Revisiting $O(N)$ sigma model at unphysical m_π and finite temperature

Zhiguang Xiao
Sichuan Univerisity
Collaborated with: Lyv Yuan-Lin, Li Qu-Zhi, Zheng Han-Qing

Based on: PRD,109,094026; arxiv:2405.11313

Talk given at USTC, 2024-10-25

2024 年 12 月 26 日

Motivation

- $\pi\pi$ scattering: important in understanding low energy QCD chiral symmetry breaking.
- Lattice calculation: phase shifts at unphysical pion mass, $m_\pi = 236$MeV[Dudek, et al., PRD86,034031(2012)]. $m_\pi = 391$MeV[Briceno,PRL118,022002(2017)].
- Hadron system with a finite temperature: Hadron gases, QCD phase information.
- These two aspects provide more information of QCD: tuning m_π and T, to learn more information about chiral symmetry, confinement ....
- By studyng the properties of resonances with varying m_π and T, one would also gain more understanding of the QCD.

Fundamentals on $\pi\pi$ scattering matrix

Full two particle scattering matrix: $S(s,t,u)$, $s+t+u=\sum_i m_i^2$.
$\pi\pi$ scattering: with isospin indices $S_{ab,cd}$

- Unitarity: $S \cdot S^\dagger = I$, $S = I + iT$. Optical theorem:

$$\mathrm{Im}\mathcal{M}(k_1,k_2 \to k_1,k_2) = 2E_{cm}p_{cm}\sigma_{\rm tot}$$

- Analyticity: $S(s,t,u)$ can be analytically continued to be an analytic function of s,t,u.
- Crossing: By analytic continuation, the same S matrix can describe different scatterings by crossing

$s > 4m_\pi^2$, $t,u<0$, s-channel, $\pi_a(p_1)+\pi_b(p_2)\to\pi_c(p_3)+\pi_d(p_4)$.
$t > 4m_\pi^2$, $s,u<0$, t-channel, $\pi_a(p_1)+\bar\pi_c(-p_3)\to\bar\pi_b(-p_2)+\pi_d(p_4)$.
$u > 4m_\pi^2$, $s,t<0$, u-channel, $\pi_a(p_1)+\bar\pi_d(-p_4)\to\pi_c(p_3)+\bar\pi_b(-p_2)$.

- Crossing and generalized Bose symmetry impose constraints between $A(s,t,u)$, $B(s,t,u)$, $C(s,t,u)$.

$$T(s,t,u) = A(s,t,u)\delta_{ab}\delta_{cd} + B(s,t,u)\delta_{ac}\delta_{bd} + C(s,t,u)\delta_{ad}\delta_{bc}$$

Fundamentals on $\pi\pi$ scattering matrix

$$T(s,t,u) = A(s,t,u)\delta_{ab}\delta_{cd} + B(s,t,u)\delta_{ac}\delta_{bd} + C(s,t,u)\delta_{ad}\delta_{bc}$$

Scattering of total angular momentum eigenstates and total Isospin eigenstates:

- Isospin projection:

$$\begin{aligned} T_s^0(s,t,u) &= 3A(s,t,u) + B(s,t,u) + C(s,t,u) \\ T_s^1(s,t,u) &= B(s,t,u) - C(s,t,u) \\ T_s^2(s,t,u) &= B(s,t,u) + C(s,t,u) \end{aligned}$$

- Partial wave projection: $S_l = 1 + 2i\rho(s)\,T_l(s)$, $\rho = \sqrt{(s-4m_\pi^2)/s}$,

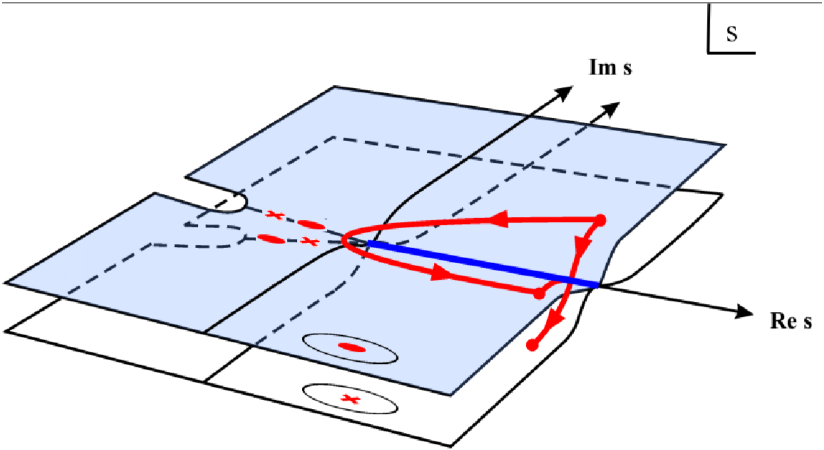
$$\begin{aligned} T_l^I(s) =& \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, P_l(\cos\theta)\, T^I(s, t(s,\cos\theta)) \\ T(s,t,u) =& 16\pi \sum_l (2l+1) P_l(\cos\theta)\, T_l^I(s) \end{aligned}$$

FUNDAMENTALS ON $\pi\pi$ SCATTERING MATRIX

- Unitarity: Single channel, for physical processes, $s > 4m_\pi^2$,

$$S_l(s)S_l^*(s) = 1, \quad \mathrm{Im}\, T_l(s) = \rho(s)|\,T_l(s)|^2.$$

- Analyticity: $S(s) \to$ analytical continuation, real analytic function of s, Schwartz reflection, $S^*(s) = S(s^*)$.
  Single channel: Double-sheeted Riemann surface, a cut above the threshold $s = 4m_\pi^2$ — unitarity cut, or physical cut.
  - $S^I = 1/S^{II}$: Zero point on the first sheet $\leftrightarrow$ poles on the second sheet.
  - On the first sheet:
    Bound state poles: on the real axis below the threshold $4m_\pi^2$.
    No other poles.
  - On the second sheet:
    Virtual state poles on the real axis below the threshold $4m_\pi^2$.
    Resonances poles, a pair of complex poles on the second sheet, symmetric w.r.t real axis.
- Crossing: The physical process of the t, u channel generate left-hand cut.


S
Im s
Re s

Chiral symmetry and σ state

2 flavor QCD: only u and d quarks,

- Chiral symmetry at chiral limit, $m_q = 0$:
  $SU_L(2) \times SU_R(2) \simeq O(4)$.

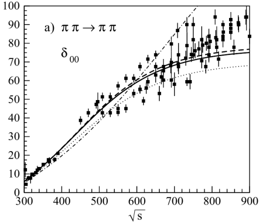
$$q_L \to e^{i\alpha_L^a \tau^a} q_L, \quad q_R \to e^{i\alpha_R^a \tau^a} q_R.$$

- SSB: quark condensate $\langle \bar{q}q \rangle \neq 0$,
  $SU_L(2) \times SU_R(2) \to SU_V(2) \to$ Goldstone: pions.
- With explicit chiral symmetry breaking: $m_q \neq 0$, $m_\pi \neq 0$,
  PCAC assumption, $\partial_\mu A^{a,\mu} = m_\pi^2 f_\pi \pi^a$.
- To realize this SSB in Low energy effective field theory:
  Linear realization $\to$ linear σ model, $\langle \sigma \rangle \neq 0 \to$ a 0^{++} scalar state. [M. Gell-Mann and M. Lévy,Nuo.Cim.16,705(1960)]
  Nonlinear realization $\to$ only pion d.o.f., no σ, ChPT.
  [Coleman,Wess,Zumino, PR,177,2239;Callan,et.al. PR177,2247;
  Gasser,Leutwyler,Ann.Phys.158,142, NPB250,465.]

$f(500)$ STATE: THE LOWEST 0^{++} STATE

- For a long time, the existence of the σ particle was in controversy.

  A broad resonance disappeared and reappeared from PDG table several times, from 1960 to 2000s.

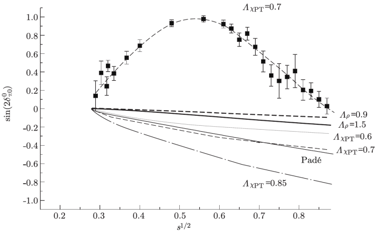


- Chiral purtabation theory (CHPT): without sigma, can reproduce the lower energy scattering length, effective range, phase shift near threshold. Quickly blow up away from threshold.
- Unitarized CHPT: IAM, a broad resonance was found — σ or f_0, but not quite reliable [Pelaez,Mod.Phy.Lett.A19,2870; A.G.Nicola,Pelaez,PRD65,054009(2002)].

σ STATE

- PKU representation with left-hand cut estimated by CHPT [ZX,Zheng,NPA695,273(2001);Z. Y. Zhou,et.al. JHEP 02, 043(2004);Zheng, et.al., NPA733,235(2004)]: respects unitarity and analyticity, crossing can be imposed by BNR relation

  Left-hand-cut contribution is always negative in $\sin 2\delta$. A broad resonance pole is needed.



- Roy equation [Colangelo,Gasser,Leutwyler NPB603,125(2001)]: model independent integral equation.
  Automatic respect unitarity, analyticity, crossing symmetry — established the existence of a broad 0^{++} resonance $f_0(500)$ [Caprini,Colangelo,Leutwyler PRL96,132001(2006)].

$$\sqrt{s_{\text{pole}}} = 441^{16}_{-8} - i272^{+9}_{-13}\text{MeV}$$

- What role does it play in the chiral SSB? Is this the sigma in LSM?

Analyzing the lattice QCD phase shift: K matrix, no crossing

$m_\pi = 236$MeV[Dudek, et al., PRD86,034031(2012)], σ resonance

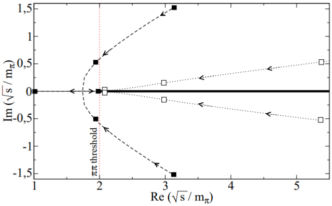
$m_\pi = 391$MeV[Briceno,PRL118,022002(2017)], σ bound state.

- PKU + crossing using BNR relation: [X. L. Gao,Z.H.Guo,ZX,ZZ, PRD 105 (2022)9,094002]

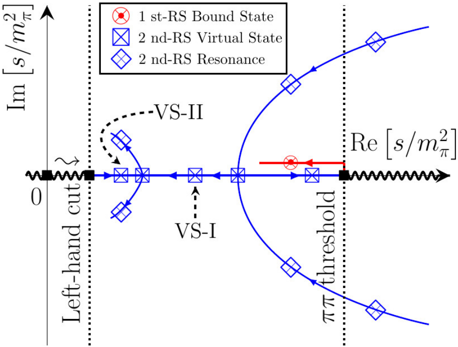
at $m_\pi = 236$MeV:$m_\sigma = 610 \pm 11$MeV,$\Gamma_\sigma = 327 \pm 8$MeV;

at $m_\pi = 391$MeV: $m_{bound} = 774 \pm 6$MeV,$m_{virtual} = 716 \pm 28$MeV

- ChPT+unitarization: [Hanhart, et al. PRL100,152001(2008)]

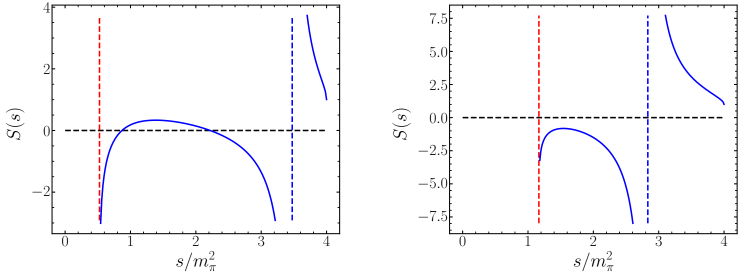


- Roy equation:[X.H. Cao, et al.,PRD108(2023)3,034009]
  at $m_\pi = 236$MeV: $\sqrt{s_\sigma} = 543 - i250$ MeV
  at $m_\pi = 391$MeV: $m_{bound} = 759$MeV, $\sqrt{s_{sub}} = (269 - i211)$ MeV
  Propose the pole trajectory:



- More recent Lattice result + K matrix [Rodas et al., PRD108,034513(2023)]:
  at $m_\pi = 330$MeV: σ is a bound state.
  at $m_\pi = 283$MeV: large statistical error, could be a virtual state or resonance.

The proposed explanation of the subthreshold resonance at $m_\pi = 391\text{MeV}$:



The reason for the appearence of the subthreshold resonance poles:

- When the VS move through the threshold and becomes a bound state, the branch point of the l.h.c. move right.
- From the positivity of the residue of the bound state pole, it can be proved that near the branch point $S(s)$ tends to negative infinty.
- For small m_π, there are two zero point of $S(s)$ below threshold. For larger m_π, $S(s)$ become smaller, no zero points $\rightarrow$ resonance pole.

Introduction: $O(N)$ sigma model

- Lagrangian: $a = 1, \ldots, N$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}\mu_0^2 \phi_a \phi_a - \frac{\lambda_0}{8N}(\phi_a \phi_a)^2 + \alpha \phi_N \,,$$

- Classical level: when $\alpha = 0$, No explicit breaking
  $\mu^2 > 0$, no SSB;
  $\mu^2 < 0$, SSB. $\phi^2 = -2\mu^2 N/\lambda \equiv \langle \phi \rangle^2, \langle \phi \rangle > 0$
- Classical level: when $\alpha \neq 0$, vacuum solution

$$(-\mu_0^2 - \frac{\lambda_0}{4N}|\phi|^2)\phi_a = 0 \Rightarrow \phi_a = 0, \quad \text{for } a = 1, \ldots, N-1$$

$$(-\mu_0^2 - \frac{\lambda_0}{4N}|\phi|^2)\phi_N + \alpha = 0$$

$\langle \phi_N \rangle \neq 0 \sim \mathcal{O}(N^{1/2})$, $\alpha \sim \mathcal{O}(N^{1/2})$.

- To count the N order: Introduce an auxiliary field χ,

  [Coleman,et.al.,PRD10,2491]

$$\begin{aligned}\mathcal{L} \to& \mathcal{L} + \frac{N}{2\lambda_0}\left(\chi - \frac{\lambda_0}{2N}\phi_a\phi_a - \mu_0^2\right)^2 \\ =& \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \alpha\phi_N + \frac{N}{2\lambda_0}\chi^2 - \frac{1}{2}\chi\phi_a\phi_a - \frac{N\mu_0^2}{\lambda_0}\chi\,,\end{aligned}$$

  Integrating out χ, it come back to the previous path integral.

- After renormalization, effective potential:

$$V(\phi,\chi) = -\alpha\phi_N + \frac{1}{2}\chi\phi^2 + \frac{N\mu^2(M)}{\lambda(M)}\chi - \frac{N}{64\pi^2}\chi^2\left(\log\frac{M^2}{\chi} + \frac{1}{2}\right)\,,$$

$\frac{\partial V}{\partial \chi} = 0$ and $\frac{\partial V}{\partial \phi_a} = 0$

$$\frac{\partial V}{\partial \chi} = 0 \Rightarrow \quad \phi_a \phi_a = \frac{2N}{\lambda}\chi - \frac{2N\mu^2}{\lambda} - \frac{N}{16\pi^2}\chi \log \frac{\chi}{M^2}\,,$$
$$\frac{\partial V}{\partial \phi_a} = 0 \Rightarrow \quad \chi\phi_a = 0 \ (a < N), \quad \chi\phi_N - \alpha = 0\,.$$

- π mass $m_\pi^2 = \chi$.
- For $N = 4$, compared with PCAC: $\partial^\mu A_\mu^a = m_\pi^2 f_\pi \pi^a$, and with explicit SSB, $\partial^\mu A_\mu^a = \alpha\pi^a$. From $\chi = \frac{\alpha}{\phi_N} = m_\pi^2$, we find

$$\langle \phi_N \rangle = f_\pi \sim \mathcal{O}(N^{1/2}).$$

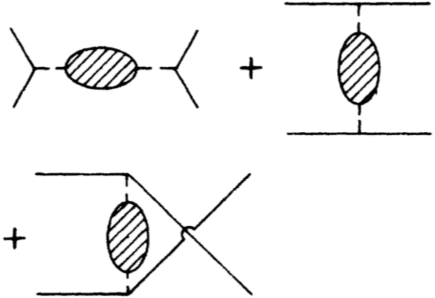


图: $O(N)$ model leading order $\pi\pi$ amplitude [Coleman,et.al.,PRD10,2491]

$\pi\pi$ amplitude to the leading $1/N$ order: (σ, τ have mixing)

$$\mathcal{T}_{\pi_a\pi_b\to\pi_c\pi_d} = iD_{\tau\tau}(s)\delta_{ab}\delta_{cd} + iD_{\tau\tau}(t)\delta_{ac}\delta_{bd} + iD_{\tau\tau}(u)\delta_{ad}\delta_{bc}\,, \tag{1}$$

$$D^{-1}(p^2) = -\,i\begin{pmatrix} p^2 - m_\pi^2 & -f_\pi \\ -f_\pi & N/\lambda_0 + NB_0(p^2, m_\pi) \end{pmatrix}\,, \tag{2}$$

$$\tau \equiv \chi - \langle\chi\rangle, \quad B_0(p^2, m_\pi) = \frac{-i}{2}\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{(\ell^2 - m_\pi^2 + i\epsilon)((\ell+p)^2 - m_\pi^2 + i\epsilon)}\,.$$

After projection to $IJ = 00$ channel, leading $\mathcal{O}(N)$ amplitude $\mathcal{T}_{00}^{LO}$ (define M such that $1/\lambda(M) = 0$), only s channel contributes

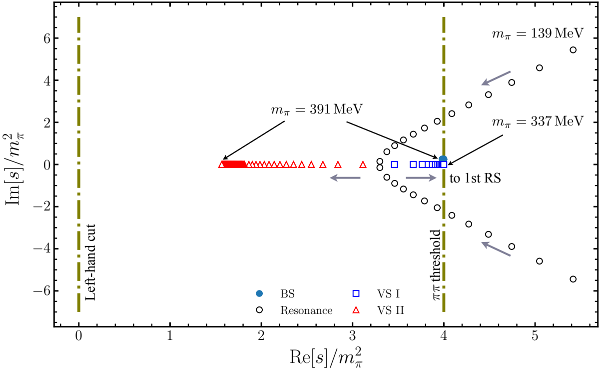
$$
\begin{aligned}
\mathcal{T}_{00}^{LO}(s) =& \frac{iND_{\tau\tau}(s)}{32\pi} = \frac{N}{32\pi}\mathcal{A}(s), \\
\mathcal{A}(s) =& \frac{m_\pi^2 - s}{(s - m_\pi^2)B(s, m_\pi, M) - f_\pi^2/N}, \\
B(s, m_\pi, M) =& \frac{1}{32\pi^2}\left(1 + \rho(s)\log\frac{\rho(s) - 1}{\rho(s) + 1} - \log\frac{m_\pi^2}{M^2}\right),
\end{aligned}
$$

- Adler zero: $s_A = m_\pi^2$.
- σ pole : solve

$$
(s - m_\pi^2)B^{II}(s, m_\pi, M) - f_\pi^2/N = 0\,.
$$

- No crossing symmetry.

m_π DEPENDENCE OF σ AT LEADING $1/N$



Fix f_π, M, changing m_π or α,

- Adjust the $M \sim 550 MeV$, at $m_\pi = 139 MeV$, $\sqrt{s_\sigma} = 356 - i148 MeV$.
- As m_π increases, $\sqrt{s_\sigma}$ moves $\rightarrow$ real axis $\rightarrow$ two virtual state poles $\rightarrow$ one virtual move left, the other move right $\rightarrow$ ($m_c \simeq 337\,\mathrm{MeV}$) one virtual, one bound
- No crossing symmetry: $\sigma \rightarrow$ bound state, no l.h.c brach point $\rightarrow$ $s = 4m_\pi^2 - m_\sigma^2$.
- Direct adding t and u channel contribution: violate unitarity.
- Unitarization method: IAM or K matrix, No control of the spurious poles, we resort to N/D.

N/D: unitarity with partial recovery of crossing

N/D Method: Basic ideas

- T matrix: $T(s) = \frac{N(s)}{D(s)}$: $N(s)$ only has left-hand cut, D only has right-hand cut.
- Since $\mathrm{Im}_R\, \mathcal{T}^{-1} = -\rho$, we have

$$\mathrm{Im}_R\, D(s) = -\rho(s) N(s)\,, \tag{3}$$

$$\mathrm{Im}_L\, N(s) = D(s)\, \mathrm{Im}_L\, \mathcal{T}(s)\,. \tag{4}$$

- Write down the dispersion relation of $N(s)$ and $D(s)$ (twice subtracted)

$$D(s) = \frac{s-s_A}{s_0-s_A} + g_D \frac{s-s_0}{s_A-s_0} - \frac{(s-s_0)(s-s_A)}{\pi} \int_R \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)(s'-s_A)} \mathrm{d}s'\,, \tag{5}$$

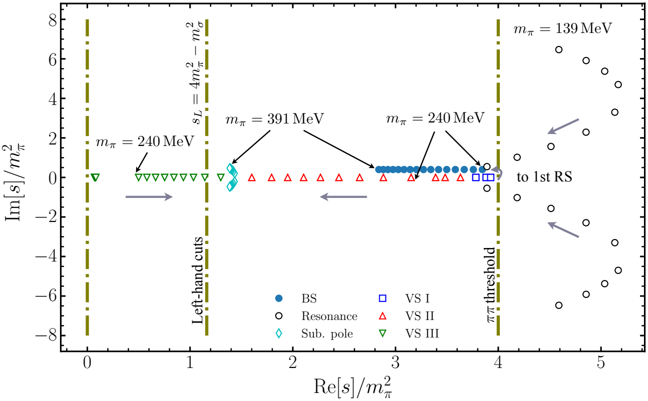
$$N(s) = b_0 \frac{s-s_A}{s_0-s_A} + g_N \frac{s-s_0}{s_A-s_0} + \frac{(s-s_0)(s-s_A)}{\pi} \int_L \frac{D(s')\mathrm{Im}_L \mathcal{T}(s')}{(s'-s)(s'-s_0)(s'-s_A)} \mathrm{d}s'\,, \tag{6}$$

- Using the $O(N)$ result $\mathrm{Im}_L \mathcal{T}(s)$: from next to leading $1/N$ order.
- Subtraction, $s_0 = 4m_\pi^2$, $s_A = m_\pi^2$, $D(s_0) = 1$: require $\mathcal{T}$ to recover $O(N)$ result at the leading $1/N$ order

$$b_0 = \mathcal{T}_{00}(s_0) = \frac{N-1}{32\pi}\mathcal{A}^{LO}(s_0) + I_{tu}(s_0)\,, \tag{7}$$

$$g_D = \frac{32\pi f_\pi^2 b_0}{N(s_0 - s_A)}\,, \qquad \frac{g_N}{g_D} = \mathrm{Re}\, I_{tu}(s_A)\,. \tag{8}$$

- When $m_\pi > m_c$, we require $D(m_\sigma^2) = 0$, for the sigma pole to be consistent with the branch point of the left hand cut.
- Solve the integral equation numerically.



- As m_π increases:

  σ approaches the real axis$\rightarrow$ two virtual states, $\rightarrow$

  one moving right $\rightarrow$ first sheet bound state

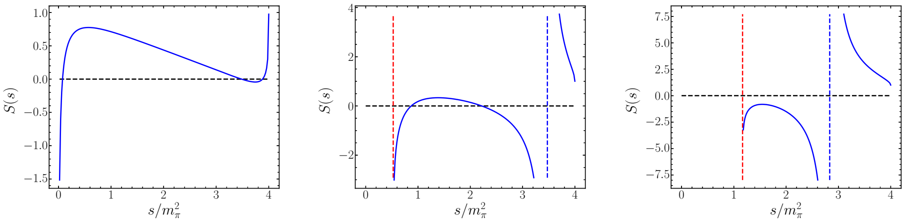
  one moving left

  l.h.c. $\rightarrow$ another virtual state

  hit $\rightarrow$ resonant poles

- The VSIII is generated when the Adler zero hit the left-hand cut

σ POLE TRAJECTORY WITH VARYING m_π

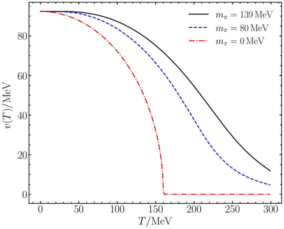


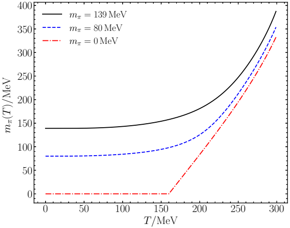
Physical-sheet S-matrix below threshold

- The left graph ($m_\pi = 207$ MeV) : two virtual states in the near threshold region and one additional virtual state pole generated close to the left-hand cut.
- The middle graph ($m_\pi = 283$ MeV) : one bound state with two virtual states.
- The right graph ($m_\pi = 391$ MeV): two virtual state poles have become a pair of resonance poles.

Temperature dependence at leading $1/N$

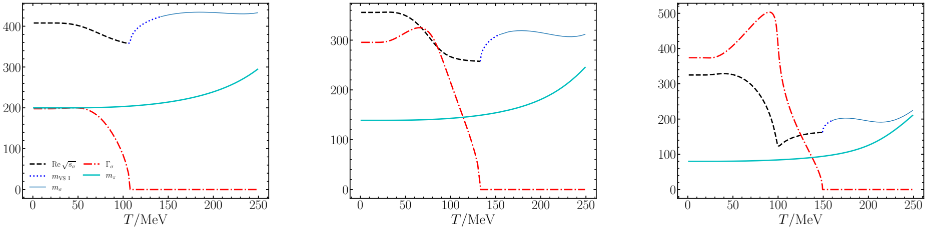
- It is well known: under high temperature, chiral symmetry recovers.[R. D. Pisarski and F. Wilczek, PRD 29,338(1984);A. Bazavov et al., PRD85,054503(2012)] $\lim_{T\to\infty} v(T) \to 0$
- It is expected that $m_\sigma \to m_\pi$ at high temperature.
- ChPT: intrincically in broken phase, break down at high temperature, $T \sim f_\pi$.
- In $O(N)$ model: v.e.v. $v(T)$ evolves with T,[J.O.Anderson, et. al. PRD70,116007]





- No explicit breaking $\alpha = 0$:at $T < T_c \sim 160\mathrm{MeV}$, $m_\pi(T) = 0$, $v(T) \neq 0$ SSB; at $T > T_c$, $v(T) = 0$, $m_\pi(T) \neq 0$.
- With $\alpha \neq 0$, $v(T) \to 0$.

At the leading $1/N$ order, $N = 4$:



From left to right, $m_\pi(0) = 200$, 139 and 80 MeV respectively.

- Scattering amplitude at T in the center of mass frame.

$$\mathcal{T}_{00}^{T}(s) = -\frac{1}{32\pi}\,\frac{s - m_\pi^2(T)}{\left(s - m_\pi^2(T)\right)\; B^T\left(s, m_\pi(T), M\right) - v^2(T)/N}\,,$$

$$B^T\left(s, m_\pi(T), M\right) = B\left(s, m_\pi(T), M\right) + B^{T\neq 0}\left(s, m_\pi(T)\right)\,,$$

$$B^{T\neq 0}\left(s, m_\pi(T)\right) = \int_0^\infty \frac{\mathrm{d}k\,k^2}{8\pi^2\omega_k^2}\, n_B(\omega_k)\left(\frac{1}{E + 2\omega_k} - \frac{1}{E - 2\omega_k}\right)\,,$$

- σ resonance on the second sheet, $\rightarrow$ virtual state, $\rightarrow$ bound state $\rightarrow$ tends to m_π.

N/D WITH TEMPERATURE

- Unitarity with two particle intermediate states for $IJ = 00$ channel,

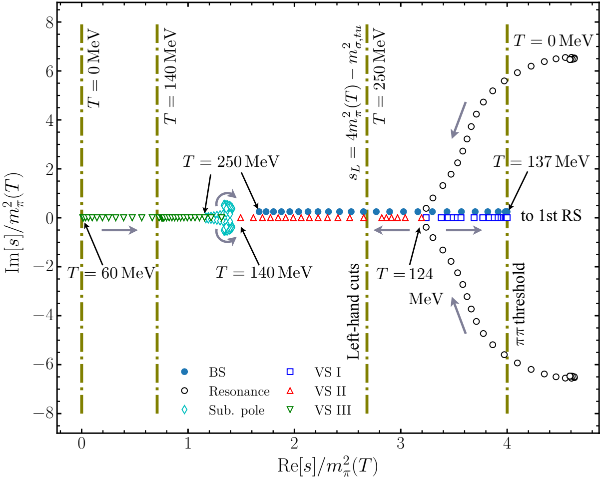
$$\operatorname{Im} \mathcal{T}^{T}(s) = \rho^{T}(s) \left|\mathcal{T}^{T}\right|^{2} , \tag{9}$$

- Lorentz symmetry is broken by the temperature: Center of mass system in s channel is different from t channel — crossing is also broken.
- $IJ = 00$ thermal amplitude:

$$\begin{aligned}
\mathcal{T}^{T}_{00}(s) &= -\frac{1}{32\pi} \frac{s - m_\pi^2(T)}{\left(s - m_\pi^2(T)\right) B^{T}\left(s, m_\pi(T), M\right) - v^2(T)/N} , \\
B^{T}\left(s, m_\pi(T), M\right) &= B\left(s, m_\pi(T), M\right) + B^{T\neq 0}\left(s, m_\pi(T)\right) , \\
B^{T\neq 0}\left(s, m_\pi(T)\right) &= \int_0^\infty \frac{\mathrm{d}k\, k^2}{8\pi^2 \omega_k^2} n_B(\omega_k) \left(\frac{1}{E + 2\omega_k} - \frac{1}{E - 2\omega_k}\right) ,
\end{aligned}$$

- N/D can be done: substitute the corresponding temperature dependent amplitudes.

N/D WITH TEMPERATURE: σ TRAJECTORY



- $T = 0,\ m_\pi = 139$MeV.
- $T = 60$MeV, VSIII generated from the l.h.c.
- σ resonance $\to$ virtual states (I, II), (T=124MeV)
- VS1$\to$ BS (T=137MeV),
- $T = 140$MeV, VSII meets VSIII$\to$ subthreshold resonance.
- $m_\sigma^2 \to m_\pi^2$.

Summary I:

- The σ ($f_0(500)$) pole trajectory with varying m_π, can be qualitatively reproduced by the N/D improved $O(N)$ linear sigma model.
- A pair of subthreshold pole is generated with crossing symmetry taken into account for large m_π.
- The $f_0(500)$ could be the σ in the linear σ model.
- The pole trajectory with varying temperature is similar to the one with varying m_π.

Vacuum structure: m_π dependence

Solve $\chi(\phi)$, insert into $V(\phi,\chi)$, to obtain the effective potential $V_{eff}(\phi)$.

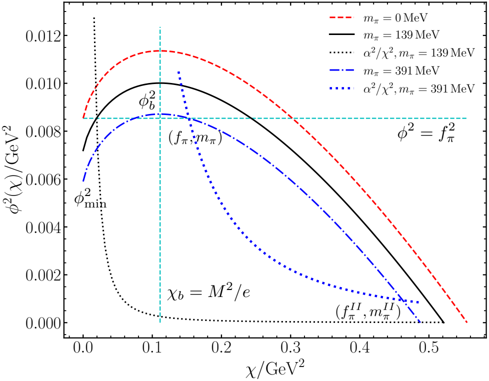
$$V(\phi,\chi) = -\alpha\phi_N + \frac{1}{2}\chi\phi^2 + \frac{N\mu^2(M)}{\lambda(M)}\chi - \frac{N}{64\pi^2}\chi^2\left(\log\frac{M^2}{\chi} + \frac{1}{2}\right),$$

$$\frac{\partial V}{\partial\chi} = 0 \Rightarrow \quad \phi^2 = f_\pi^2 + \frac{N}{16\pi^2}\left(m_\pi^2\log\frac{m_\pi^2}{M^2} - \chi\log\frac{\chi}{M^2}\right),$$

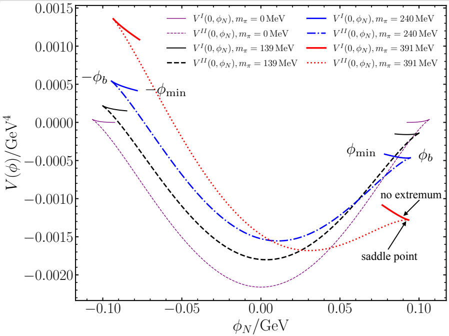
$$\frac{\partial V}{\partial\phi_a} = 0 \Rightarrow \quad \chi\phi_a = 0\ (a<N), \quad \chi\phi_N - \alpha = 0\,.$$

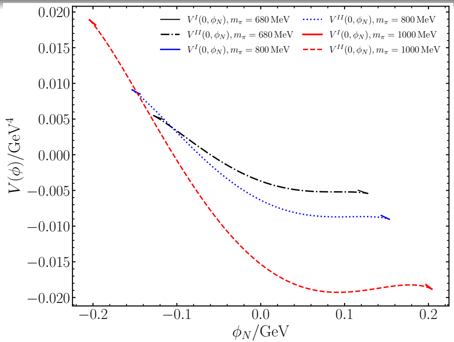
Fix f_π, M, changing m_π or α,

- Two solutions branches: separated at χ_b
- Left one: With Chiral SSB in the Chiral limit, determine f_π (fixed), m_π.
- Right one : No chiral SSB in the Chiral limit.
- V become complex for $|\phi|^2 > \phi_b^2$: the system not stable. $\phi^2 < \phi_{min}^2$.



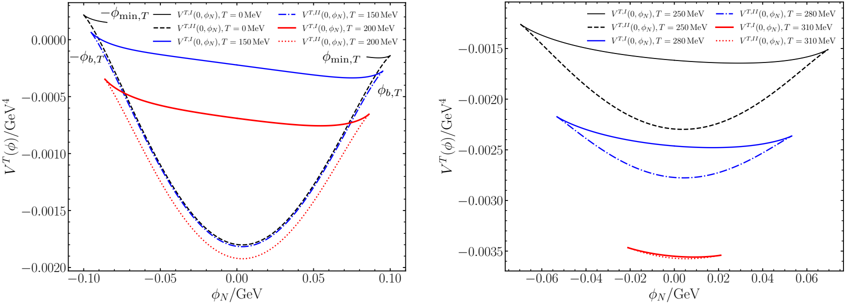
Two branches of the vacuum



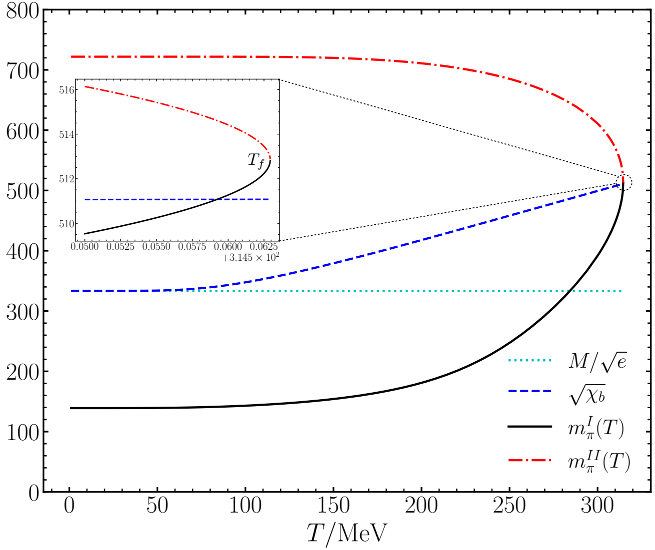


- $m_\pi^2 < \chi_b \sim 333\mathrm{MeV}$, solution I: the local minimum on the first branch, false vacuum. There is a tachyon.
- Global minimum: Solution II on the second branch.
- As m_π increases, solution I moves towards the second branch.
- $m_\pi^2 > \chi_b$, no local minimum on the first branch. Solution I moves on the second branch $\rightarrow$ saddle point.
- $m_\pi > (32\pi^2 f_\pi^2/(Ny_0))^{1/2} \sim 680\mathrm{MeV}$: solution I $\leftrightarrow$ Solution II.

Vacuum structure: Finite temperature



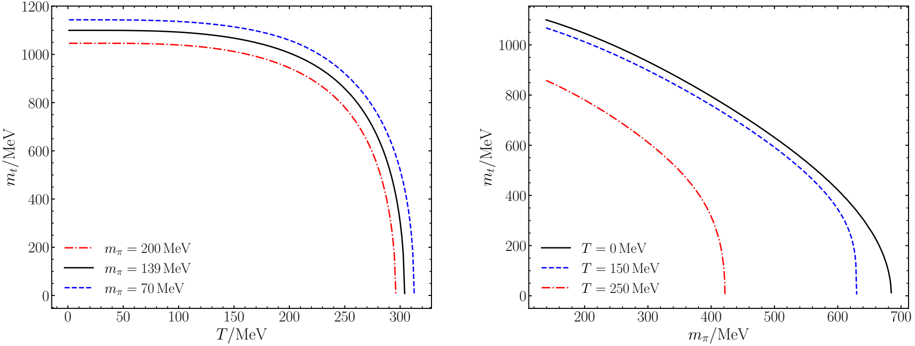
- T increases: $|\phi_b|$ and $|\phi_{min}| \to$ smaller.
- T_f: there is no solution for the gap equations. $T_f \sim 314$MeV
- T_b: $\phi_b = 0$.
- The two branches: Effective potential V get closer.
- $T_c < T_f < T_b$. $T > T_f$, no vacuum, the system is already unstable. Difference $T_b - T_f \sim$ keV.



At high temperature: the Solution I will move to the second branch, becoming a saddle point.

Tachyon: m_π and T dependence

There could be a tachyon for solution I.



- Plays a role of another cutoff of the theory: $-m_t^2 \ll s \ll s + m_t^2$. ($m_t \sim 1.1$GeV for physical mass, $T = 0$) [R. S. Chivukula and M. Golden, PLB 267, 233]
- $s = -m_t^2 < 0$: m_t decreases with temperature and m_π.
- Tachyon has positive residue in the $\sigma - \sigma$ propogator, similar to bound state.
- Tachyon $\rightarrow$ bound state transition $\leftrightarrow$ the point of exchanging the two solution.

Summary

- σ pole trojectory in leading $O(N)$ and N/D modified $O(N)$: with varying m_π and temperature.
- Subthreshold resonance pole generation: After crossing symmtry partially recovered.
- Vacuum structure: with varying m_π and temperture. Phenominological favored one is the first branch.

m_π (MeV)	139	239	283	330	391
$O(N)$ (LO)	$356 - i148$	$448 - i57$	558(VS I) 438(VS II)	660(VS I) 451(VS II)	780(BS) 489(VS II)
N/D modified $O(N)$	$348 - i180$	469(BS) 426(VS II) 168(VS III)	527(BS) 422(VS II) 264(VS III)	585(BS) $396 - i28$ (Sub. pole)	658(BS) $466 - i77$ (Sub. pole)
lattice + K-matrix		$(487 \sim 809$ $-i136 \sim 304)$[49, 51]	$(476 \sim 579$ $-i0 \sim 129)$[51]	657^{+3}_{-4}(BS)[51]	758 ± 4(BS)[49]
lattice + Roy Eq.		$(416 \sim 644$ $-i176 \sim 307)$[57, 58]	$522 \sim 562$ (VS I&II)[58] [a]		759^{+7}_{-16}(BS)[57] $269^{+40}_{-25} - i211^{+26}_{-23}$ (Sub. pole)[57]

[a] Additionally, there is a third though "noisy" pole close to the left-hand cut on the second sheet, which could correspond to the virtual state pole VS III appearing in the N/D modified $O(N)$ model analysis. However, a definitive conclusion about whether the σ is a virtual state or a subthreshold resonance at this m_π value cannot be made, owing to large statistical uncertainties in the results, see Ref. [58] for details.

TABLE I. Comparison of the pole positions ($\sqrt{s_{\rm pole}}$) for $O(N)$ model, lattice + K-matrix [49, 51] and lattice + Roy equation [57, 58]. When $m_\pi = 391$ MeV, the subthreshold (Sub.) pole close to the left-hand cut in Ref. [57] can also be found (qualitatively) within the N/D modified $O(N)$ model discussed in this work.