Generalized cross ratio parametrization of finite type cluster algebras and its applications

USTC 2021/05/06

Cluster Algebra

Initial Seed=(Initial variables {x1,…,xk,…xn}, Initial Exchanges Matrix B)

Mutation at k

$$xk' = \frac{\prod_{B_{ij}>0} xi^{B_{ik}} + \prod_{B_{ij}<0} xi^{-B_{ik}}}{xk}$$

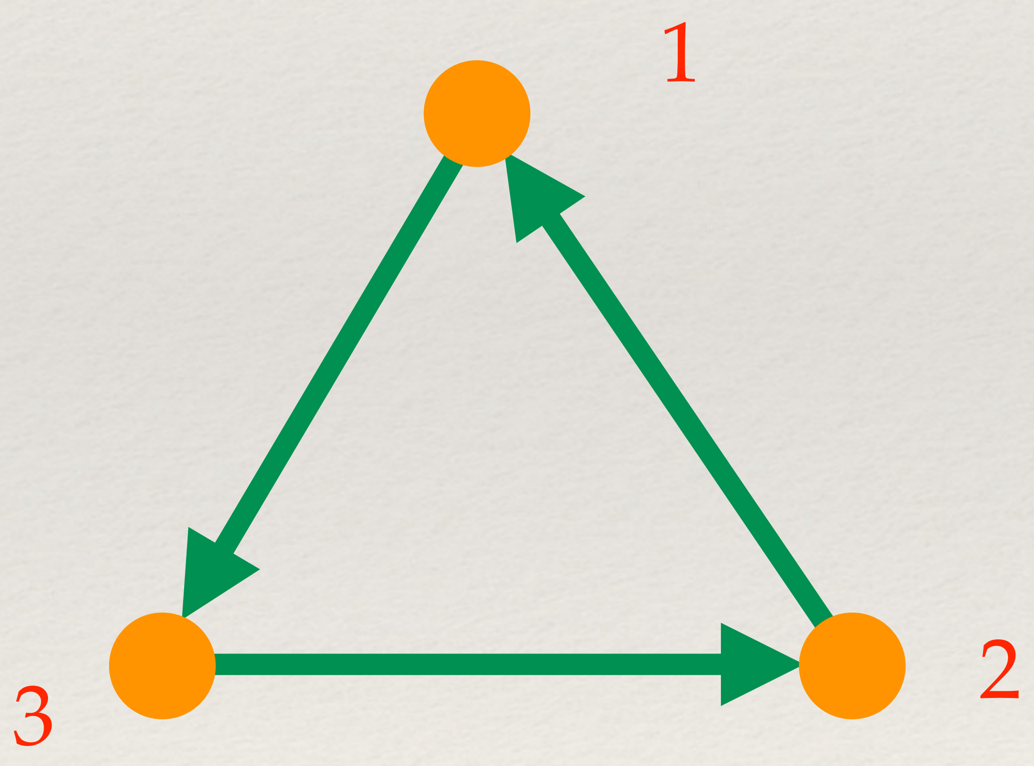
$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k \\ B_{ij} + B_{ik}B_{kj} & \text{if } B_{ik} > 0 \text{ and } B_{kj} > 0 \\ B_{ij} - B_{ik}B_{kj} & \text{if } B_{ik} < 0 \text{ and } B_{kj} < 0 \\ B_{ij} & \text{otherwise} \end{cases}$$

Seed=({x1,…,xk′,…xn}, B′)

Quiver mutation

Exchange Matrix $\longrightarrow$ Quiver

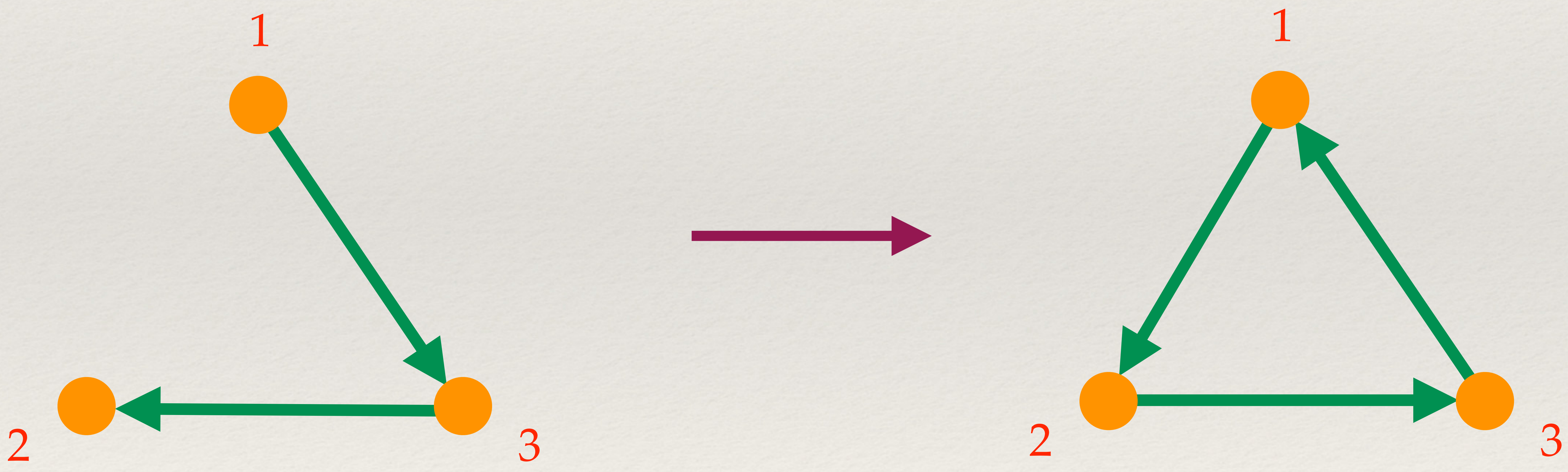
$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \longrightarrow$$



Quiver mutation

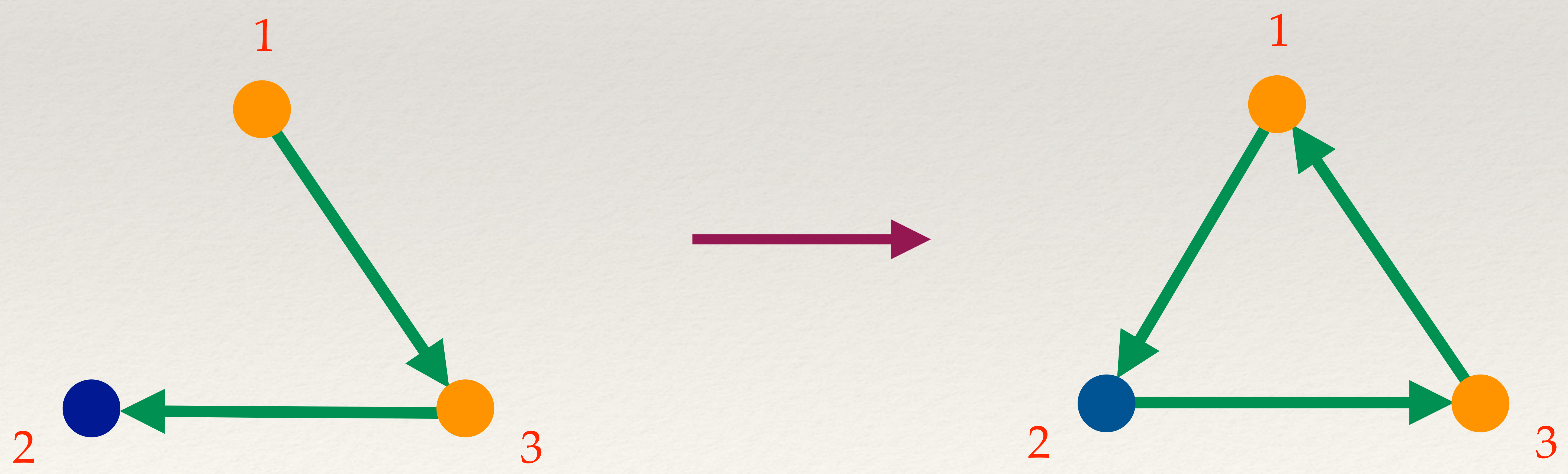
$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k \\ B_{ij} + B_{ik}B_{kj} & \text{if } B_{ij} > 0 \text{ and } B_{kj} > 0 \\ B_{ij} - B_{ik}B_{kj} & \text{if } B_{ij} < 0 \text{ and } B_{kj} < 0 \\ B_{ij} & \text{otherwise} \end{cases}$$

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$



Mutable and Frozen Vertices

$$\begin{pmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{pmatrix}$$



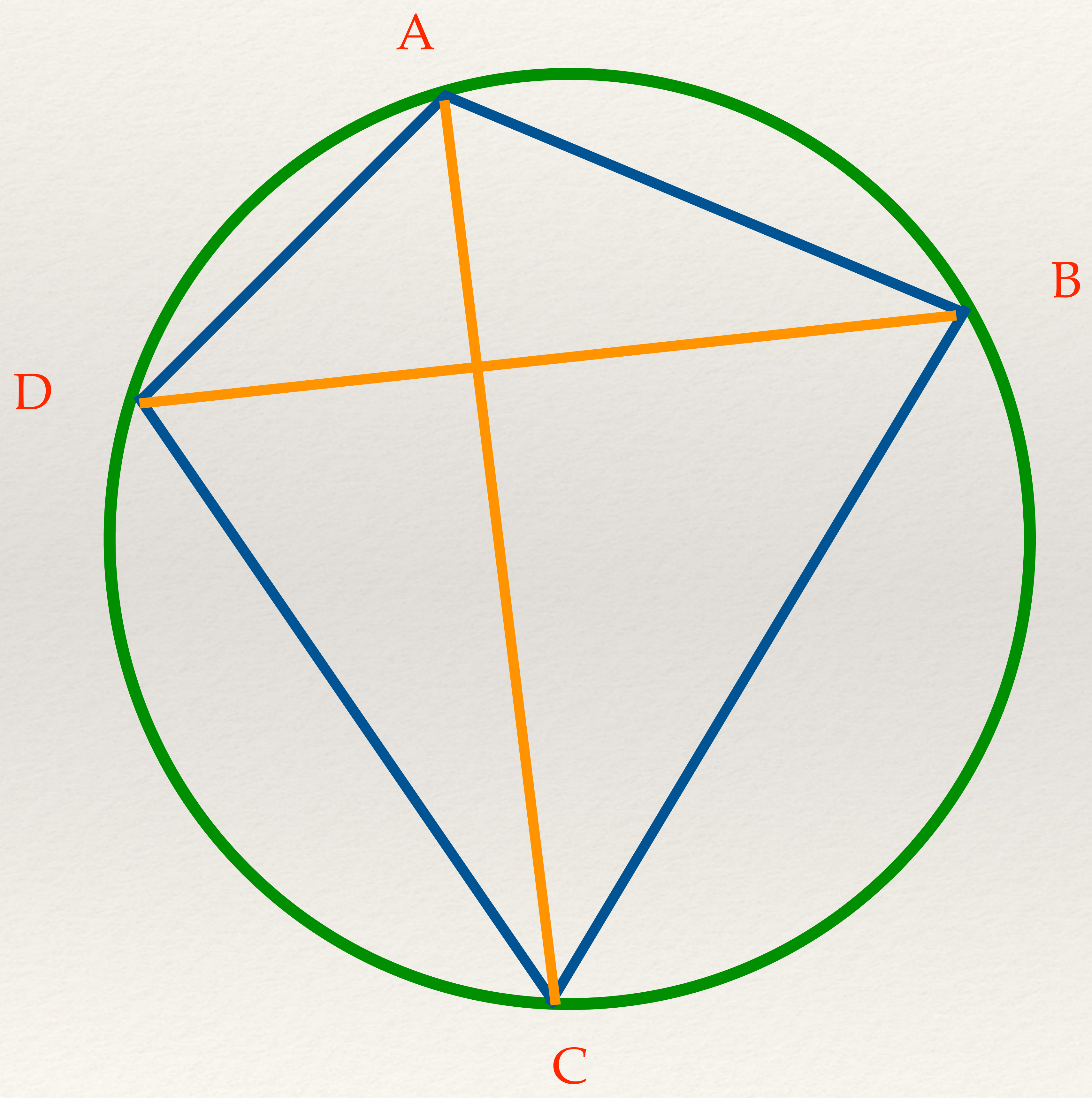
Motivations and Applications for Cluster Algebra(2002,2003,2007)

- Zamolodchikov Y-system (1991)

- Lusztig Canonic basis for quantum algebra (1990)

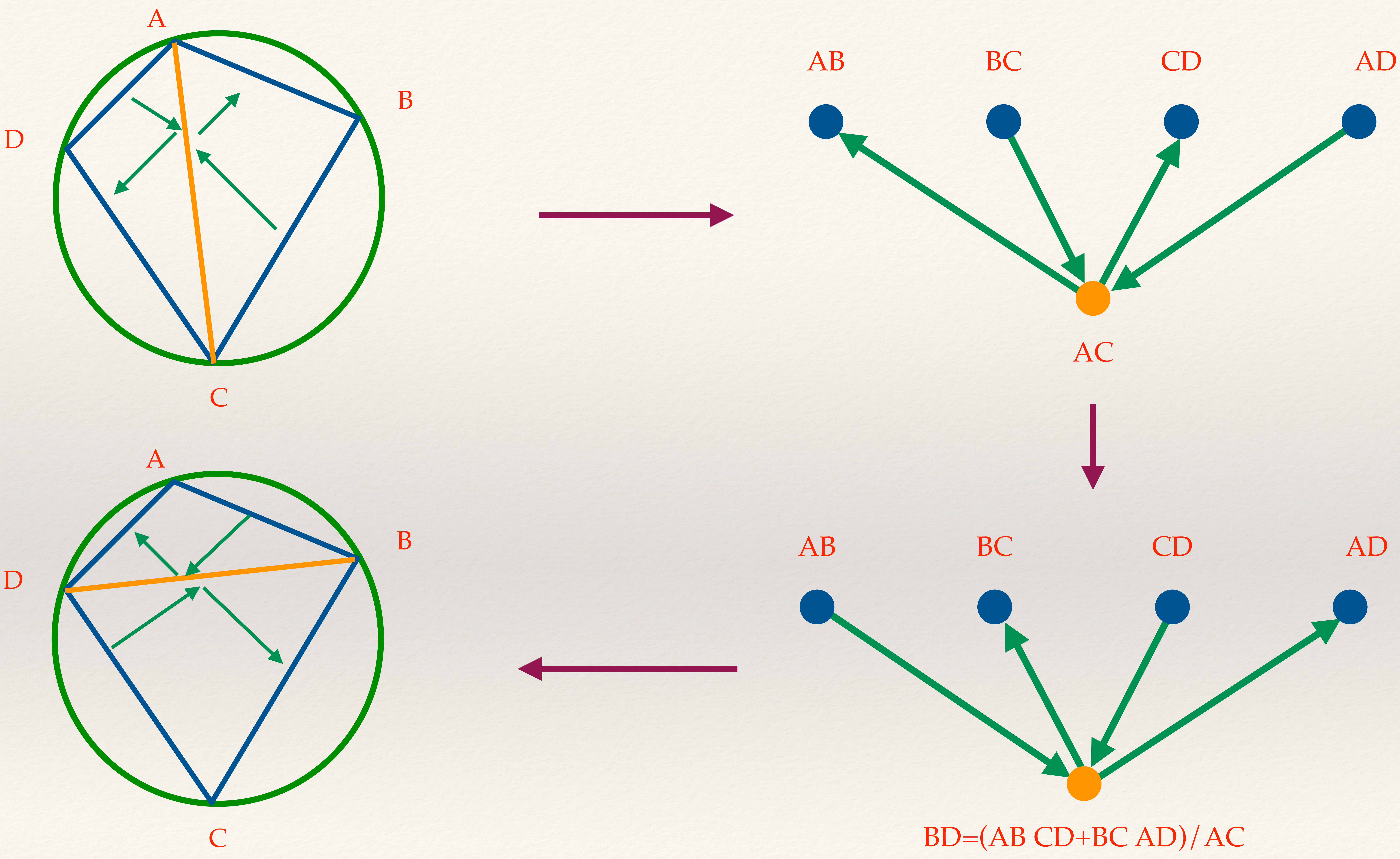
- Positive Grassmannian (Williams 2003,Postnikov 2006)

Ptolemy's Theorem(~150A.D.)

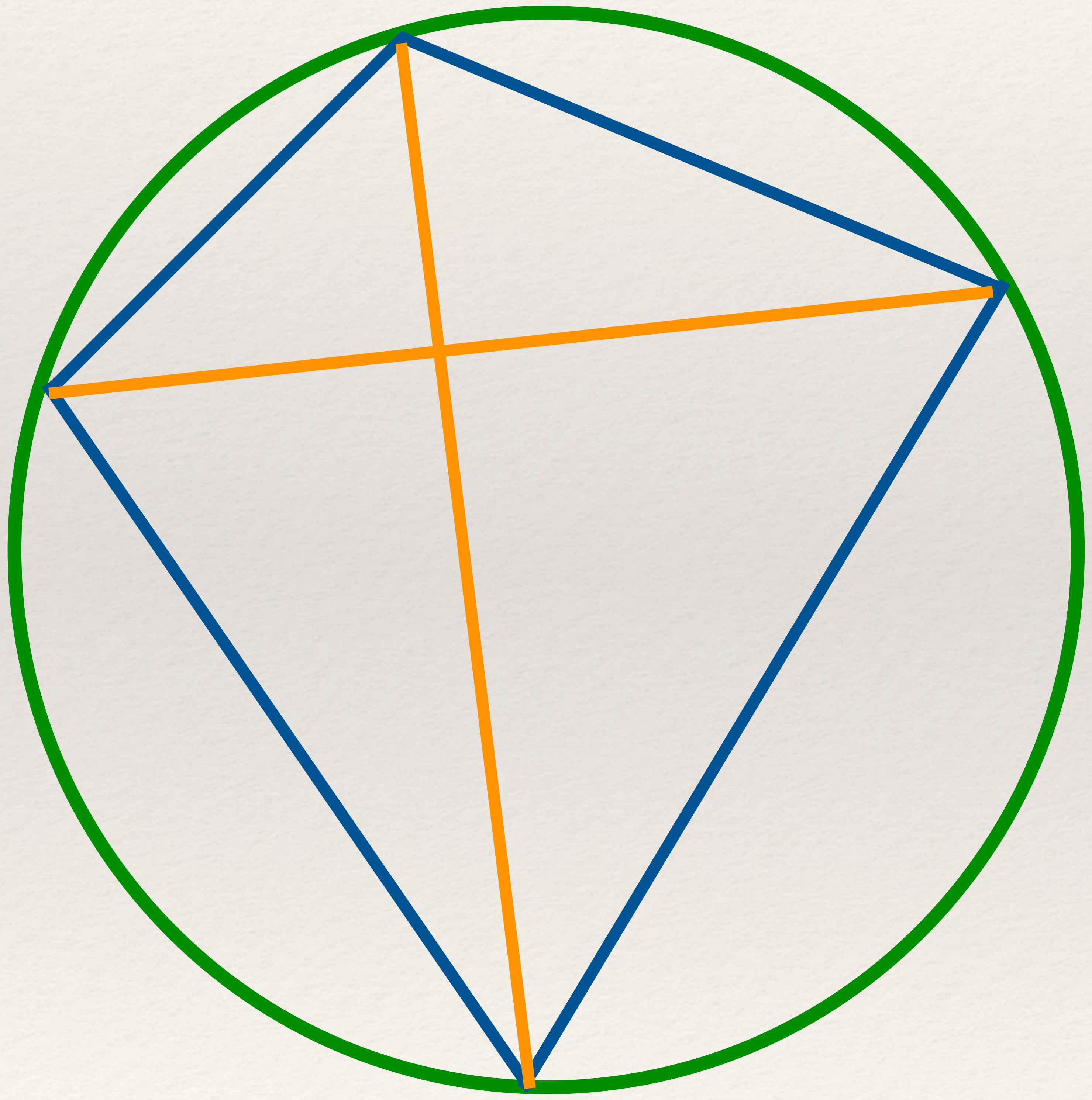


$$|AB|\,|CD| + |AD|\,|BC| = |AC|\,|BD|$$

$$|BD| = \frac{|AB|\,|CD| + |AD|\,|BC|}{|AC|}$$


A
B
D
C
AB
BC
CD
AD
AC
A
B
D
C
AB
BC
CD
AD
BD=(AB CD+BC AD)/AC

Ptolemy's Theorem and Plücker relation



$$x = x0 + r\cos\theta$$
$$y = y0 + r\sin\theta$$

$$\begin{pmatrix} \sin\theta A/2 & \sin\theta B/2 & \sin\theta B/2 & \sin\theta B/2 \\ \cos\theta A/2 & \cos\theta B/2 & \cos\theta B/2 & \cos\theta B/2 \end{pmatrix}$$

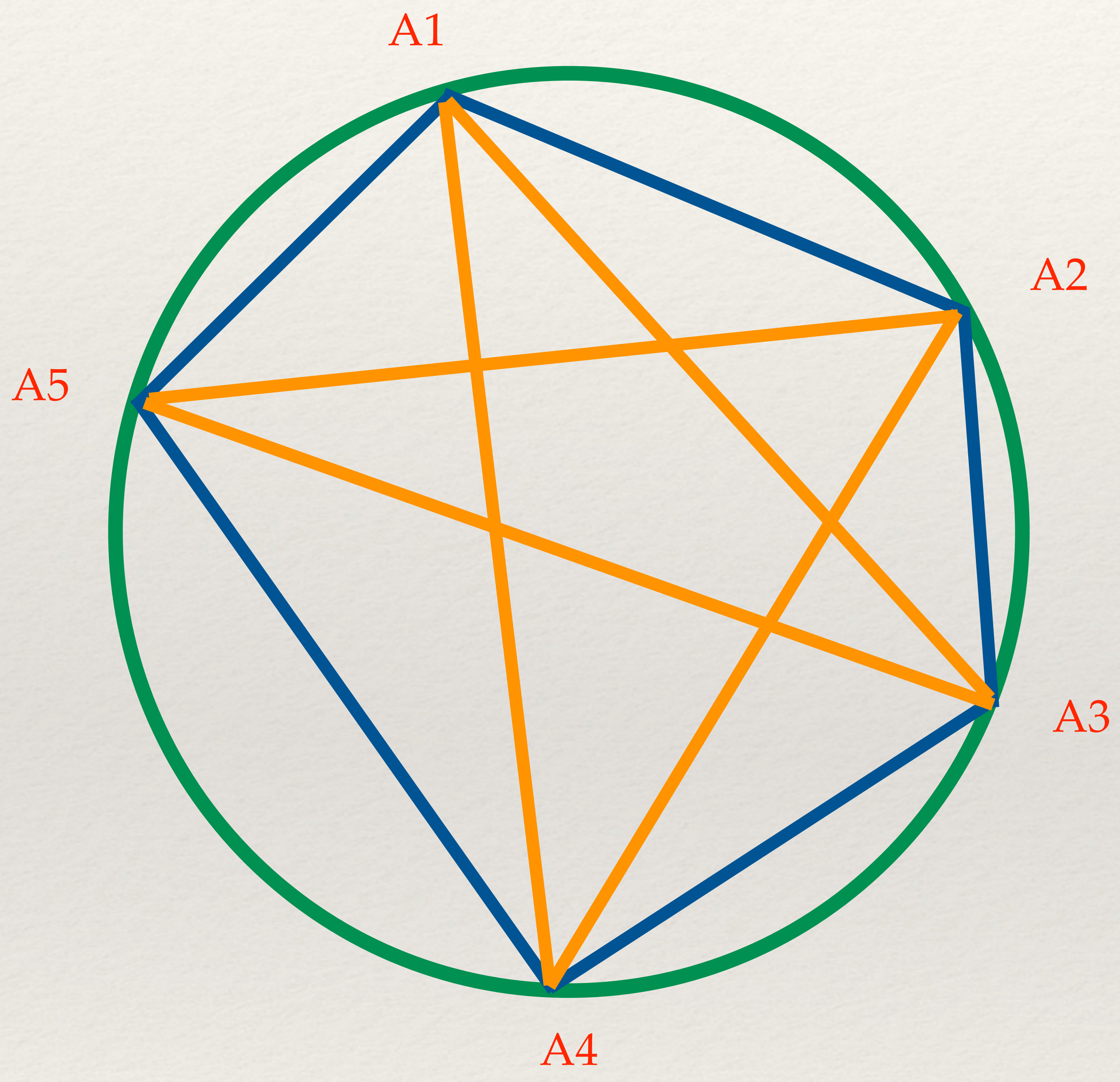
$$|AB| = 2r\Delta_{AB}$$

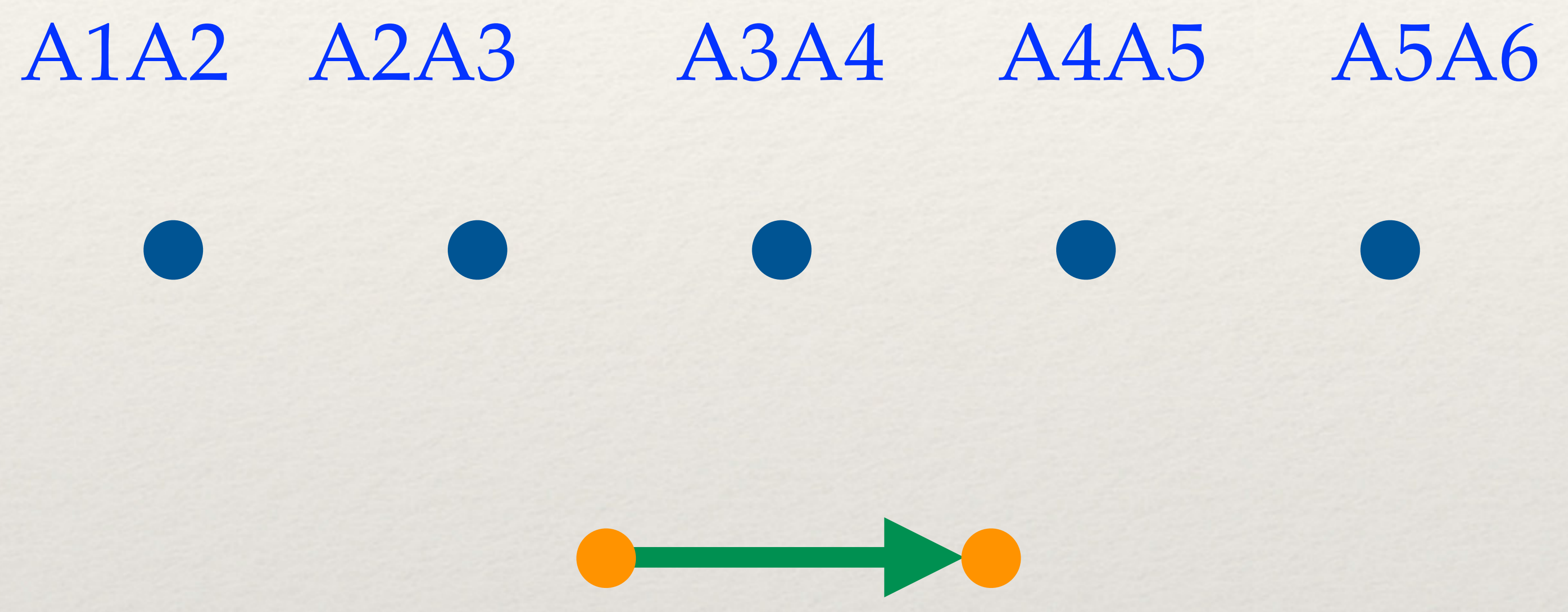
$$|AB|\,|CD| + |AD|\,|BC| = |AC|\,|BD|$$

↓

$$\Delta_{AB}\Delta_{CD} + \Delta_{BC}\Delta_{AD} = \Delta_{AC}\Delta_{BD}$$

Example: diagonals of pentagon, A2 cluster algebra, and G(2,5)

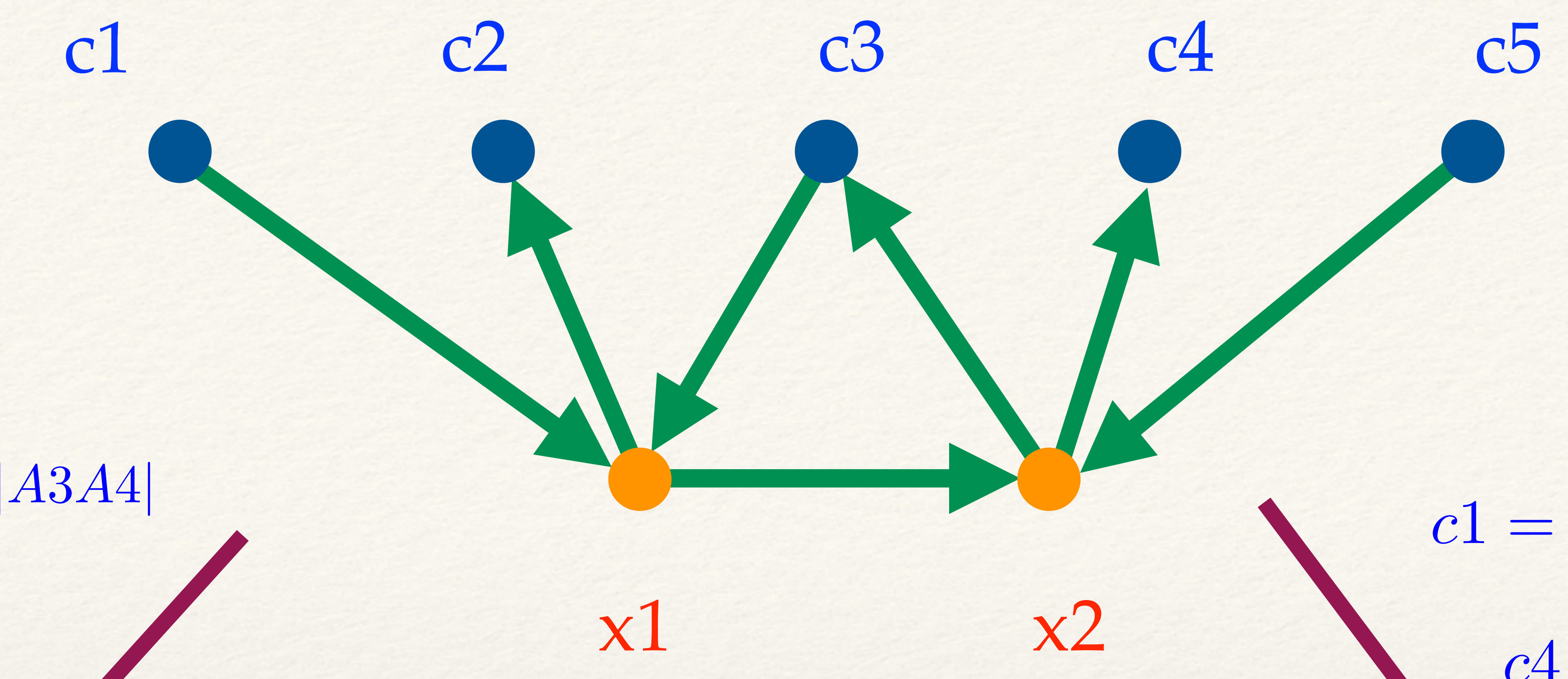




$$\begin{pmatrix} x1 & x2 & x3 & x4 & x5 \\ y1 & y2 & y3 & y4 & y5 \end{pmatrix}$$

$\Delta_{1,2}, \Delta_{2,3}, \Delta_{3,4}, \Delta_{4,5}, \Delta_{5,1}$

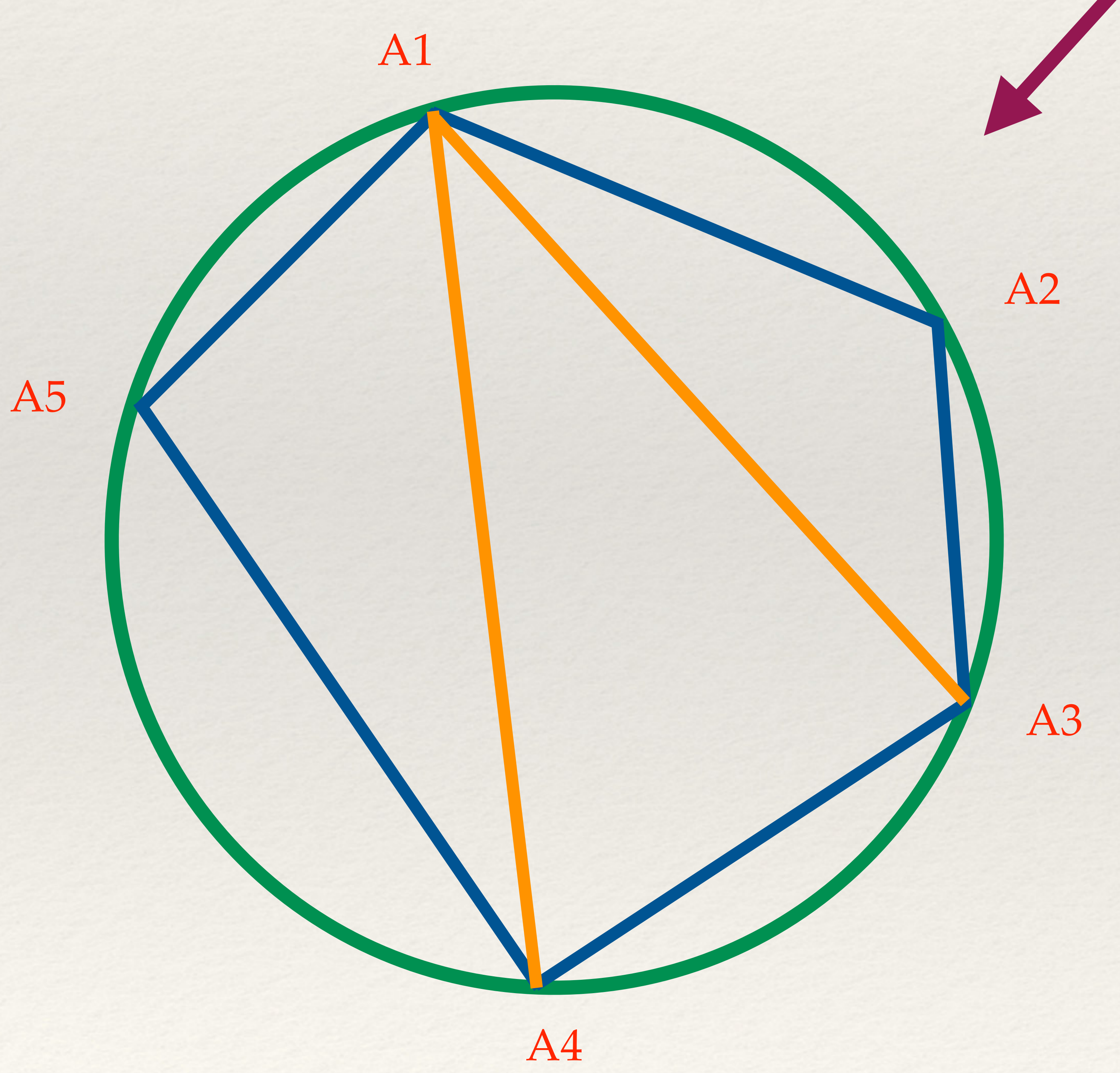
$\Delta_{1,3}, \Delta_{1,4}, \Delta_{3,5}, \Delta_{2,5}, \Delta_{2,4}$



$c1 = |A1A2|\,, c2 = |A2A3|\,, c3 = |A3A4|$

$c4 = |A4A5|\,, c5 = |A1A5|$

$x1 = |A1A3|\,, x2 = |A1A4|$



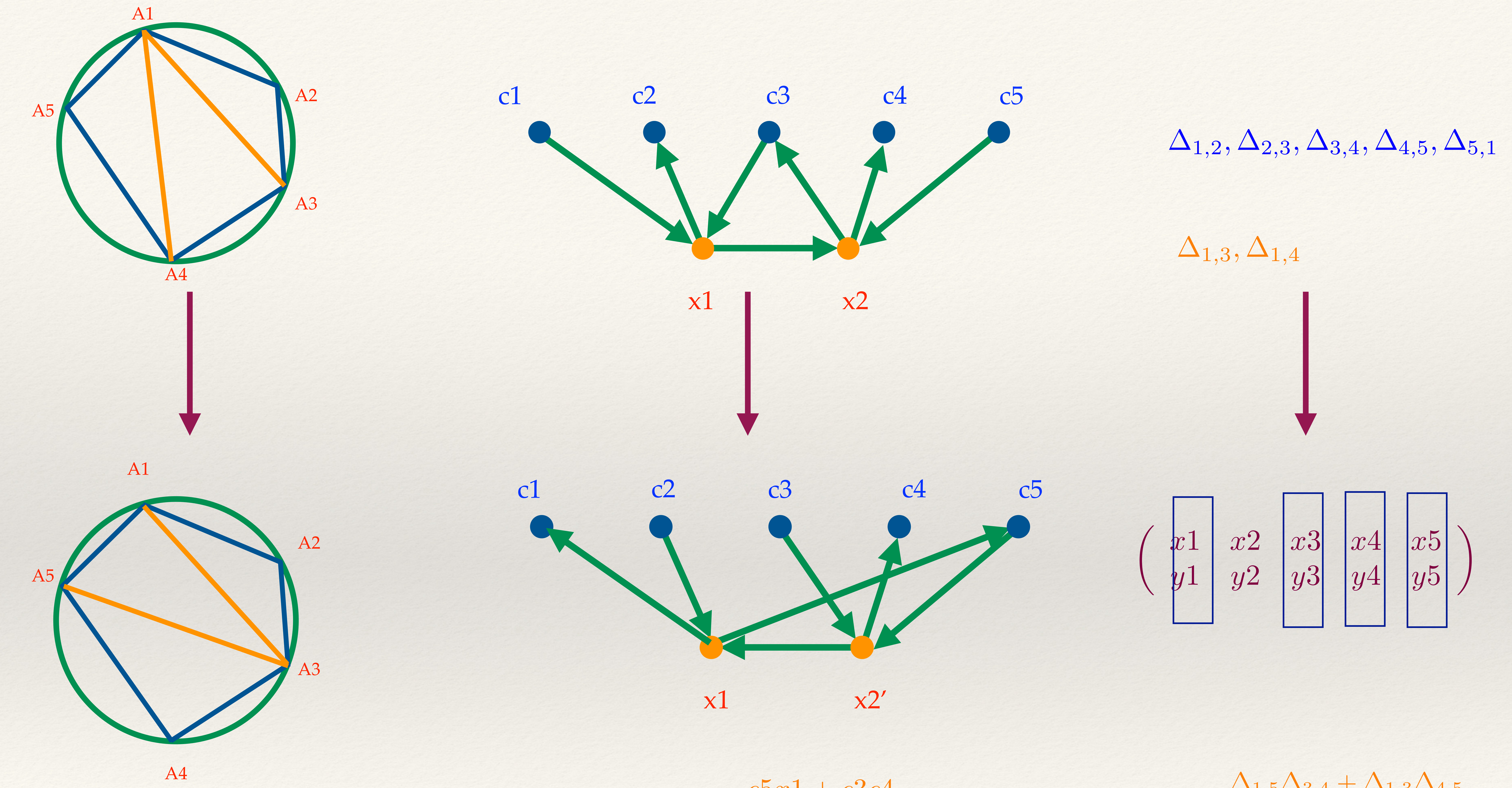
$c1 = \Delta_{1,2}, c2 = \Delta_{2,3}, c3 = \Delta_{3,4}$

$c4 = \Delta_{4,5}, c5 = \Delta_{5,1}$

$x1 = \Delta_{1,3}, x2 = \Delta_{1,4}$

$\Delta_{1,2}, \Delta_{2,3}, \Delta_{3,4}, \Delta_{4,5}, \Delta_{5,1}$

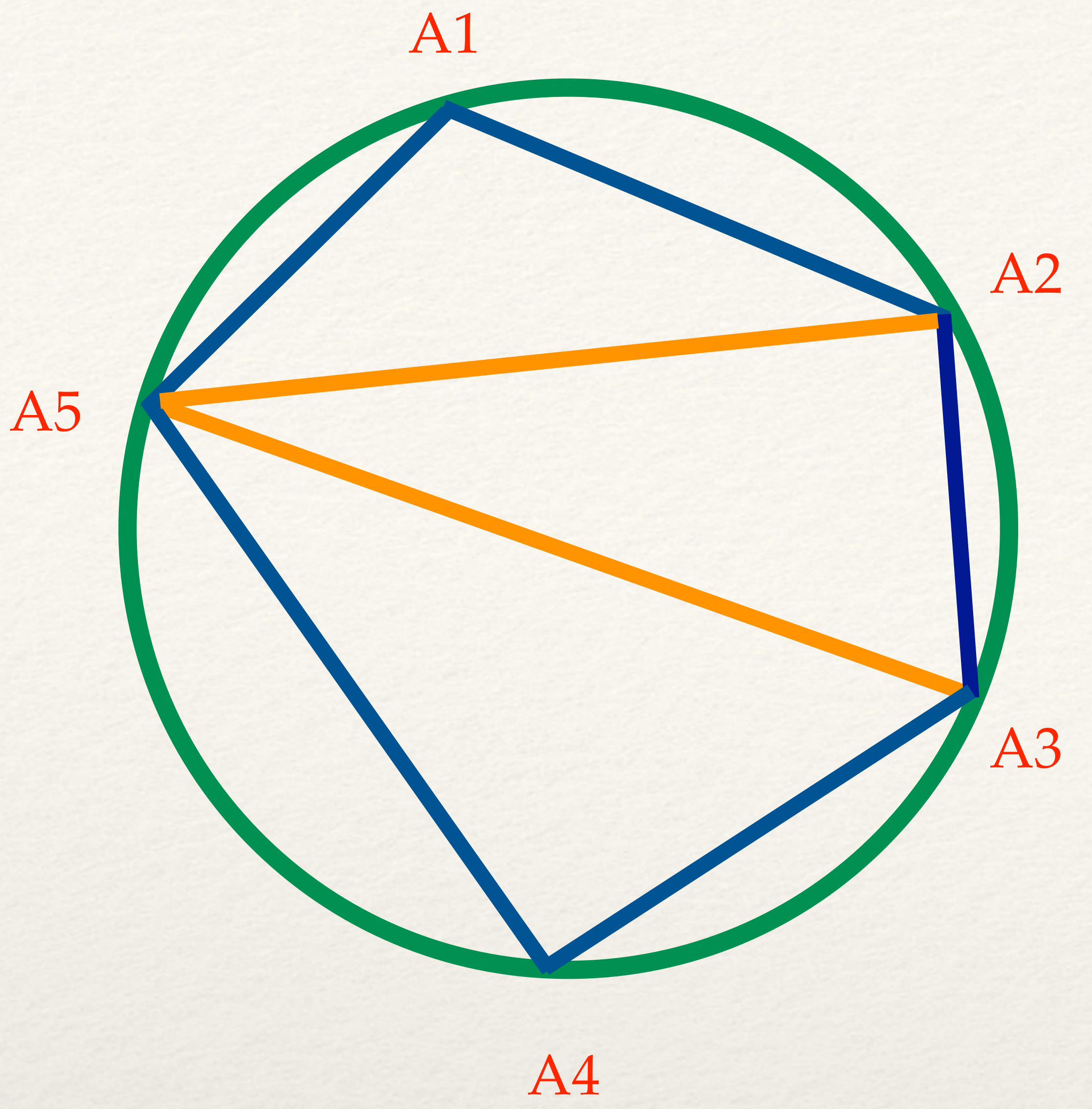
$\Delta_{1,3}, \Delta_{1,4}$

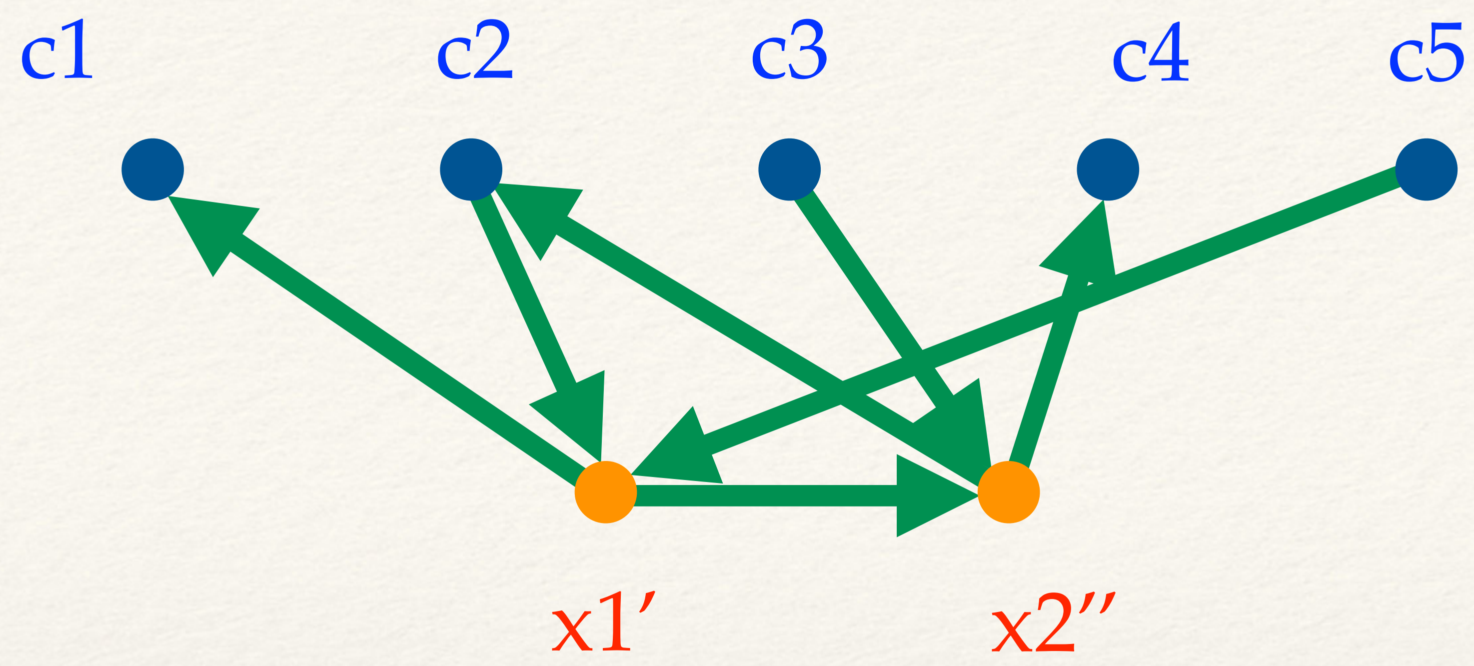


$$|A3A5| = \frac{|A1A5|\,|A3A4| + |A1A3|\,|A4A5|}{|A1A4|}$$

$$x2' = \frac{c5x1 + c3c4}{x2}$$

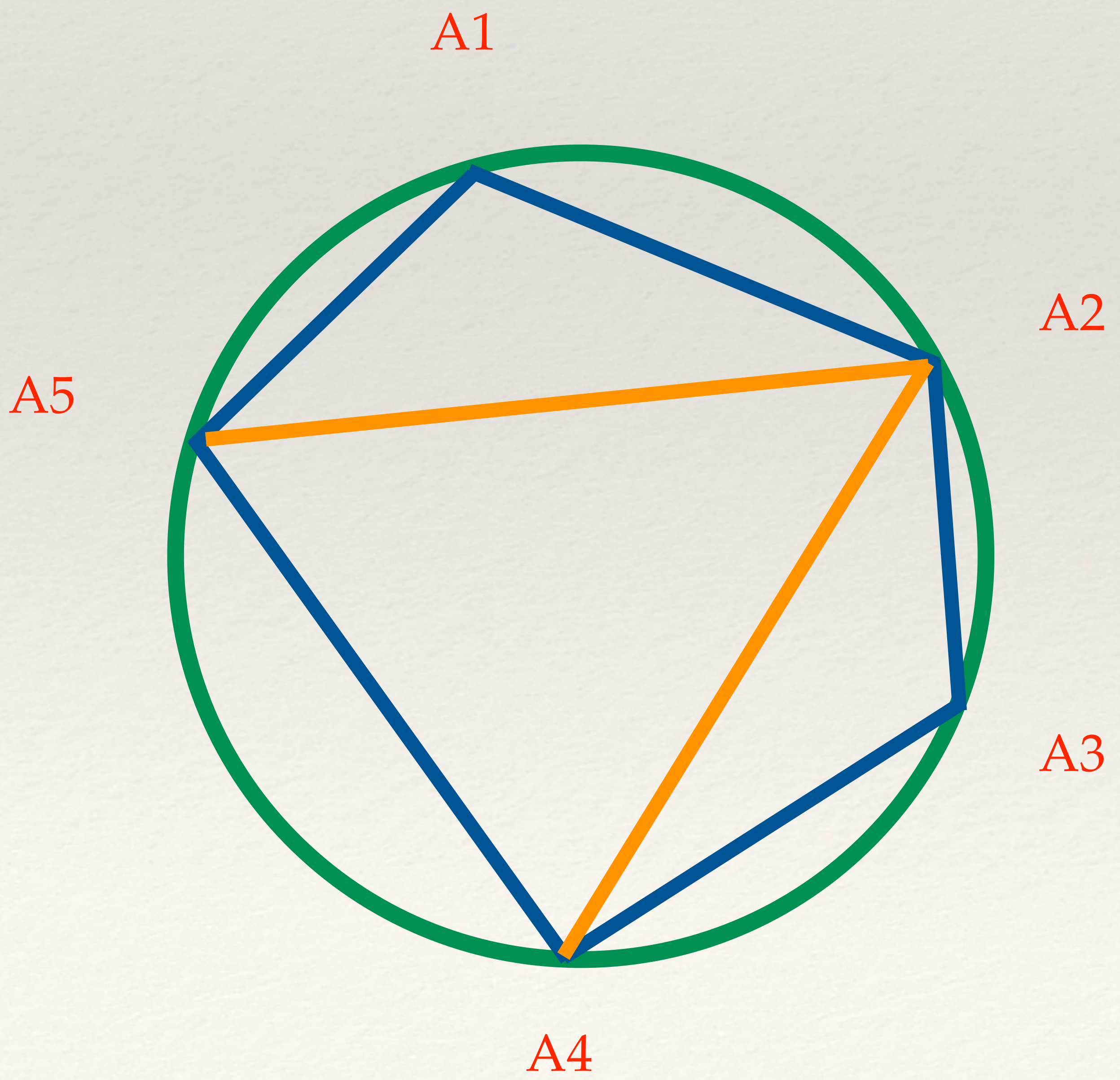
$$\Delta_{3,5} = \frac{\Delta_{1,5}\Delta_{3,4} + \Delta_{1,3}\Delta_{4,5}}{\Delta_{1,4}}$$

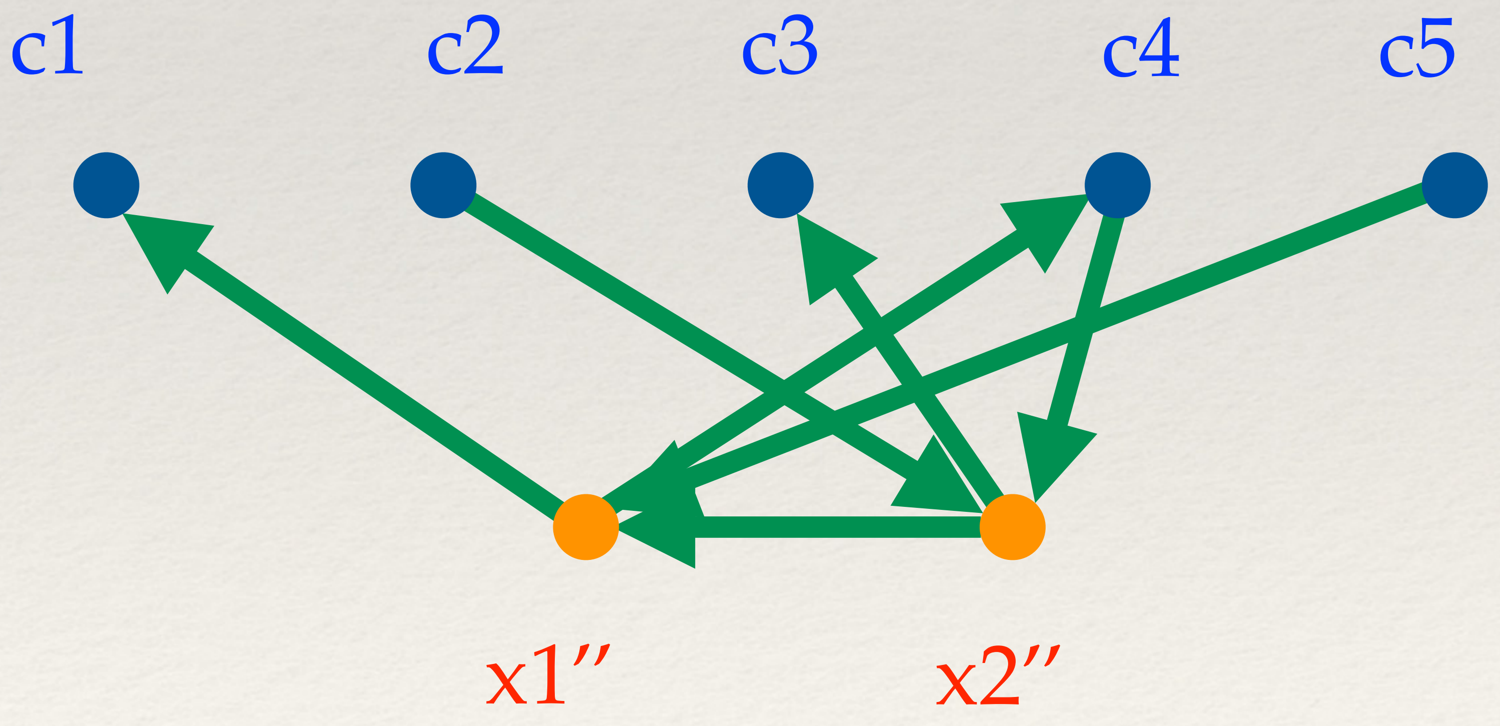




$$\begin{pmatrix} \boxed{\begin{matrix} x1 \\ y1 \end{matrix}} & \boxed{\begin{matrix} x2 \\ y2 \end{matrix}} & \boxed{\begin{matrix} x3 \\ y3 \end{matrix}} & \begin{matrix} x4 \\ y4 \end{matrix} & \boxed{\begin{matrix} x5 \\ y5 \end{matrix}} \end{pmatrix}$$

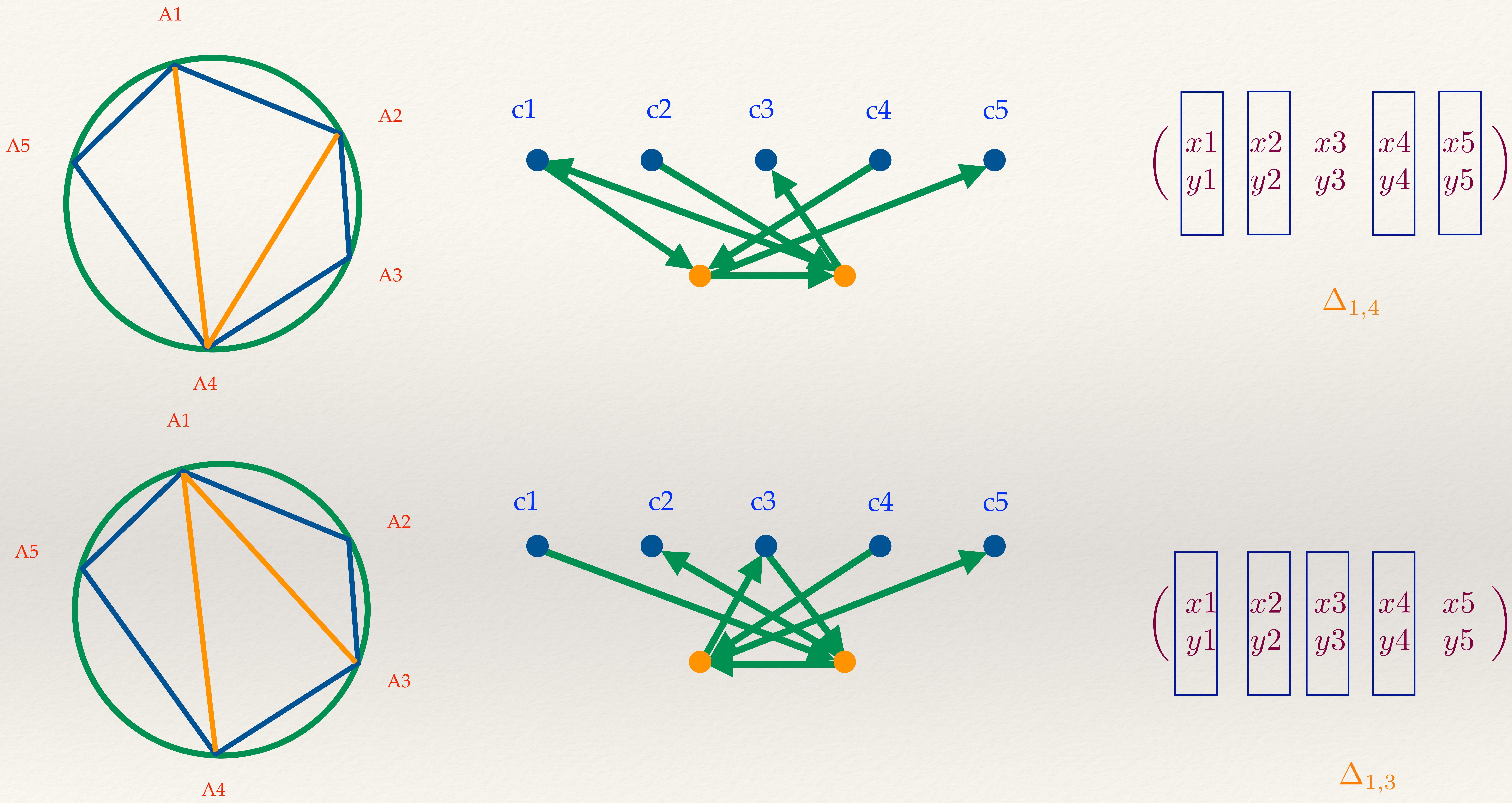
$\Delta_{2,5}$



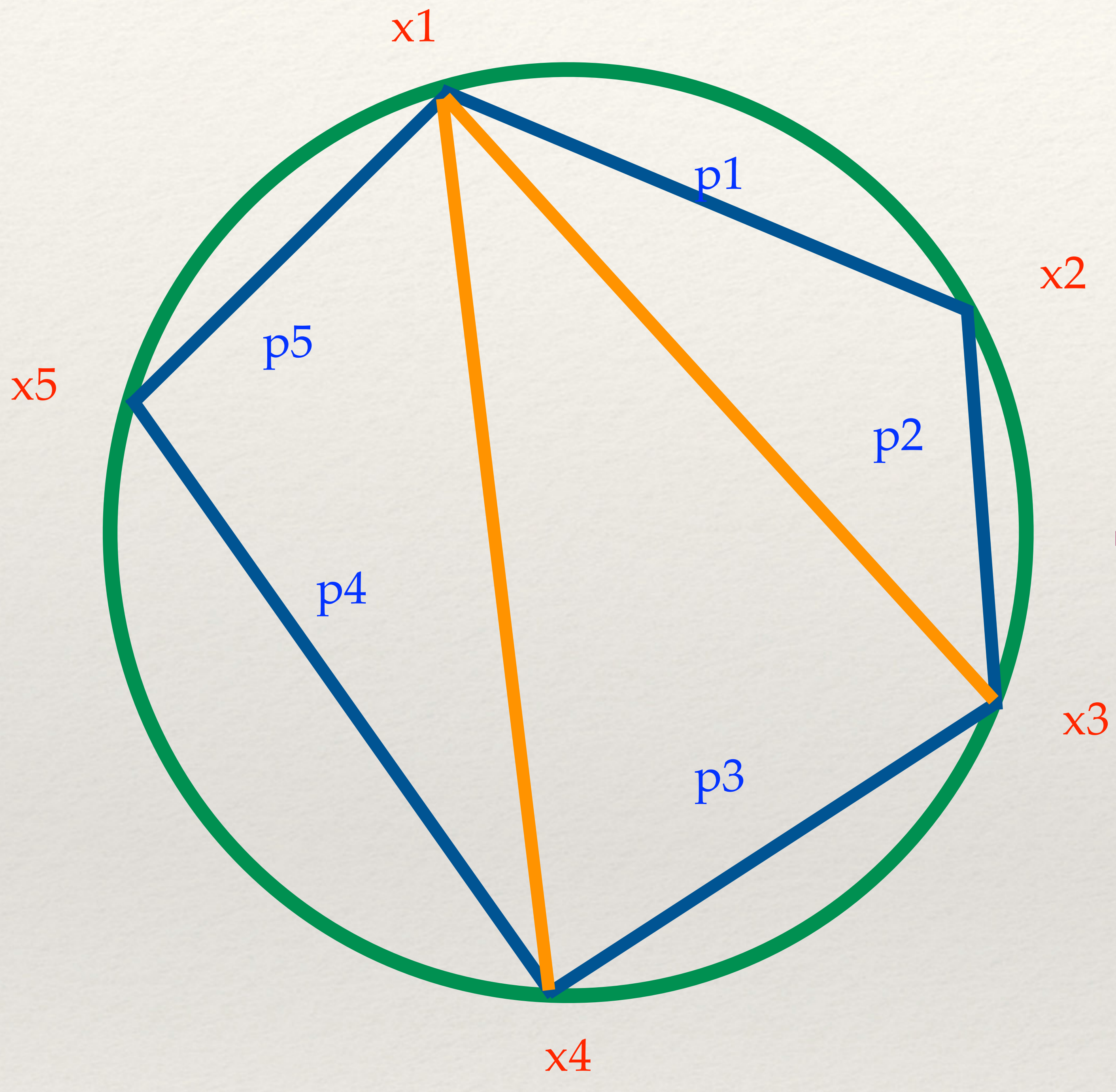


$$\begin{pmatrix} \begin{matrix} x1 \\ y1 \end{matrix} & \boxed{\begin{matrix} x2 \\ y2 \end{matrix}} & \boxed{\begin{matrix} x3 \\ y3 \end{matrix}} & \boxed{\begin{matrix} x4 \\ y4 \end{matrix}} & \boxed{\begin{matrix} x5 \\ y5 \end{matrix}} \end{pmatrix}$$

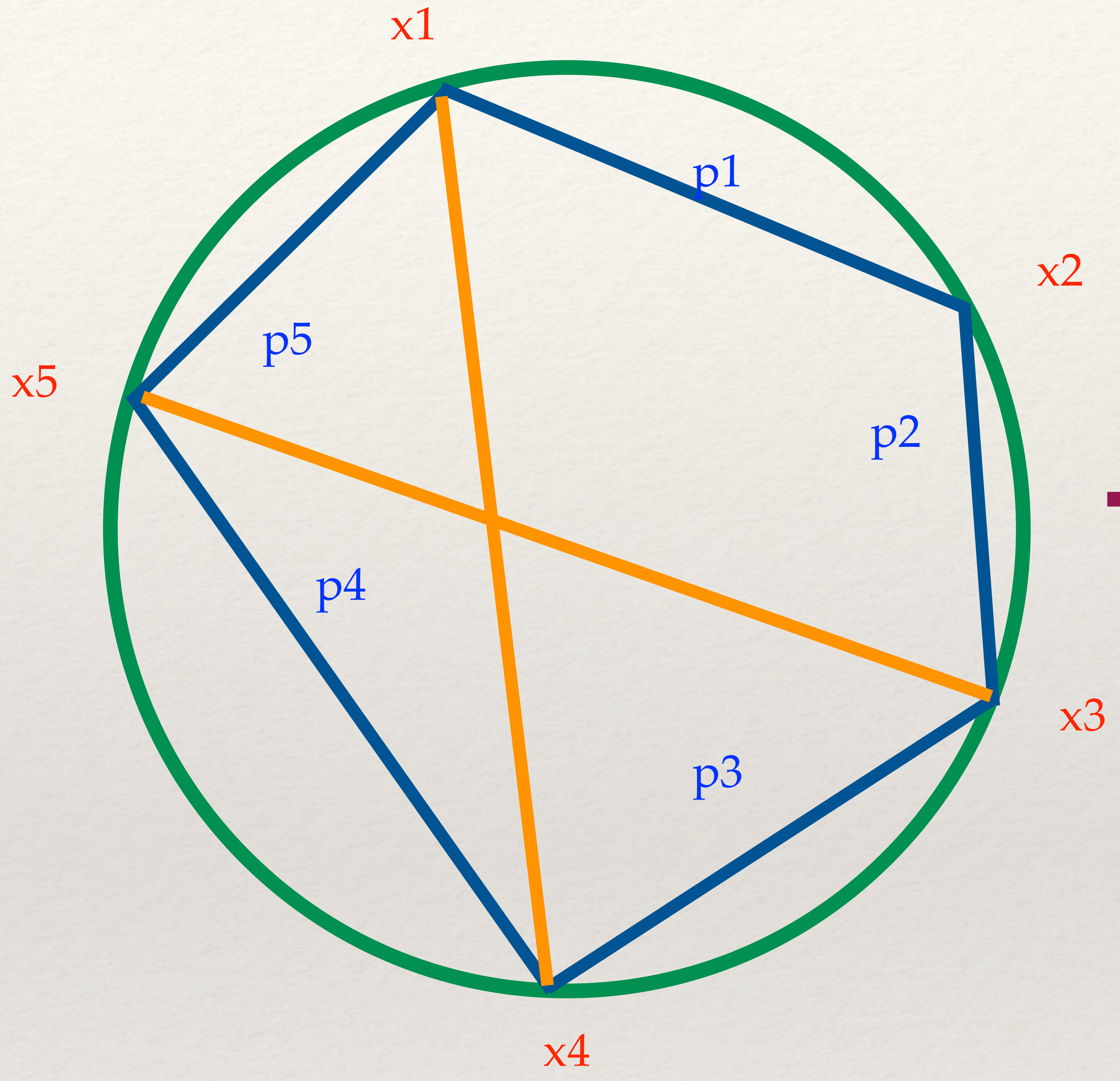
$\Delta_{2,4}$


A1
A2
A3
A4
A5
c1
c2
c3
c4
c5
x1 x2 x3 x4 x5
y1 y2 y3 y4 y5
Δ1,4
A1
A2
A3
A4
A5
c1
c2
c3
c4
c5
x1 x2 x3 x4 x5
y1 y2 y3 y4 y5
Δ1,3

Applications in Scatttering Amplitudes-Adjacency (compatibility)



$$\frac{1}{s_{1,2}s_{1,2,3}}$$



$$\frac{1}{s_{3,4}}\frac{1}{s_{4,5}}$$

J. Drummond, J. Foster, and O. Gurdogan (2017,2018)
Dmitry Chicherin and Johannes M. Henn, (2020)

Applications in Scatttering Amplitudes-Dilogarithm Identity

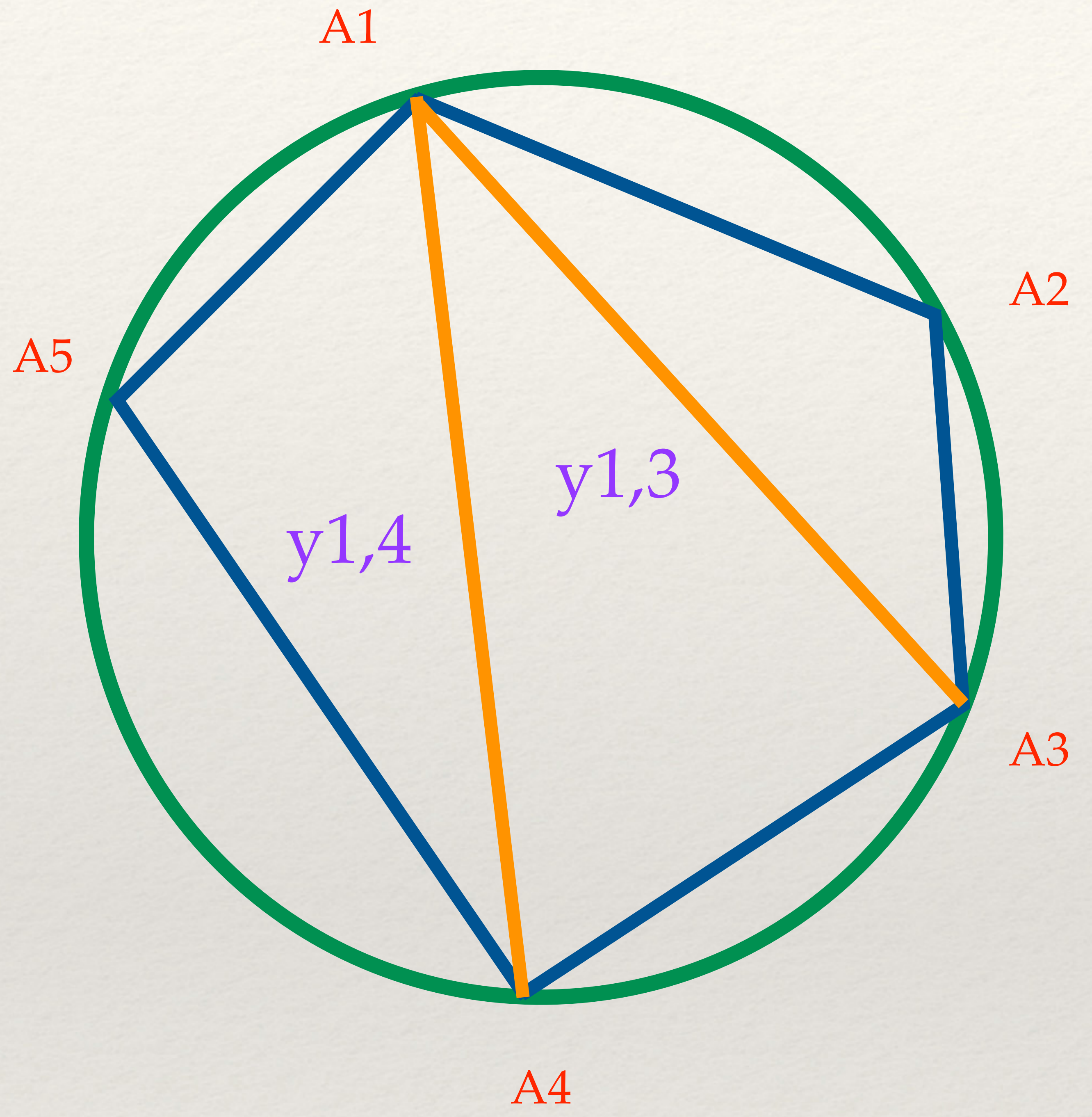
Cluster $\mathcal{X}$-variables $\quad y_i = \prod_{l=1}^{r+nf} x_l^{B_{li}} \qquad y_i' = \begin{cases} 1/y_i & k = i \\ y_i \left(1 + y_k^{-sgn(B_{k,i})}\right)^{-B_{k,i}} & k \neq i \end{cases}$ Y-system

$$L(x) + L(y) + L(1-xy) + L\left(\frac{1-x}{1-xy}\right) + L\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2}$$

$$\frac{6}{\pi^2} \sum_{a \in I, 1 \leq m \leq l-1} L\left(\frac{Y_m^{(a)}}{1+Y_m^{(a)}}\right) = \frac{l \dim \mathfrak{g}}{h+l} - r$$

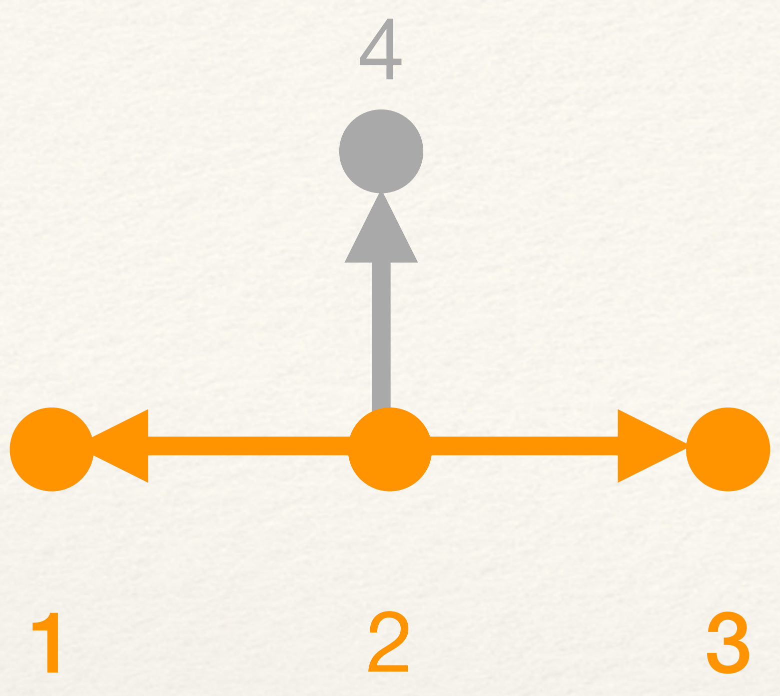
D4 generalization: 40-term identity for Li4 John Golden, Miguel F. Paulos, Marcus Spradlin, and Anastasia Volovicha(2013)

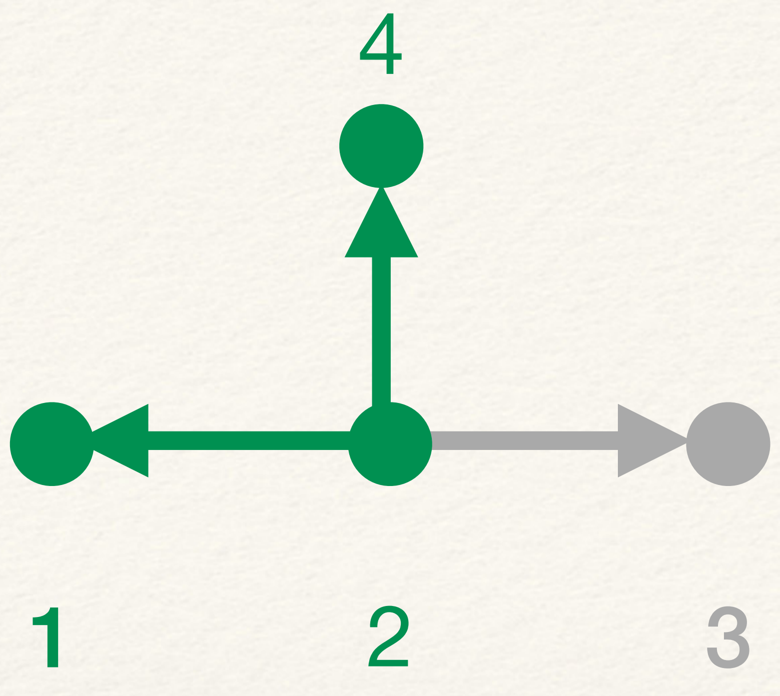
Y-system and cross ratio

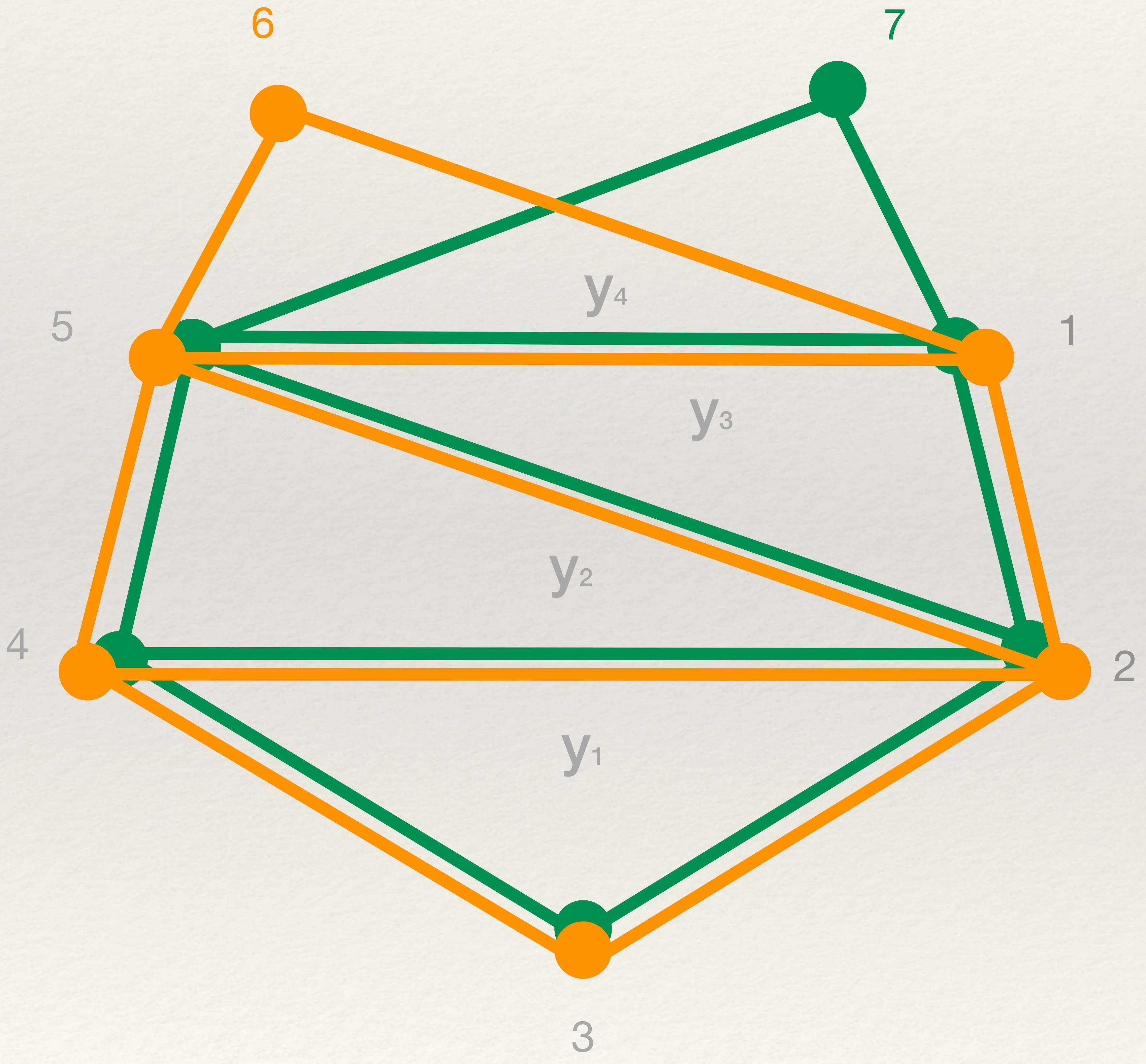


$$y_{ij} = \frac{(z_i - z_{j-1})(z_{i-1} - z_j)}{(z_i - z_{i-1})(z_j - z_{j-1})} \longleftarrow Y_{i,j} = \frac{\Delta_{i,j-1}\Delta_{i-1,j}}{\Delta_{i,i-1}\Delta_{j,j-1}}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix}$$







$$\det\begin{pmatrix} 1 & 1 & 1 \\ z_1 + z_2 & z_3 + z_4 & z_5 + z_6 \\ z_1 z_2 & z_3 z_4 & z_5 z_6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & & 1 \\ \sum_{i\in I_1} z_i & \sum_{i\in I_2} z_i & \cdots & \sum_{i\in I_n} z_i \\ \vdots & \vdots & & \vdots \\ \prod_{i\in I_1} z_i & \prod_{i\in I_2} z_i & & \prod_{i\in I_n} z_i \end{pmatrix}$$

Thank you~ ~