# Holographic Entanglement Entropy Inequality

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# Content

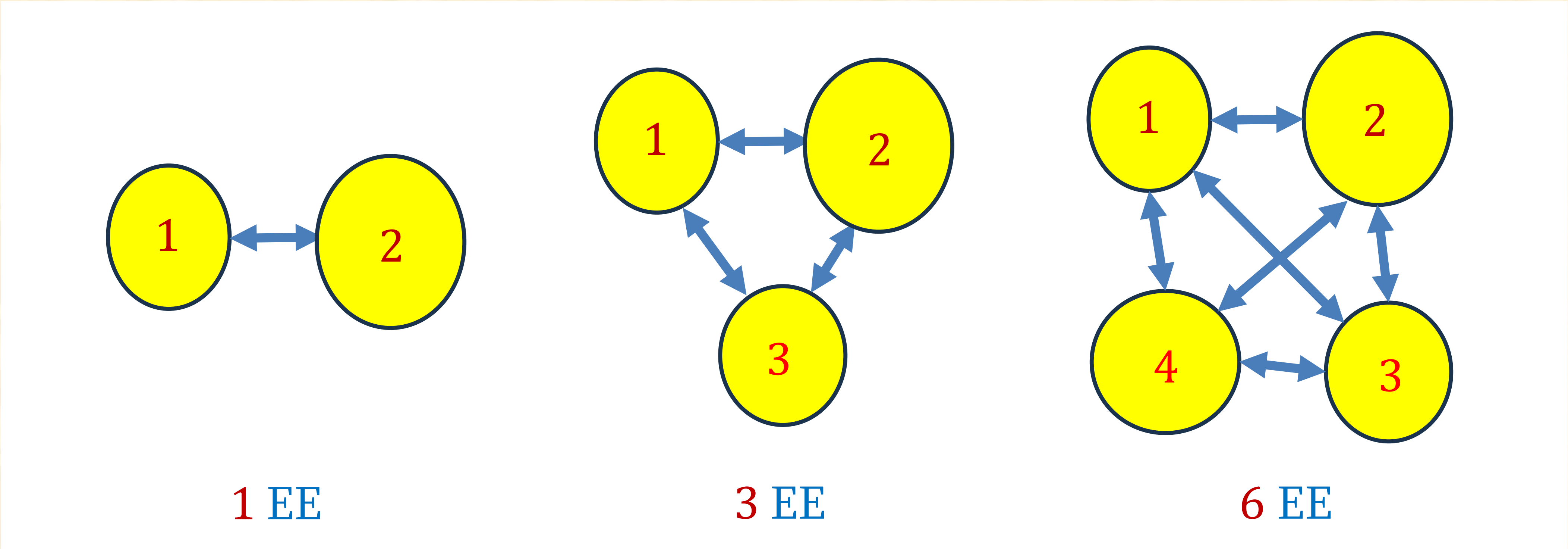
- Holographic entanglement entropy inequality (HEI)
- Prove a HEI
- Independent HEI

# Entanglement Entropy

- Hilber space of a bipartite system: $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$
- Density matrix of a quantum state: $\rho_{12} = |\Psi_{12}\rangle\langle\Psi_{12}|$
- Reduced density matrix: $\rho_1 = \mathrm{Tr}_2 \rho_{12}$
- Entanglement entropy: $S_1 = -\mathrm{Tr}(\rho_1 \ln \rho_1)$
- Rényi entropy: $S_1^{(q)} = \frac{1}{1-q} \log \mathrm{Tr}\left(\rho_1^q\right)$, $\quad S_1 = \lim_{q-1} S_1^{(q)}$
- Replica trick: path integral on $q$ copies of word sheet

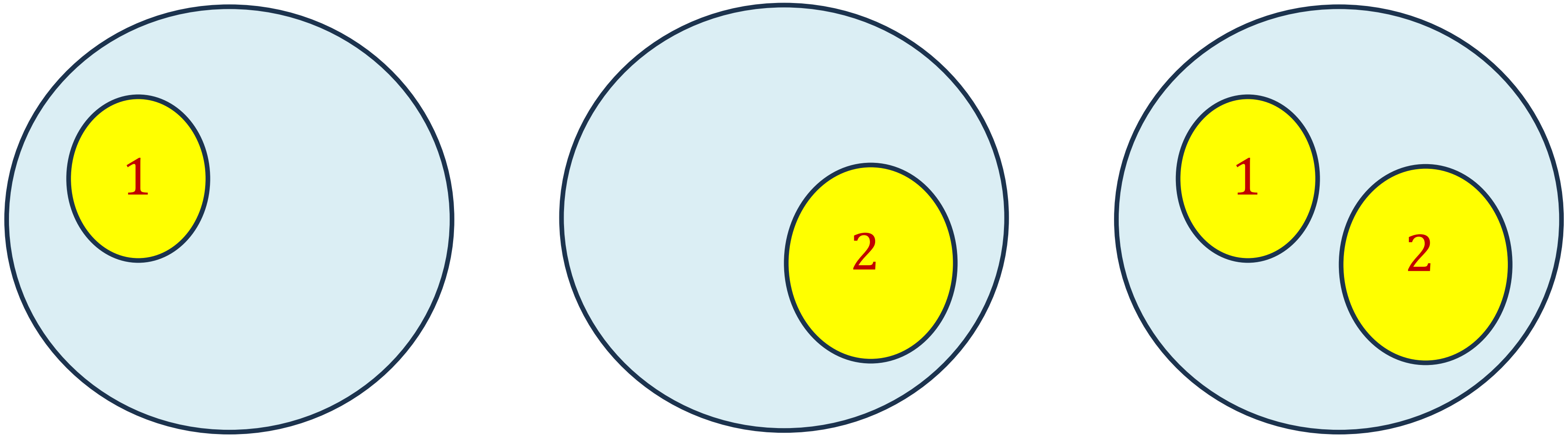
# EE in Multipartite Systems

- Entanglement between two quantum states
- $\frac{n(n-1)}{2}$ EE in a $n$-partite system: constraints



# Subadditivity (SA)

$$S_1 + S_2 \geq S_{12}$$



# Entanglement Entropy Inequality

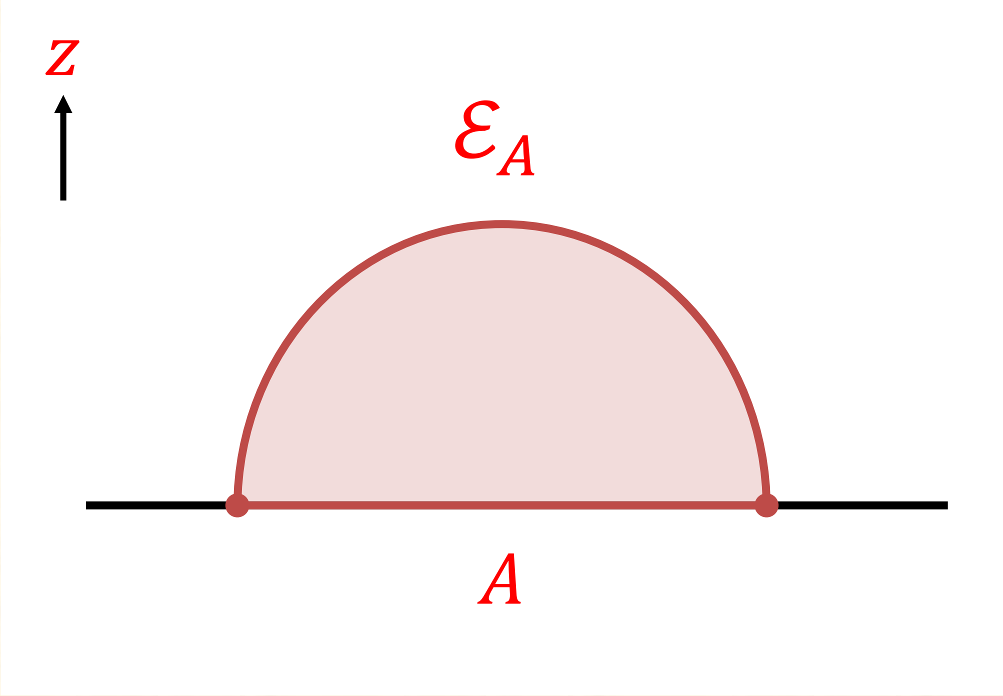
- Strong Subadditivity for black hole information paradox
- Quantum information, quantum transportations…
- It is a challenge to prove multipartite HEIs in QFT
- Holography offers a geometric approach

# Holographic Entanglement Entropy

- Ryu-Takayanagi surface

$$S_A = \min_{\mathcal{E}_A} \frac{Area(\mathcal{E}_A)}{4G_N}$$

- Homology
- Divergence



# Holographic SA: Simplex Basis

$S_1 = S_{[11]}$

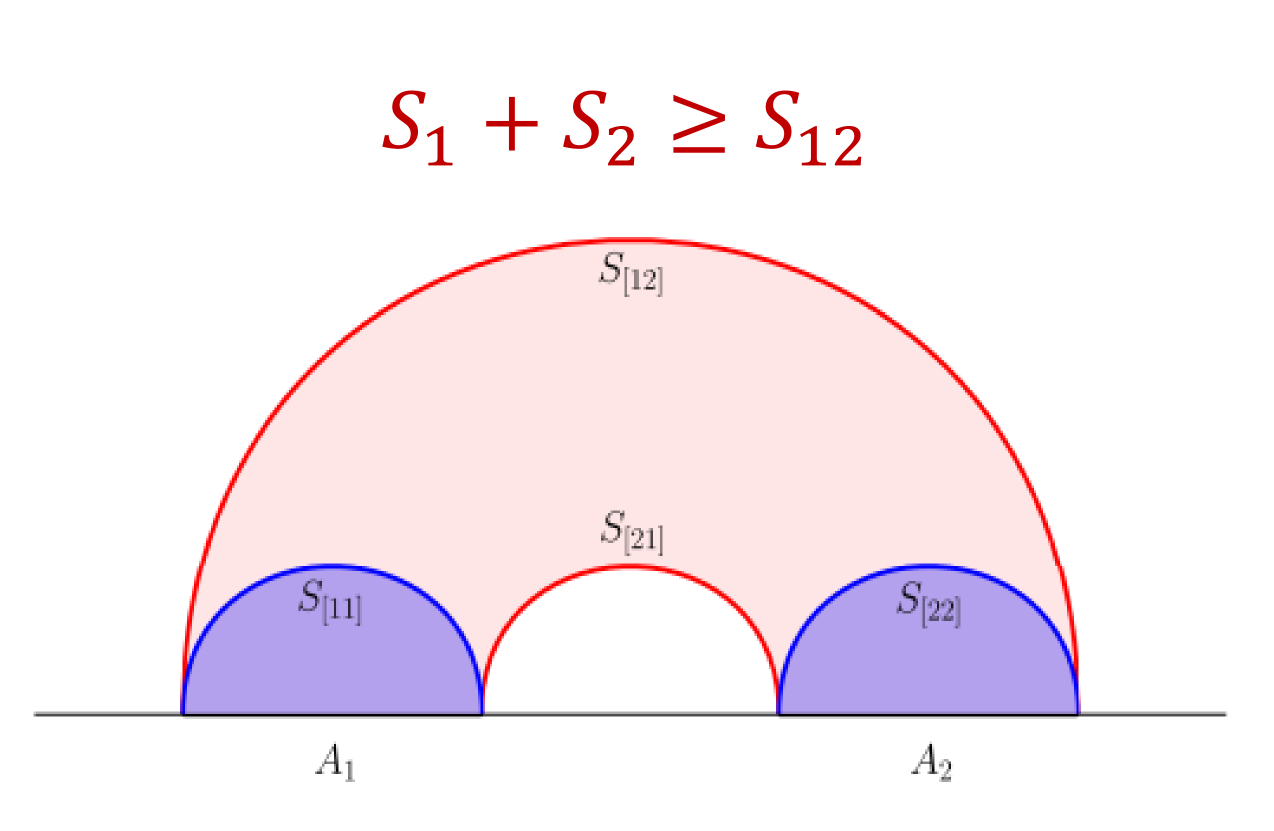
$S_2 = S_{[22]}$

$S^d_{1,2} = S_{[11]} + S_{[22]} = S_1 + S_2$

$S^c_{12} = S_{[12]} + S_{[21]}$

$S_{12} = \min(S^d_{1,2}, S^c_{12})$

- Divergence cancellation
- Balanced HEI



# Multipartite HEI: $n = 2, 3$

- Balanced HEIs:

  Subadditivity (SA): $S_1 + S_2 \geq S_{12}$

  Strong Subadditivity (SSA): $S_{12} + S_{23} \geq S_2 + S_{123}$

  Monogamy of Mutual Information (MMI):

  $S_{12} + S_{13} + S_{23} \geq S_1 + S_2 + S_3 + S_{123}$

- Configurations

# Configurations

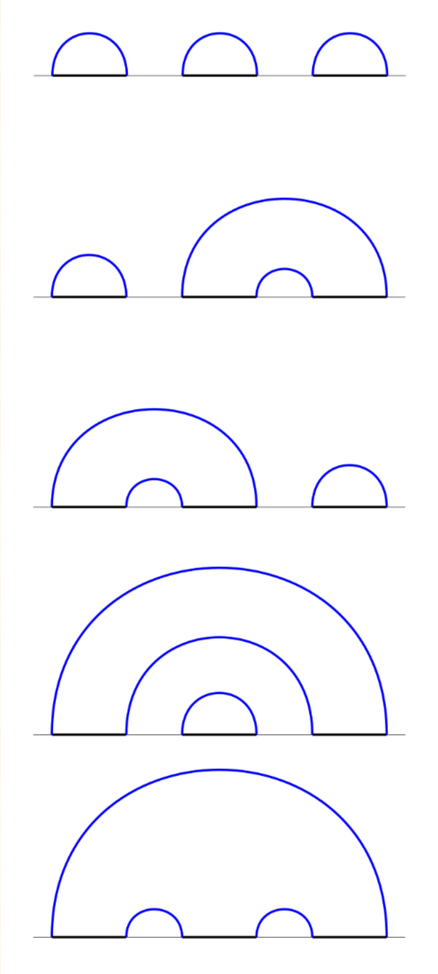
$$S^{d}_{1,2,3} = S_1 + S_2 + S_3 = S_{[11]} + S_{[22]} + S_{[33]}$$

$$S^{d}_{1,23} = S_1 + S^{c}_{23} = S_{[11]} + S_{[23]} + S_{[32]}$$

$$S^{d}_{12,3} = S^{c}_{12} + S_3 = S_{[12]} + S_{[21]} + S_{[33]}$$

$$S^{d}_{13,2} = S^{c}_{13} + S_2 = S_{[13]} + S_{[31]} + S_{[22]}$$

$$S^{c}_{123} = S_{[13]} + S_{[32]} + S_{[21]}$$

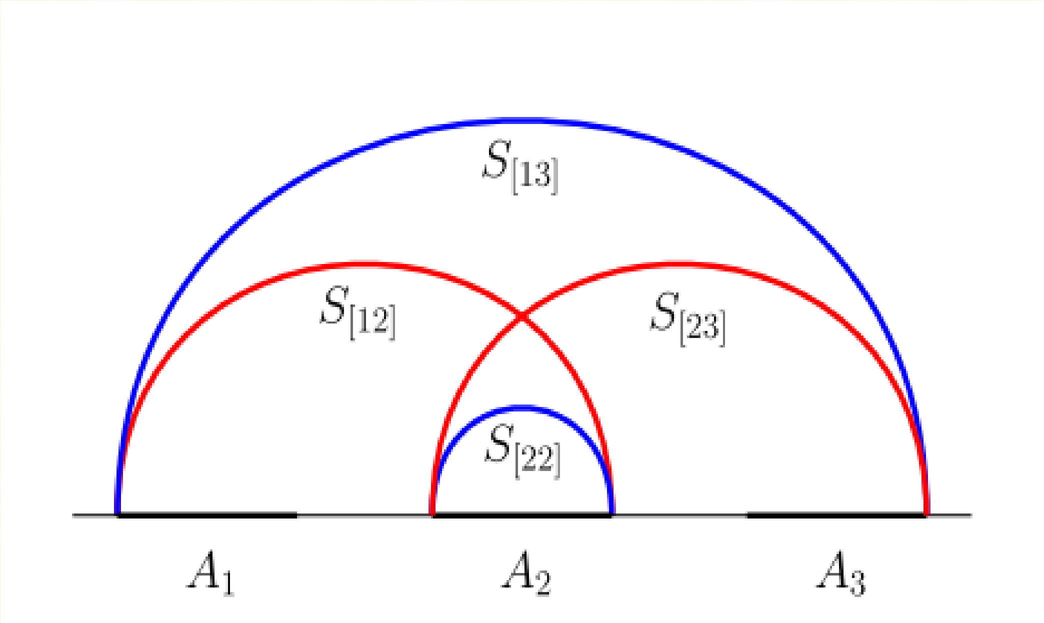


# Completed Connected Configuration

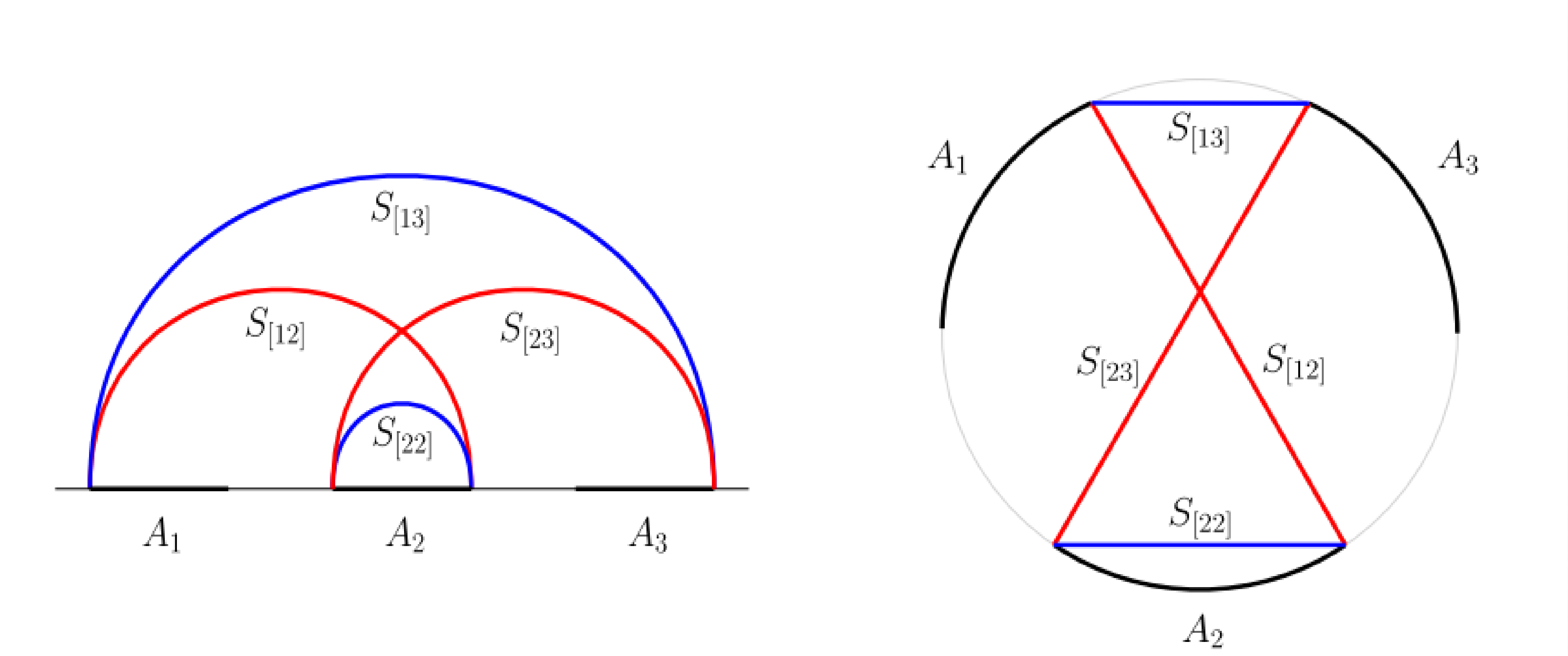
SSA: $S_{12} + S_{23} \geq S_2 + S_{123} \rightarrow S^c_{12} + S^c_{23} \geq S_2 + S^c_{123}$

$$\left(S_{[12]} + S_{[21]}\right) + \left(S_{[23]} + S_{[32]}\right) \geq S_{[22]} + \left(S_{[13]} + S_{[32]} + S_{[21]}\right)$$

$$S_{[12]} + S_{[23]} \geq S_{[22]} + S_{[13]}$$



# Circular Graph: Cross Inequality

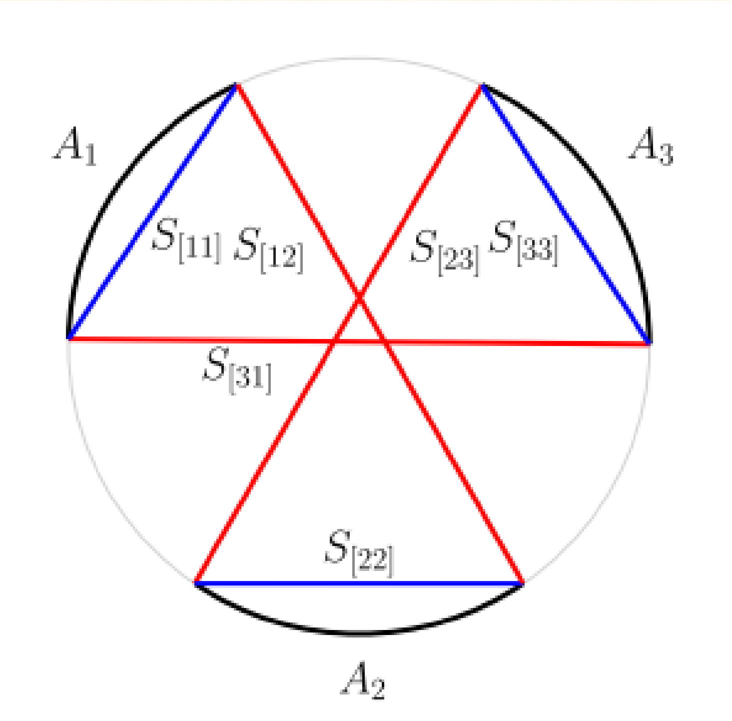


# Tripartite System: MMI

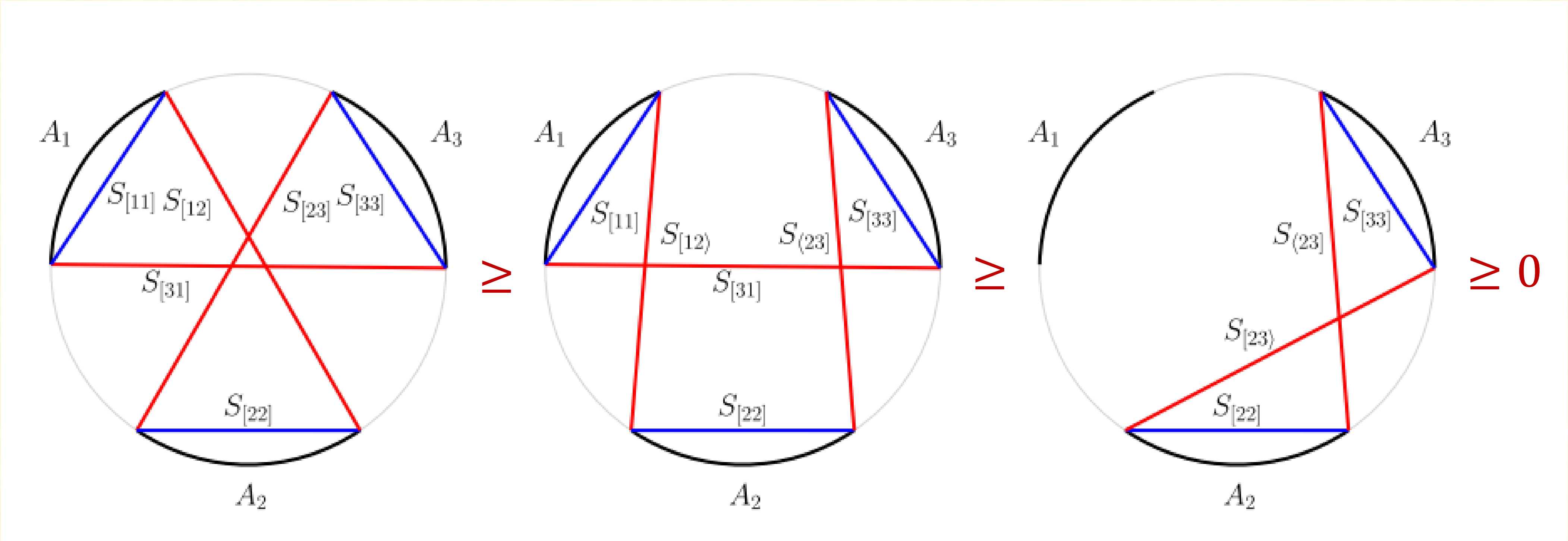
$$S^c_{12} + S^c_{13} + S^c_{23} \geq S_1 + S_2 + S_3 + S^c_{123}$$

$$\left(S_{[12]} + S_{[21]}\right) + \left(S_{[13]} + S_{[31]}\right) + \left(S_{[23]} + S_{[32]}\right)$$
$$\geq S_{[11]} + S_{[22]} + S_{[33]} + \left(S_{[13]} + S_{[32]} + S_{[21]}\right)$$

$$S_{[12]} + S_{[23]} + S_{[31]} \geq S_{[11]} + S_{[22]} + S_{[33]}$$



# Prove by Cross Inequalities



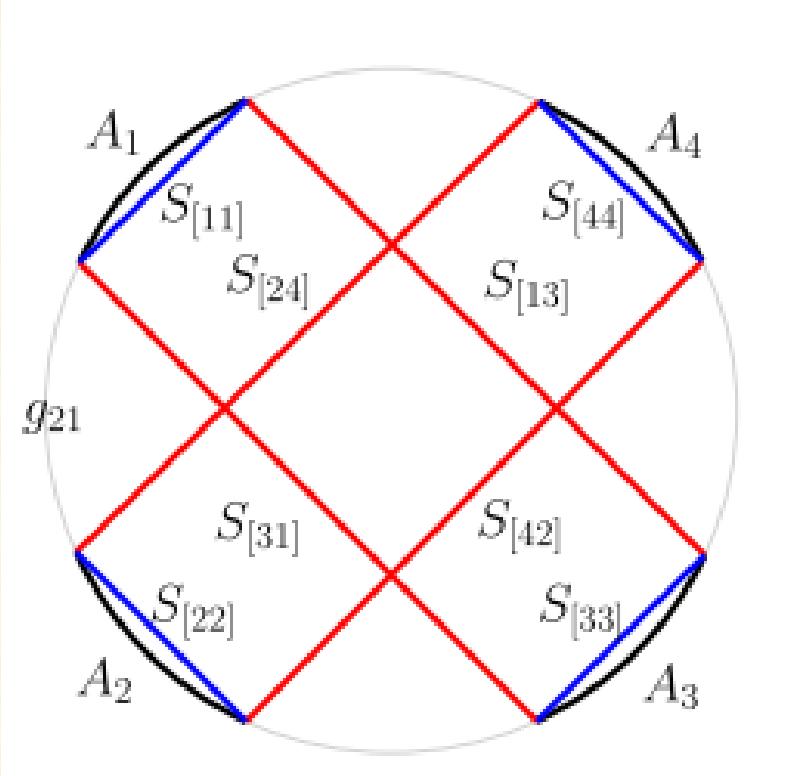
# 4-partite System: Compatible

$S^c_{13} \Rightarrow S_{[11]} + S_{[33]} \geq S_{[13]} + S_{[31]}$

$S^c_{24} \Rightarrow S_{[22]} + S_{[44]} \geq S_{[24]} + S_{[42]}$

$$S_{[11]} + S_{[22]} + S_{[33]} + S_{[44]}$$

$$\geq S_{[13]} + S_{[24]} + S_{[31]} + S_{[42]}$$



# Compatible Theorem

**Theorem:** For two HEEs $S_X$ and $S_{X'}$, if

1. $X \cap Y = \emptyset$
2. RT sufaces of $S_X$ and $S_{X'}$ are nonplanar

Then, $S_X$ and $S_{X'}$ are incompatible.

# Allowed Configurations

- Incompatible pairs in a 5-partite system

| | $S_{124}$ | $S_{134}$ | $S_{135}$ | $S_{235}$ | $S_{245}$ |
|---|---|---|---|---|---|
| $S_{13}$ | | | | | ✓ |
| $S_{14}$ | | | | ✓ | |
| $S_{24}$ | | | ✓ | | |
| $S_{25}$ | | ✓ | | | |
| $S_{35}$ | ✓ | | | | |

- Compatible Completed Connected (CCC) configurations

# A HEI of a 5-partite System in CCC

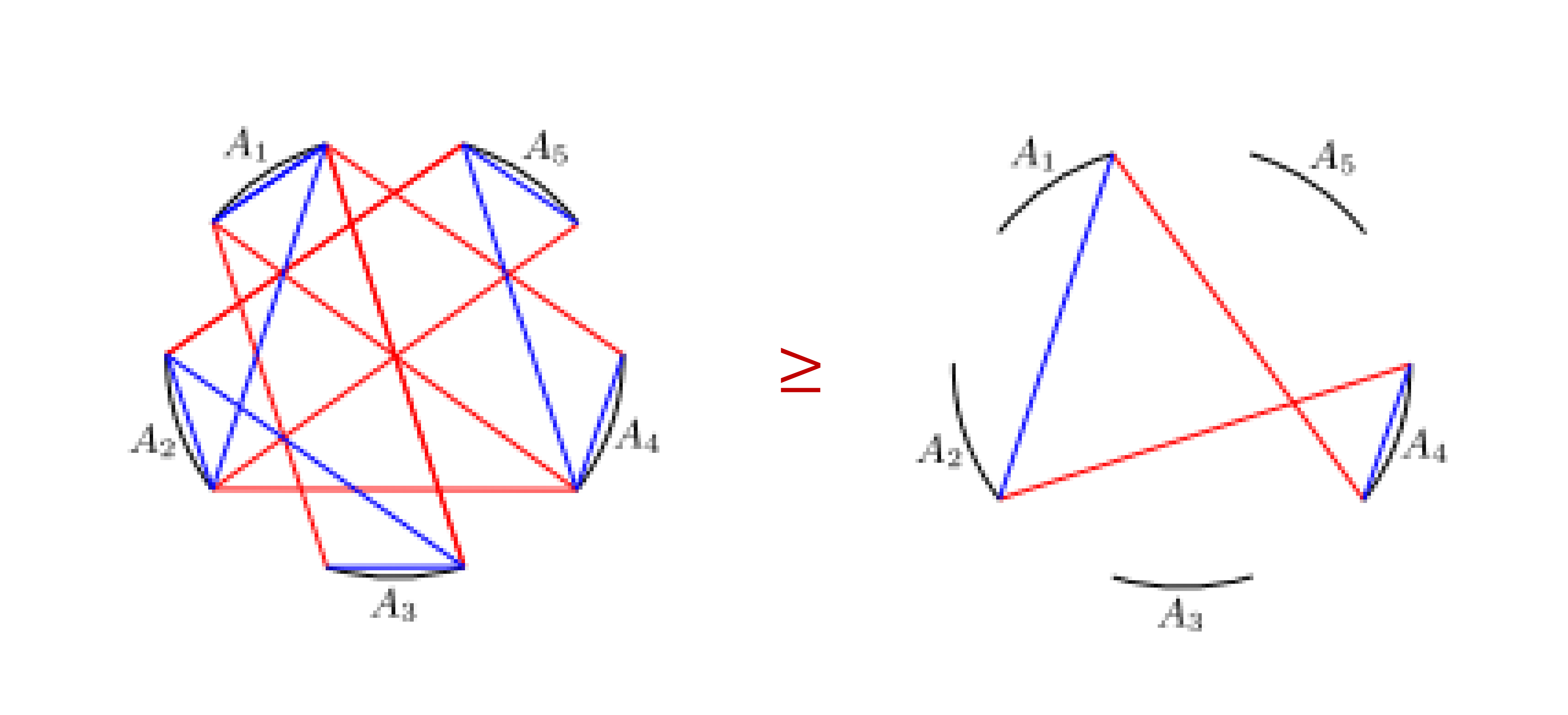
$$2S^c_{123} + S^c_{124} + S^c_{125} + S^c_{134} + S^c_{145} + S^c_{235} + S^c_{245} \geq$$

$$S^c_{12} + S^d_{13} + S^d_{14} + S^c_{23} + S^d_{25} + S^c_{45} + S^c_{1234} + S^c_{1235} + S^c_{1245}$$

$$2S_{[13]} + S_{[14]} + 2S_{[25]} + S_{[31]} + S_{[41]} + S_{[42]} + S_{[52]} \geq$$

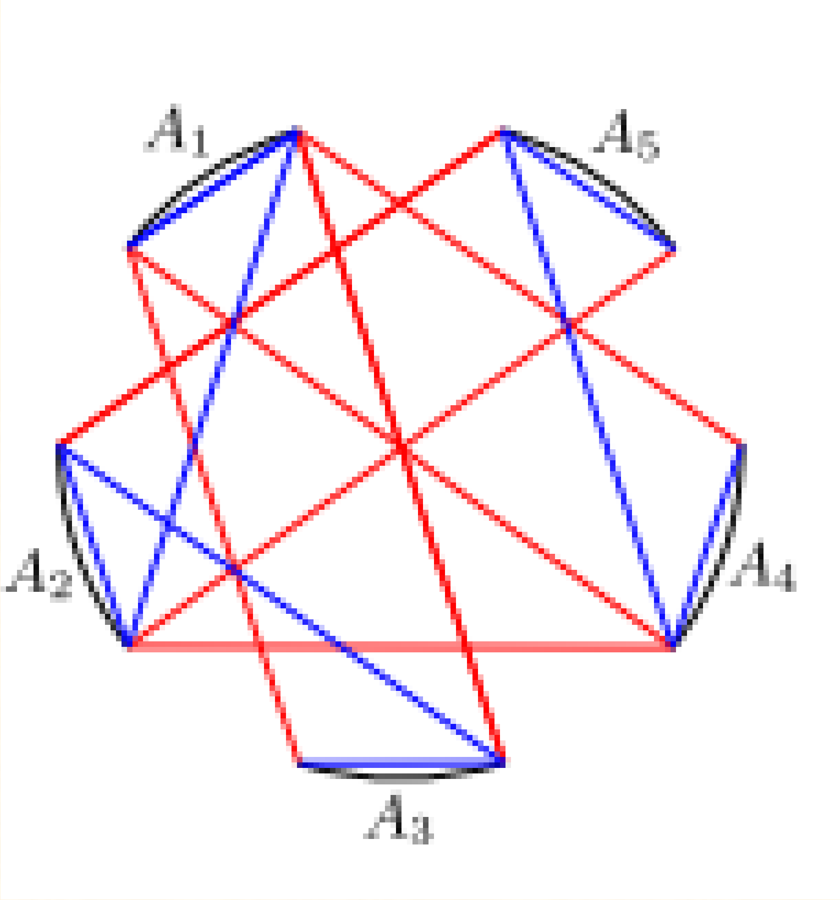
$$2S_{[11]} + S_{[12]} + S_{[22]} + S_{[23]} + S_{[33]} + S_{[44]} + S_{[45]} + S_{[55]}$$

# Circular Graph

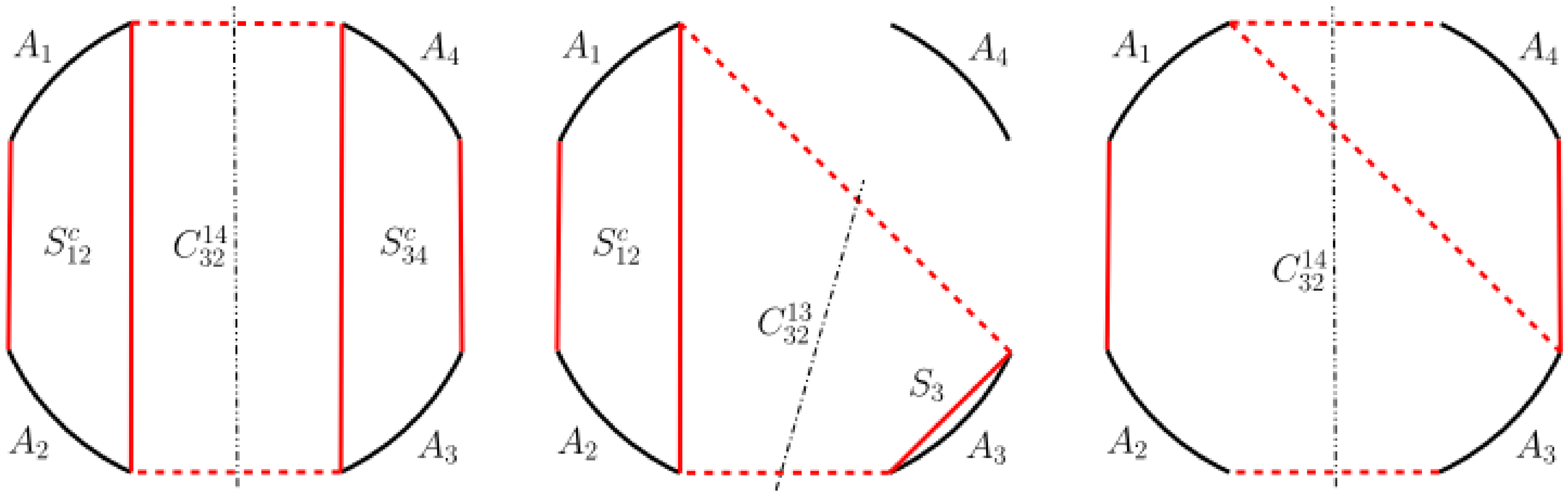


# Steps to Prove a HEI in CCC

- Write the HEI in CCC in terms of simplex basis
- Draw its circular graph
- Reduce the circular graph by cross inequalities



# Non CCC configurations: Cuts




(a) $S^c_{1234}(C^{14}_{32}) = S^c_{12} + S^c_{34}$ (b) $S^c_{123}(C^{13}_{32}) = S^c_{12} + S_3$ (c) $C^{14}_{32}$

# Cut Theorem

**Theorem:** A cut $C_{kj}^{il}$ split $S_{i'\dots j'k'\dots l'}^{c}$ into $S_{i'\dots j'}^{c}$ and $S_{k'\dots l'}^{c}$,

if $[i',j'] \subseteq [i,j]$ and $[k',l'] \subseteq [k,l]$.

Thus, $C_{kj}^{il}$ induces $C_{k'j'}^{i'l'}$.

$C_{42}^{14}$: $S_{124}^{c} \rightarrow S_{12}^{c} + S_4$, $S_{14}^{c} \rightarrow S_1 + S_4$, $S_{24}^{c} \rightarrow S_2 + S_4$

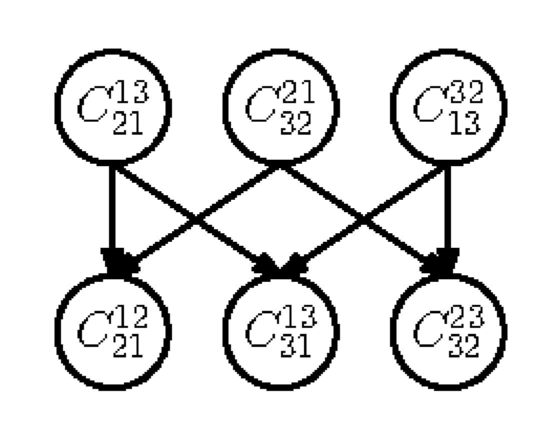
$C_{42}^{14} \rightarrow C_{41}^{14} \quad C_{42}^{24}$

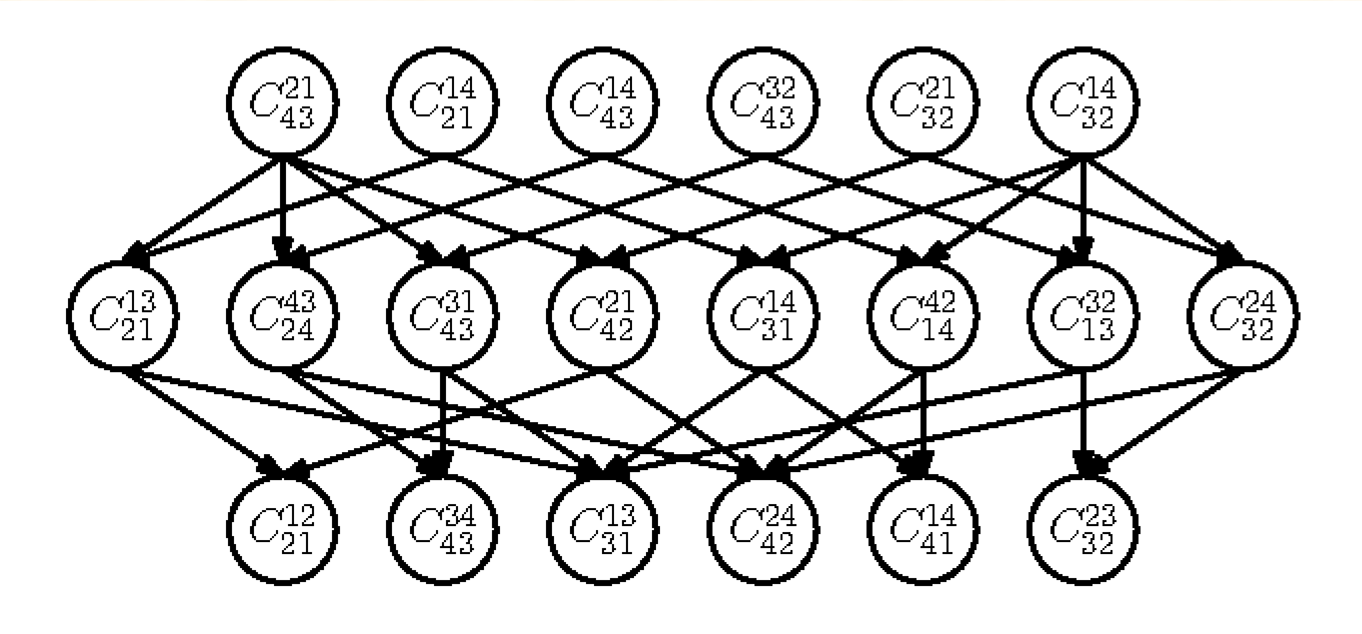
# Cuts in a $n$-partite System

- All the cuts belong to $(n-1)$ levels
- A level-$k$ cut induces some level-$(k-1)$ cuts
- There are $\frac{nk(n-k)}{2}$ cuts in level-$k$
- There are total $\sum_{k=1}^{n-1} \frac{nk(n-k)}{2} = \frac{n^2(n^2-1)}{12}$ cuts

# Dependency Acyclic Graph of Cuts

$$\frac{n^2(n^2-1)}{12}$$

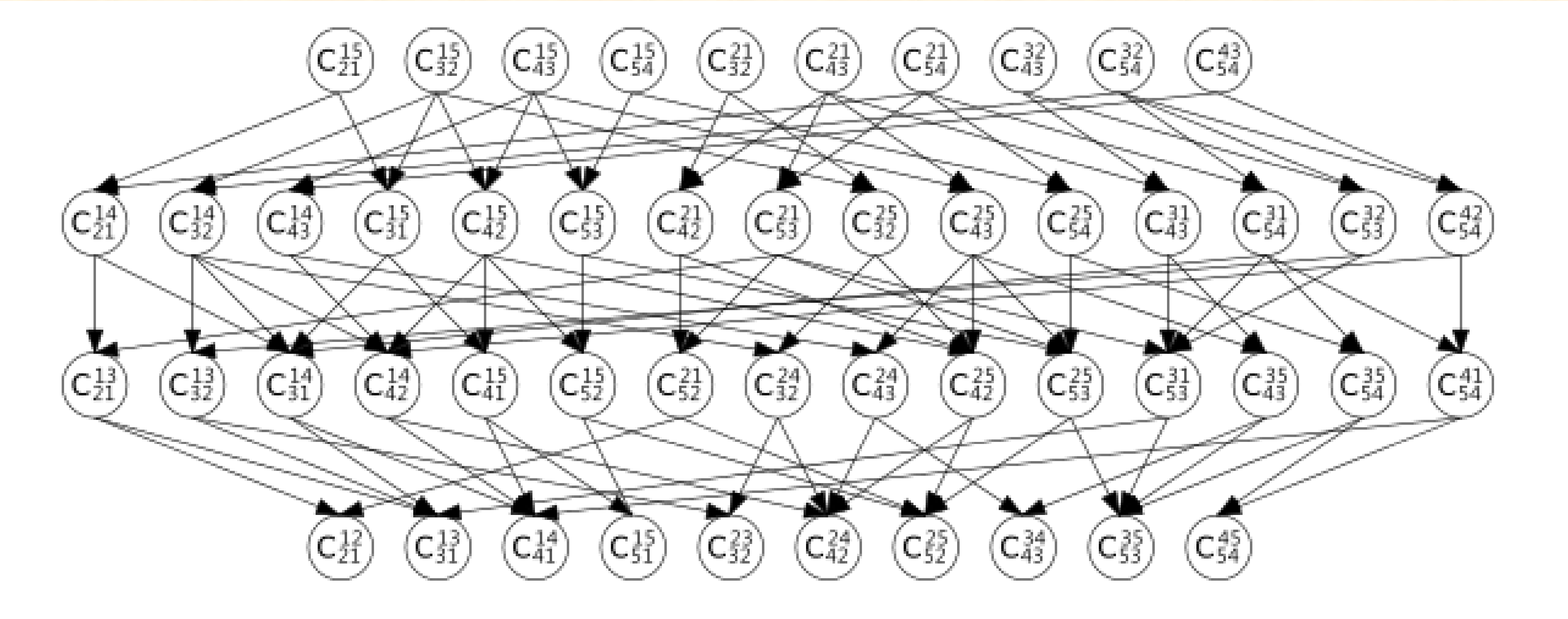



$n = 3 \Rightarrow 6$ cuts
$2^6 = 64 \rightarrow 17$
configurations

$n = 4 \Rightarrow 20$ cuts
$2^{20} = 1048576 \rightarrow 1570$
configurations

# Dependency Acyclic Graph of Cuts




$n = 5 \Rightarrow 50$ cuts

$2^{50} = 1125899906842624 \rightarrow 2{,}864{,}048$

configurations

# Configuration Theorem

**Theorem:** If a balanced HEI is valid in CCC configurations, then it is valid in all configurations.

# HEI in Multipartite Systems

- How many HEIs in a $n$-partite system?
- $n = 2$: $S_i \geq 0$, $S_{12} \geq 0$, $S_1 + S_2 \geq S_{12}$ (SA)
- $n = 3$: $S_K \geq 0$, $S_1 + S_{23} \geq S_{123}$ (enhanced SA), SSA, MMI
- $n = 4$: $S_K \geq 0$, enhanced SA/SSA/MMI, … ?
- How many independent HEIs in a $n$-partite system?

# Finite HEI

- Divergence and trivial HEI: $S_K \to \infty \Rightarrow S_K \geq 0$

- Balance condition and finite HEI

$S_1 + S_2 \geq S_{12}$ (SA)

$S_{12} + S_{23} \geq S_2 + S_{123}$ (SSA)

$S_{12} + S_{13} + S_{23} \geq S_1 + S_2 + S_3 + S_{123}$ (MMI)

# Enhanced HEI

- Enhanced from lower partite system

$$S_1 + S_2 \geq S_{12} \to S_1 + S_{23} \geq S_{123}$$

$$\to S_{12} + S_{367} \geq S_{12367}$$

# Dependent HEI

SA: $S_1 + S_3 \geq S_{13}$

SSA: $S_{12} + S_{23} \geq S_2 + S_{123}$

MMI: $S_{12} + S_{13} + S_{23} \geq S_1 + S_2 + S_3 + S_{123}$

MMI + SA = SSA

# Nontrivial Independent HEI

- $n = 2$: SA
- $n = 3$: ~~SSA~~, MMI
- $n = 4$: no new independent HEI
- $n \geq 5$ ?

# I-symble

$$I_K = \sum_{J\subseteq K}(-1)^{|J|+1} S_J$$

$$I_{12} = S_1 + S_2 - S_{12} \geq 0 \quad \text{(SA)}$$

$$I_{123} = S_1 + S_2 + S_3 - S_{12} - S_{13} - S_{23} + S_{123} \leq 0 \quad \text{(MMI)}$$

- Balance condition: finite HEI
- Superbalance condition: independent HEI

# Independent HEI for $n = 5$

$$-I_{123} - I_{124} - I_{135} + I_{1234} + I_{1235} \geq 0$$

$$-I_{123} - I_{124} - I_{125} - I_{345} + I_{1235} + I_{1245} \geq 0$$

$$-I_{123} - I_{124} - I_{125} - I_{345} + I_{1234} + I_{1235} + I_{1245} \geq 0$$

$$-I_{123} - I_{124} - I_{135} - I_{245} + I_{1234} + I_{1235} + I_{1245} \geq 0$$

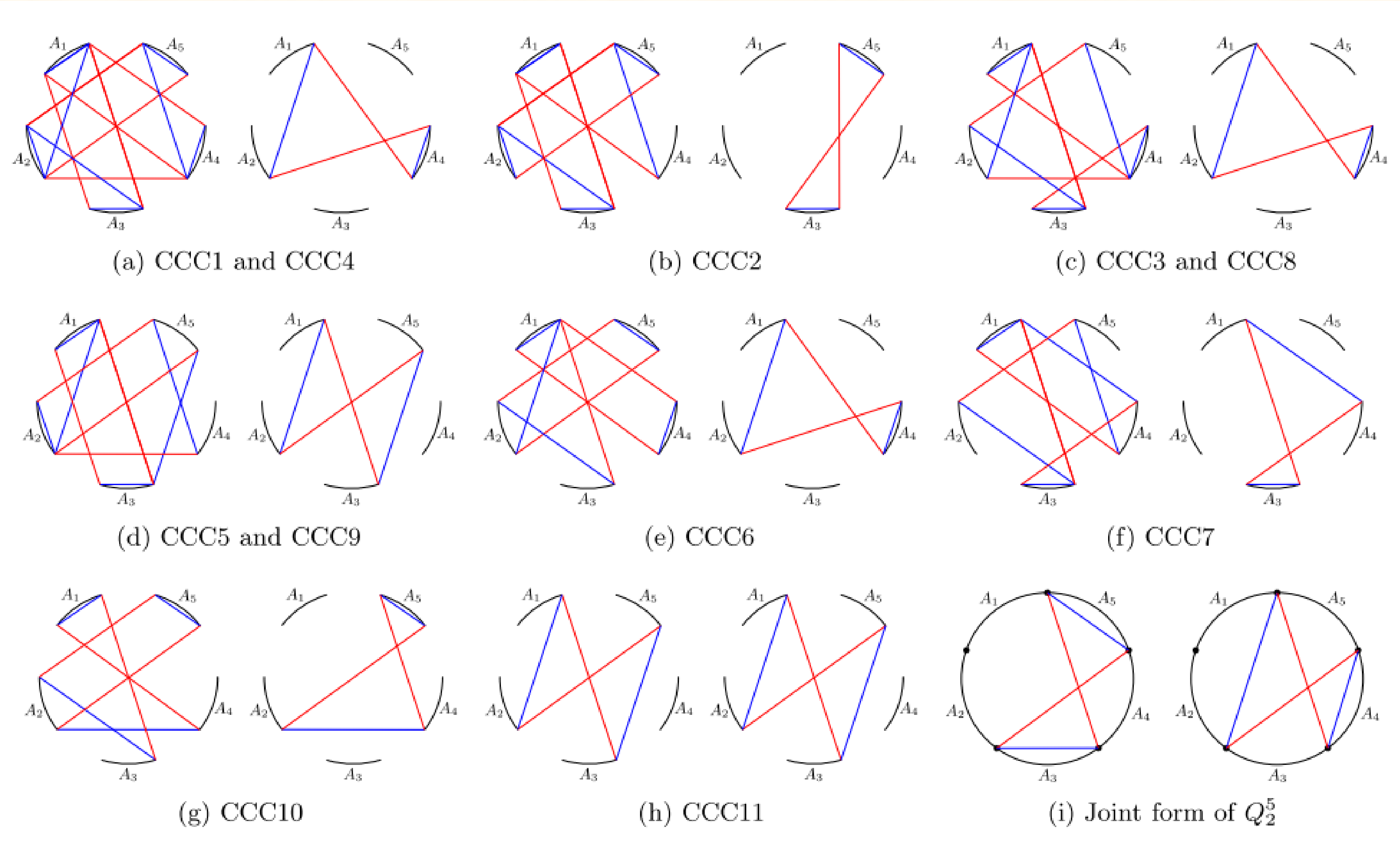
$$-I_{123} - I_{124} - I_{135} - I_{145} + I_{1234} + I_{1235} + I_{1245} + I_{1345} - I_{12345} \geq 0$$

$$-2I_{123} - I_{124} - I_{134} - I_{145} - I_{235} - 2I_{245} + 2I_{1234} + I_{1235} + 2I_{1245} + I_{2345} \geq 0$$

$$-I_{123} - I_{124} - I_{135} - I_{245} - I_{345} + I_{1234} + I_{1235} + I_{1245} + I_{1345} + I_{2345} - I_{12345} \geq 0$$

# Independent HEI for $n = 5$

$$I_{1234} + I_{1235} + I_{1245}$$
$$\geq I_{123} + I_{124} + I_{135} + I_{245}$$



(a) CCC1 and CCC4 (b) CCC2 (c) CCC3 and CCC8
(d) CCC5 and CCC9 (e) CCC6 (f) CCC7
(g) CCC10 (h) CCC11 (i) Joint form of $Q_2^5$

# Examples for $n = 6, 7$

$$-I_{123} - I_{124} - I_{134} - I_{135} - I_{136} - I_{246} - I_{256} - I_{346}$$
$$+I_{1234} + I_{1236} + I_{1246} + I_{1346} + I_{1356} + I_{2456} - I_{12346} \geq 0$$

$$-I_{123} - I_{145} - I_{147} - I_{156} - I_{245} - I_{247} - I_{256} - I_{345} - I_{347} - I_{356}$$
$$+I_{1245} + I_{1247} + I_{1256} + I_{1345} + I_{1347} + I_{1356} + I_{1456} + I_{1457}$$
$$+I_{2345} + I_{2347} + I_{2356} + I_{2456} + I_{2457} + I_{3456} + I_{3457}$$
$$-I_{12456} - I_{12457} - I_{13456} - I_{13457} - I_{23456} - I_{23457} \geq 0$$

# Summary

- Holographic entanglement entropy inequality (HEI)
- Simplex basis and CCC configurations
- Circular graph and cross inequality
- Cuts and Configuration Theorem for non-CCC HEI
- Independent HEI: I-basis and superbalance