Constraints about the number of horizons by energy conditions

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Based on arXiv: 2104.03012, 2009.05520

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- No inner horizon theorem for charged scalar field
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Motivation and main results

- We know in detail about the exterior of a black hole.

Black hole thermodynamics, various inequalities, LIGO, Event-Horizon Telescope and so on

- Though the interior a black hole cannot be observed for us, it may be crucial in understanding black hole physics, gravitation and quantum physics.

- Black hole information paradox may be solved by including ``island'' that lies in the interior of black hole.

Motivation and main results

- A basic question: how many horizons can appear in a black hole?

- Schwarzschild black hole has one horizon; Kerr-Newman black hole has at most two horizons; dS-RN black hole has 3 horizons;

- We also can construct multiple horizons in various gravity theories coupling with various matters

It seems that there is no universal restriction on the number of inner horizons!

Main results: Einstein-Maxwell-scalar theory

- In planar or spherically symmetric case, if the event horizon has positive temperature, then there is no inner horizon;

- In hyperbolically symmetric case, if the event horizon has positive temperature, then there are at most two horizons. In addition, the second horizon must have nonzero surface gravity.

- In zero temperature case, if the black hole is charged, the event horizon is a double root and there is no inner horizon.

All these results do not depend on the scalar potential V and factor Z

Main results: Part II-general case

Assume Γ is a spacelike cross-section of black hole event horizon. If the Einstein's equation and SEC are satisfied, the spacetime is static or stationary axisymmetric with ``$t - \phi$'' reflected isometry

Then there is at most one odd-order inner horizon inside every connected branch of black hole event horizon.
In addition, if a connected branch of black hole event horizon is even-order, then there is no inner horizon.

Part I: black hole with charged scalar field

- We consider following model

$$S=\frac{1}{2\kappa_N^2}\int d^{d+2}x\sqrt{-g}\left[\mathcal{R}+\mathcal{L}_M\right],$$

$$\mathcal{L}_M=-\frac{Z(|\Psi|^2)}{4}F_{\mu\nu}F^{\mu\nu}-(D_\mu\Psi)^*D^\mu\Psi-V(|\Psi|^2)$$

Here Z>0, V is arbitrary analytical function.

- The metric has following ansatz

$$ds^2=\frac{1}{z^2}\left[-f(z)e^{-\chi(z)}dt^2+\frac{dz^2}{f(z)}+d\Sigma_{d,k}^2\right]$$

$$\Psi=\psi(z)\qquad A=A_t(z)dt$$

The gauge symmetry and EoM insure the phase of scalar is constant.

Method and basic idea

Our main tools are following lemmas:

- **Lemma1**: *For hairy charged black hole (i.e, ψ and A_t are not zero somewhere), $A_t(z_i)$ must be zero if $f(z_i) = 0$*

- **Lemma2:** *Following quantity is radial-independent*

$$\mathcal{Q}(z) = z^{2-d} e^{\chi/2} \left[z^{-2} (f e^{-\chi})' - Z A_t A_t' \right] + 2k(d-1) \int^z y^{-d} e^{-\chi(y)/2} dy$$

In this talk I only focus on non-zero temperature case. For zero-temperature, one can refer to our paper 2009.05520

Proofs for planar/spherical case

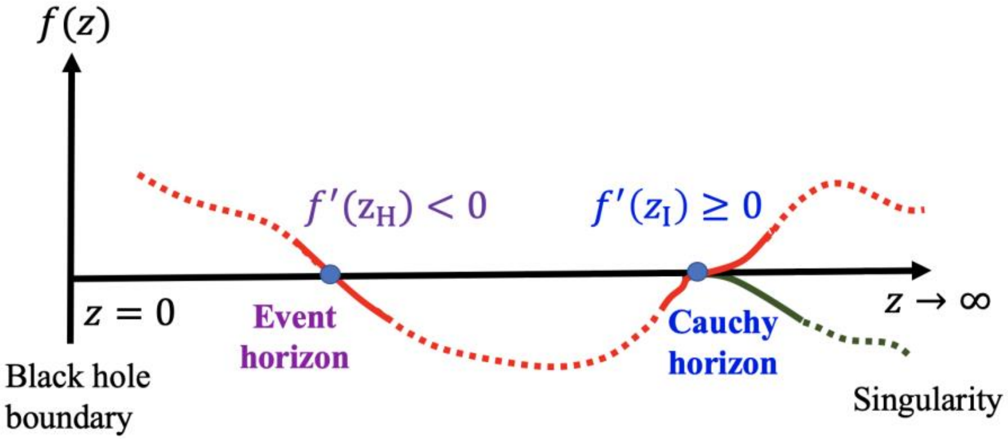
- At horizons, we have

$$\mathcal{Q}(z_j) = \frac{f'(z_j)}{z_j^d} e^{-\chi(z_j)/2} + 2k(d-1)\int^{z_j} y^{-d} e^{-\chi(y)/2} dy$$

- If there are two horizons, we then have

$$\frac{f'(z_H)}{z_H^d} e^{-\chi(z_H)/2} - \frac{f'(z_I)}{z_I^d} e^{-\chi(z_I)/2}$$

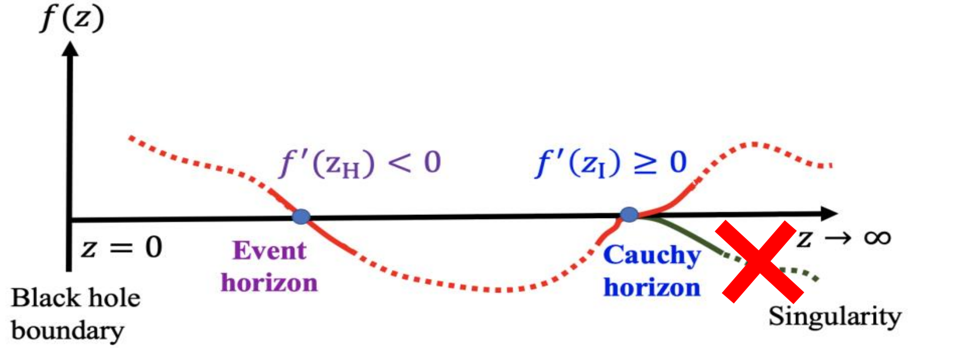
$$= 2k(d-1)\int_{z_H}^{z_I} y^{-d} e^{-\chi(y)/2} dy$$



- In the case k⩾0, this is impossible!

Proof in hyperbolic case

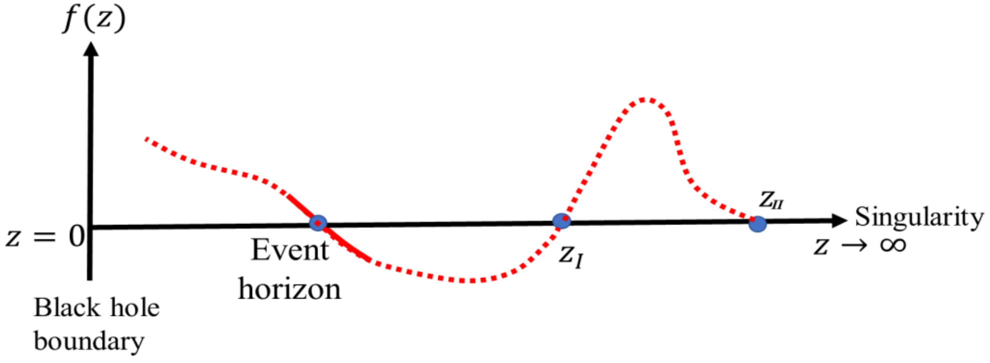
- We first prove, if inner horizon appear, then $f'(z_I)>0$. Otherwise, we have



$$Q'(z_I) = \frac{f''(z_I)}{z_I^d} e^{-\chi(z_I)/2} - Z z_I^{2-d} e^{\chi(z_I)/2} A_t'(z_I)^2 - 2(d-1) z_I^{-d} e^{-\chi(z_I)/2} < 0\,.$$

- Then we can prove there is no the third horizon. Otherwise,

$$\frac{f'(z_I)}{z_I^d} e^{-\chi(z_I)/2} - \frac{f'(z_{II})}{z_{II}^d} e^{-\chi(z_{II})/2} = -2(d-1) \int_{z_I}^{z_{II}} y^{-d} e^{-\chi(y)/2} dy$$



- This is impossible because the two sides have different signs

Diving into a holographic superconductor

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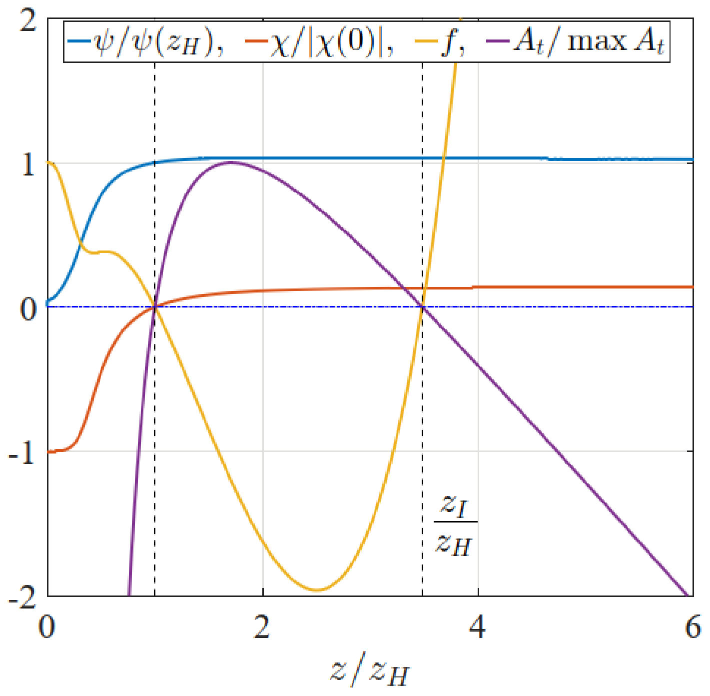
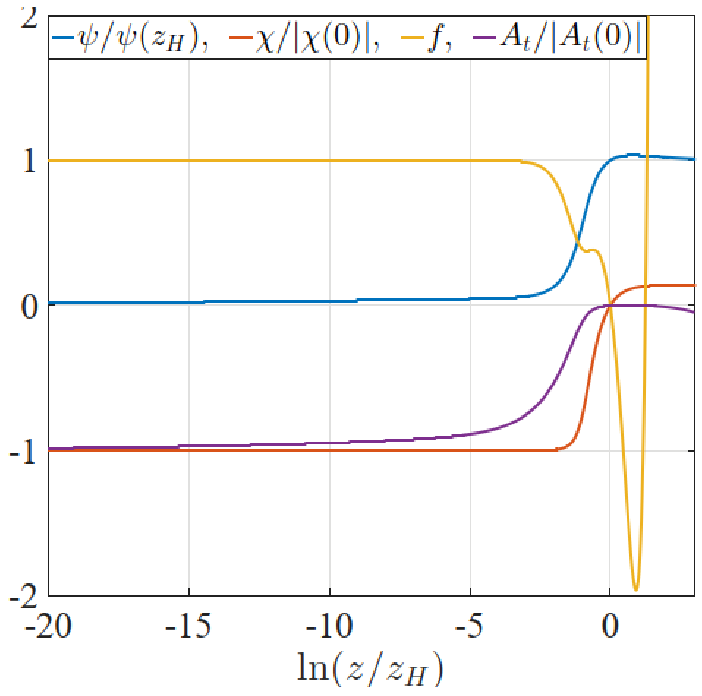
Abstract

Charged black holes in anti-de Sitter space become unstable to forming charged scalar hair at low temperatures $T < T_c$. This phenomenon is a holographic realization of superconductivity. We look inside the horizon of these holographic superconductors and find intricate dynamical behavior. The spacetime ends at a spacelike Kasner singularity, and there is no Cauchy horizon. Before reaching the singularity, there are several intermediate regimes which we study both analytically and numerically. These include strong Josephson oscillations in the condensate and possible 'Kasner inversions' in which after many e-folds of expansion, the Einstein-Rosen bridge contracts towards the singularity. Due to the Josephson oscillations, the number of Kasner inversions depends very sensitively on T, and diverges at a discrete set of temperatures $\{T_n\}$ that accumulate at T_c. Near these T_n, the final Kasner exponent exhibits fractal-like behavior.

Example of double horizons when k=-1

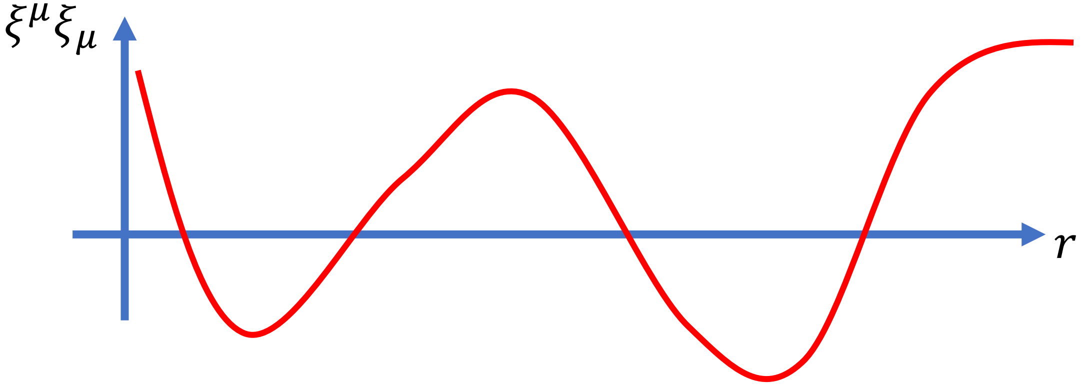




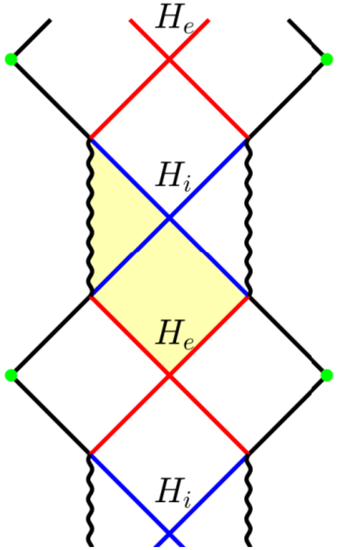
Part II:
Theorem of no-multiple inner horizons for static black holes

Basic assumptions and conceptions

- In static black hole, one may study how many foots of $\xi^\mu \xi_\mu$ may appear



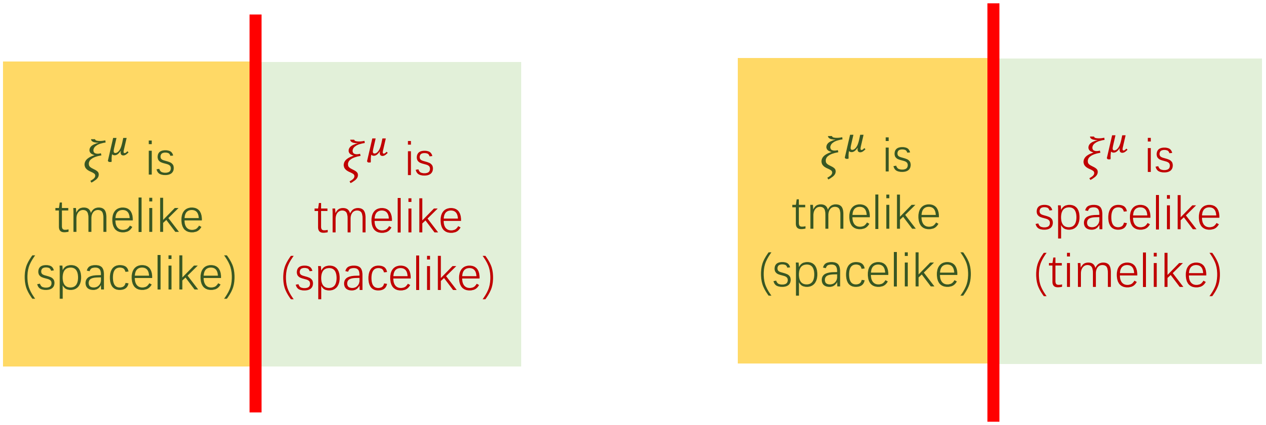
- Maximally analytically continued RN black hole has infinite horizons



It is more suitable to ask the number of horizons inside a connected branch of black hole event horizon.

Basic assumptions and conceptions

- There is one linear-independent Killing vector ξ^μ standing for static symmetry;
- "Horizons" always stand for ``Killing horizons'';
- Even-order and odd-order horizons;

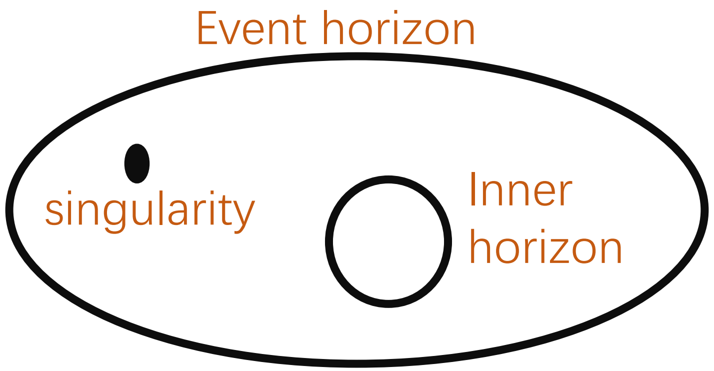


Even-order horizons Odd-order horizons

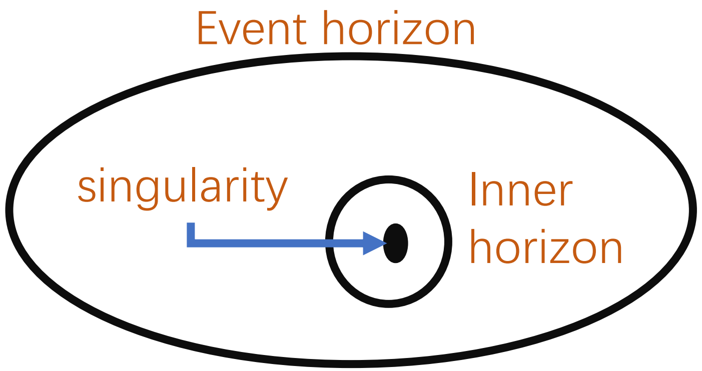
- Odd-order horizon may have zero surface gravity;

Basic assumptions and conceptions

- Every connected branch of event horizon is simply connected;
- Weak cosmic censorship is true: all singularities hide in the event horizon;
- We assume a stronger version: all singularities hide in the innermost horizon;

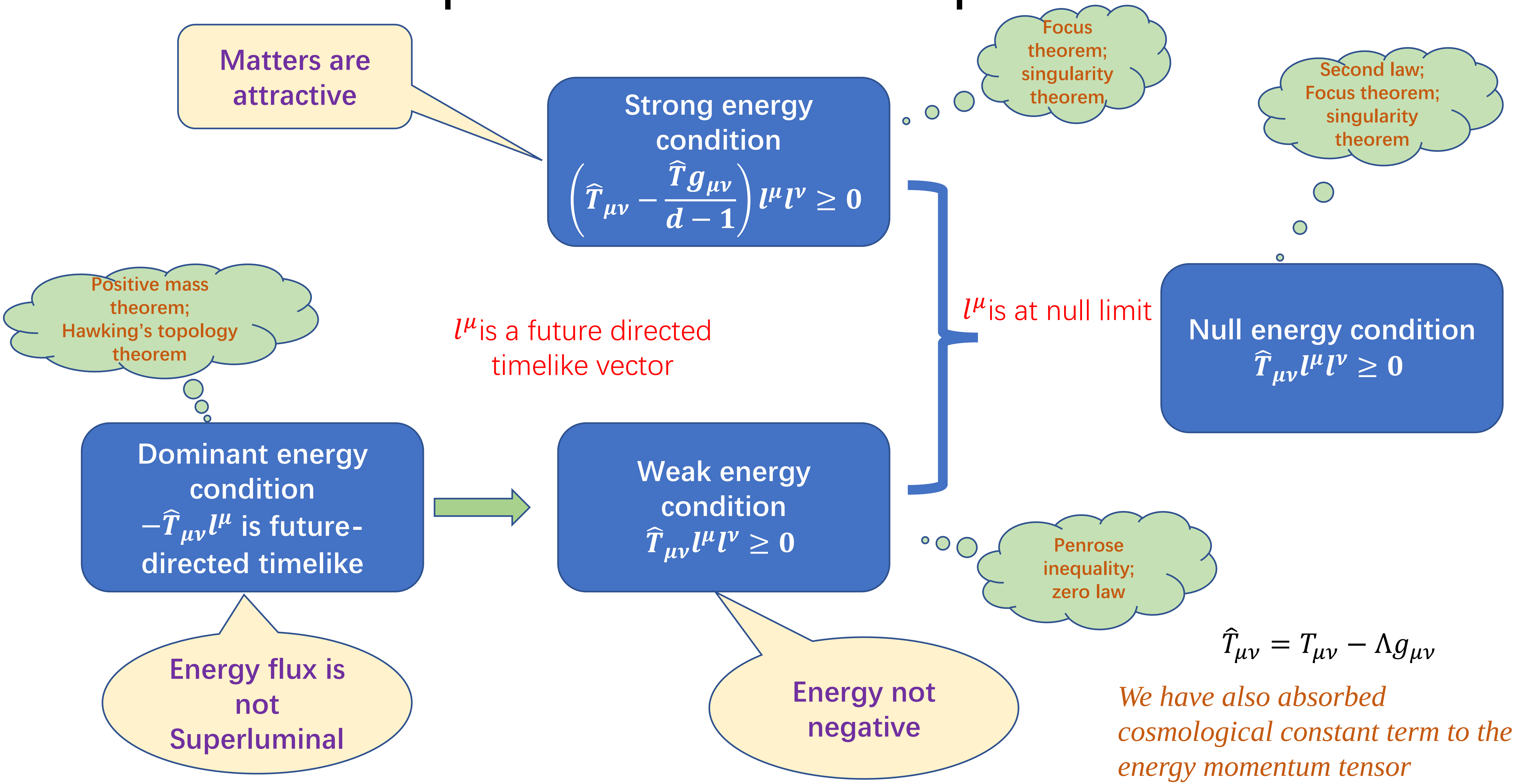


Weak cosmic censorship



Strongger weak cosmic censorship

Basic assumptions and conceptions

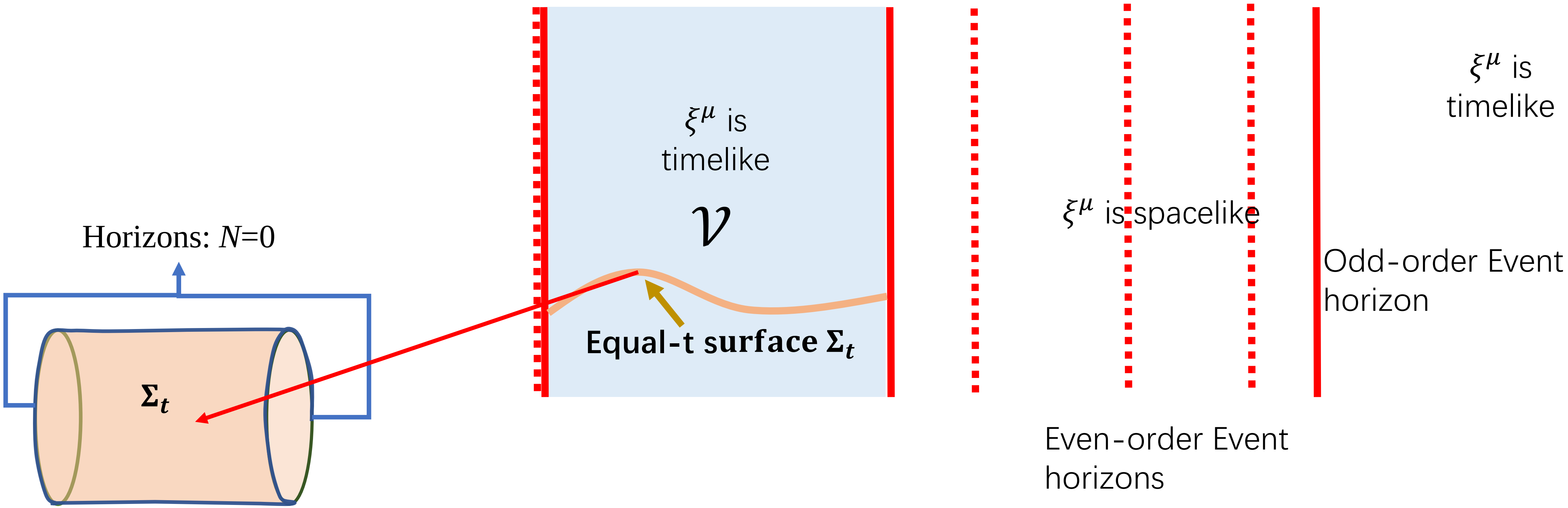


$$\hat{T}_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu}$$

We have also absorbed cosmological constant term to the energy momentum tensor

Proof 1: compact horizon

- If there are more than two odd-order inner horizons inside odd-order event horizon



- The metric in the spacetime region $\mathcal{V}$ has following (d+1) decomposition

$$ds^2 = -N^2 dt^2 + h_{ab} dx^a dx^b$$

Proof 1: compact horizon

- In the spacetime region $\mathcal{V}$

$$^{(d)}R = 2\hat{\rho}$$

$$^{(d)}R_{ab} = N^{-1}D_aD_bN + \left[\hat{\mathcal{T}}_{ab} + \frac{h_{ab}}{d-1}(\hat{\rho} - \hat{\mathcal{T}})\right]$$

Hamiltonian constraint and evolutional equation of extrinsic curvature

$^{(d)}R_{ab}$ is the spatial Ricci tensor and $\hat{\mathcal{T}}_{ab}$ is the spatial components of stress tensor

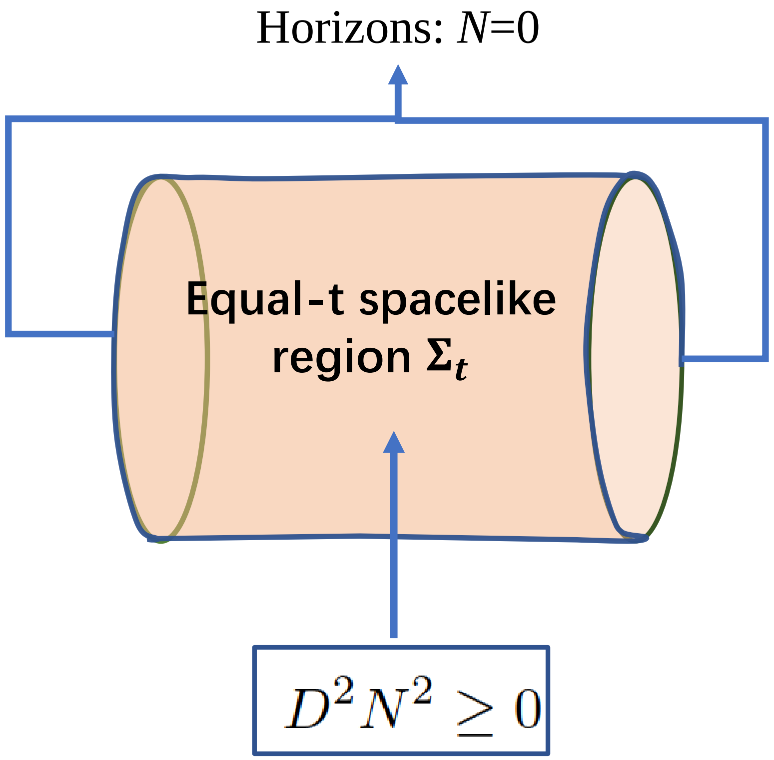
- These two equations contain all information of Einstein's equation.

$$\Rightarrow \quad D^2N^2 = 2N^2[\hat{\rho} + \hat{\mathcal{T}}/(d-1)] + 2h^{ab}(\partial_a N)(\partial_b N)$$

SEC insures it nonnegative

$$\Rightarrow \quad \boxed{D^2N^2 \geq 0}$$

Proof 1: compact horizon



- The maximum principle shows that the maximum of N^2 must be at the boundaries of Σ_t

$$\max N^2|_{\Sigma_t} = \max N^2|_{\partial\Sigma_t} = 0$$

- This is contradictory to fact that ξ^μ is timelike inside Σ_t.

- Other way:

$$\int_{\Sigma_t} D^2N^2 = \int_{\partial\Sigma_t} 2ND_aN\, dS^a = 0$$

$$D^2N^2 = 2N^2[\hat{\rho} + \hat{T}/(d-1)] + 2h^{ab}(\partial_a N)(\partial_b N)$$

SEC insures it nonnegative

$\Longrightarrow$ $$\int_{\Sigma_t} h^{ab}\partial_a N \partial_b N = 0$$ $\Longrightarrow$ N is constant and so N=0

Axisymmetric stationary spacetime

- Two commutative linear independent Killing vector fields: ξ^μ and Ψ^μ;
- Assume a "t-φ" reflected isometry;
- Under such assumption the metric will have following form

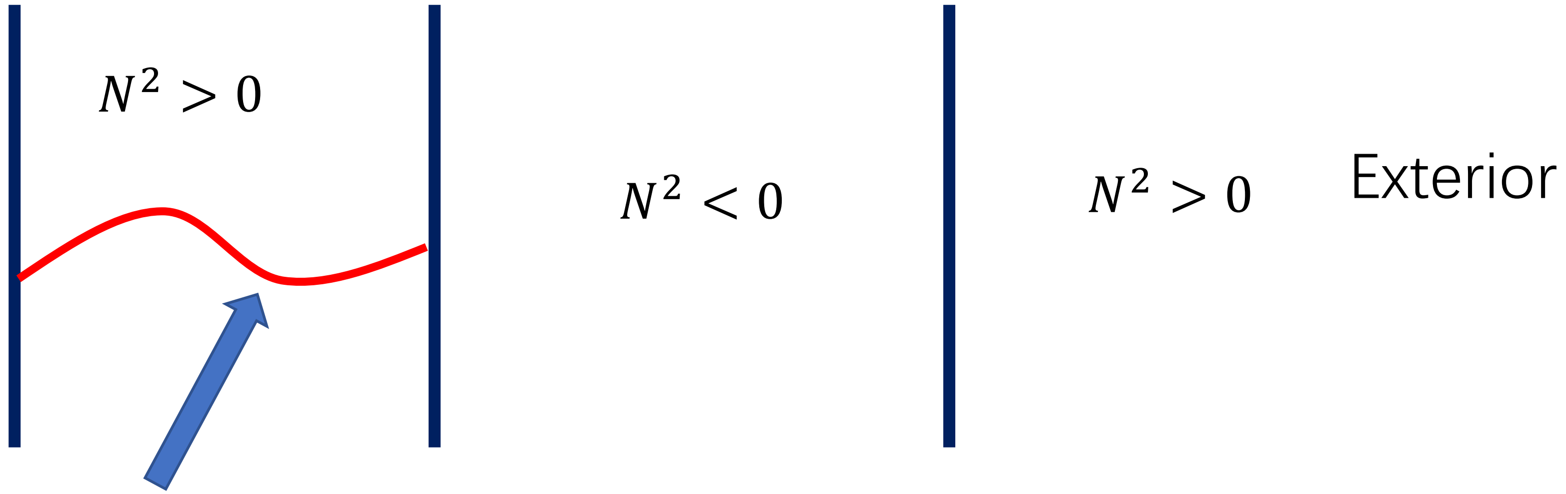
$$\mathrm{d}s^2 = -N^2\mathrm{dt}^2 + \gamma^2(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + q_{AB}\mathrm{d}x^A\mathrm{d}x^B$$

- We can prove

If H is a Killing horizon, then N^2 must be zero at H and vice versa.

ADM decomposition

- Assume there are three Killing horizon, then there is a spacetime region V bounded by N=0;



Equal-t surface, where $g^{\mu\nu}(\mathrm{d}t)_\mu(\mathrm{d}t)_\nu = g^{tt} = -N^2 < 0$

$$\mathrm{d}s_\Sigma^2 = h_{ab}\mathrm{d}x^a\mathrm{d}x^b = \gamma^2\mathrm{d}\phi^2 + q_{AB}\mathrm{d}x^A\mathrm{d}x^B$$

ADM decomposition of Einstein equation

- The Hamiltonian constraint and a momentum constraint read

$$^{(d)}R + K^2 - K_{ab}K^{ab} = 2\hat{\rho}\,,$$

$$D^a K_{ab} - D_b K = -\hat{J}_b$$

- Due to stationary the evolutional equations of extrinsic curvature and induced metric read $\partial_t K_{ab} = \partial_t h_{ab} = 0$, which gives us following two equations

$$^{(d)}R_{ab} - 2K_{ac}K^c{}_b = \frac{1}{N}\mathcal{L}_\beta K_{ab} + N^{-1}D_aD_bN + \left[\hat{\mathcal{T}}_{ab} + \frac{h_{ab}}{d-1}(\hat{\rho} - \hat{\mathcal{T}})\right] \qquad 2NK_{ab} + \mathcal{L}_\beta h_{ab} = 0$$

Here $\beta^a = -\omega\Psi^a$ is the shift vector field.

Extrinsic curvature

- The extrinsic curvature of equal-t surface is

$$K_{ab} = -\frac{1}{2N}(D_a\beta_b + D_b\beta_a) = \frac{1}{2N}(\Psi_a D_b\omega + \Psi_b D_a\omega)$$

- The we see $K = N^{-1}\Psi^a D_a\omega = 0$, so equal-t surface is extreme

$$0 = \mathcal{L}_\beta(h^{ab}K_{ab}) = h^{ab}\mathcal{L}_\beta K_{ab} + K^{ab}\mathcal{L}_\beta h_{ab} = h^{ab}\mathcal{L}_\beta K_{ab} + 2NK^{ab}K_{ab}$$

$$^{(d)}R_{ab} - 2K_{ac}K^c{}_b = \frac{1}{N}\mathcal{L}_\beta K_{ab} + N^{-1}D_aD_bN + \left[\hat{\mathcal{T}}_{ab} + \frac{h_{ab}}{d-1}(\hat{\rho} - \hat{\mathcal{T}})\right]$$

$$^{(d)}R = N^{-1}D^2N + \left[\hat{\mathcal{T}} + \frac{d}{d-1}(\hat{\rho} - \hat{\mathcal{T}})\right]$$

Constraint equation for lapse function

- Now we have two equations about scalar curvature

$$^{(d)}R = N^{-1}D^2N + \left[\hat{\mathcal{T}} + \frac{d}{d-1}(\hat{\rho} - \hat{\mathcal{T}})\right]$$

$$^{(d)}R + K^2 - K_{ab}K^{ab} = 2\hat{\rho}$$

$$D^2N^2 = 2N^2[\hat{\rho} + \hat{\mathcal{T}}/(d-1)] + 2h^{ab}(\partial_a N)(\partial_b N) + 2N^2 K_{ab}K^{ab}$$

Nonnegative due to SEC

Nonnegative

$$D^2N^2 \geq 0$$

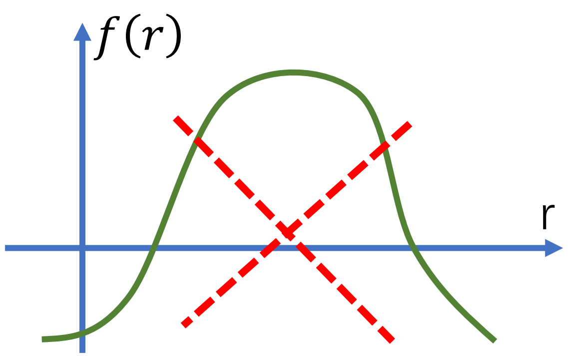
Generalization in asymptotically de-Sitter case

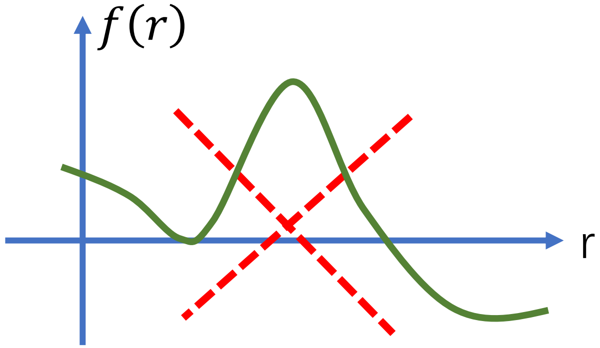
- One significant difference in the dS case is that there is a positive cosmological constant which itself violates the SEC.

if the conditions are satisfied inside the cosmological horizon, then

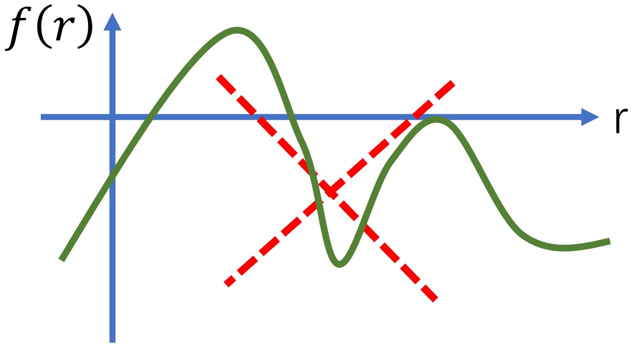
- there is no horizon inside every odd-order connected branch of cosmological horizon and

$$f(r) = -\xi^{\mu}\xi_{\nu}$$





- there is at most one odd-order horizon inside every degenerated connected branch of cosmological horizon.



We have absorbed the cosmological constant term to the energy momentum tensor!

Generalization in asymptotically de-Sitter case

- In the asymptotically dS case, the cosmological constant always dominates at infinity and break the SEC at infinity;
- However, in compact horizon case, we only requires SEC to be satisfied inside the cosmological horizon.
- In RN-dS black hole, there is an event horizon as well as an inner horizon behind the event horizon when the U(1) charge is small.
- When the charge is large enough to dominate the interior, there is no horizon except the cosmological horizon.

Physical significance

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - T^{\mu\nu}k^{\mu}k^{\nu}$$

- NEC and SEC will lead to the caustic singularity and so the formation of event horizon;

- We show that this is **not** the entire story: classical matter will also prevent formation of inner horizons.

Summary

- With quite general conditions, this work shows that the number of horizons is highly constrained by classic matters.