Aspects on Partial Entanglement Entropy

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Based on my recent work:

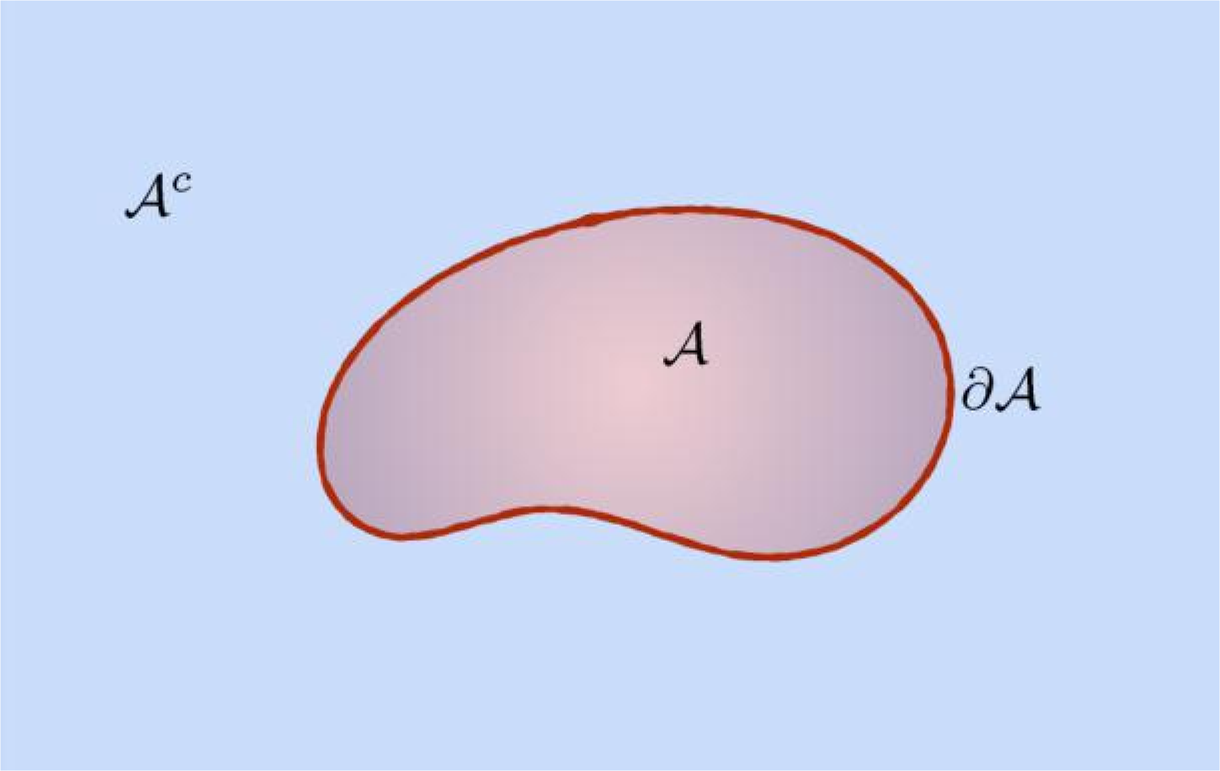
Balanced Partial Entanglement and the Entanglement Wedge Cross Section, Qiang Wen, JHEP 04 (2021)

- Fine structure in holographic entanglement and entanglement contour, Qiang Wen, Phys.Rev.D 98 (2018)
- Towards the generalized gravitational entropy for spacetimes with non-Lorentz invariant duals, Qiang Wen, JHEP 01 (2019) 220
- Entanglement contour and modular flow from subset entanglement entropies, Qiang, JHEP 05 (2020)
- Formulas for Partial Entanglement Entropy, Qiang Wen, Phys.Rev.Res. 2 (2020)

Outline

- **Introducing the partial entanglement entropy**
- **Several approaches to PEE**
- **PEE in condensed matter and quantum information**

Entanglement Entropy



- **Reduced density matrix** $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{A}^c}\left(|\Psi\rangle\langle\Psi|\right)$
- **Entanglement entropy** $S_{\mathcal{A}} = -\mathrm{Tr}_{\mathcal{A}}\left(\rho_{\mathcal{A}}\log\rho_{\mathcal{A}}\right)$

The Simplest Example: two spins (2 qubits)

(i) $|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \otimes \left[|\uparrow\rangle_B + |\downarrow\rangle_B\right]$

$$\Rightarrow \rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\cdot\left[\langle\uparrow|_A + \langle\downarrow|_A\right] \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Not Entangled

$$S_{\mathrm{A}} = 0$$

(ii) $|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_{\mathrm{B}}\right] / \sqrt{2}$

$$\Rightarrow \rho_A = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\right] \cong \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

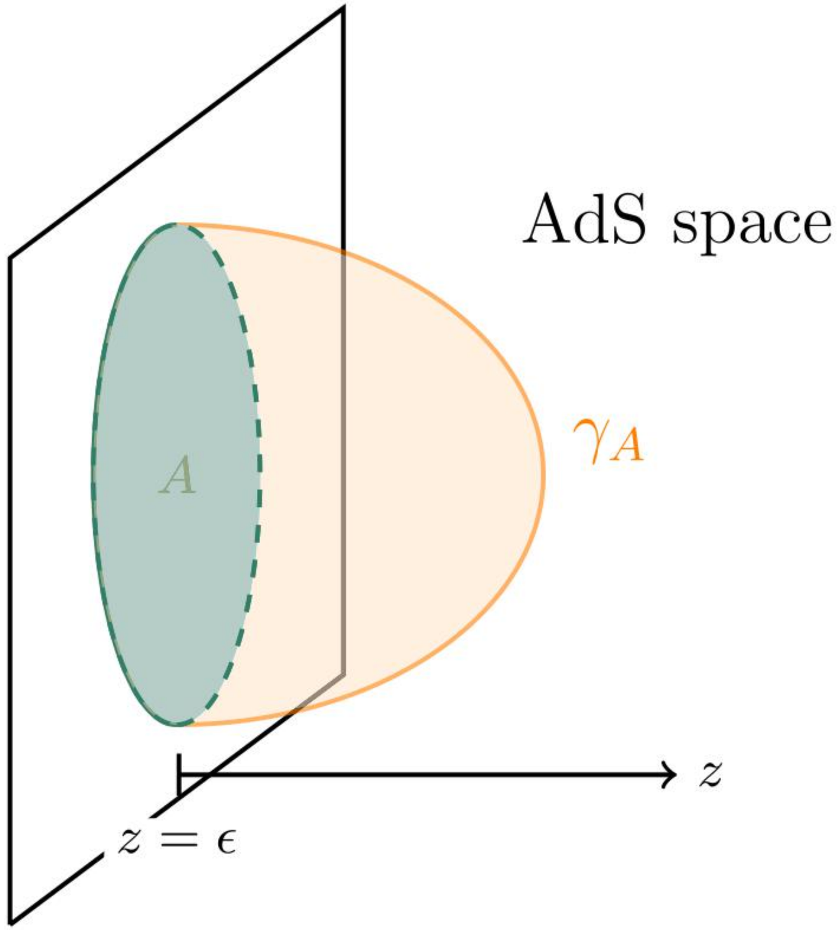
Entangled

$$S_{\mathrm{A}} = -2\cdot\log\frac{1}{2} = \log 2.$$

Holographic entanglement entropy in AdS /CFT

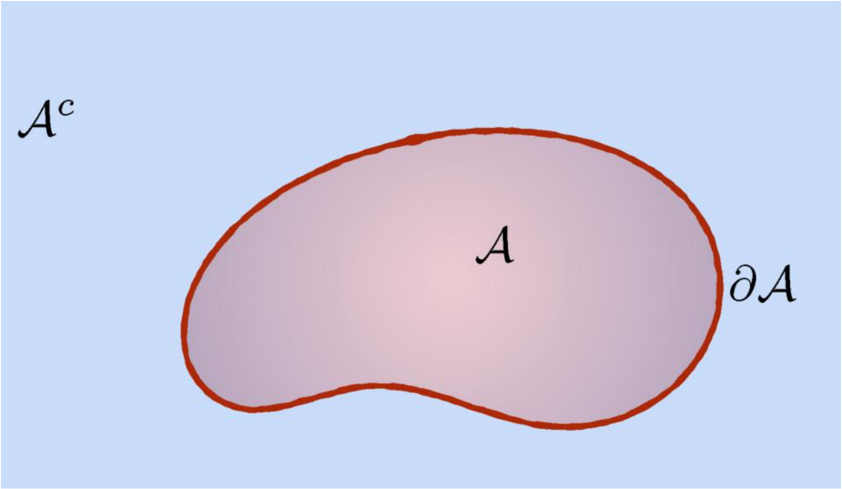
Ryu-Takayanagi formula 06’

$$S_{EE} = \frac{Area(\mathcal{E}_{\mathcal{A}})}{4G}\,.$$



Quantum entanglement ⟷ Bulk geometry

Entanglement contour quantifies the contribution distribution from each degree of freedom in the region to the total entanglement entropy.



Entanglement contour function

$$S_{\mathcal{A}} = \int_{\mathcal{A}} f_{\mathcal{A}}(x_1, \cdots x_{d-1}) dx_1 \cdots dx_{d-1}$$

Partial entanglement entropy

$$s_{\mathcal{A}}(\mathcal{A}_2) = \int_{\mathcal{A}_2} f_{\mathcal{A}}(x_1, \cdots x_{d-1}) dx_1 \cdots dx_{d-1}$$

Physical requirements for the contour

Y. Chen and G. Vidal, 2014'; QW 2019'

So far there is no fundamental definition for this function!!!

1. *Additivity*: if $A_i^a \cup A_i^b = A_i$ and $A_i^a \cap A_i^b = \varnothing$, by definition we should have

$$s_A(A_i) = s_A(A_i^a) + s_A(A_i^b)\,. \tag{1.5}$$

2. *Invariance under local unitary transformations*: $s_A(A_i)$ should be invariant under any local unitary transformations inside A_i or $\bar{A}$.

3. *Symmetry*: for any symmetry transformation $\mathcal{T}$ under which $\mathcal{T}A = A'$ and $\mathcal{T}A_i = A_i'$, we have $s_A(A_i) = s_{A'}(A_i')$.

4. *Normalization*: $S_A = s_A(A_i)|_{A_i \to A}\,.$

5. *Positivity*: $s_A(A_i) \geq 0$.

6. *Upper bound*: $s_A(A_i) \leq S_{A_i}\,.$

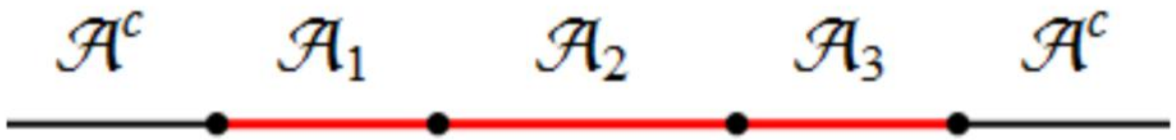
7. *Symmetry under the permutation*: $\mathcal{I}(\bar{A}, A_i) = \mathcal{I}(A_i, \bar{A})$, which implies $s_A(A_i) = s_{\bar{A}_i}(\bar{A})$.

4 Proposals

- The Gaussian Formula — Chen, Vidal 14’
- The additive linear combination proposal — WQ 18’
- PEE from the fine structure of the entanglement wedge — WQ 18’
- Determine PEE for Poincare invariant theories — WQ 19’; Huerta, Casini 08’

Proposal 1： The additive linear combination proposal for PEE

QW, 1803.05552 PRD
QW, 1902.06905 JHEP



$$s_{\mathcal{A}}(\mathcal{A}_2) = \frac{1}{2}\left(S_{\mathcal{A}_1\cup\mathcal{A}_2} + S_{\mathcal{A}_2\cup\mathcal{A}_3} - S_{\mathcal{A}_1} - S_{\mathcal{A}_3}\right)$$

$$s_{\mathcal{A}}(\mathcal{A}_2^a) = \frac{1}{2}\left(S_{\mathcal{A}_1\cup\mathcal{A}_2^a} + S_{\mathcal{A}_2\cup\mathcal{A}_3} - S_{\mathcal{A}_1} - S_{\mathcal{A}_2^b\cup\mathcal{A}_3}\right)$$

$$s_{\mathcal{A}}(\mathcal{A}_2^b) = \frac{1}{2}\left(S_{\mathcal{A}_1\cup\mathcal{A}_2} + S_{\mathcal{A}_2^b\cup\mathcal{A}_3} - S_{\mathcal{A}_1\cup\mathcal{A}_2^a} - S_{\mathcal{A}_3}\right)$$

$$\Rightarrow \quad s_{\mathcal{A}}(\mathcal{A}_2) = s_{\mathcal{A}}(\mathcal{A}_2^a) + s_{\mathcal{A}}(\mathcal{A}_2^b)$$

- For a general quantum system, the entanglement entropies will always obey certain inequalities.

(a) *Subadditivity*: $S(A) + S(B) \geq S(AB)$,
(b) *Araki-Lieb*: $S(AB) \geq |S(A) - S(B)|$,
(c) *Strong subadditivity 1*: $S(AB) + S(BC) \geq S(ABC) + S(B)$,
(d) *Strong subadditivity 2*: $S(AB) + S(BC) \geq S(A) + S(C)$.

Lieb and Ruskai, 1973'

- **Positivity**: The strong subadditivity $S_{\mathcal{A}_1\cup\mathcal{A}_2}+S_{\mathcal{A}_2\cup\mathcal{A}_3}-S_{\mathcal{A}_1}-S_{\mathcal{A}_3}\geq 0$ for any three regions indicates $s_{\mathcal{A}}(\mathcal{A}_2)\geq 0 \Rightarrow f_{\mathcal{A}}(x_1,\cdots x_{d-1})\geq 0$
- **Normalization**: $s_{\mathcal{A}}(\mathcal{A}_2)|_{\mathcal{A}_2\to\mathcal{A}}=S_{\mathcal{A}}$
- **Invariance under local transformations**: All the subset entanglement entropies are invariant under local transformations that only act on $\mathcal{A}_2$, so $s_{\mathcal{A}}(\mathcal{A}_2)$ is also invariant.
- **Upper bound**: subadditivity $S_{\mathcal{A}_1}+S_{\mathcal{A}_2}\geq S_{\mathcal{A}_1\cup\mathcal{A}_2}$ and $S_{\mathcal{A}_2}+S_{\mathcal{A}_3}\geq S_{\mathcal{A}_2\cup\mathcal{A}_3}$ indicates

$$s_{\mathcal{A}}(\mathcal{A}_2)\leq S_{\mathcal{A}_2}\,.$$

- **Symmetry**: Since $\mathcal{T}$ is a symmetry, the subsets $\mathcal{A}_i$ and $\mathcal{A}'_i$ should play the equivalent role, in other words we have $S_{\mathcal{A}_i}=S_{\mathcal{A}'_i}, S_{\mathcal{A}_i\cup\mathcal{A}_j}=S_{\mathcal{A}'_i\cup\mathcal{A}'_j}$. This means

$$\begin{aligned}S_{\mathcal{A}_1\cup\mathcal{A}_2}+S_{\mathcal{A}_2\cup\mathcal{A}_3}-S_{\mathcal{A}_1}-S_{\mathcal{A}_3}&=S_{\mathcal{A}'_1\cup\mathcal{A}'_2}+S_{\mathcal{A}'_2\cup\mathcal{A}'_3}-S_{\mathcal{A}'_1}-S_{\mathcal{A}'_3}\\ \Rightarrow s_{\mathcal{A}}(\mathcal{A}_2)&=s_{\mathcal{A}}(\mathcal{A}'_2)\,.\end{aligned}$$

Comments on the ALC proposal

- Applies for general theory (holographic, CFT, non CFT, lattice model…)
- Relies on a definite order of the degrees of freedom in the region
- Very powerful in 2-dimensional system and can be generalized to higher dimensions with enough symmetries
- Works at the full quantum level in holography

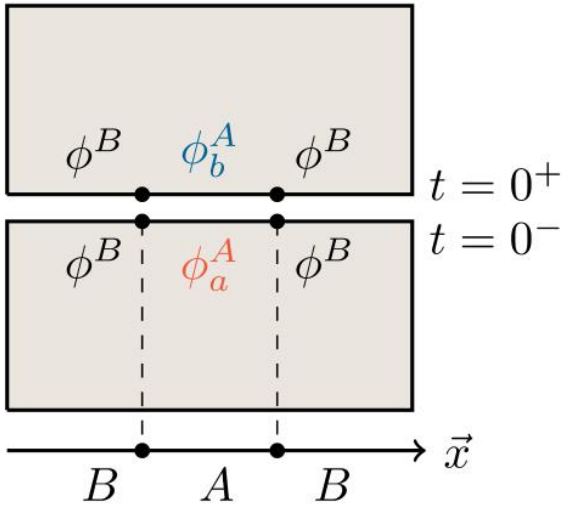
Proposal 2: PEE from the fine structure of the entanglement

Reduced density matrix in path integral

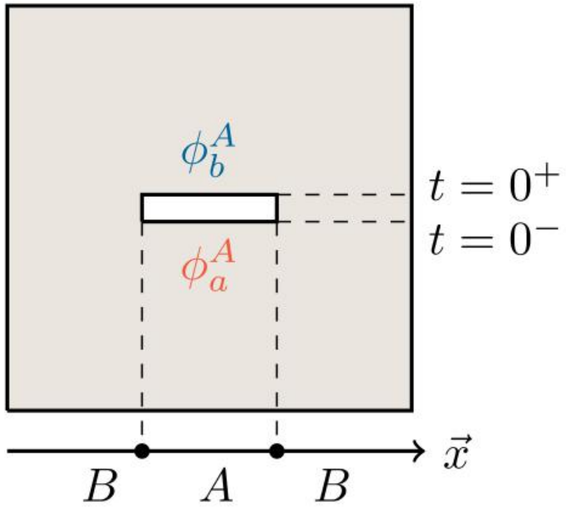
$$[\rho_A]_{ab}$$

$$= \frac{1}{Z}\int [\mathcal{D}\phi^B(t=0,\vec{x}\in B)]\ \left(\langle\phi_a^A|\langle\phi^B|\right)|\Psi\rangle\langle\Psi|\left(|\phi_b^A\rangle|\phi^B\rangle\right)\ ,$$

$$= \frac{1}{Z}\int [\mathcal{D}\phi^B(t=0,\vec{x}\in B)]$$

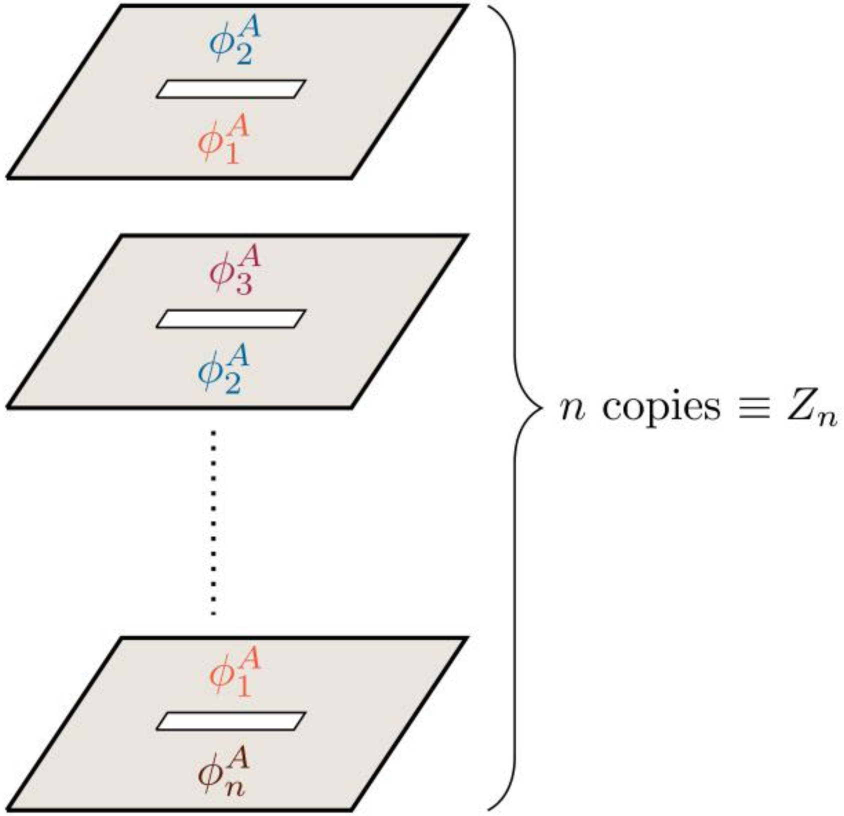


$$= \frac{1}{Z}$$



Replica trick

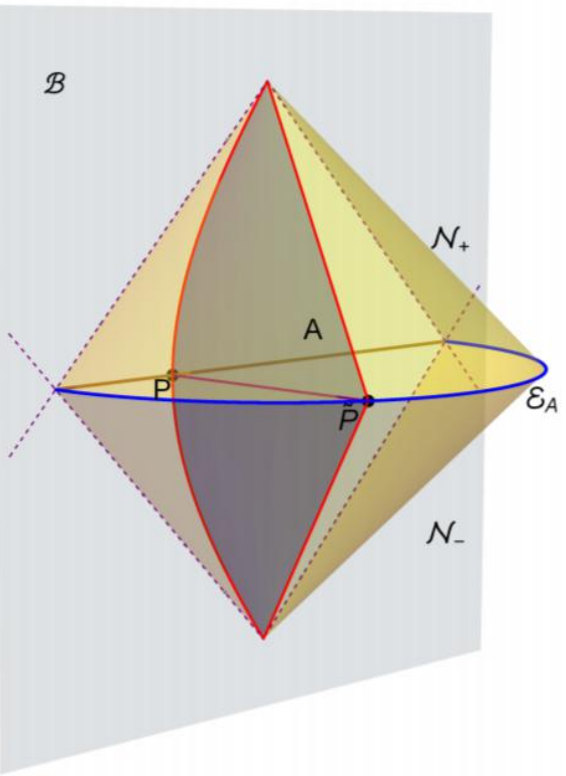
$$\mathrm{tr}_A(\rho_A^n) = \frac{1}{(Z)^n}$$

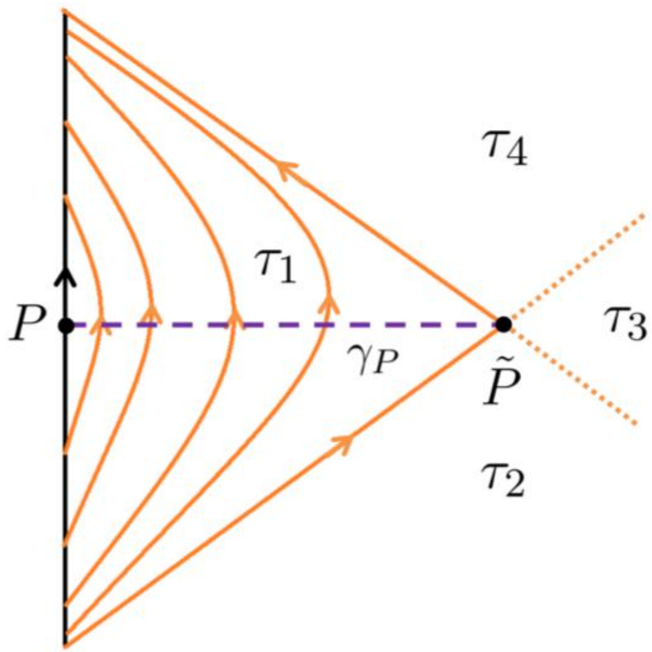


$$= \frac{Z_n}{(Z)^n} \ ,$$

$$S_n = \frac{1}{1-n}\log\left(\frac{Z_n}{Z^n}\right)$$

Modular slice

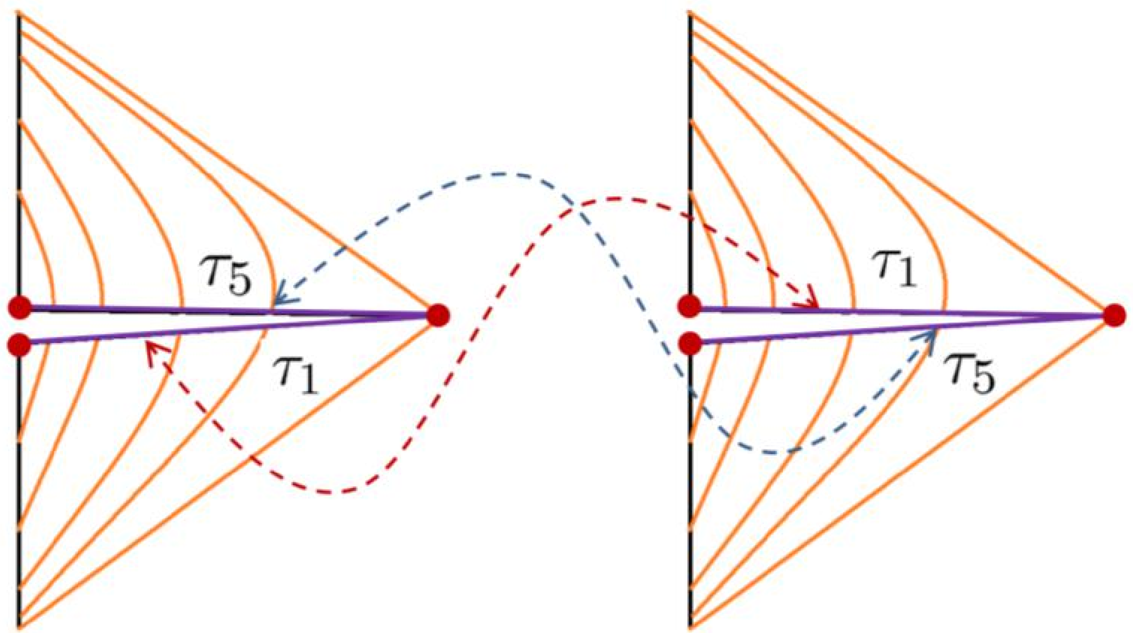




1, Each point in A determines a modular slice

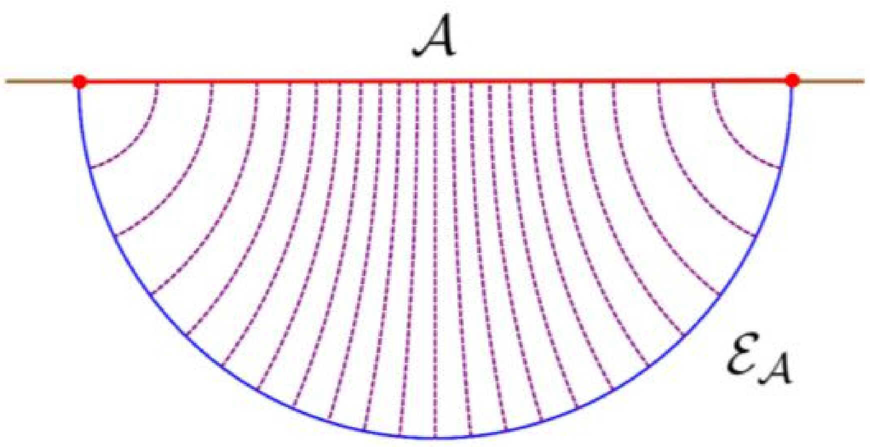
2, The entanglement wedge is a slicing of the modular slices

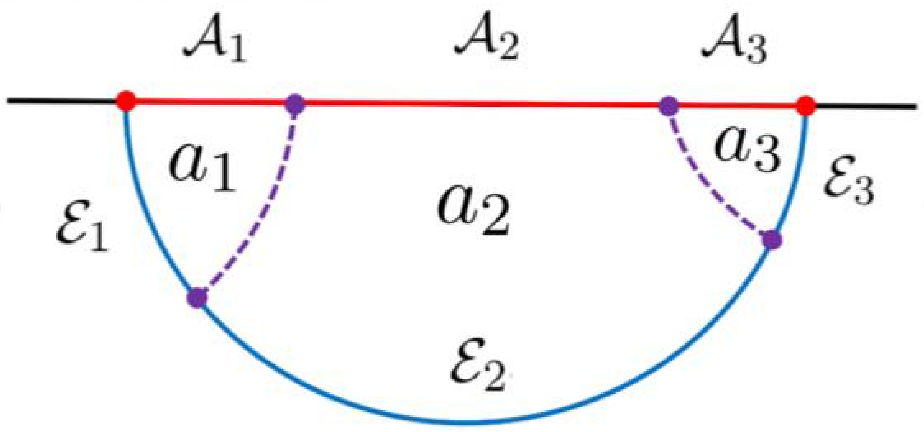
Replica trick on the modular slice with n=2



The replica on a point P in A turns on the contribution of a partner point in the RT surface

A time slice of the entanglement wedge





$$s_A(A_i) = \frac{Length\,(\mathcal{E}_i)}{4G} ,$$

Proposal 3: Solving all the requirements

Additivity | Symmetry under permutation

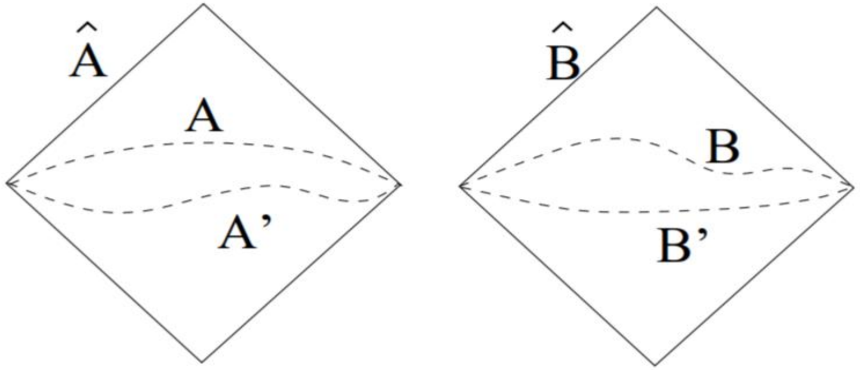
⇩

$$\mathcal{I}(\bar{\mathcal{A}}, \mathcal{A}_i) = \int_{\bar{\mathcal{A}}} d\sigma_{\mathbf{x}} \int_{\mathcal{A}_i} d\sigma_{\mathbf{y}}\, J(\mathbf{x}, \mathbf{y})$$

$$\mathcal{I}(\bar{\mathcal{A}}, \mathcal{A}_i) = \sum_{i \in \bar{\mathcal{A}}} \sum_{j \in \mathcal{A}_i} J_{ij}$$

Casini and Huerta 0812.1773
Derivation for extensive mutual information

Invariance under local unitary transformations



$$\mathcal{I}(A,B) = \int_A d\sigma_{\mathbf{x}}{}^{\mu} \int_B d\sigma_{\mathbf{y}}{}^{\nu} J_{\mu\nu}(\mathbf{x},\mathbf{y}),$$

With $\partial_\mu J^{\mu\nu}(\mathbf{x},\mathbf{y}) = 0$

Poincare invariant theories

- Invariance under Poincare symmetry:

$$J^{\mu\nu}(\mathbf{x},\mathbf{y}) = \frac{(\mathbf{x}-\mathbf{y})^{\mu}(\mathbf{x}-\mathbf{y})^{\nu}}{(\mathbf{x}-\mathbf{y})^{2d}} G(l) - \frac{g^{\mu\nu}}{(\mathbf{x}-\mathbf{y})^{2(d-1)}} F(l)$$

- Conservation: ⇨ $$[G(l)-F(l)]' = -(d-1)\frac{2F(l)-G(l)}{l}$$

- **Positivity** → $${\sigma_{\mathbf{x}}}^{\mu}{\sigma_{\mathbf{y}}}^{\nu}J_{\mu\nu}(\mathbf{x},\mathbf{y})\geq 0$$
  for any time-like vectors $\vec{\sigma}_{\mathbf{x}}$ and $\vec{\sigma}_{\mathbf{y}}$

- Furthermore implies: $$2F(l)\geq G(l)\geq 0$$

- It is convenient to define $C(l)=G(l)-F(l)$, thus $C'(l)\leq 0$

- Which implies C(l) deceases under the RG flow, hence is a c-function.

- Then we have

$$F(l) = -\frac{lC'(l)}{d-1} + C(l)\,, \qquad G(l) = -\frac{lC'(l)}{d-1} + 2C(l)$$

Then it is convenient to define another function $H(l)$ by

$$C(l) = (d-1)l^{2d-3}H'(l)\,.$$

Thus

$$J_{\mu\nu}(l) = -\partial_\mu\partial_\nu H(l) + g_{\mu\nu}\partial_\alpha\partial^\alpha H(l)\,.$$

At last, after we applied the Stokes' theorem we arrive at the following formula for PEE

$$\mathcal{I}(A,B) = \int_{\partial A}\int_{\partial B} d\vec{\eta}_{\mathbf{x}} \cdot d\vec{\eta}_{\mathbf{y}}\, H(|\mathbf{x}-\mathbf{y}|)\,,$$

General Formula!

where $\vec{\eta}_{\mathbf{x}}$ and $\vec{\eta}_{\mathbf{y}}$ are the infinitesimal subsets on the boundaries ∂A and ∂B with an outward pointing direction in the system and normal to ∂A and ∂B.

PEE in conformal field theories

Things become much more determined in the case of conformal field theories. Since $C(l)$ is a c-function, it should be a constant in CFTs. Let us define $C(l) = 2C_d(d-1)(d-2)$, then we have

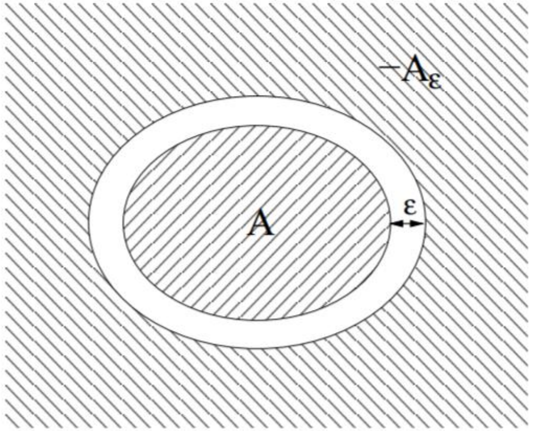
$$H(|\mathbf{x}-\mathbf{y}|) = -\frac{C_d}{|\mathbf{x}-\mathbf{y}|^{2d-4}}\,, \qquad d > 2\,, \tag{4.16}$$

Cd is a constant that depend on the theory and dimension

When d=2 H(l) just gives the entanglement entropy for the single interval with length l.

The impact of the PEE in other fields

- Primitive evaluation from PEE respect the same features as the EE regulated by scale.

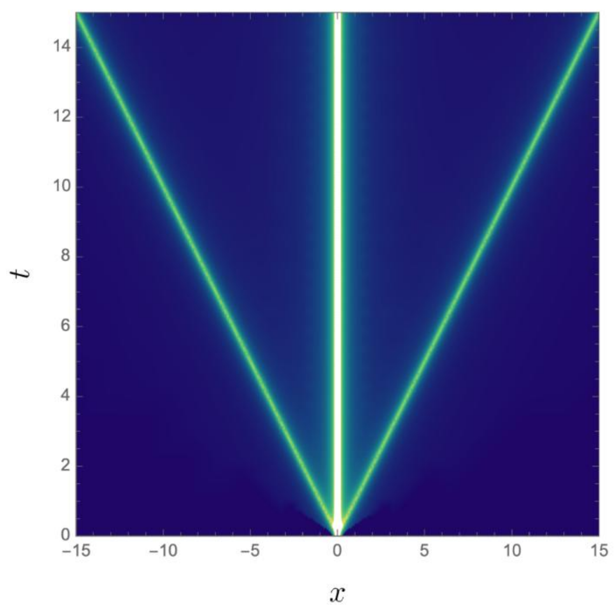


Very powerful for the cases of corners and cones !

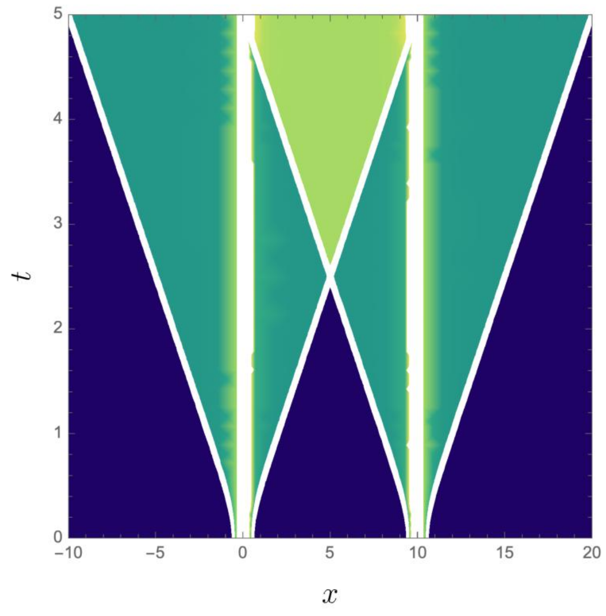
Bueno, Myers and Witczak-Krempa, Universality of corner entanglement in conformal field theories Phys. Rev. Lett., [1505.04804].
Bueno, Myers and Witczak-Krempa, Universal corner entanglement from twist operators, JHEP , [1507.06997].
Bueno, H. Casini and Witczak-Krempa, Generalizing the entanglement entropy of singular regions in conformal field theories, JHEP, [1904.11495].

Kudler-Flam, MacCormack, Ryu, J.Phys.A, 1902.04654

Entanglement contour in dynamical systems



The contour under a local quench



The contour under thermalization (global quench), **entanglement tsunami**

Directly address the question of how quantum information locally flows in time

- Also a similar contour construction for entanglement negativity is investigated by the group of Ryu.

Kudler-Flam, Shapourian, Ryu, **1908.07540,** SciPost Phys
“The negativity contour: a quasi-local measure of entanglement for mixed states”

- The entanglement contour has been investigated for on-thermalizing phases with novel properties of entanglement spreading beyond the measure of the out-of-time-ordered correlator (OTOC).

MacCormack, Tan, Kudler-Flam, Ryu, **2001.08222**
“Operator and entanglement growth in non-thermalizing systems: many-body localization and the random singlet phase ”

Interaction with other information quantities

The Markov gap for geometric reflected entropy

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A canonical purification for the entanglement wedge cross-section

Souvik Dutta and Thomas Faulkner

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conformal field theories.

JHEP03(20

Conclusion

- We introduce the concept of PEE and its physical requirements
- We introduce the approaches to construct the PEE in different theories
- However the fundamental definition for PEE is still not clear
- The study of PEE is still at a primitive stage

Thank you!