

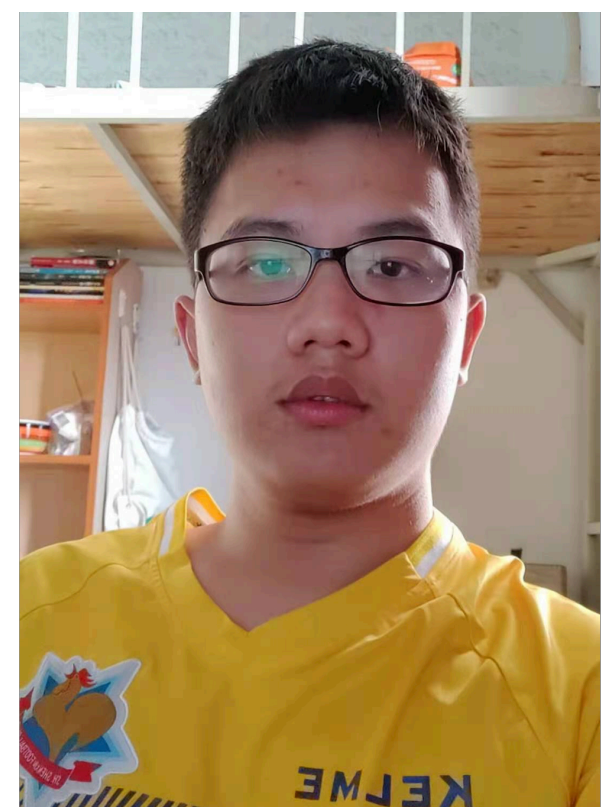
Scattering amplitudes, Feynman integrals & cluster algebras

何颂 (Institute of Theoretical Physics, CAS)

Based on works with

- Z. Li, C. Zhang 1911.01290 (PRD), 2009.11471 (JHEP)
- Z. Li, Y. Tang, Q. Yang 2012.13094 (JHEP)
- Z. Li, Q. Yang, C. Zhang 2012.15092 (PRL)
- Z. Li, Q. Yang 2103.02796, 2106.09314, 2108.07959 (JHEP)

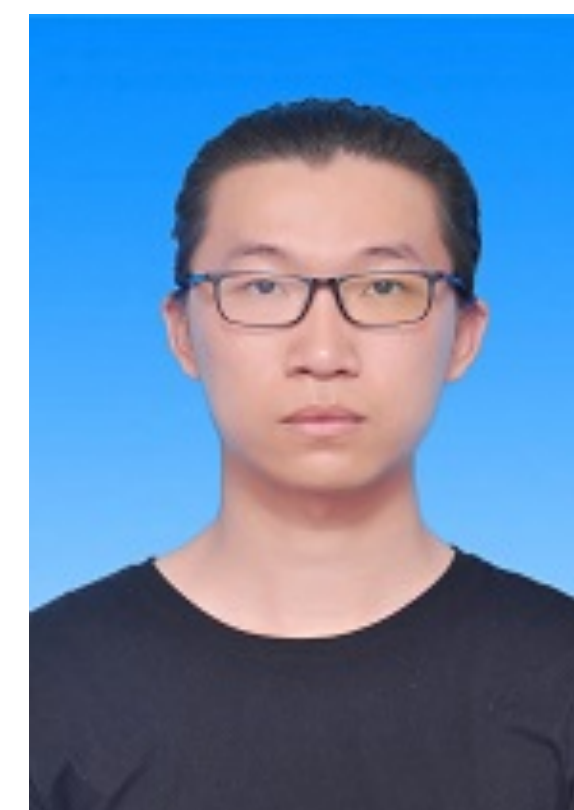
Peng Huanwu center for
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March 10, 2021



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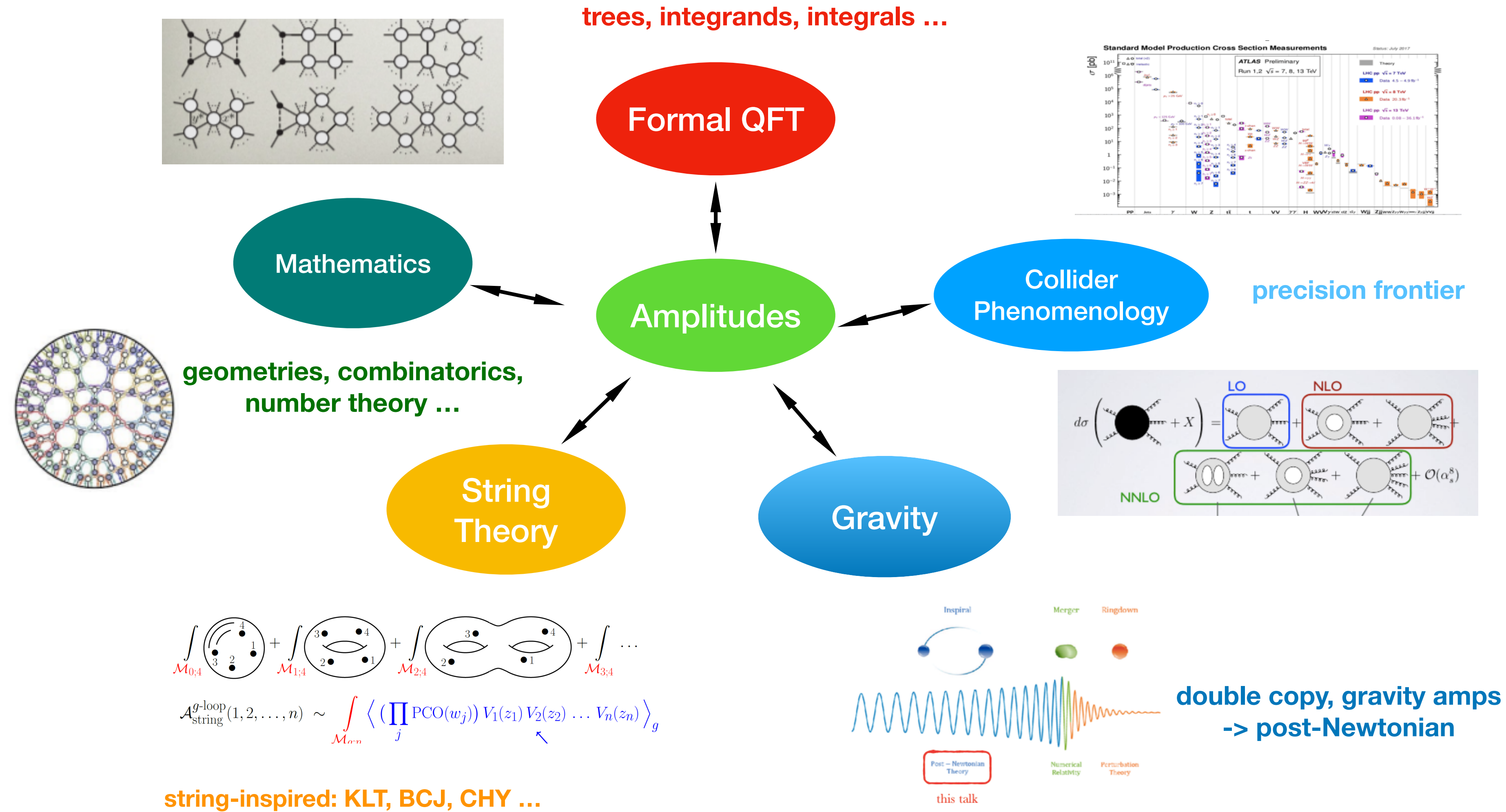


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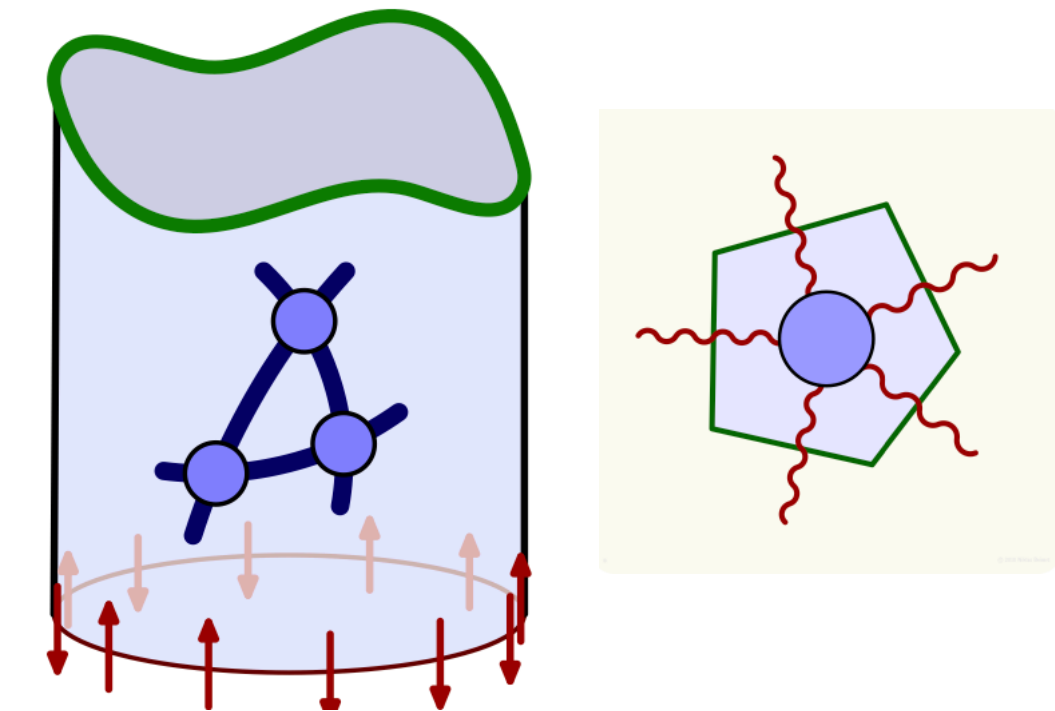
“Amplitudes” today



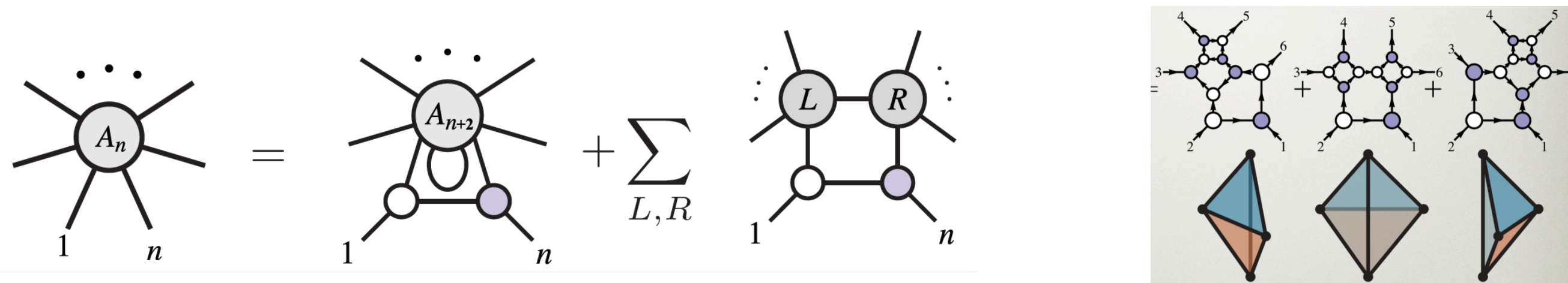
The simplest QFT

Harmonic oscillator of 21st century: hidden simplicity & structure in $\mathcal{N} = 4$ SYM (planar limit)

Integrability (planar limit): strong coupling via AdS/CFT, Wilson loops & OPE
Yangian symmetry ... Ising model of gauge theories!



All-loop integrands \leftrightarrow positive Grassmannian + amplituhedron [Arkani-Hamed, Trnka]



(Integrated) amplitudes + Feynman Integrals: extremely rich laboratory for perturbative QFT!
iterated integrals (polylogs & beyond), symbology, cluster algebras, differential eqs, bootstrap + Qbar,...

Amplitudes/WL in N=4 SYM

Wilson loop & symmetries

MHV amplitudes (tree stripped) = **null polygonal** Wilson loops (strong+ weak coupling)

[Alday, Maldacena][Brandhuber, Heslop, Travaglini] [Drummond, Henn, Korchemski, Sokatchev][...]

$$A_n(p_1, p_2, \dots, p_n) \leftrightarrow W_n(x_1, x_2, \dots, x_n) \sim \langle \text{Tr} \mathcal{P} \exp (i \oint \mathbf{A} \cdot dx) \rangle$$

super-amplitudes $\mathcal{A}_{n,k}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = \text{super-Wilson loops } \mathcal{W}[\mathbf{A} := \mathbf{A}^{\alpha, \dot{\alpha}} + \bar{\psi}^{\dot{\alpha}} \theta^\alpha + \dots]$ [Mason, Skinner] [Caron-Huot]

dual space w. $\mathcal{N} = 4$ SUSY extension: $(x_{i+1} - x_i)^{\alpha \dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}$, $(\theta_{i+1} - \theta_i)^{\alpha A} = \lambda_i^\alpha \eta_i^A$

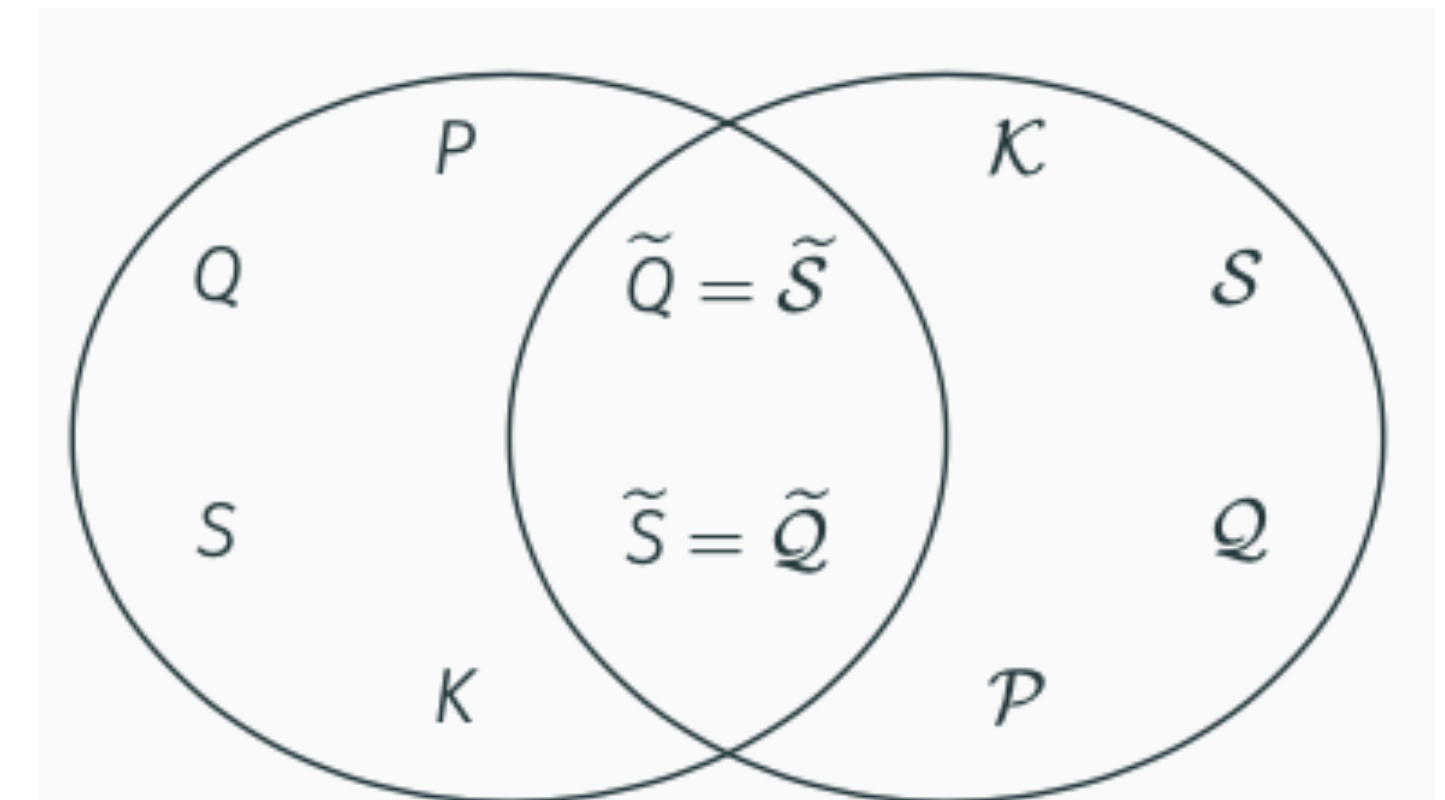
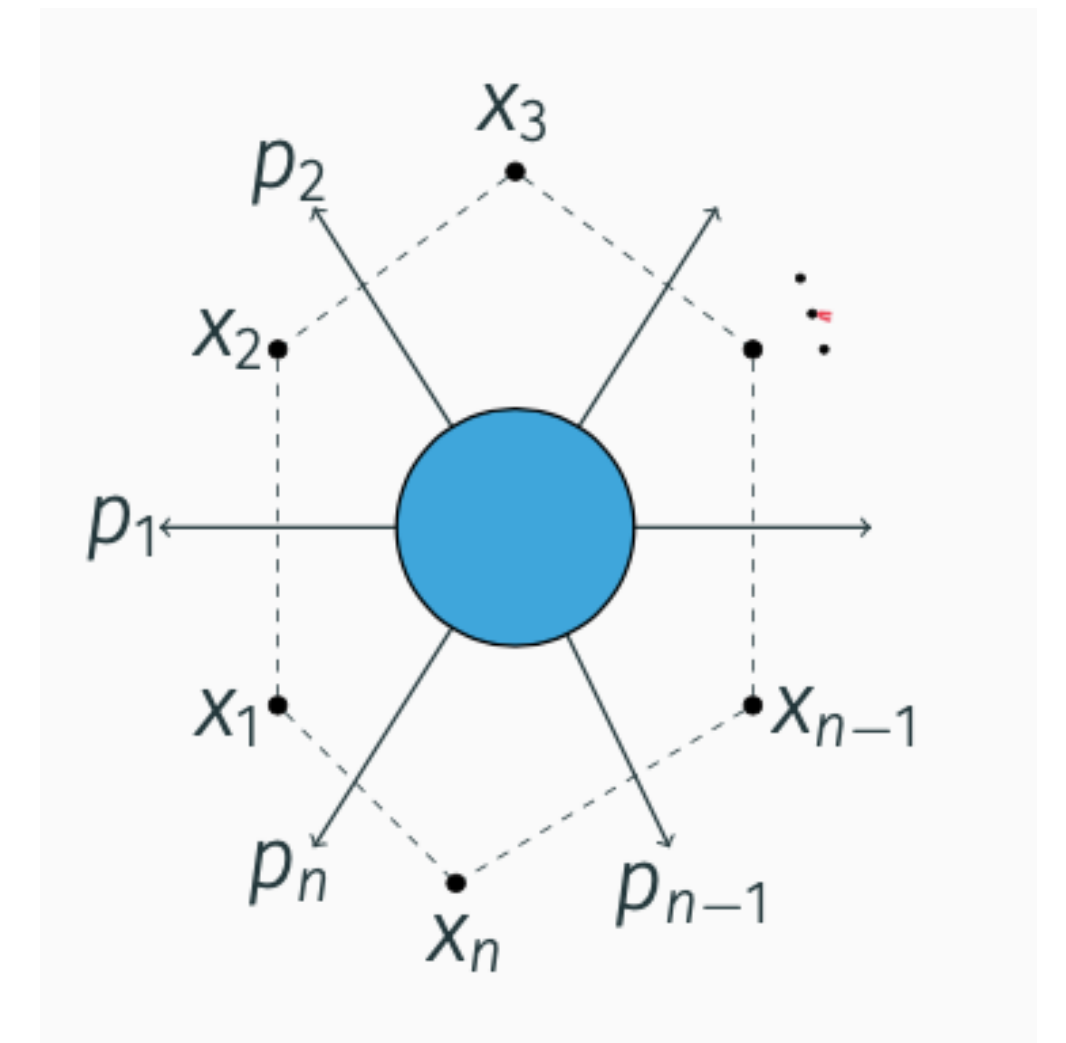
superconformal (amps) + dual superconformal (WL)

→ **Yangian symmetry** (infinite dim.) → integrability

[Drummond, Korchemski, Sokatchev; + Henn, Smirnov] [Drummond, Henn, Plefka]

PSU(2,2|4)

Loop-level: **symmetry broken** by IR/UV divergence!



Loop amplitudes

BDS ansatz [Bern, Dixon, Smirnov]: $A_n^{\text{BDS}} \sim \exp\left(\frac{1}{4}\Gamma_{\text{cusp}}F_n^{1-\text{loop}}\right) \implies$ **BDS-normalized amps:** $R_{n,k} = \mathcal{A}_{n,k}/A_n^{\text{BDS}}$

- **Dual conformal invariant** (DCI) function of $3(n-5)$ cross-ratios ($n=4,5$ trivial)

interestingly only invariant under chiral half of dual SUSY, not the other half!

- natural separation into transcendental (w. discontinuities) & algebraic part (only poles)

$$R_{n,k} \sim \underbrace{(\text{Yangian invariants})}_{\text{helicity, "rational/algebraic"}} \times \underbrace{(\text{Transcendental functions})}_{\text{DCI, "uniform weight"}=2 L}$$

Yangian inv. classified, e.g. MHV, $R_{n,0} \sim$ pure functions, similarly for NMHV

These two cases expected to be simplest: only generalized polylogarithms!

for $k \geq 2$ (NNMHV...): elliptic integrals etc. will appear [Bourjaily et al] [A. Kristensson, M. Wilhelm, C. Zhang]

Momentum twistors [Hodges]

- **Unconstrained** variables for any massless kinematics (useful for QCD etc), naturally a planar ordering
- “**Light-rays**” of dual space, **linearly realize** dual symmetry $SL(4|4)$: $\mathcal{F}_i = (Z_i^a \mid \chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha, \dot{\alpha}} \lambda_{i, \alpha} \mid \theta_i^{\alpha, A} \lambda_i^\alpha)$
- Basic $SL(4)$ invariant: **4-bracket** $\langle ijkl \rangle := \varepsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d$ e.g. $\langle i-1 \ i \ j-1 \ j \rangle \propto (x_i - x_j)^2$
- With DCI, only cross-ratios of invariants appear! **configuration space**: $G(4, n)/T$ (note $G(2, n)/T \sim \mathcal{M}_{0, n}$)
- **dual superconformal symmetries**: $K_b^a = \sum_i Z_i^a \frac{\partial}{\partial Z_i^b}$, $R_B^A = \sum_i \chi_i^A \frac{\partial}{\partial \chi_i^B}$, $Q_A^a = (Q_a^\alpha, \bar{S}_A^{\dot{\alpha}}) = \sum_i Z_i^a \frac{\partial}{\partial \chi_i^A}$
 annihilate $R_{n, k}$, but **not** $\bar{Q}_a^A = (\bar{Q}_\alpha^A, S_{\dot{\alpha}}^A) = \sum_i \chi_i^A \frac{\partial}{\partial Z_i^a}$ (**chiral nature** of super WL!)

Symbology & bootstrap

Multiple polylogs: $G(\mathbf{a}, t_0) = \int_0^{t_0} \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_2}{t_2 - a_2} \dots \int_0^{t_{w-1}} \frac{dt_w}{t_w - a_w} \rightarrow$ **symbol & letters** [Goncharov, Spradlin, Vergu, Volovich]

$$dG^{(w)} = \sum_i G_i^{(w-1)} d \log x_i \implies \mathcal{S}(G^{(w)}) = \sum_i \mathcal{S}(G_i^{(w-1)}) \otimes x_i \quad \text{e.g. } \mathcal{S}(\log(x)) = x, \mathcal{S}(\text{Li}_2(x)) = -(1-x) \otimes x$$

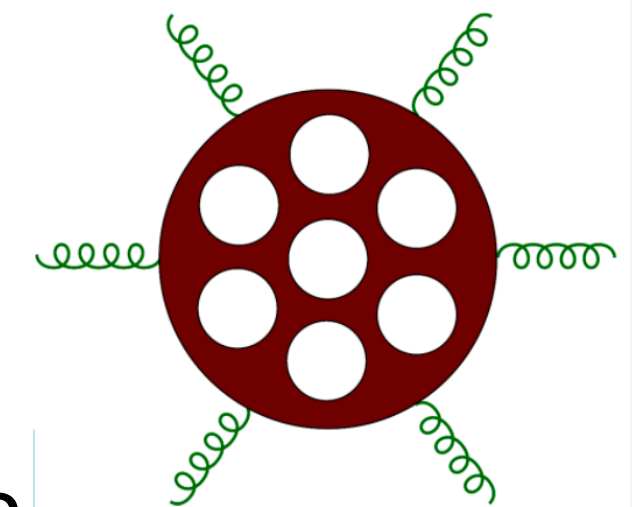
trivialize polylog relations; 1st entry: physical discontinuities, last entry: differential

For $n=6,7$: only 9 & 42 letters! **conjecture: cluster variables** of $G_+(4,n)$

finite-type (Dynkin) cluster algebras A_3 for $n=6$, E_6 for $n=7$ [Golden et al][...]

hexagon/heptagon bootstrap: ansatz with alphabet + symmetry conditions (Qbar + collinear etc.)
 -> unique answer to very high loops [Dixon et al] [Caron-Huot, Dixon, Dulat, McLeod, von Hippel, Papathanasiou][...]

starting $n=8$: infinite cluster algebra for $G(4,n)$, how to obtain finite alphabet? square roots & more?



Grassmannian cluster algebras

Symbol alphabet for $n=6,7$ (to high loops) given by cluster variables of $G(4,n)$ [Golden et al][Dixon et al][Drummond et al]...

Cluster algebras: mutations from an initial quiver $\rightarrow a_i$ (cluster variables) grouped into overlapping $\{a_1, \dots, a_d\}$ (clusters) [Fomin, Zelevinski]
 \rightarrow only finite for Dynkin-type quiver, $A_d(d(d+3)/2)$, $B_d \sim C_d(d(d+1))$, $D_d(d^2)$, $E_{6,7,8}(42,70,128)$, $F_4(28)$, $G_2(8)$!

For $G(4,n)$ only finite for $n=6,7$ [Speyer, Williams]: A_3 & E_6 cluster-algebra alphabets!

hexagon: 6 choose 4=15 $\langle a b c d \rangle$ (9 unfrozen + 6 frozen)

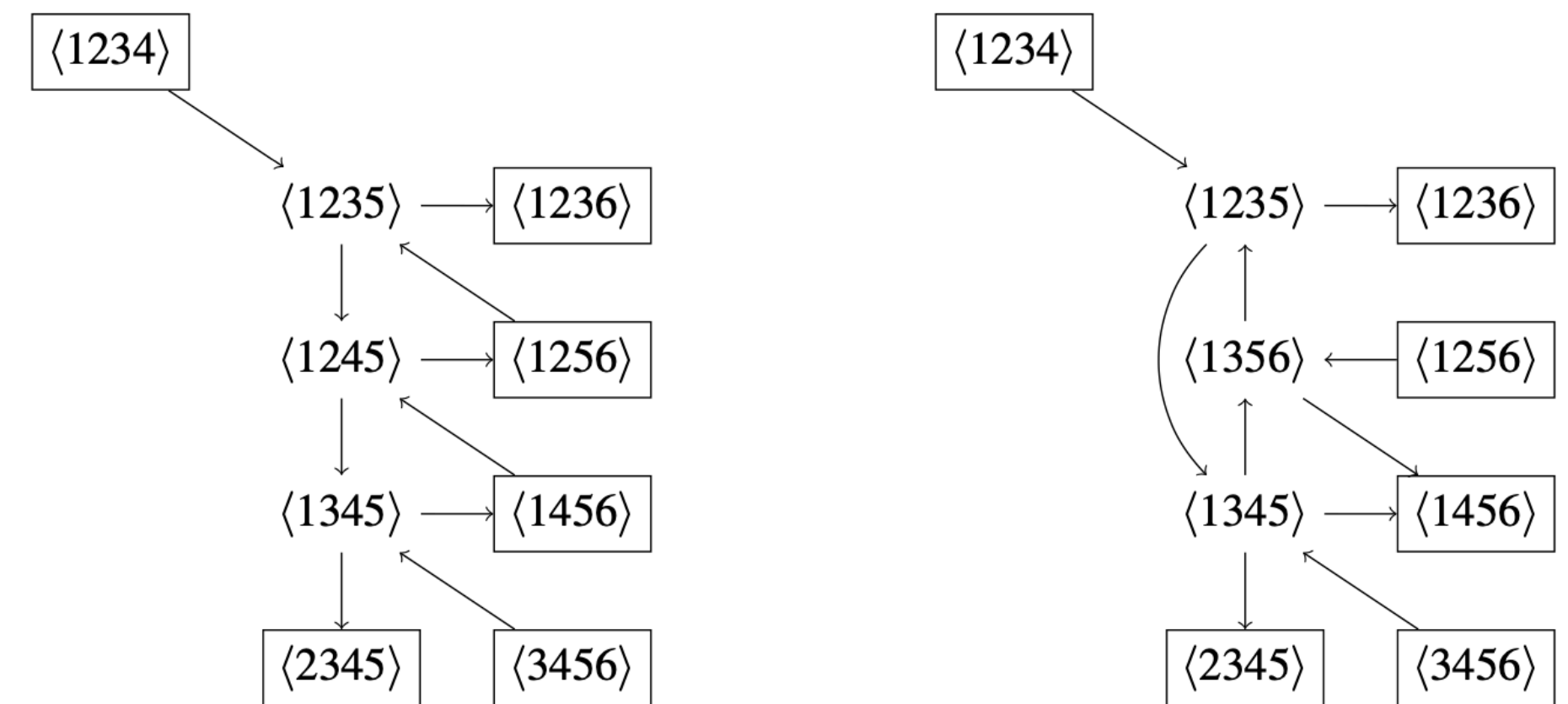
heptagon: 7 choose 4=35 $\langle a b c d \rangle$ (28 unfrozen + 7 frozen)

+ 14 degree-2: $\langle 1(23)(45)(67) \rangle$ + cyclic & $\langle 1(27)(34)(56) \rangle$ + cyclic

$$\langle a(bc)(de)(fg) \rangle \equiv \langle abde \rangle \langle acfg \rangle - \langle abfg \rangle \langle acde \rangle$$

\Rightarrow DCI letters: 9/42 X coordinates of A_3/E_6 cluster algebra

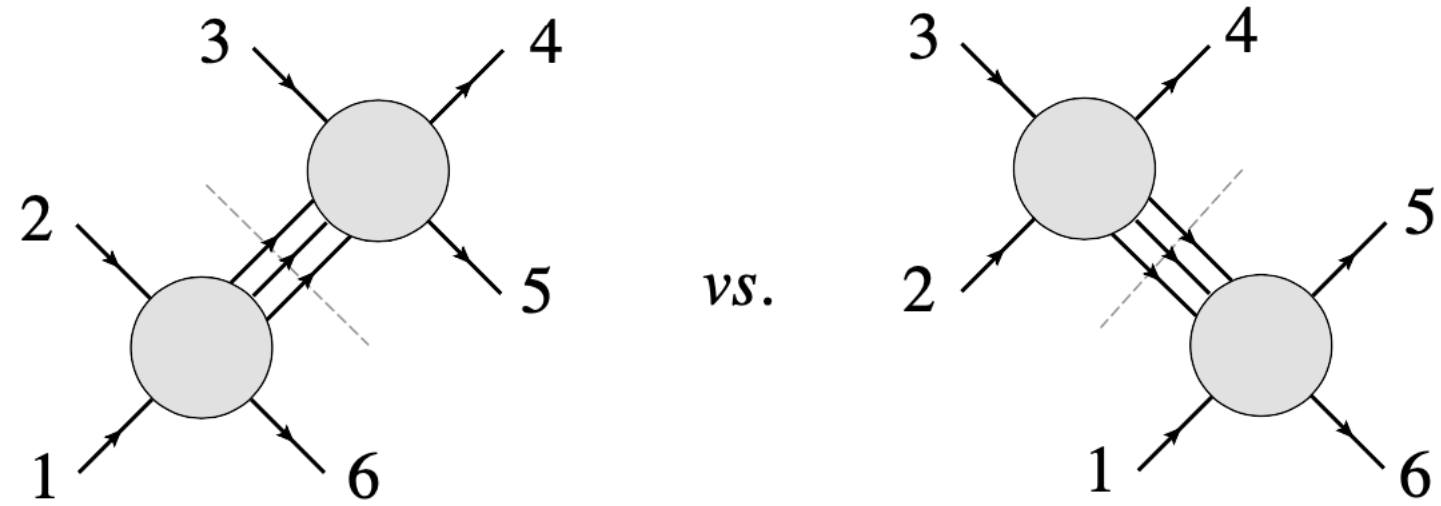
$$\langle 1245 \rangle \rightarrow \frac{\langle 1235 \rangle \langle 1456 \rangle + \langle 1345 \rangle \langle 1256 \rangle}{\langle 1245 \rangle} = \langle 1356 \rangle,$$



[from 2005.06735 by Caron-Huot et al]

(Extended) Steinmann relations

Basic properties of (massless) QFTs [Steinmann]: double discontinuities of any Feynman diagram (amplitude) vanish @ overlapping channels (no two overlapping $s_{a,\dots,b-1}$ for planar) e.g. for 3-particle case:



$$\text{Disc}_{s_{j,j+1,j+2}} \left(\text{Disc}_{s_{i,i+1,i+2}} (A_{n,k}) \right) = 0, \quad \text{for } j = i \pm 1, i \pm 2,$$

[from 2005.06735 by Caron-Huot et al]

For $n=6$: $\langle 1245 \rangle \approx \langle 2356 \rangle \approx \langle 3461 \rangle$ no pair for first 2 entries; even stronger constraints for $n=7$!

For divergent amps/integrals, need suitable subtraction due to 2-pt singularities (frozen)

Remarkably, data for $n=6,7$ reveals **extended Steinmann relations**: adjacent entries satisfy same constraints!

A new property of general QFT (for $n=6,7$) [Caron-Huot et al] more amps & integrals @ higher n/L ?

For hexagon/heptagon, ES relations imply **cluster adjacency**: two cluster variables can appear consecutively only if there exist a cluster with both [Drummond et al]

Steinmann cluster bootstrap

hexagon/heptagon bootstrap: compute amplitudes for $n=6,7$ without computing integrands/integrals at all

- (1). hexagon/heptagon space: all integrable symbols (+ more) with **A3/E6 alphabet**
- (2). first entry, Steinmann & ES relations/cluster adjacency \rightarrow **huge reduction** of the space
- (3). physical constraints: **final-entry, collinear, multi-Regge limits/OPE...**

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
First entry	1	3	9	26	75	218	643	1929	5897	?	?	?	?	?
Steinmann	1	3	6	13	29	63	134	277	562	1117	2192	4263	8240	?
Ext. Stein.	1	3	6	13	26	51	98	184	340	613	1085	1887	3224	5431

Table 1: The dimensions of the hexagon, Steinmann hexagon, and extended Steinmann hexagon spaces at symbol level.

weight n	0	1	2	3	4	5	6	7
First entry	1	7	42	237	1288	6763	?	?
Steinmann	1	7	28	97	322	1030	3192	9570
Ext. Stein.	1	7	28	97	308	911	2555	6826

Table 2: The dimensions of the heptagon, Steinmann heptagon, and extended Steinmann heptagon spaces at symbol level.

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. \mathcal{H}_6	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	(0*,0*)	(0*,2*)	(1* ³ ,5* ³)	(6* ² ,17* ²)
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0*,0*)	(1* ² ,2* ²)
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0*,0*)	(1*,0* ²)
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0*)
8. N ³ LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Amp/WL from Qbar equations [Caron-Huot, SH]

$$\bar{Q}_a^A R_{n,k} = a \operatorname{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3} \mathcal{Z}_{n+1} \right)_a^A [R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\text{tree}}] + \text{cyclic},$$

loop parameter $a := \frac{1}{4} \Gamma_{\text{cusp}} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 + \dots,$

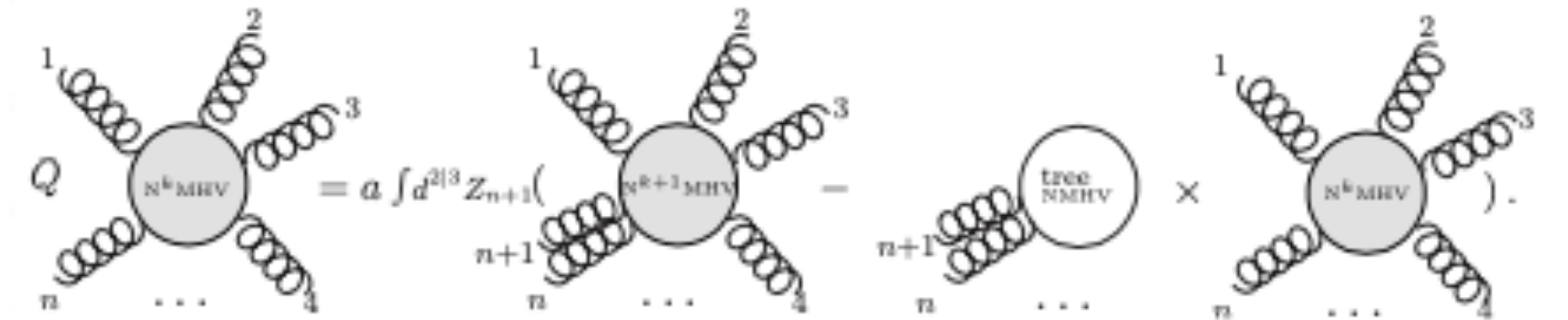


Figure 1. All-loop equation for planar $\mathcal{N} = 4$ S-matrix.

1st order diff eqs for all-loop S-matrix: determine **MHV & NMHV** amps uniquely, given lower-loop amps

Last entries for all loops (**important for bootstrap**) MHV and NMHV (to all n) [SH, Z. Li, C. Zhang]

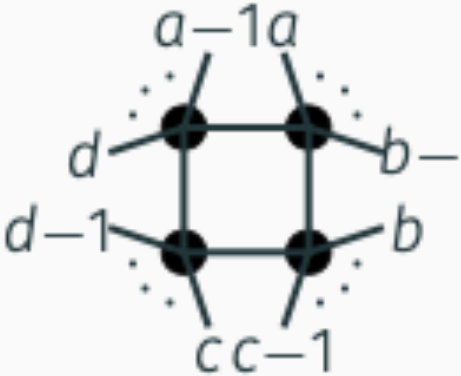
computed 2-loop NMHV heptagon \rightarrow 3-loop MHV hexagon [Caron-Huot, SH] + 2-loop MHV to all n

nice observation: even for $n > 7$, 2-loop MHV have letters that are (simple) cluster variables of $G(4, n)$!

However, for $n > 7$, $k+l > 2$ (3-loop MHV, 2-loop NMHV, even 1-loop NNMHV): **algebraic letters** (square roots) \rightarrow rationalization on RHS \rightarrow new data for **symbol alphabet & structures** !

2-loop NMHV + 3-loop MHV [SH, Z. Li, C. Zhang]

$$\begin{aligned}
 & \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^{c-1} \otimes \mathcal{X}_{a,b,c,d}^{c-1} [a-1 a b-1 b c-1] \\
 & - \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^c \otimes \mathcal{X}_{a,b,c,d}^c [a-1 a b-1 b c] \\
 & + \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^{b-1} \otimes \mathcal{X}_{a,b,c,d}^{b-1} [a-1 a b-1 c-1 c] \\
 & - \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^b \otimes \mathcal{X}_{a,b,c,d}^c [a-1 a b c-1 c] \\
 & + \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^{a-1} \otimes \mathcal{X}_{a,b,c,d}^{a-1} [a-1 b-1 b c-1 c] \\
 & - \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^a \otimes \mathcal{X}_{a,b,c,d}^a [a b-1 b c-1 c],
 \end{aligned}$$



$$\begin{cases}
 u_{abcd} = \frac{x_{ab}^2 x_{cd}^2}{x_{ac}^2 x_{bd}^2}, & v_{abcd} = \frac{x_{ad}^2 x_{bc}^2}{x_{ac}^2 x_{bd}^2}, & \Delta_{abcd} = \sqrt{(1-u-v)^2 - 4uv} \\
 z_{abcd} = \frac{1}{2}(1+u-v+\Delta), & \bar{z}_{a,b,c,d} = \frac{1}{2}(1+u-v+\Delta),
 \end{cases}$$

$$\mathcal{X}_{a,b,c,d}^* := \frac{(x_{a,b,c,d}^* + 1)^{-1} - \bar{z}_{d,a,b,c}}{(x_{a,b,c,d}^* + 1)^{-1} - z_{d,a,b,c}}, \quad \tilde{\mathcal{X}}_{a,b,c,d}^* := \frac{(x_{a,b,c,d-1}^* + 1)^{-1} - z_{d,a,b,c}}{(x_{a,b,c,d-1}^* + 1)^{-1} - \bar{z}_{d,a,b,c}}$$

with 6 choices $a-1, a, b-1, b, c-1, c$ of the superscript, where

$$\begin{aligned}
 x_{a,b,c,d}^a &= \frac{\langle \bar{d}(c-1c) \cap (ab-1b) \rangle}{\langle \bar{d}a \rangle \langle b-1bc-1c \rangle}, & x_{a,b,c,d}^{a-1} &= x_{a,b,c,d}^a |_{a \leftrightarrow a-1} \\
 x_{a,b,c,d}^b &= \frac{\langle \bar{d}(c-1c) \cap (a-1ab) \rangle}{\langle \bar{d}(a-1a) \cap (bc-1c) \rangle}, & x_{a,b,c,d}^{b-1} &= x_{a,b,c,d}^b |_{b \leftrightarrow b-1} \\
 x_{a,b,c,d}^c &= \frac{\langle \bar{d}c \rangle \langle a-1ab-1b \rangle}{\langle \bar{d}(a-1a) \cap (b-1bc) \rangle}, & x_{a,b,c,d}^{c-1} &= x_{a,b,c,d}^c |_{c \leftrightarrow c-1}
 \end{aligned}$$

Remarkably constrained & compact “**algebraic part**”: 4-mass \otimes algebraicⁱ \otimes finalⁱ $\times R_i$ (all correlated!)

$n = 8$: $\Delta_{1,3,5,7}$ & $\Delta_{2,4,6,8}$, 9+9 independent algebraic letters (+180 rational letters)

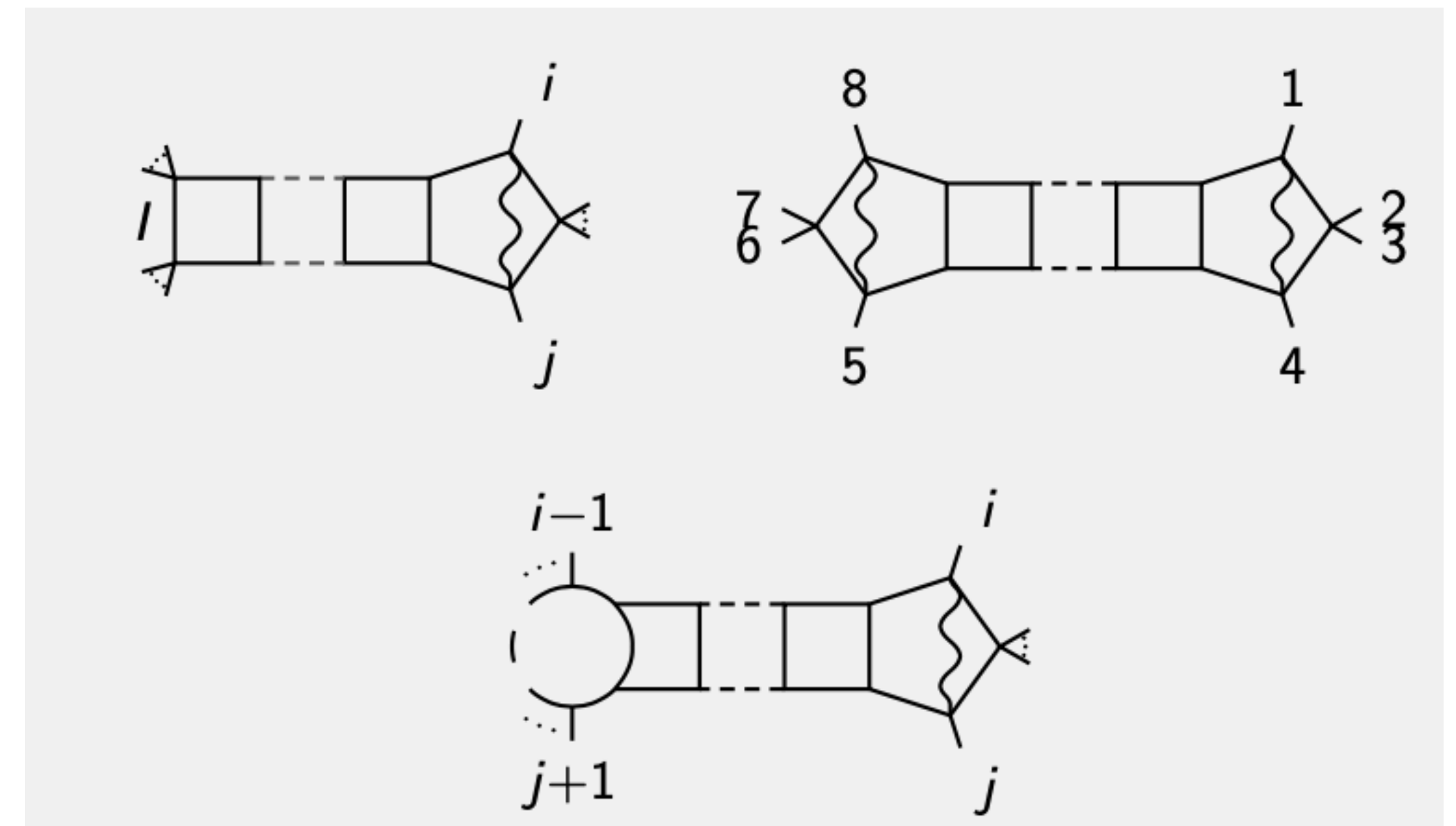
9 Δ for $n=9$, 11*9 independent algebraic letters... -> 3-loop $n=8$ MHV: still 9+9 (+204 rational)! [Z. Li, C. Zhang]

Origin of alphabet: Landau equations [Spradlin et al] tropical $G_+(4,8)$ etc. [Drummond, et al] [Arkani-Hamed, Lam, Spradlin] poles/“letters” of Yangian invariants [Mago, Schreiber, Spradlin, Volovich] [SH, Z. Li] [...]

Feynman integrals from WLS

Uniform transcendental integrals

$$\begin{aligned}
 \text{1-loop MHV amp.} &= \sum_{i < j < l} \text{Diagram} \\
 \text{2-loop MHV amp.} &= \sum_{i < j < k < l < i} \text{Diagram} \\
 \text{2-loop NMHV} \Big|_{\chi_i \chi_j \chi_k \chi_l} &= \text{Diagram} - \text{Diagram} \quad (i, j, k, l \text{ non-adjacent})
 \end{aligned}$$



(double-) pentagons for MHV/NMHV amps [Arkani-Hamed et al]

(e.g. numerators $\langle \ell_1 \bar{i} \cap \bar{j} \rangle$, $\langle \ell_2 \bar{k} \cap \bar{l} \rangle$)

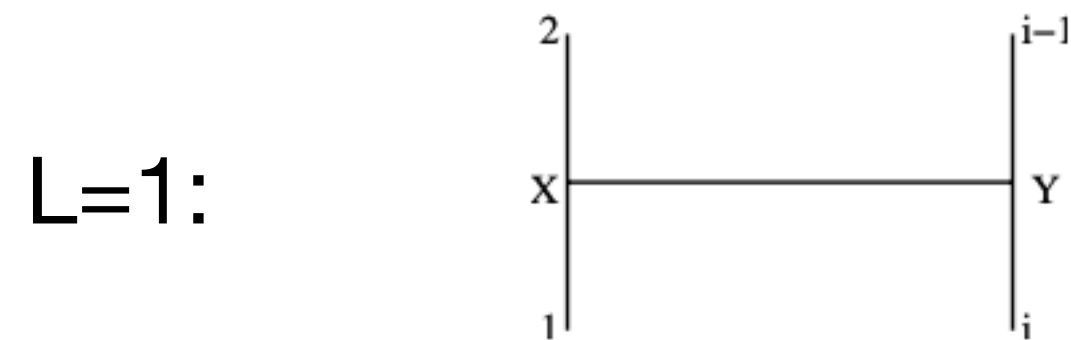
(double-) penta-ladder [Drummond, Henn, Trnka] & more

uniform transcendentality vs. dlog (focus on **IR finite** integrals e.g. $i < j - 1$ & $k < l - 1$ for dp)
 -> a class of integrals in N=4 SYM, also play an important role in general (**master integrals**)

Feynman integrals from WL

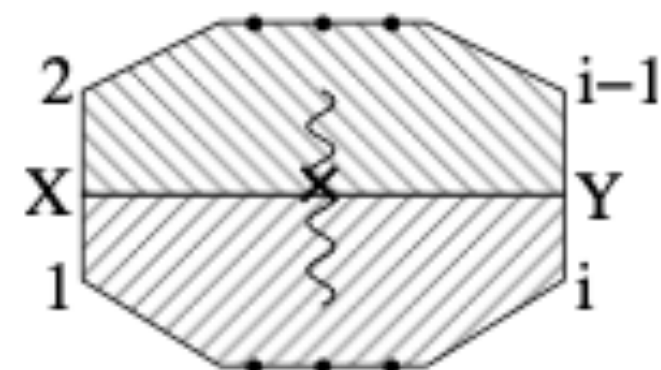
WL powerful for not only full amps, but also its building blocks: WL diagrams = Feynman integrals

How we originally computed 2-loop MHV: $dR_{n,0} = \sum_{i < j} C_{i,j} d \log \langle \bar{i}j \rangle$ w. $C_{i,j}$ (super-) WL diagrams



$$C_{2,i} = \int_0^\infty d\tau_X d\tau_Y \frac{\langle \bar{2}i \rangle \langle \bar{i}2 \rangle}{\langle XY \rangle^2} = \log u_{2,i-1,i,1}$$

L=2: 1-d τ -integral of box integrals



simplest NMHV: difference of two WL diagrams = double pentagon!

$$\mathcal{W}_{n,k=1}^{(2)} \Big|_{x_i^A x_j^B x_k^C x_l^D} = \text{Diagram 1} - \text{Diagram 2}$$

FIG. 2. NMHV component of super-WL as difference of two diagrams, each equals to a double-pentagon integral.

WL $d \log$: 1-loop examples [SH, Z. Li, Y. Tang, Q. Yang]

Why useful? swap order of integrals, left with simple line integrals (“smart parametrization”)

chiral pentagon: $\frac{1}{\langle \ell i - 1i \rangle \langle \ell i i + 1 \rangle} = \int_0^\infty \frac{d\tau}{\langle \ell i X(\tau) \rangle^2}$, $X(\tau) := Z_{i-1} + \tau_X Z_{i+1}$ “fermions” at $x := (iX)$ & $y := (jY)$

$$\implies I_{\text{pent.}} = \int d\tau_X d\tau_Y \int \frac{d^4 \ell \langle \ell \bar{i} \cap \bar{j} \rangle \langle Iij \rangle}{\langle \ell i X \rangle^2 \langle \ell j Y \rangle^2 \langle \ell I \rangle} = \int d^2 \tau \frac{\langle I \bar{i} \cap \bar{j} \rangle \langle Iij \rangle}{\langle IiX \rangle \langle IjY \rangle \langle iXjY \rangle} \quad (\text{star-triangle identity})$$

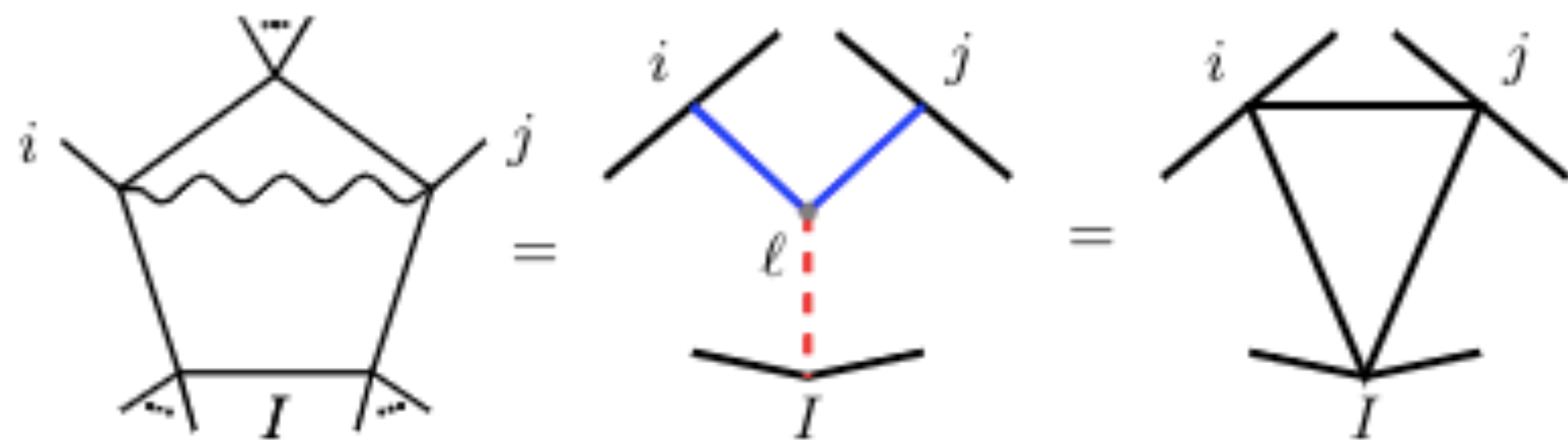


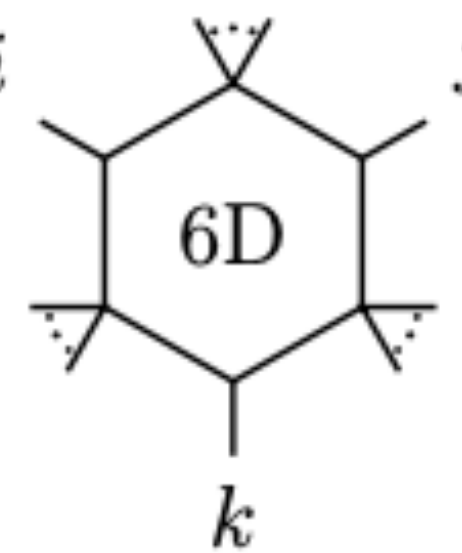
FIG. 3. The chiral pentagon written as a WL diagram, and loop integral performed using “star-triangle” identity.

Nice $d \log$ 2-form: $\int_{(i,j)} d \log \frac{\langle IjY \rangle}{\langle \bar{i}(jY) \cap (iI) \rangle} d \log \frac{\langle iXjY \rangle}{\langle IiX \rangle}$

Trivially give well-known dilog (manifest DCI + weight-2)

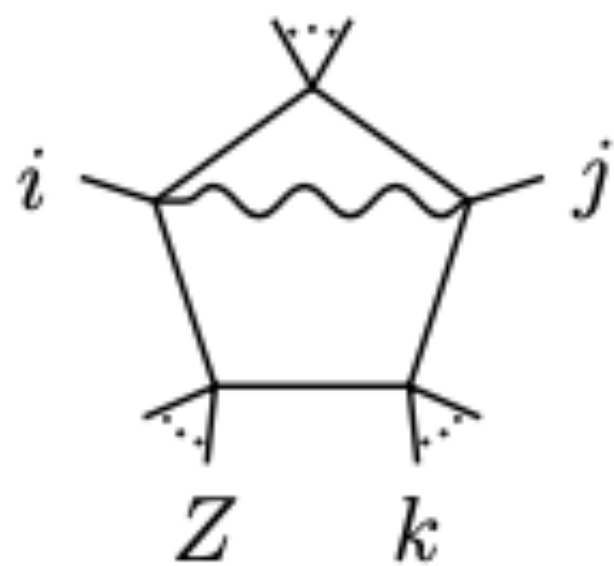
Geometry: integrating $\Omega(\Delta')$ in Δ (similar to Aomoto)

6d 3-mass-easy hexagon [Del Duca et al]
 (weight-3 polylog of 9 cross-ratios)

$$\Omega_1^{(6D)}(i, j, k) := \text{Diagram} = \int \frac{d^6 x_0}{\pi^3} \frac{x_{i,j+1}^2 x_{j,k+1}^2 x_{k,i+1}^2 \sqrt{\Delta_9}}{x_{0,i}^2 x_{0,i+1}^2 x_{0,j}^2 x_{0,j+1}^2 x_{0,k}^2 x_{0,k+1}^2}.$$


momentum twistors G(4,n): all square roots disappear

nice 3-fold dlog integral
 (1-d integral of “deformed” pentagon)

$$\Omega_1^{(6D)}(i, j, k) = \int_0^\infty d\tau_z \frac{\langle (ijk)\bar{i} \cap \bar{j} \cap \bar{k} \rangle}{\langle kZ\bar{i} \cap \bar{j} \rangle \langle kZij \rangle} \text{Diagram} = \int_{\mathbb{R}_{\geq 0}^3} d \log \frac{\langle kZij \rangle}{\langle kZ\bar{i} \cap \bar{j} \rangle} \left(d \log \frac{\langle jYkZ \rangle}{\langle jYi(kZ) \cap \bar{i} \rangle} d \log \frac{\langle iXjY \rangle}{\langle iXkZ \rangle} \right).$$


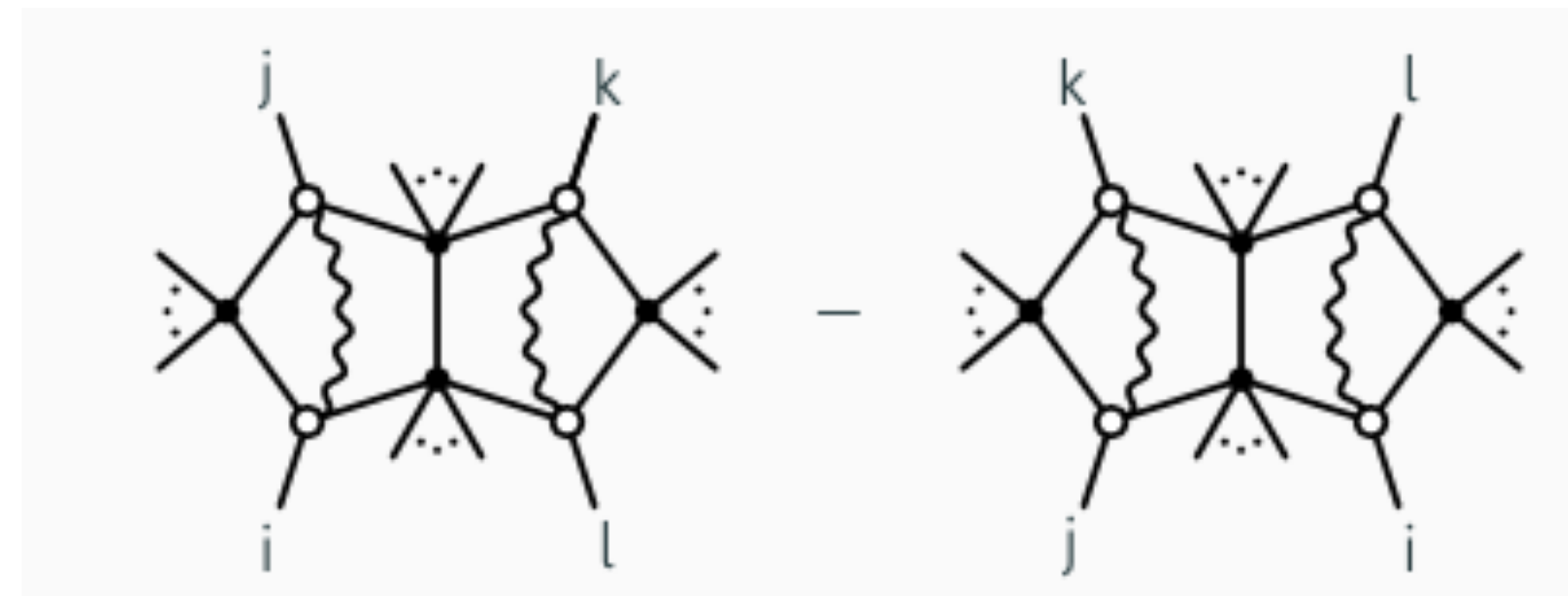
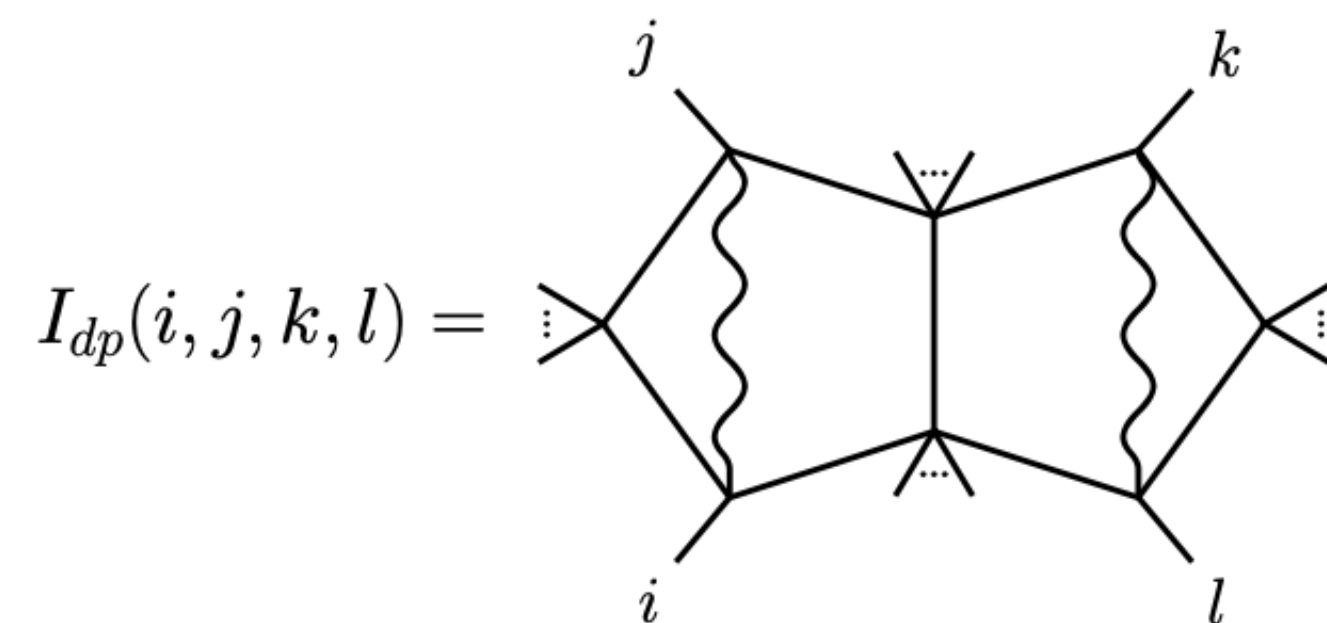
using **symbol integration** [Caron-Huot SH][...]
 straightforward to get weight-3 symbol
 without performing any integral

$$\int_a^b d \log(t+c) (F(t) \otimes w(t)) \implies \{F(t) \otimes w(t) \otimes (t+c)\} \Big|_{t=a}^{t=b}, \left(\int_a^b d \log(t+c) F(t) \right) \otimes w, \left(\int_a^b d \log \frac{t+c}{t+d} F(t) \right) \otimes (c-d)$$

Two-loop amps from integrals

Longstanding problem: compute generic double-pentagon analytically (12 legs, lots of square roots)

All we need for MHV: $A_{n,\text{MHV}}^{2\text{-loop}} = \sum_{i < j < k < l} I_{\text{dp}}(i, j, k, l)$; how to see cancellation of square roots?



Component $\chi_i^1 \chi_j^2 \chi_k^3 \chi_l^4$ for non-adjacent i, j, k, l (vanishes for $L=0,1$) given by **2** double-pentagon integrals

Surprise: \bar{Q} result for the components free of roots (algebraic words vanish)!

For $n=8$: also observed in [\[Bourjaily et al\]](#) by evaluating $I_{\text{dp}}(1,3,5,7)$ at a numeric point? Why cancel?

Generic double pentagon [SH, Z. Li, Q. Yang, C. Zhang]

Exactly the same method: 2d integral of a hexagon
 ($k = j + 1, l = i - 1$: chiral hexagon)

$$X := Z_{i-1} - \tau_X Z_{i+1}, \quad Y := Z_{j-1} - \tau_Y Z_{j+1}$$

New: hexagon not “pure”, 15 boxes w. 2-form “leading singularities”;
 need **rationalization** for computing τ -integrals

weight-3 symbol: “4-mass box \otimes algebraic letter”
 final-entry **free of square roots** (as 2-loop NMHV/3-loop MHV!)

compact **algebraic words** : sum of 16 simple blocks
 (algebraic) $\big|_{(i,j,k,l)-(j,k,l,i)} = 0$! **cancellation of square roots**

$$\sum_{\sigma_a \in \{0,1\}} (-)^{\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4} S^{4-m}(i + \sigma_1, j + \sigma_2, k + \sigma_3, l + \sigma_4) \otimes W_{\sigma_1, \dots, \sigma_4}^{i,j,k,l}$$

$$= \int \frac{d^2 \tau \langle ijkl \rangle}{\langle iXjY \rangle} \int ([x, x_k] I_{x, x_k} - (k-1 \leftrightarrow k+1) - (\bar{k} \leftrightarrow \bar{l}) + [x, y] I_{x, y})$$

$$[x, y] = d \log \frac{\langle iXkl \rangle}{\langle iXjY \rangle} d \log \frac{\langle \bar{l}(jY) \cap (ikl) \rangle}{\langle jYkl \rangle},$$

$$[x, x_k] = d \log \frac{\langle jYil \rangle}{\langle jYkl \rangle} d \log \frac{\langle iXjY \rangle}{\langle l(iX)(jY)(kk+1) \rangle},$$

$$I_{x, x_k} := \tilde{F}(x, y, x_{k+1}, x_l) - \tilde{F}(x, y, x_{k+1}, x_{l+1})$$

$$- L_2(l+1, x, y, l) + L_2(l+1, x, k+1, l)$$

$$- L_2(l+1, y, k+1, l) + \log u_{l+1, x, y, l} \log u_{x, y, k+1, l+1},$$

$$I_{x, y} := L_2(x, k, k+1, l) - L_2(x, k, k+1, l+1)$$

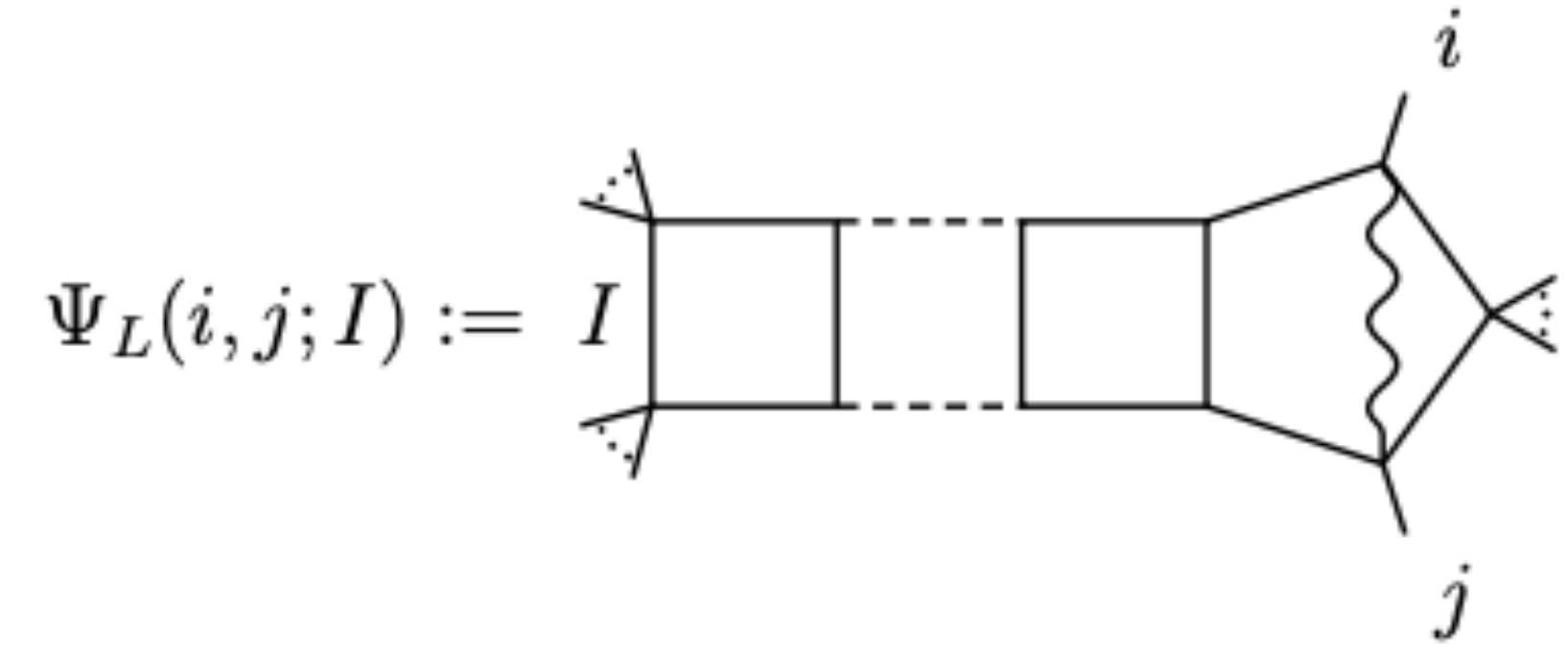
$$- L_2(l+1, x, k, l) + L_2(l+1, x, k+1, l)$$

$$- L_2(l+1, k, k+1, l) + \log u_{l+1, x, k, l} \log u_{x, k, k+1, l+1}$$

$$L_2(a, b, c, d) := \text{Li}_2(1 - u_{a,b,c,d}).$$

Recursion for all-loop ladders [SH, Z. Li, Y. Tang, Q. Yang]

Simplest multi-loop application: penta-box ladder integral



recursion: L-loop as 2-fold dlog integral of deformed (L-1)-loop
 1-loop pentagon=2 dlog \rightarrow 2L-fold dlog integral

$$u = \frac{\langle i-1iI \rangle \langle jj+1ii+1 \rangle}{\langle i-1ijj+1 \rangle \langle Iii+1 \rangle}, \quad v = \frac{\langle jj+1I \rangle \langle i-1ij-1j \rangle}{\langle jj+1i-1i \rangle \langle Ij-1j \rangle}, \quad w = \frac{\langle i-1ijj+1 \rangle \langle j-1jii+1 \rangle}{\langle i-1ij-1j \rangle \langle jj+1ii+1 \rangle}$$

$$\Psi_L(i, j, I) = \int \left[\prod_{a=1}^{L-1} d \log \langle i-1ijY_a \rangle d \log \frac{\langle iX_a jY_a \rangle}{\tau_{X_a}} \right] d \log \frac{\langle jY_L I \rangle}{\langle jY_L iI \cap \bar{i} \rangle} d \log \frac{\langle iX_L jY_L \rangle}{\langle iX_L I \rangle}$$

nice DCI form: simple deform & “odd” weight objects in between (tree=1 - u - v + uvw)

$$\Psi_{L+\frac{1}{2}}(u, v, w) = \int d \log \frac{\tau_X + 1}{\tau_X} \Psi_L \left(\frac{u(\tau_X + w)}{\tau_X + uw}, v, \frac{w(\tau_X + 1)}{\tau_X + w} \right)$$

possible to get symbol to all loops w. 9 letters
 $\{u, v, w, 1-u, 1-v, 1-w, 1-uw, 1-vw, 1-u-v+uvw\}$

$$\Psi_{L+1}(u, v, w) = \int d \log(\tau_Y + 1) \Psi_{L+\frac{1}{2}} \left(u, \frac{v(\tau_Y + 1)}{v\tau_Y + 1}, \frac{\tau_Y + w}{\tau_Y + 1} \right)$$

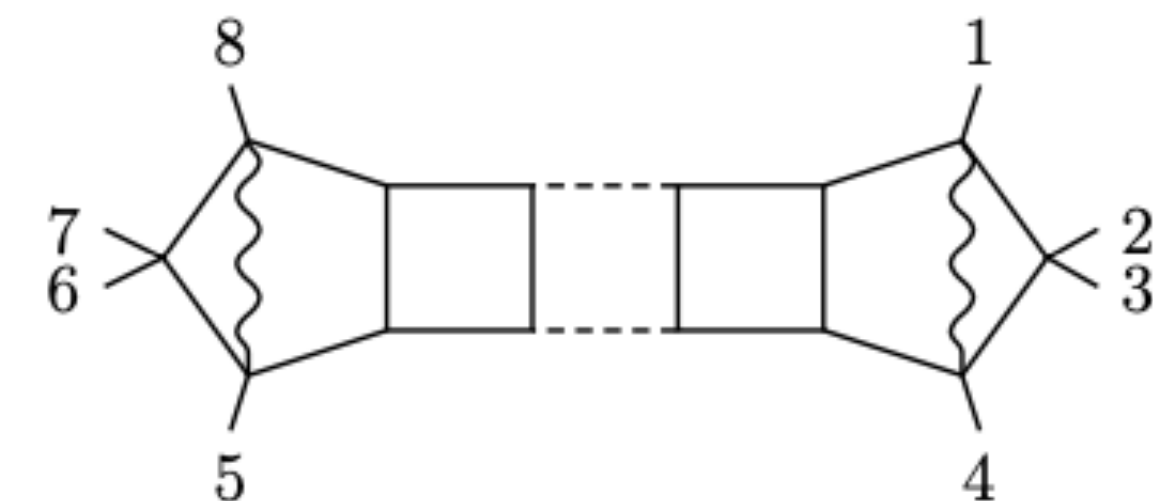
Generalized penta ladders [SH, Z. Li, Y. Tang, Q. Yang]

applicable to a large class of integrals w. “pentagon handles” (or similar 1-loop) → reduce loop orders in particular a recursion for “generalized penta ladders”: $2(L-1)$ -fold dlog of some 1-loop integrals

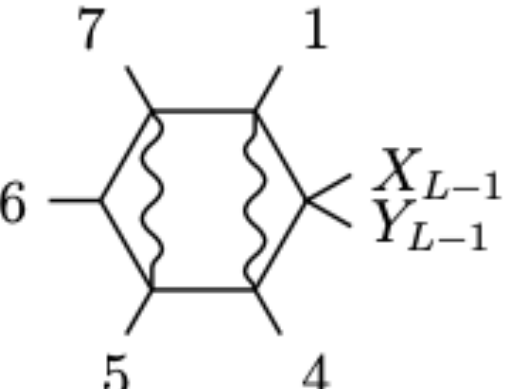
$$\text{Diagram} = \int_{\mathbb{R}_{\ge 0}^2} d \log \langle i-1ijY_1 \rangle d \log \frac{\langle iX_1jY_1 \rangle}{\tau_{X_0}} \times \text{Diagram}$$

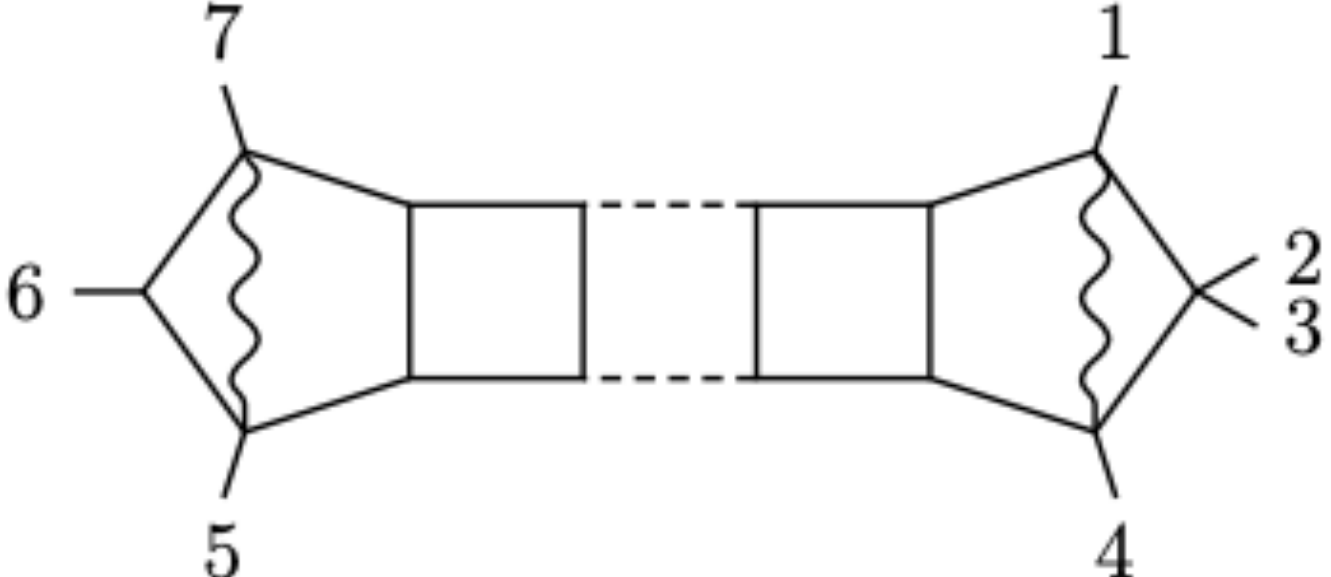
straightforward to obtain symbol if no square roots involved but need “rationalization” otherwise

e.g. $\Omega_L(1,4,5,8)$ involves square root $\Delta_{2,4,6,8} = \sqrt{(1-u-v)^2 - 4uv}$



focus on $\Omega_L(1,4,5,7)$: $2(L-1)$ fold d log integral of 1-loop (7-pt) hexagon

$$\Omega_L(1,4,5,7) = \int \prod_{a=1}^{L-1} d \log \langle 147Y_a \rangle d \log \frac{\langle 1X_a4Y_a \rangle}{\tau_{X_a}} \times \text{Diagram}$$


$$\Omega_L(1,4,5,7) = \text{Diagram}$$


$$u_1 = \frac{\langle 1245 \rangle \langle 5671 \rangle}{\langle 1256 \rangle \langle 4571 \rangle}, \quad u_2 = \frac{\langle 3471 \rangle \langle 4567 \rangle}{\langle 3467 \rangle \langle 4571 \rangle}, \quad u_3 = \frac{\langle 1267 \rangle \langle 3456 \rangle}{\langle 1256 \rangle \langle 3467 \rangle}, \quad u_4 = \frac{\langle 1234 \rangle \langle 4571 \rangle}{\langle 1245 \rangle \langle 3471 \rangle}.$$

beautiful DCI form: (rescaled) deformation

-> symbol to all loops w. 16 letters

$$u_1, u_2, u_3, u_4, 1 - u_1, 1 - u_2, 1 - u_3, 1 - u_4, \\ 1 - u_1 u_4, 1 - u_2 u_4, 1 - u_3 - u_1 u_4, 1 - u_3 - u_2 u_4; y_1, y_2, y_3, y_4$$

$$\Omega_{L+\frac{1}{2}}(u_1, u_2, u_3, u_4) = \int d \log \frac{\tau_X + 1}{\tau_X} \Omega_L \left(\frac{u_1(\tau_X + u_4)}{\tau_X + u_1 u_4}, u_2, \frac{\tau_X u_3}{\tau_X + u_1 u_4}, \frac{u_4(\tau_X + 1)}{\tau_X + u_4} \right), \\ \Omega_{L+1}(u_1, u_2, u_3, u_4) = \int d \log(\tau_Y + 1) \Omega_{L+\frac{1}{2}} \left(u_1, \frac{u_2(\tau_Y + 1)}{u_2 \tau_Y + 1}, \frac{u_3}{1 + \tau_Y u_2}, \frac{\tau_Y + u_4}{\tau_Y + 1} \right),$$

nicely, alphabet of D_4 cluster algebra [SH, Z. Li, Q. Yang] also appear for 6d 1-mass hexagon [Chicherin, Henn, Papathanasiou]

w. $u_3 \rightarrow 0$ back to the 9 letters of Ψ_L : sub-algebra $D_3 = A_3$

Cluster algebras & adjacency

Cluster algebras for integrals [SH, Z. Li, Q. Yang]

Not only (n=6,7) amplitudes, but a large class of integrals (beyond N=4 SYM) have alphabet of cluster algebra!

e.g. all-loop n=6 ladder integrals $\rightarrow A_3$, certain n=7 integral $\rightarrow E_6$, ...

Systematically studied in a general setting: [Chicherin, Henn, Papathanasiou] e.g. 4-pt with a off-shell leg:

C_2 “folded” from $A_3 : \{z_1, z_2, z_3, 1 - z_1, 1 - z_2, 1 - z_3\}$ ($z_1 + z_2 + z_3 = 1$) 2d harmonic multi-polylogs

various 1-loop integrals w. A, C, D cluster algebras & 5-pt alphabet from limit of $G(4,8)/T!$

Natural question: more evidence for higher n (still DCI) & higher loops? algebraic letters?

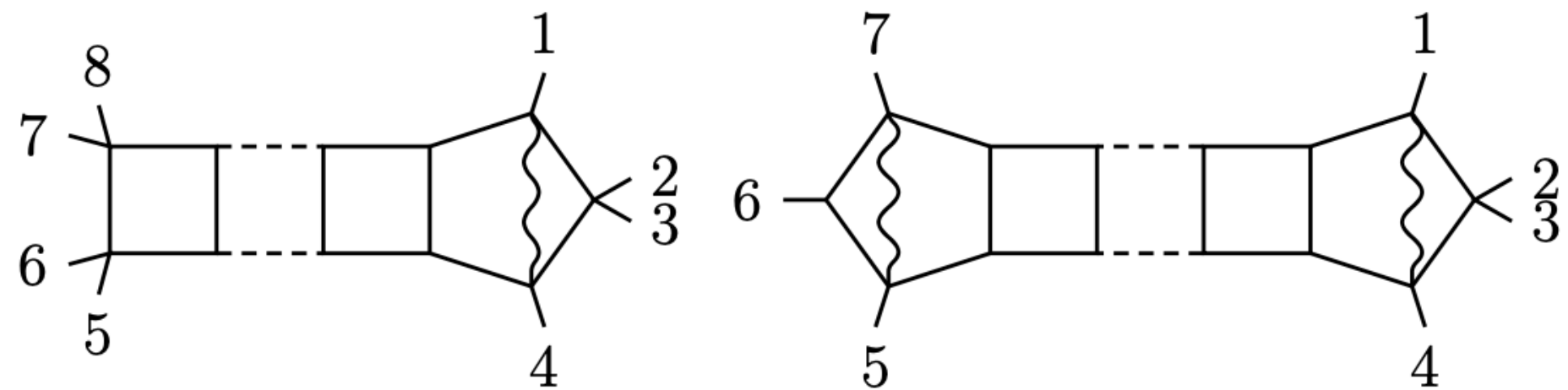
(trivial) example: 8-pt box ladders: $\{z, \bar{z}, 1 - z, 1 - \bar{z}\} \sim D_2 = A_1^2$

n=8 amplitudes/integrals in $\mathbb{R}^{1,1}$ kinematics: $C_2 = \text{overlap } A_2 + A_2 \{v, w, 1 + v, 1 + w, v - w, 1 - vw\}$

much simpler than 4d \rightarrow used to bootstrap octagons up to 3 loops! [w. Z. Li, Y. Tang, Q. Yang]

Observation: at least for ladder integrals, alphabet does not grow beyond e.g. 2 loops, independent of details (numerators...) → possibly fixed by kinematics (external dual points)!

e.g. 8-pt penta-box ladder $D_3 = A_3$ (not hexagon A_3), 7-pt double-penta ladder D_4 (emb. in heptagon E_6)



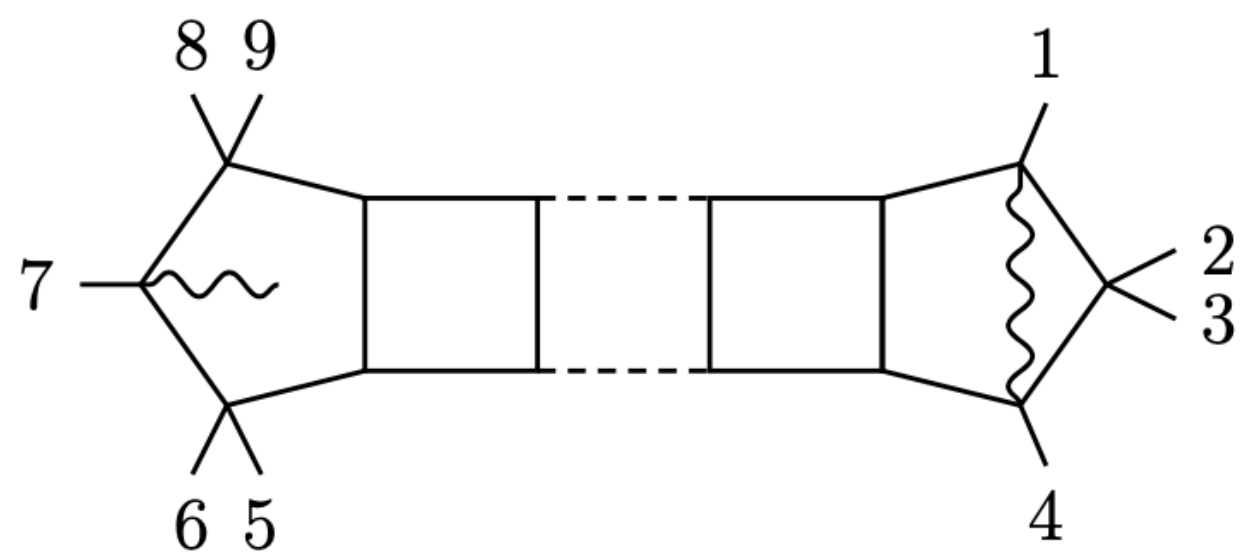
“good” variables: $z_1 = 1 - u^{-1}, z_2 = 1 - v^{-1}, z_3 = 1 - w$ for D_3

$$D_3 = \{z_1, z_2, z_3, 1 + z_1, 1 + z_2, 1 + z_3, z_1 - z_2, z_1 - z_3, z_1 + z_2 z_3\}$$

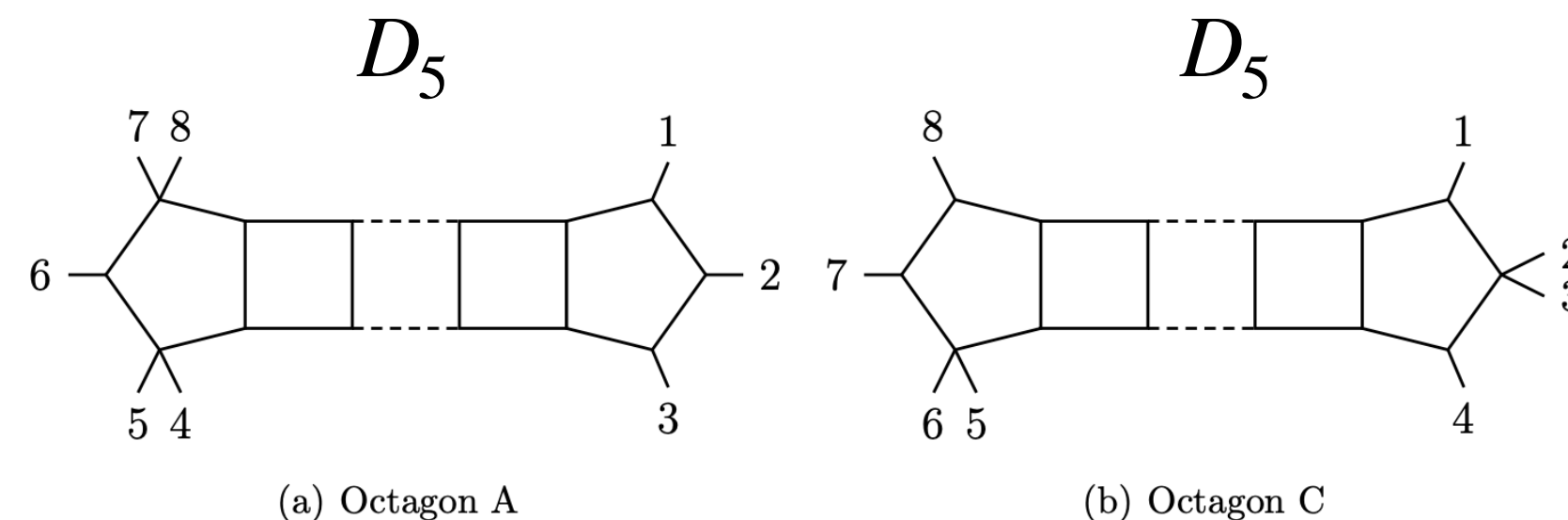
similarly for D_4 : introduce z_1, \dots, z_4 , same para. as [Chicherin, Henn, Papathanasiou] :

$$D_4 = \{z_1, \dots, z_4, 1 + z_1, \dots, 1 + z_4, z_1 - z_2, z_1 - z_3, z_1 - z_4, z_2 - z_3, z_2 - z_4, z_1 + z_3 z_4, z_2 + z_3 z_4, \dots\}$$

many more cases: D_6 ($n=9$) & deg. to D_5, D_4 etc.

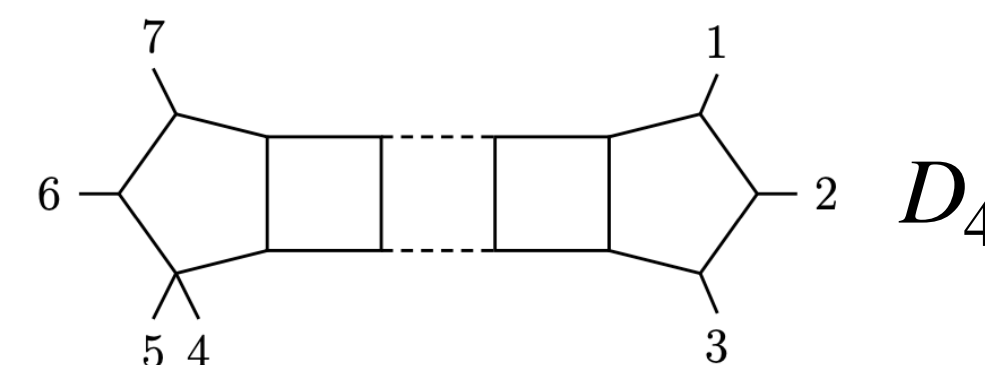


D_6



(a) Octagon A

(b) Octagon C



D_4

Cluster algebras from “kinematic quiver” [SH, Z. Li, Q. Yang]

A possible origin: (truncated) cluster algebra from mutating quivers associated with the kinematics

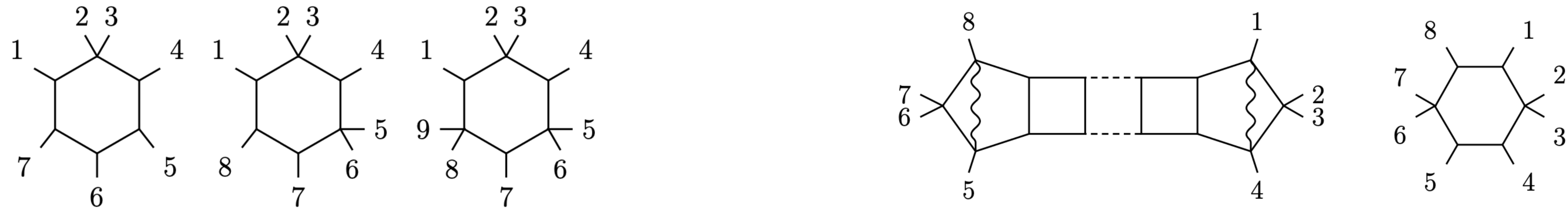
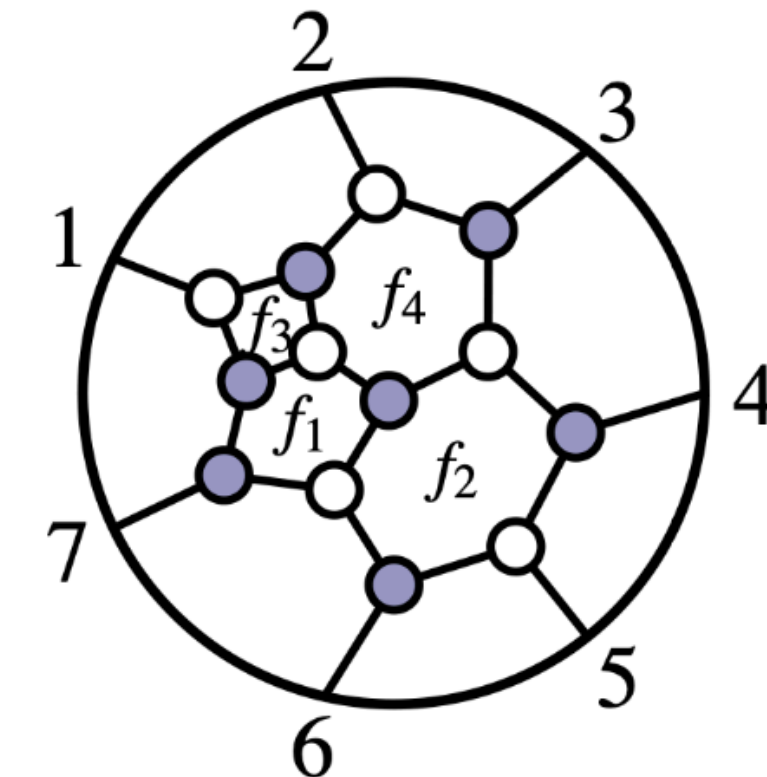
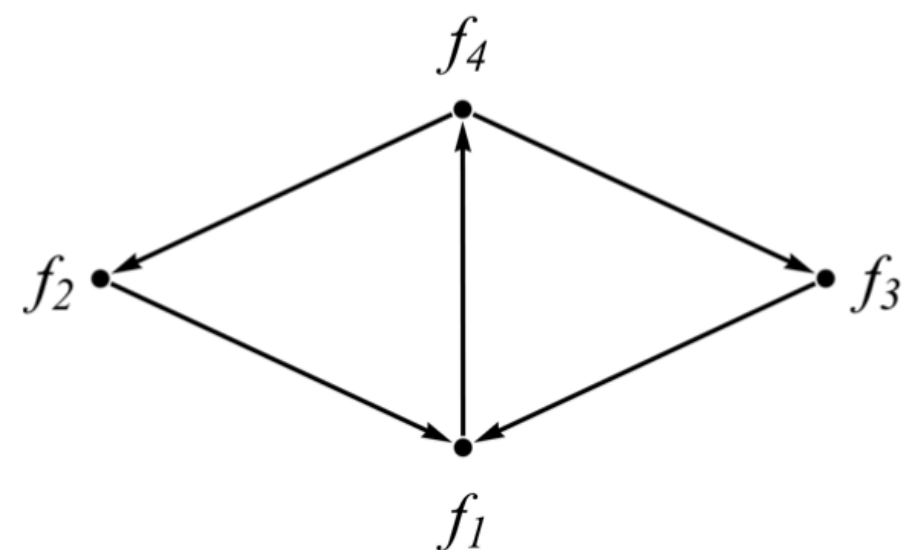
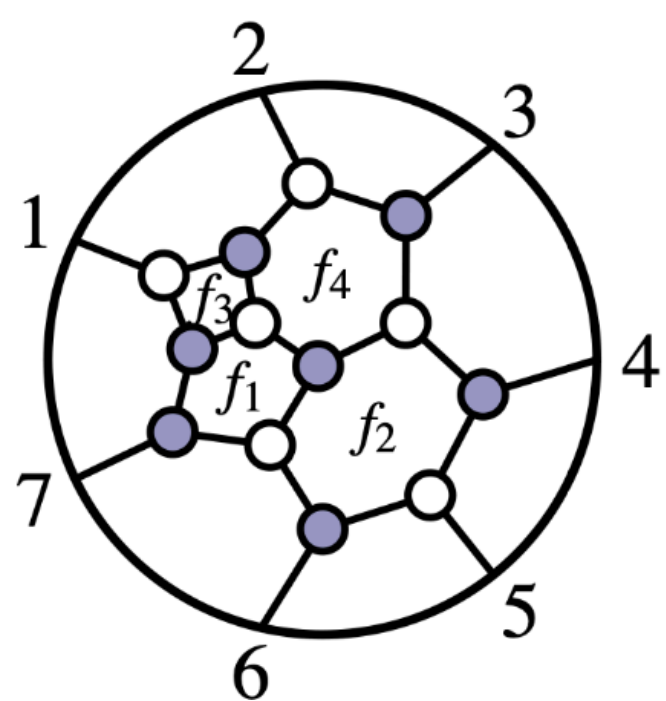


Figure 1: One-, two-, three-mass-easy hexagon kinematics with $n = 7, 8, 9$ legs

nicely, our cases correspond to lower-dim cell of $G_+(4, n)/T$ described by **plabic graph**

- Algorithm:**
1. find plabic graph for the kinematics (labelled by **face var.**)
 2. draw the **dual quiver**, apply mutations \rightarrow cluster algebras
 3. if it's finite type \rightarrow the **alphabet**, otherwise need truncation!
 4. similar to tropical $G(4, n)$ for $n > 8$: **Minkowski sum** from Pluckers etc.?

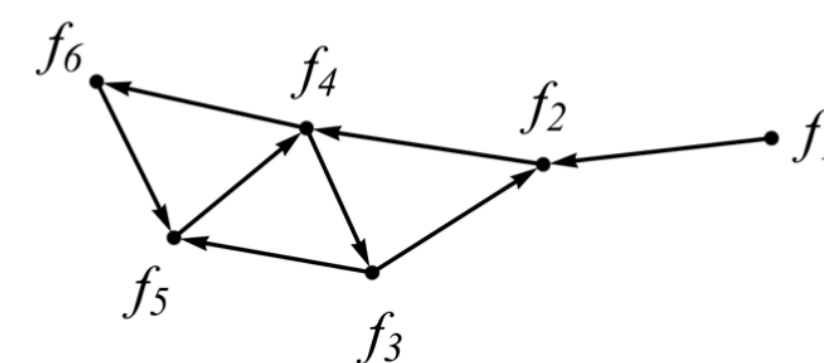
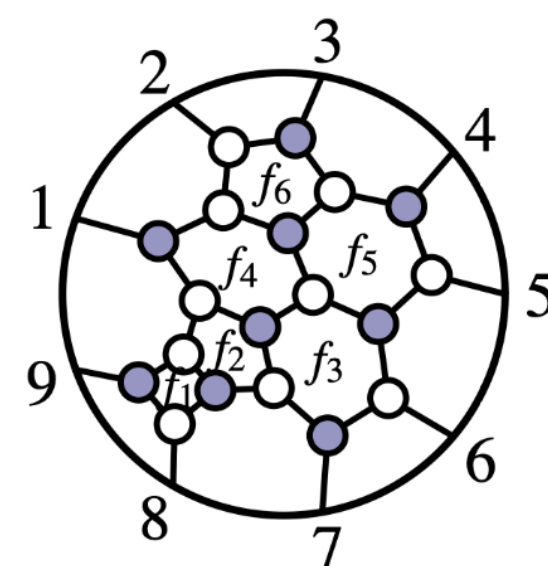
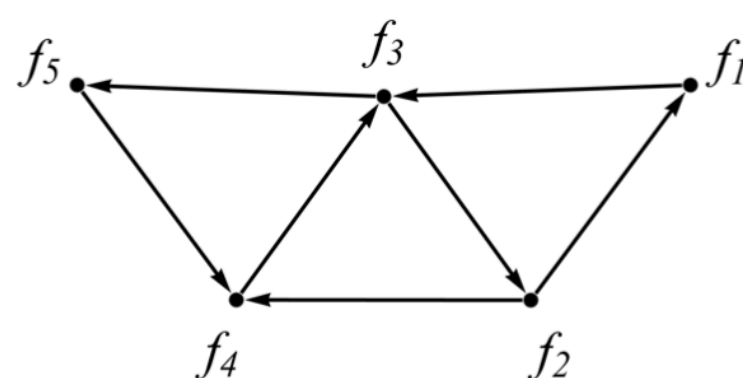
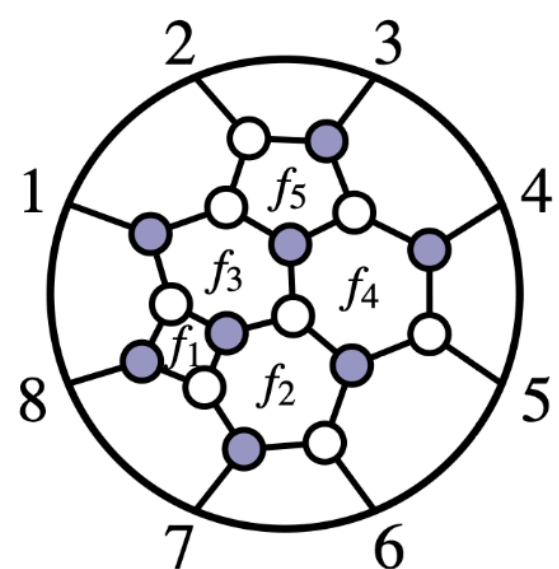




$$\begin{pmatrix} f_3 f_4 (1 + f_3) f_4 & 1 + f_4 + f_3 f_4 & 1 & 0 & 0 & 0 \\ 0 & f_1 f_2 f_4 & f_2 (1 + f_1 + f_1 f_4) & 1 + f_2 + f_1 f_2 & 1 & 0 & 0 \\ 0 & 0 & f_2 & 1 + f_2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

D4: 16 letters

$\{1 + f_1, 1 + f_2, 1 + f_3, 1 + f_4, 1 + f_2 + f_1 f_2, 1 + f_3 + f_1 f_3, 1 + f_1 + f_1 f_4, 1 + f_4 + f_2 f_4, 1 + f_4 + f_3 f_4, 1 + f_2 + f_3 + f_2 f_3 + f_1 f_2 f_3, 1 + f_4 + f_2 f_4 + f_3 f_4 + f_2 f_3 f_4 + f_1 f_2 f_3 f_4\}$



$$\begin{pmatrix} f_5 & f_5 & 1 + f_5 & 1 & 0 & 0 & 0 & 0 \\ 0 & f_1 f_2 f_3 f_4 f_5 & f_2 f_4 (1 + f_1 + f_1 f_3 + f_1 f_3 f_5) & 1 + f_2 + f_1 f_2 + f_2 f_4 + f_1 f_2 f_4 + f_1 f_2 f_3 f_4 & 1 + f_2 + f_1 f_2 & 1 & 0 & 0 \\ 0 & 0 & f_2 f_4 & 1 + f_2 + f_2 f_4 & 1 + f_2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

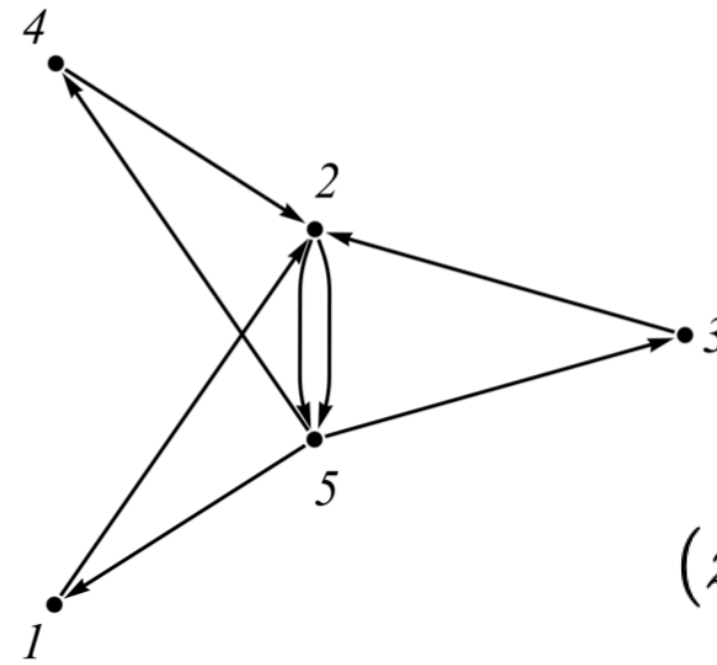
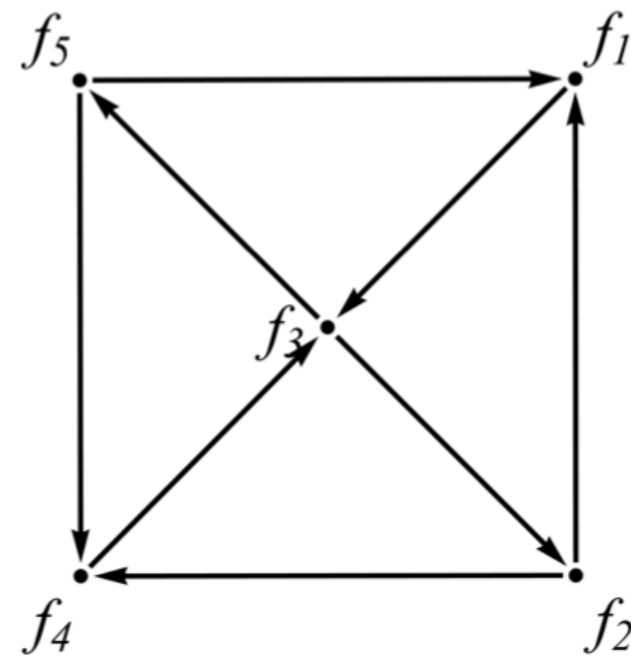
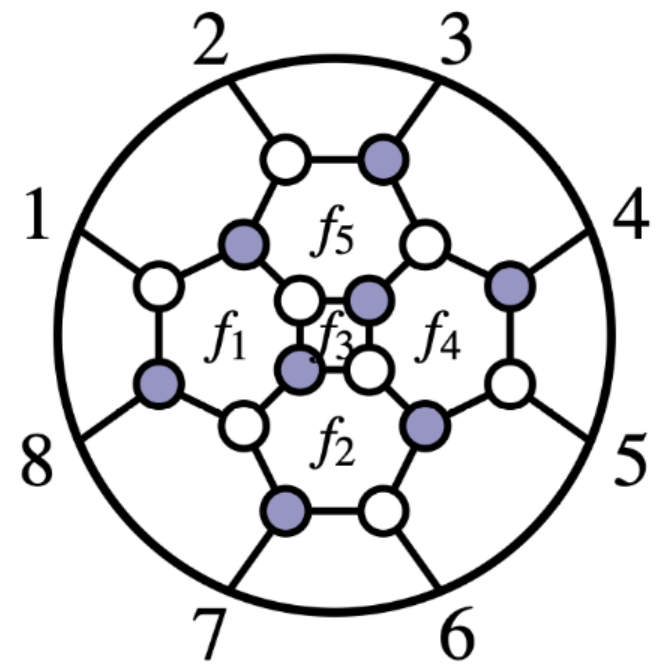
$$\begin{pmatrix} f_6 & f_6 & 1 + f_6 & 1 & 0 & 0 & 0 & 0 \\ 0 & f_1 f_2 f_3 f_4 f_5 f_6 & f_3 f_5 (1 + f_1 + f_1 f_2 + f_1 f_2 f_4 + f_1 f_2 f_4 f_6) & * & 1 + f_1 + f_3 + f_1 f_3 + f_1 f_2 f_3 & 1 + f_1 & 0 & -1 & 0 \\ 0 & 0 & f_3 f_5 & 1 + f_3 + f_3 f_5 & 1 + f_3 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

D5: 25 letters

D6: 36 letters

Bootstrap Feynman integrals [SH, Z. Li, Q. Yang]

For $\Omega_L(1,4,5,8)$: affine D4 infinite cluster algebra! natural truncation \rightarrow 38 cluster variables + 1 **limit ray**



5-dim space of algebraic letters

$$u = \frac{x_{24}^2 x_{68}^2}{x_{26}^2 x_{48}^2} = z\bar{z}, \quad v = \frac{x_{46}^2 x_{82}^2}{x_{26}^2 x_{48}^2} = (1-z)(1-\bar{z}).$$

$$(z-a)/(\bar{z}-a) \quad a=0, 1, \frac{\langle 1234 \rangle \langle 1568 \rangle}{\langle 1256 \rangle \langle 1348 \rangle}, \frac{\langle 1234 \rangle \langle 4578 \rangle}{\langle 1245 \rangle \langle 3478 \rangle}, \left(1 - \frac{\langle 1578 \rangle \langle 3456 \rangle}{\langle 1345 \rangle \langle 5678 \rangle}\right)^{-1}$$

Use this 38+5 alphabet \rightarrow bootstrap to weight 8 (without Steinmann): nicely locate integrals by DE!

conditions	# free parameters
weight-6 function space	3585
last entry	257
symmetry $z_3 \leftrightarrow z_4$	146
symmetry $z_1 \leftrightarrow z_2$	56
DE	3
boundary conditions	0

unique **algebraic building block** @ two-loop (weight-4):

$$\mathcal{S}_{2,4,6,8} := \mathcal{S}(F_{2,4,6,8}) \otimes \left(\frac{L_2 L_5}{L_1 L_3} \otimes z_1 + \frac{L_2 L_5}{L_1 L_4} \otimes z_2 + \frac{L_5}{L_1^2 L_3 L_4} \otimes z_3 + \frac{L_5}{L_1} \otimes z_4 + \frac{L_1^2 L_3 L_4}{L_2 L_5^2} \otimes z_5 \right) + \text{rational}$$

conjecturally **all-loop algebraic structure** (checked to 4 loops):

$$\sum_{i=1}^5 \mathcal{S}(F(2,4,6,8)) \otimes L_i \otimes \mathcal{S}(F_i)$$

Extended Steinmann relations universally [\[SH, Z. Li, Q. Yang\]](#)

Remarkably, ES relations hold for all known **finite** integrals, NMHV finite amps & MHV (w. subtraction)!!!

e.g. all ladder integrals (n=8,9, etc., up to 4 loops): cancellation between rational & algebraic letters

$$\text{Disc}_{x_{ij}^2=0}^s(\text{Disc}_{x_{kl}^2=0}^s(\mathcal{S}(F))) = 0 \quad \text{for any } 1 \leq s \leq w - 1 \text{ and any two overlapping channels } x_{ij}^2, x_{kl}^2, \text{ i.e. } (ij) \not\sim (kl).$$

similarly, first finite amplitudes (NMHV components) @ 2 loops satisfy ES relations for all n

after subtraction, even **3-loop MHV octagon** [\[Z. Li, C. Zhang\]](#) satisfies ES relations!

$$\mathcal{S}(E_8^{(3)}) = \mathcal{S}\left(R_8^{(3)} + \frac{F_8^3}{6} + R_8^{(2)} F_8\right)$$

can also check rational letters satisfy cluster adjacency of $G(4,8)$ using Sklyanin brackets [\[Golden et al\]](#)

possible to bootstrap for n=8 (with alphabet of ~200 rational letters + 18 algebraic ones):
easy to get 2-loop MHV octagon (no alg. letters), what about 3 loops etc.? [\[Z. Li et al, WIP\]](#)

Conclusions

- **Wilson loops** powerful for perturbative computations: amplitudes, Feynman integrals, ...
- **Yangian anomaly eqs** (\bar{Q} + parity): all-loop equations, higher loops? kinematic limits?
- **WL rep for integrals**: relation to diff. eqs & resum? apply to more integrals (IR divergent, mass etc.)
- **Steinmann cluster bootstrap**: extremely nice for $n=6,7$; extend to higher n with algebraic letters!
- **Mathematical structures**: cluster algebras, extended Steinmann/cluster adjacency \rightarrow general QFT?

Thank You!!!