1</sub>), (1*,n*+1)

$$
\Phi^{(n)}[u,v](\rho^{(0,1)}_v\otimes\rho^{(1,n)}_u)\Delta(g(z))=\rho^{(1,n+1)}_{-uv}(g(z))\Phi^{(n)}[u,v]
$$

 *<sup>±</sup>*(*z*) *x*(*w*) = *g* **dal algebra?** What is a quantum toroidal algebra?



#### quantum toroidal algebra (of gl<sub>1</sub>)  $C_{\mathbf{q},t}(\mathcal{Y} \mathfrak{t}_1)$  [Dina lot  $\mathfrak{p}$ quantum toroidal algebra (of gl1)  $\qquad$  [ *g* (ˆ<sup>1</sup>*z/w*) <sup>2</sup> *z/w*⌘  $\alpha$  quantum toroidal algeb *g (z) (dal algebro*  $\mathbf{u}$  (of gl<sub>1</sub>)  $I$   $\bar{I}$  *<sup>±</sup>*(*z*) *x*<sup>+</sup>(*w*) = *g* ⇣ ˆ*<sup>±</sup>* <sup>1</sup> <sup>2</sup> *z/w*⌘ **(Ding-Iohara-Miki algebra)***x*<sup>+</sup>(*w*) *<sup>±</sup>*(*z*)

of ⌘(*z*) and ⇠(*z*).  $U_{q,t}(\mathfrak{gl}% _{4})\simeq\mathfrak{gl}_{q,t}(\mathfrak{gl}% _{4})\simeq\mathfrak{gl}_{q,t}(\mathfrak{gl}_{2})$  $\mathfrak{g}$  $\left( \begin{array}{c} 1 \end{array} \right)$ 

 $\mathcal{O}^{\star}$  /  $[$  Ding, Iohara (1997)] **[Miki (2007)]**

$$
[\psi^{\pm}(z), \psi^{\pm}(w)] = 0,
$$
  
\n
$$
\psi^{+}(z) \psi^{-}(w) = \frac{g(\hat{\gamma}z/w)}{g(\hat{\gamma}^{-1}z/w)} \psi^{-}(w) \psi^{+}(z)
$$
  
\n
$$
\psi^{\pm}(z) x^{+}(w) = g(\hat{\gamma}^{\pm \frac{1}{2}} z/w) x^{+}(w) \psi^{\pm}(z)
$$
  
\n
$$
\psi^{\pm}(z) x^{-}(w) = g(\hat{\gamma}^{\mp \frac{1}{2}} z/w)^{-1} x^{-}(w) \psi^{\pm}(z)
$$
  
\n
$$
x^{\pm}(z) x^{\pm}(w) = g(z/w)^{\pm 1} x^{\pm}(w) x^{\pm}(z)
$$
  
\n
$$
[x^{+}(z), x^{-}(w)] = \frac{(1-q_{1})(1-q_{2})}{(1-q_{3}^{-1})} (\delta(\hat{\gamma}w/z) \psi^{+}(\hat{\gamma}^{\frac{1}{2}}w) - \delta(\hat{\gamma}^{-1}w/z) \psi^{-}(\hat{\gamma}^{-\frac{1}{2}}w)),
$$
  
\nwhere

**where**

$$
g(z) = \frac{(1 - q_1 z)(1 - q_2 z)(1 - q_3 z)}{(1 - q_1^{-1} z)(1 - q_2^{-1} z)(1 - q_3^{-1} z)}
$$
  $q_1 q_2 q_3 = 1$ 

$$
(q_1 = t, q_2 = q^{-1})
$$

# $\hat{\gamma}\;$  :center of the group

 $\psi_0^+$  / ?  $\overline{\ }$ *i s* also  $\overline{Q}$ a cer  $\overline{a}$ *i* Q  $\psi_0^{+}/\psi_0^{-}$  is also a center

# **A unifying framework of dualities**



**Relation with W-algebras**



**• Ding-Iohara-Miki algebra on the tensor product of N Fock**  spaces contains a U(1)x (q-deformed) W<sub>N</sub> algebra.

**[Feigin, Hoshino, Shibahara, Shiraishi, Yanagida, 2010]**



**After all, what we want to study is SO(N) theory.**

As we mentioned before, the integral is difficult to perform for SO(N) **and Sp(N) theories.**

 $0$  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 D5 *• • • • • •* NS5 *• • • • • •* 7-brane *• • • • • • • •* O5*<sup>±</sup> • • • • • •*  $\begin{array}{|c|c|c|c|c|}\n\hline\n\text{DR} & \text{e} &$ brane web diagram is obtained by putting a stack of *N* D5-branes on the top of an O5− plane. Hence we can directly apply the O-vertex proposed in section 2.2.

According to SO or Sp gauge group, we add O5<sup>+</sup> or O5<sup>-</sup> orientifold the direction branes stretch along, and dot means the point-like direction for **(O-plane).**  The first example is the simple state of  $\mathcal{G}$  and  $\mathcal{G}$  and  $\mathcal{G}$  are solved by the solven by the  $\mathcal{G}$ 



**However, the brane construction is simple. 3.1 SO**(2*N*) **gauge theories**

# **Proposal**

where we used the complete the complete the complete the Frobenius basis to sum over the Frobenius basis to sum over the Frobenius basis to sum over the Young diagrams of the Young diagrams of the Young diagrams of the You **Proposal [Hayashi, RZ (2020)] [Nawata, RZ (wip)]**

*.* (4.10)

**The topological vertex seems to be applicable to brane webs with O-plane,**  Increpting for vertex because to be applicable to braile weberman in plane,<br>except for the intersection point of 5-brane and orientifold. I he topological vertex seems Since (4.8) gives a part of the perturbative part of the part of the part of the part of the pure of the pure o<br>In the pure of the pure of

![](_page_29_Figure_3.jpeg)

We propose a new topological vertex (O-vertex) for this intersection point. *Prose a new top* new topo<mark>l</mark>ogica  $\mathrm{O5}^+$   $\phantom{-}\mathrm{O5}^ \phantom{-}$   $\phantom{-}$   $\phantom{-$ 

$$
V_{\nu}(-P)^{|\nu|}=\left\langle 0\right|\mathbb{O}(P,q)\left|\nu\right\rangle .
$$

**Expectation value of a vertex-operator,**  *PANNEX* **EXPECTATION VALUE OF A VERTEX-OPERATOR,** 

$$
\mathbb{O}(P,q) = \exp\left(\sum_{n=1}^{\infty} \left(-\frac{P^{2n}(1+q^n)}{2n(1-q^n)}J_{2n} + \frac{P^{2n}}{2n}J_nJ_n\right)\right).
$$

O(*P, q*) = % *|µ| Vµψ*<sup>∗</sup> *<sup>µ</sup>,* (4.2) **It directly follows that Vν vanishes for odd-size Young diagram.** 

We can compute the explicit expression of the O-vertex. *V*<sup>*ν*</sup> = *Σ* (*q*; *q*)*|ν|/*<sup>2</sup> *,* (2.30)

$$
V_{\nu} = \frac{P_{\nu}(q)}{(q;q)_{|\nu|/2}},
$$

*and it seems that*  $P$ *<sup><i>v***</sup> is a polynomial.** only be non-zero when *|ν|* is even. Some of the explicit expressions of *P<sup>ν</sup>* are listed below, In this appendix, we provide the explicit form of the O-vertices *Vy* defined in (2.28) we can consider the original in (2.28) we can consider the original in (2.28) we can consider the original in (2.28) we can consider and it seems that  $P_{\nu}$  is a polyr

$$
P_{(2)} = -q
$$
,  $P_{(1,1)} = 1$ ,  
\n $P_{(4)} = q^3$ ,  $P_{(3,1)} = -q$ ,  $P_{(2,2)} = 1 + q^3$ ,  $P_{(2,1,1)} = -q^2$ ,  $P_{(1,1,1,1)} = 1$ ,

$$
P_{(6)} = -q^{6}, \t P_{(5,1)} = q^{3}, \t P_{(4,2)} = -(q+q^{5}+q^{6}),
$$
  
\n
$$
P_{(4,1,1)} = q^{4} + q^{5}, \t P_{(3,3)} = 1 + q^{4} + q^{5}, \t P_{(3,2,1)} = 0,
$$
  
\n
$$
P_{(3,1,1,1)} = -(q+q^{2}), \t P_{(2,2,2)} = -(q+q^{2}+q^{6}), \t P_{(2,2,1,1)} = 1 + q + q^{5},
$$
  
\n
$$
P_{(2,1,1,1,1)} = -q^{3}, \t P_{(1,1,1,1,1,1)} = 1
$$

 $\frac{10}{6}$  de not beye salesed form formule for them We do not have a closed-form formula for them.

**Reasoning behind this proposal: 3.1.1 Pure SO(4) gauge theory**  $\overline{a}$ *N v*−(*Q* i<sup>i</sup> *Q* i<sup>*a*</sup> *d* i<sup>*a*</sup> *d* i<sup>*a*</sup> *d* i<sup>*d*</sup> *d* i*a d* i*a d* i*a d* i*a d* 

e.g. in SO(4) theory It is possible to write (4.5) in terms of the expectation value of vertex operators. First note <sup>=</sup> ⟨0*<sup>|</sup>* <sup>O</sup>(*P, q*)*V*−(*q*−*ρ*)

![](_page_31_Figure_2.jpeg)

**To reproduce the one-loop part, we need to require** = ⟨0 *|* O(*P, q*)*| ν*⟩ % *ν* & & *<sup>V</sup>*−(*q*−*ρ*) & & *η*<sup>1</sup>  $\frac{1}{2}$  *To reproduce the one-loon part, we peed to require* 

$$
P.E. \left( \frac{P^2 q}{2(1-q)^2} \left( -\left(1 + \sum_{i=1}^N \tilde{Q}_i^2\right) + \left(1 + \sum_{i=1}^N \tilde{Q}_i\right)^2 \right) \right)
$$
  
=  $\left\langle 0 \middle| \mathbb{O}(P, q) \exp \left( \sum_{n=1}^\infty \frac{1}{n} \frac{(1 + \sum_{i=1}^N \tilde{Q}_i^n) q^{\frac{n}{2}}}{1 - q^n} J_{-n} \right) \middle| 0 \right\rangle$ 

with *l*<sup>1</sup> ≤ *k*1*, l*<sup>2</sup> ≤ *k*2*, ···* . Similarly applying the O-vertex (2.37) to (3.9) yields *Z*SO(4) right <sup>=</sup> *P.E.* ! *<sup>q</sup>* me O-vertex (up to terms annihilated by an analysis of the Voltar True Computer of the Voltar True of the Voltar<br>and Young diagrams annihilated by an analysis to sum of the Young diagram of the Young diagram of the Young from the second line to the third line. this determines the O-vertex (up to terms annihilated by<br>the vacuum state).<br> **the vacuum state).**

**However, in the computation of partition function, the vertex operator is enough.**  $\blacksquare$  $\mathcal{L}_{\mathbf{p}}$  is suppose that the *chi*cago  $\mathcal{L}_{\mathbf{p}}$  is given by  $\mathcal{L}_{\mathbf{p}}$ ,  $\mathcal{L}_{\mathbf{p}}$ , the  $\mathcal{L}_{\mathbf{p}}$ 

part that involves O-vertex in the calculat **The part that involves O-vertex in the calculation:**  *I<sub>ne</sub> part tr*  $\frac{1}{2}$  *J*  $\frac{1}{2}$  **Q**  $\frac{1}{2}$  **D**  $\frac{1}{2}$  . This simplifies the  $\frac{1}{2}$  partition function  $\frac{1}{2}$  partition function function

 $\sum$ *ν,η*1*,η*2*,...η<sup>N</sup>*  $V_\nu(t,q)(-P)^{|\nu|}\bigg(\prod^N$ *i*=1  $Q_i^{|\eta_i|}$  $\overline{\phantom{0}}$  $s_{\nu/\eta_1}(q^{-\rho})s_{\eta_1/\eta_2}(q^{-\rho})\ldots s_{\eta_{N-1}/\eta_N}(q^{-\rho})s_{\eta_N}(q^{-\rho})$ =  $\overline{1}$ 0 ' ' ' ' '  $\mathbb{O}(P,t,q)\exp\left(\frac{\infty}{\sum_{n=1}^{\infty}}\right)$ *n*=1 1 *n*  $(1 + \sum_{i=1}^{N} \tilde{Q}_i^n) q^{\frac{n}{2}}$ 2  $1 - q^n$ *J*−*<sup>n</sup>*  $\setminus$ |
| |
| |
| ' 0  $\sqrt{2}$ *,*  $\nu, \eta_1, \overline{\eta_2}, ... \eta_N$ *˛ ⁄* q*<sup>N</sup> <sup>s</sup>*=1 *<sup>|</sup>⁄*(*s*)*<sup>|</sup> Q<sup>|</sup>*2*⁄*(1)*<sup>|</sup>* Ÿ  $i$  $r+\sum$ *a*<sup>*y*</sup> *c r* 3)*⁄*<sub>(</sub>*r*) *N s*=1  $\overline{a}$ (5≠2*s*)*Ÿ*(*⁄*(*s*))  $\left( \sum_{n=1}^{n} n \right)$   $1-q^n$   $\left( \sum_{n=1}^{n} n \right)$ 

**and it is not hard to evaluate with BCH formula.** 1Æ*t<u*Æ*N*

*Vν*(*t, q*)(−*P*) We define on M feeter **We define an M-factor**

$$
M_{\vec{\lambda}}(\vec{A}) = \frac{\langle 0 | \mathbb{O}(A_1, q) \prod_{s=1}^N V_{-}(A_s A_1^{-1} q^{-\rho - \lambda^{(s)}}) | 0 \rangle}{\langle 0 | \mathbb{O}(A_1, q) \prod_{s=1}^N V_{-}(A_s A_1^{-1} q^{-\rho}) | 0 \rangle}.
$$

**P.E.** a closed-form formula free of SO(N) theories are written in terms Nekrasov factor and M-factor.  $\overline{\phantom{a}}$ <sup>2</sup> M-factor  $\mathbf{r}$  $\mathbf{h}$  $\mathbf{L}$ **the partition function** We have a closed-form formula for M-factor, and the partition function

# **More precisely**

![](_page_33_Figure_1.jpeg)

#### **Consistency checks:**  $\mathbf{e}_{\mathbf{u}}$ *<u>istency</u>*  $\overline{\mathbf{a}}$

**SO(4) theory**  $S<sub>0</sub>(A)$  theory.

![](_page_34_Figure_2.jpeg)

![](_page_35_Figure_0.jpeg)

![](_page_35_Figure_1.jpeg)

**hint: SU(3) subgroup**  $\mathcal{S}(\cdot)$  gauge theory can be realized on the following brane diagram,  $\mathcal{S}(\cdot)$ 

![](_page_35_Figure_3.jpeg)

![](_page_35_Picture_4.jpeg)

![](_page_35_Figure_5.jpeg)

*P.* **(2.29) with those known in the literature. We checked that our results match** 

**three instantons one instanton** ☑ **two instantons** The Coulomb branch moduli *a*1*, a*2*, a*3*, a*<sup>4</sup> are the height of the color D5-brane. Hence

![](_page_36_Figure_0.jpeg)

We checked SO(5) and SO(7) theories matching with the known results in the literature up to three instantons. we checked  $SU(5)$  and  $SU(7)$  theories matching with the known  $\mathbf{h}$ **We checked SO(5) and SO(7) theories matching with the known results in the literature up to three instantons.** 

**Remark:**

![](_page_37_Figure_1.jpeg)

 $\mathbf{r}$ **Figure 3**. (a): the brane web by sending *T* → 1 from the diagram in figure 1(b). (b): the diagram **We need to take a special limit** *T***→1 at the end. It appears to be difficult to do, since the computation involves**

![](_page_37_Picture_3.jpeg)

ons,<br>sed, and furthermore, in the vertex-operator formalism we proposed, We can use this technique to compute the partition functions of SO(2*N* + 1) gauge **but the summation converges with finite non-zero contributions, we have a closed-form expression and the limit is straightforward to take.**

**G2 gauge theory**

**No ADHM construction known, but there is brane construction proposed, and there is also a proposal for its blow-up equation to determine the Nekrasov partition function recursively.** wo ADI in voltal dotion Known, but there is brane constructs We the second consider a proposal in the signal vertex as  $\mu$  and  $\mu$ also realizes the SO(7) gauge theory with a hypermultiplet in the spinor representation. One can perform the Higgs to the Higgs of the Diagram of the diagram.<br>The proposed and there is also a proposal for its blow-up equation  $t$   $\frac{1}{3}$ 

![](_page_38_Figure_2.jpeg)

**Potential application:** This theory may be obtained by a circle compact of the obtained by a circle compact of th

we the a twist of the  $\frac{1}{3}$  gauge theory  $\frac{1}{3}$  and  $\frac{1}{3}$   $\frac{1}{3}$ . A  $\frac{1}{3}$  and  $\frac{1}{3}$   $\frac{1}{3}$ reposed branc diagram for od ee p) theory at enemi**proposed brane diagram for 5d SU(3) theory at Chern-Simons level 9.**

![](_page_39_Figure_2.jpeg)

an S-dual effective web diagram can be computed with our O-vertex

![](_page_39_Figure_4.jpeg)

### **Comment:**

**As far as we computed, our prescription literally reproduces each contribution in the localization integral.**

In the unrefined limit, all JK poles are labeled by Young diagrams.

For example, we can effectively choose

$$
\phi_i = \pm a_j + (r-1)\epsilon_1 + (s-1)\epsilon_2 + \frac{\epsilon_1 + \epsilon_2}{2}.
$$

**So the refinement is not straightforward to do in this framework.**

**On the other hand, Sp(N) theories are more complicated.**

**Kim-Yagi's prescription NSTERNES. WE REVIEW THE FORMALISM BY USING AN EXAMPLE WHICH WE WILL USING WHICH WE WILL US AN EXAMPLE WE WILL U** 

![](_page_41_Figure_1.jpeg)

way to calculate **way to as the one for the pure for** 

![](_page_41_Figure_3.jpeg)

Namely we take a mirror image and sum over  $\lambda$ . **use a mirror image and sum over λ.**

### **S-dual description** The brane construction with ON<sup>0</sup> for *D*-type quiver gauge theories was originally reported in

U duar acsoription<br>
[Bourgine, Fukuda, Matsuo, RZ (2017)]

construction proposed in the second in the second in  $\mathbf{W}$ We again use a resolved brane diagram.

![](_page_42_Figure_3.jpeg)

**Reflection state in quantum toroidal algebra** *Comparent* (*comparent* (*comparent* (*c)* (*c)*  $\alpha$ <sup>2</sup> Reflection sta ate in (<br> *l*<br>
1 *q*<br>
1 *a*<br>
1 *d*<br>
1 *d*<br>
3 *d* ction state in quantum toroidal algebra *n* ус :<br>Ul d*x* ual<br>T ary<del>c</del>r *<sup>Y</sup>* (*xq*1*q*2) + <sup>1</sup>

$$
\sum_{\lambda} |v, \lambda\rangle \otimes |v\gamma, \lambda\rangle \qquad \qquad \longrightarrow \qquad \qquad \qquad \overline{\qquad \qquad } \qquad \gamma = \sqrt{t/q}
$$

**We can use a reflection state in the quantum toroidal algebra** 2 we can use a renection state in the quality of  $\exp\left(-x\right) \approx 1 + 1 \approx x \pm (x)$ 

$$
(x^\pm(z)\otimes 1+1\otimes x^\mp(z))\ket{\Omega}\rangle=0
$$

**where we showed the property in the unrefined limit (for simplicity)**

$$
|\Omega\rangle\rangle:=\sum_{\lambda} |v,\lambda\rangle\otimes|v,\lambda\rangle
$$

juilly upservation nere is that the renection state above<br>a the boundary state in the 1d limit  $a\rightarrow 1, \;t=a^\beta$  $\mathbf{r}$ (*x±*(*z*) ⌦ 1+1 ⌦ *x*⌥(*z*))*|*⌦ii = 0 (65) **• One intriguing observation here is that the reflection state above** reduces to the boundary state in the 4d limit  $q \rightarrow 1$ ,  $t = q^{\beta}$ . • One intriguing observation nere is that the reflect<br>reduces to the boundary state in the 4d limit  $q \rightarrow$ 

$$
(L_n \otimes 1 - 1 \otimes L_{-n})|\Omega\rangle = 0.
$$

(⇢(0*,m*) (*x*(*z*)) ⌦ 1)*|*⌦ii <sup>=</sup> (1 ⌦ ⇢(0*,m*) *x±*(*z*) (66) **"2d" picture of this construction?**In the second term, *L<sup>n</sup>* = (*Ln*)*†* corresponds to the contragredient action of Virasoro generators. **\*Sp(N) theory in ADHM basis**

In the ADHM construction, we have O(k) theory on D1 branes.

**The Nekrasov partition function differs in the pieces of O(k)+ and O(k)-.**

$$
Z_{Sp(N), \theta=0,\pi}^{(k)} = \frac{Z_{O(k)+} \pm Z_{O(k)-}}{2}.
$$

**The integral expression of the partition function differs for instanton number k odd and even.**

In total, we have four different contributions.

**We gave an analytic expression for each piece of the contribution in terms of the M-factor.**

**[Nawata, RZ (wip)]**

**But what is the corresponding brane-web realization?**

**Future directions:**

- **• adding matter fundamental matter** ☑ **(straightforward) spinor? • qq-characters [work in progress with S. Nawata] • topological vertex for Sp(N) in ADHM basis?**  $\int$  $\bigg\}$ :
- **• meaning in the context of topological string?**
- **• O7-plane**
- **• algebraic description of 3d N=2\* web?**
- **• AGT for SO/Sp gauge theories (algebraic structure, integrability)**

# **Summary**

- **• beautiful results known for U(N) theories, but not so many things known for other gauge groups.**
- **• The main reason is the difficulty to perform the localization integral.**

- **• We used an extended version of the topological vertex formalism to write down an analytic expression of SO(N) (and some other) gauge theories.**
- **• We expect our new formula to be useful in the analysis of properties, e.g. algebraic structure, in gauge theories beyond U(N).**