An improved Amati correlation from Gaussian copula

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Contents

01 Motivation

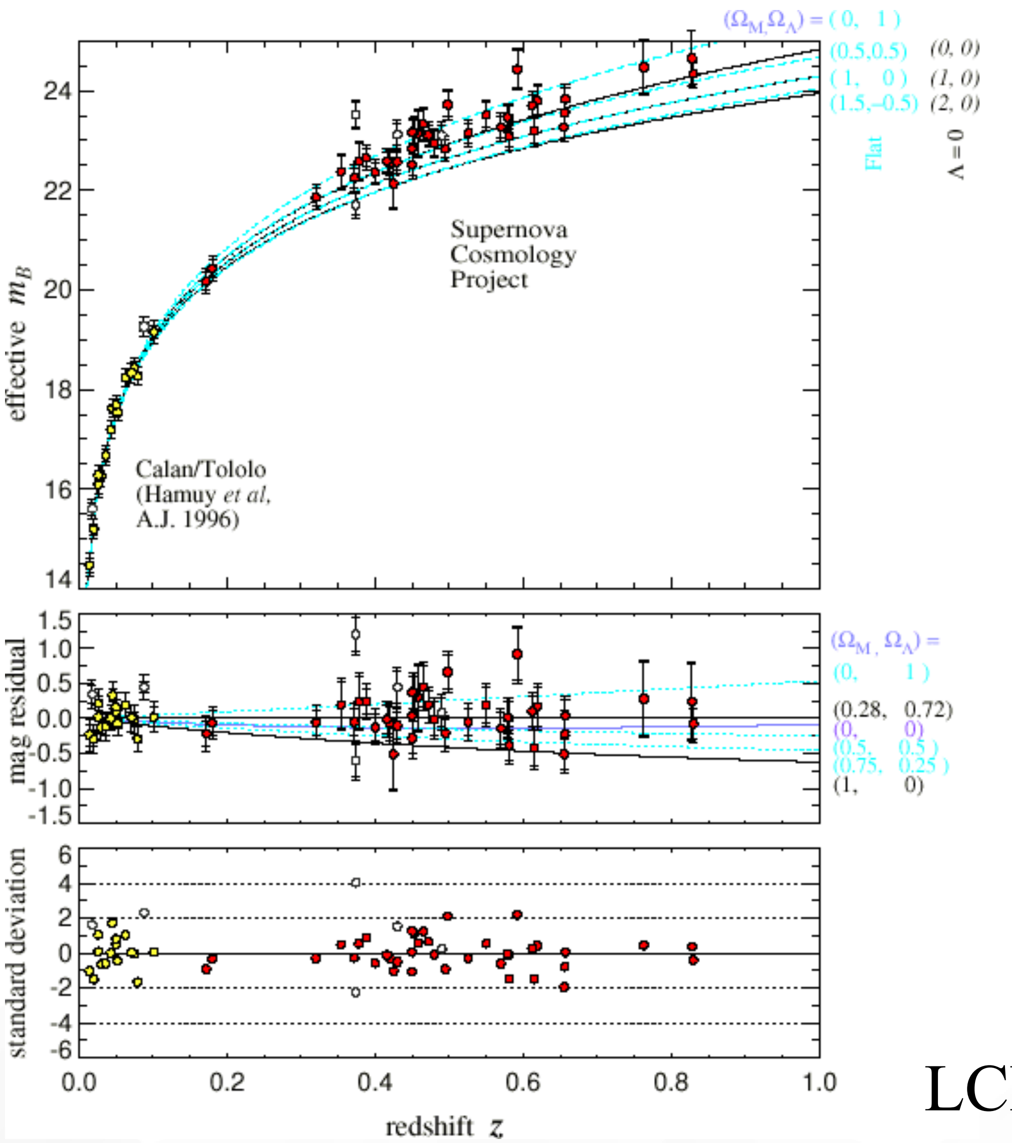
02 Copula function

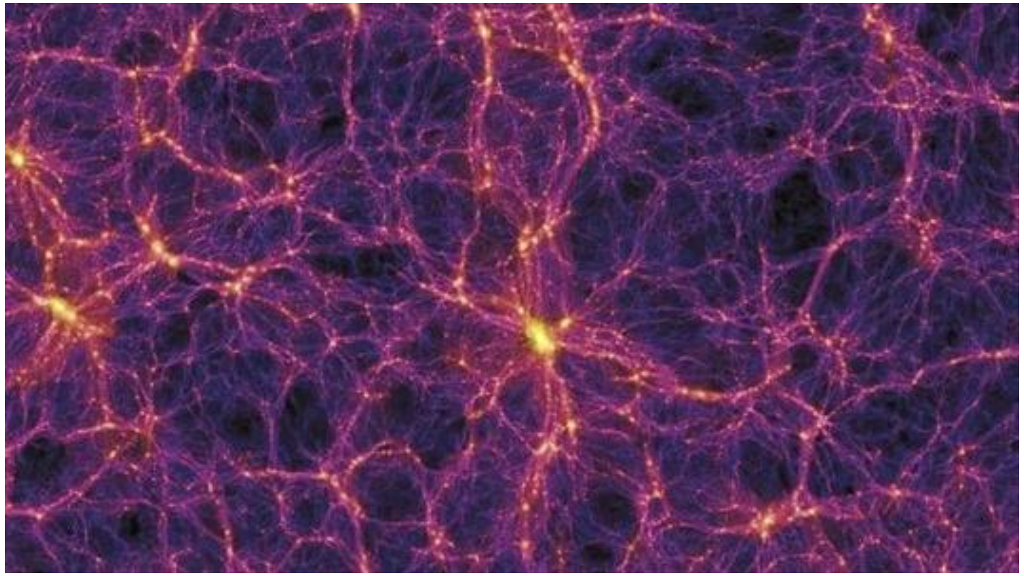
03 An improved Amati correlation

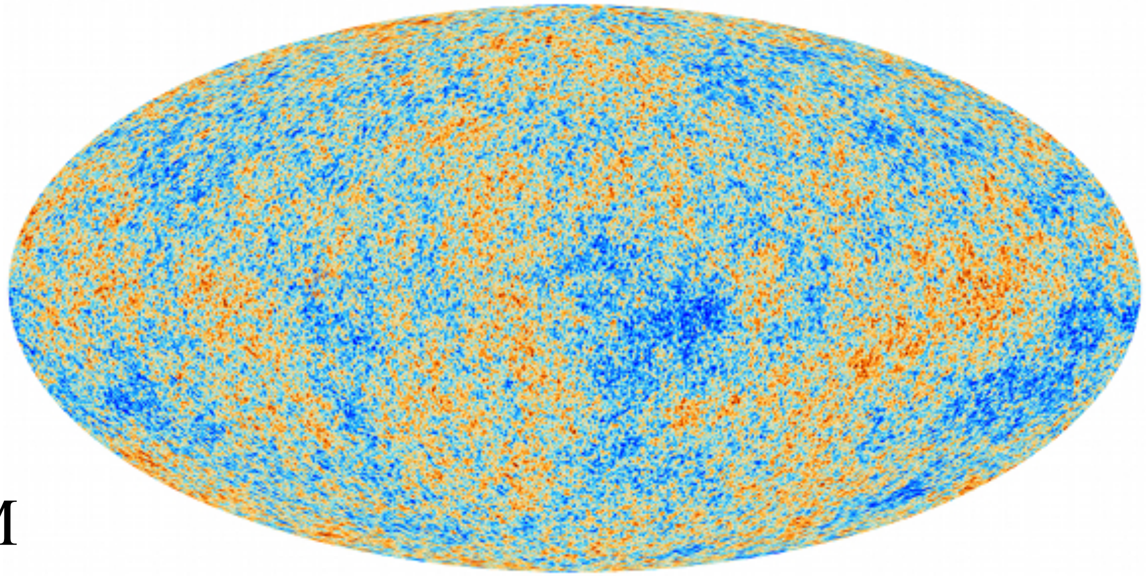
04 Conclusions

Motivation

Accelerated cosmic expansion

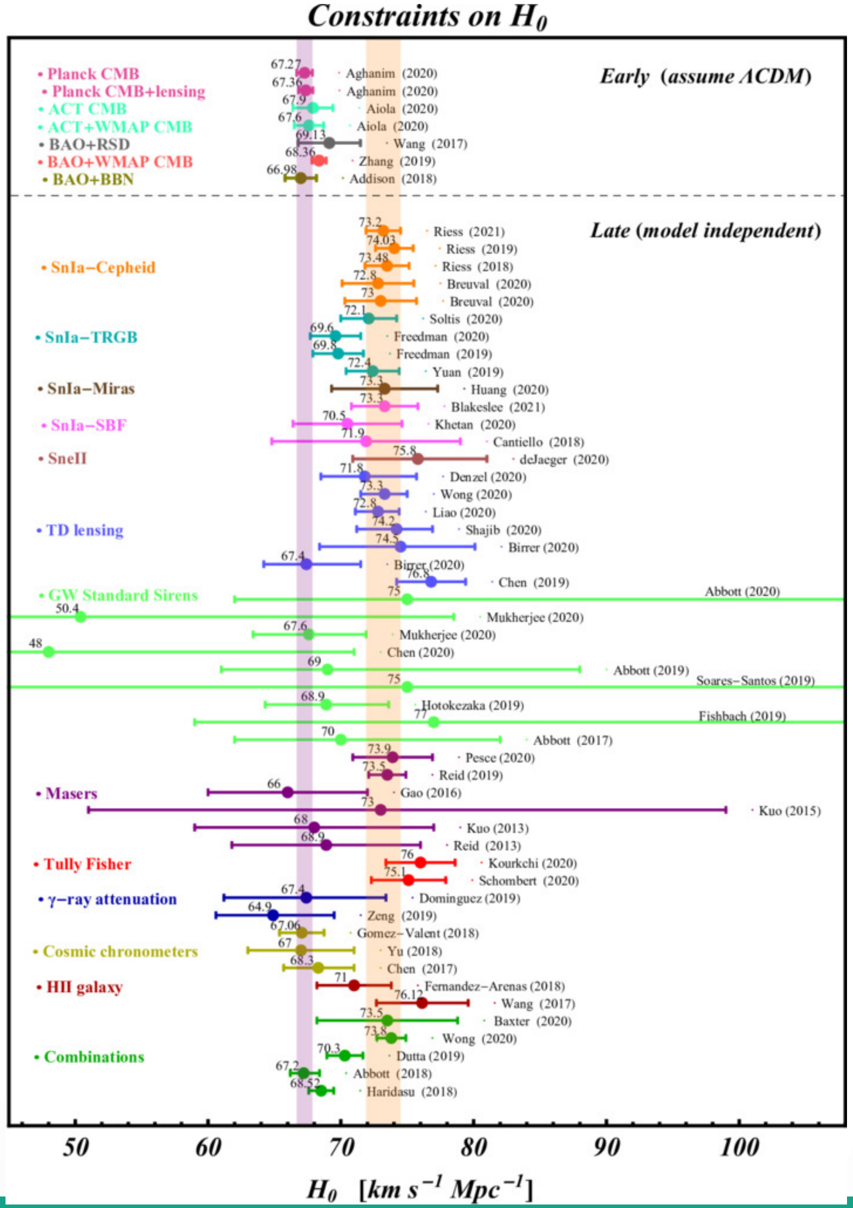






LCDM

Motivation



H_0 tension

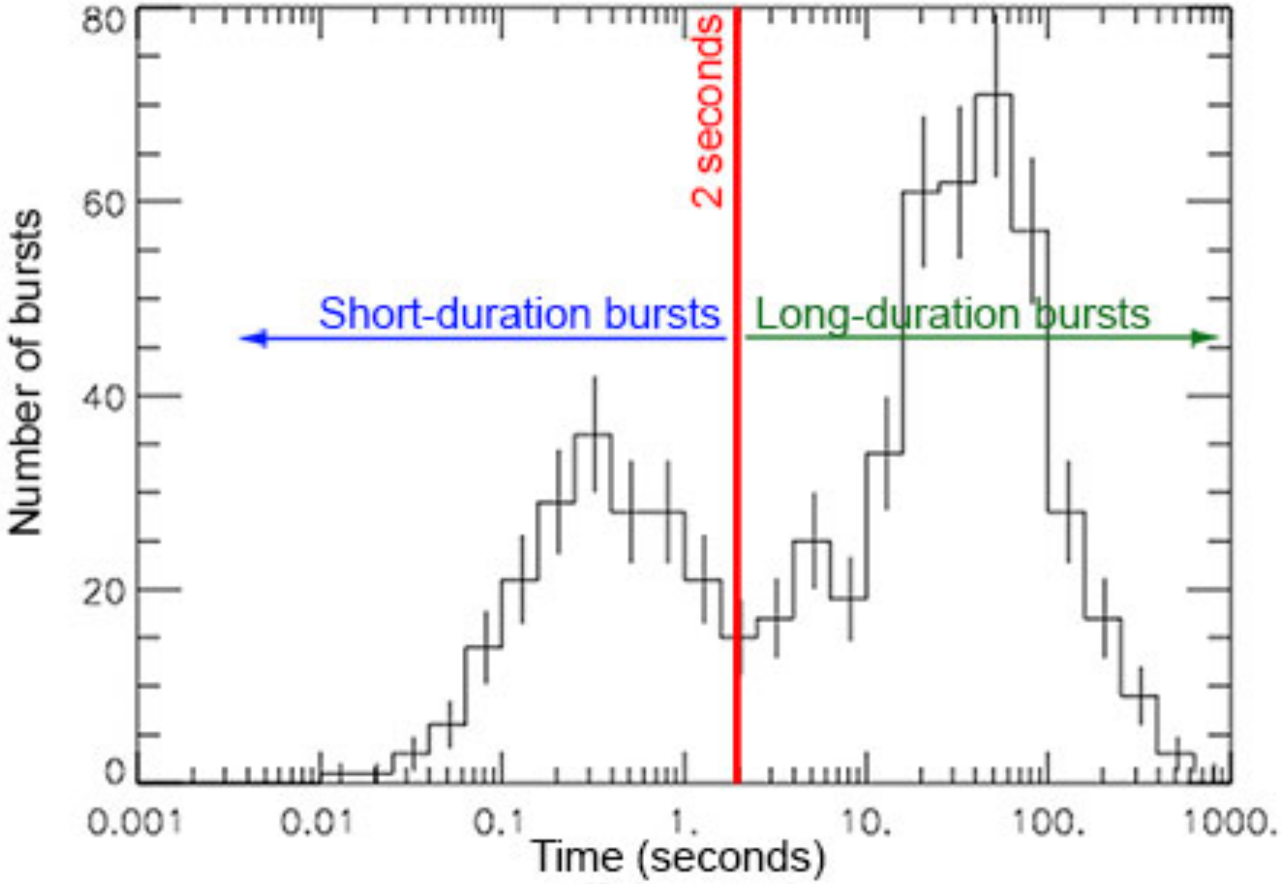
红移沙漠：

$$2 < z < 1100$$

伽马射线暴：

$$0 < z < 10$$

Gamma ray bursts (GRBs)



The luminosity correlations:

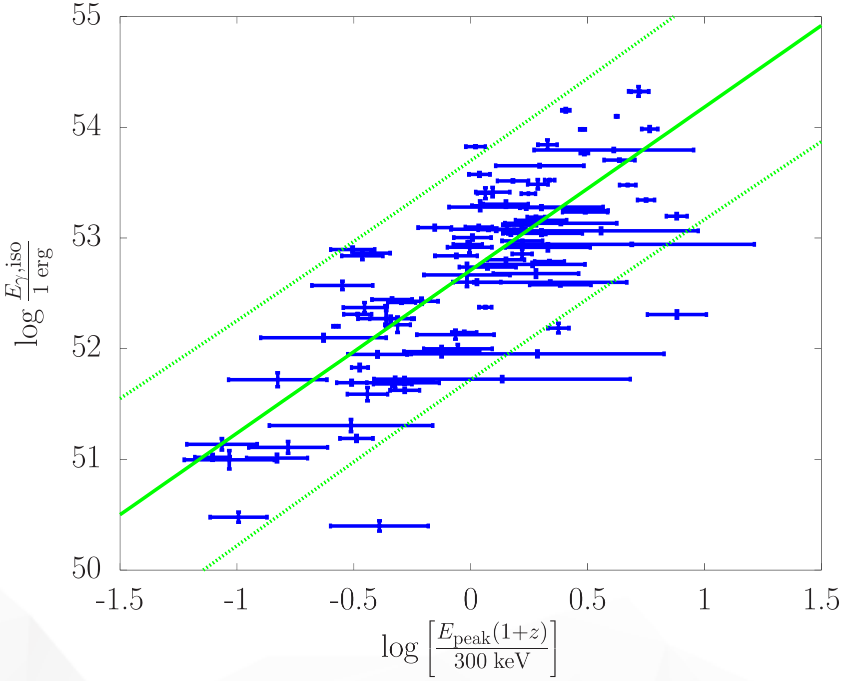
- $L_{\rm iso} - \tau_{\rm lag}$ correlation
- Ghirlanda correlation
- Liang-Zhang correlation
- Amati correlation

....

Motivation

Amati correlation:

$$\log \frac{E_{iso}}{1\mathrm{erg}} = a + b \log \frac{E_p}{300\mathrm{keV}}$$



Extended Amati correlation:

$$\log \frac{E_{iso}(1+z)^{-k_{iso}}}{1\mathrm{erg}} = a + b \log \frac{E_p(1+z)^{-k_p}}{300\mathrm{keV}}$$

$$\log \frac{E_{iso}}{1\mathrm{erg}} = \left(a + \alpha \frac{z}{1+z}\right) + \left(b + \beta \frac{z}{1+z}\right) \log \frac{E_p}{300\mathrm{keV}}$$

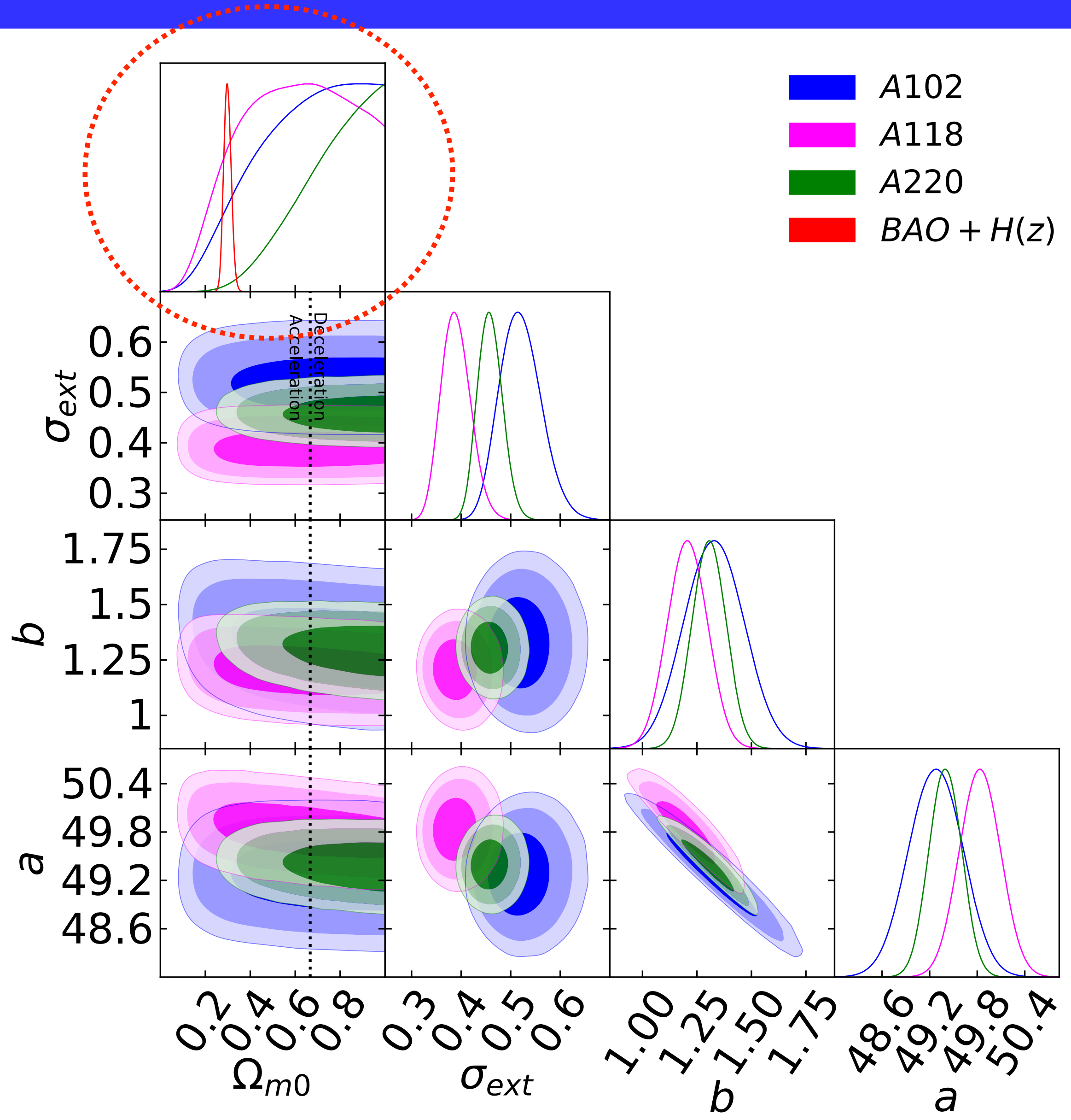
E_{iso}: the isotropic energy

E_p : the peak spectral energy

$E_{iso} = 4\pi d_L^2 S_{bolo}(1+z)^{-1}$

S_{bolo}: the bolometric fluence

Motivation



Khadka, et al. arXiv: 2105.12692

Copula

What are copulas?

The word copula was first employed in a mathematical or statistical sense by Abe Sklar (1959) in the theorem describing the functions that "**join together**" **one-dimensional distribution functions to form multivariate distribution functions.**

Type of copulas

t-copula

Gaussian copula

Frank copula

…….

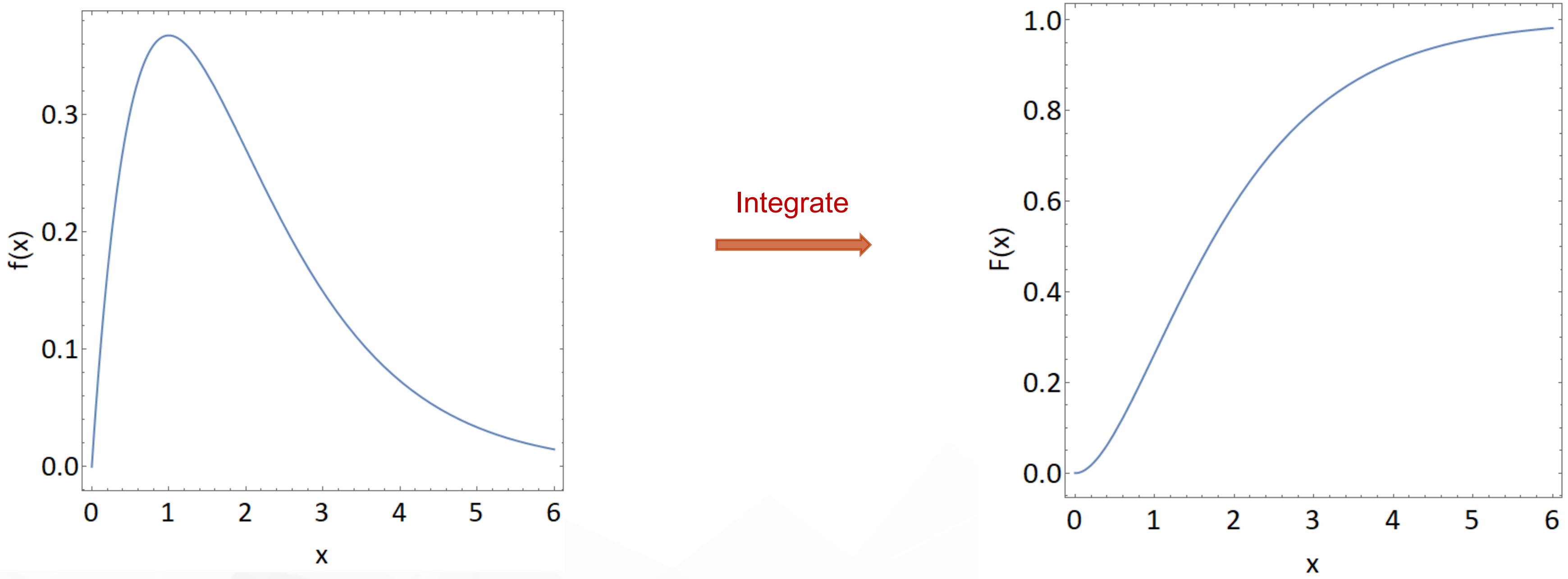
Copula

How does copula work?

Assuming a variable *x*, its probability density function (PDF) and cumulative distribution function (CDF), respectively, are *f*(*x*) and *F*(*x*):

$$F(x) = \int_{-\infty}^{x} f(x')dx'.$$

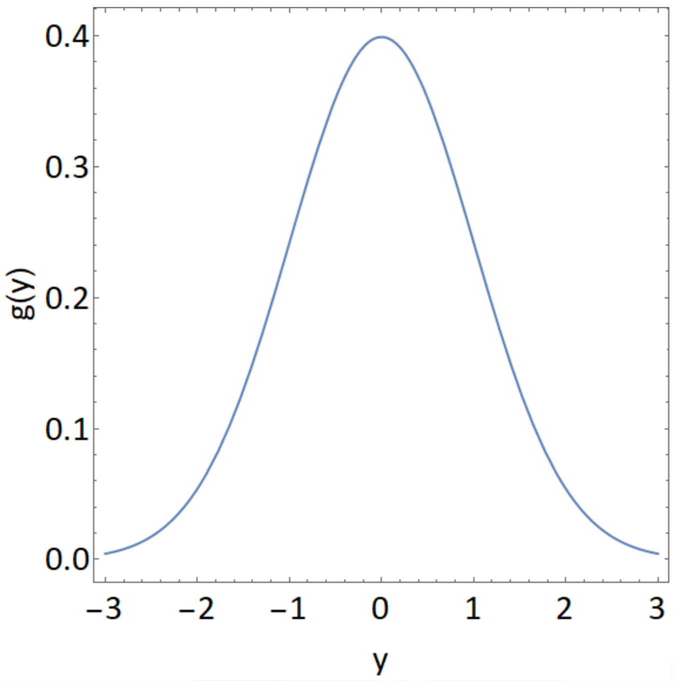
例如： $f(x) = xe^{-x}$

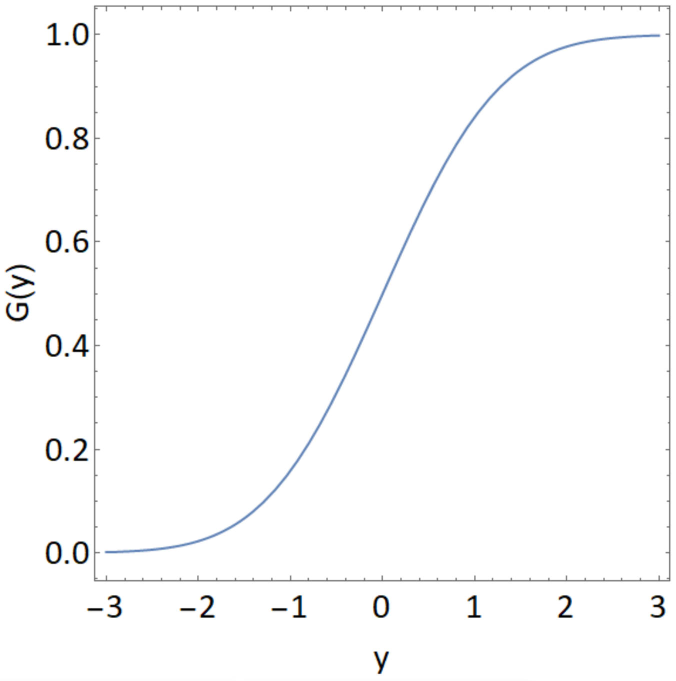


Copula

Now, we introduce other variable y which obeys a standard Gaussian distribution. Its the PDF and CDF are:

$$g(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}, \; G(y) = \int_{-\infty}^{y} g(y')dy'.$$

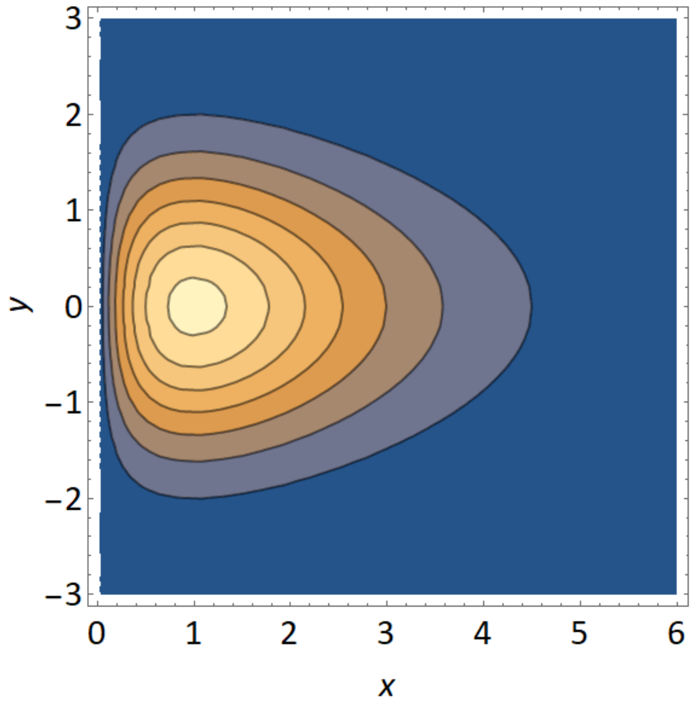




Copula

If x and y are independent of each other, the joint CDF of variables x and y is

$$H(x, y) = F(x)G(y),$$



For the case that x is related to y, we can use copula to obtain the joint distribution of variables x and y

$$H(x, y; \rho) = C(F(x), G(y); \rho).$$

Copula

Gaussian copula function

$$C(u, v; \rho) = \Psi_2 \left[\Psi_1^{-1}(u), \Psi_1^{-1}(v); \rho\right]$$

$u=F(x)$ $v=G(y)$

ψ_2 is the 2-dimensional standard Gaussian CDF

ψ_1 is the 1-dimensional standard Gaussian CDF

The PDF of joint distribution $H(x,y;\rho)$ is

$$h(x, y; \rho) = \frac{\partial^2 H(x, y; \rho)}{\partial x \partial y} = \frac{\partial^2 C(u, v; \rho)}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} = c(u, v; \rho) f(x) g(y).$$

Here $c(u,v;\rho)$ is the density function of Gaussian copula,

Copula

$c(u,v;\rho)$ has the form

$$c(u,v;\rho)=\frac{\partial^2\Psi_2[\Psi_1^{-1}(u),\Psi_1^{-1}(v);\rho]}{\partial u\partial v}$$

$$=\frac{1}{\sqrt{\det\boldsymbol{\Sigma}}}\exp\left\{-\frac{1}{2}\left[\boldsymbol{\Psi}^{-1^T}\left(\boldsymbol{\Sigma}^{-1}-\mathbf{I}\right)\boldsymbol{\Psi}^{-1}\right]\right\},$$

where $\boldsymbol{\psi}^{-1}\equiv[\psi_1^{-1}(u),\psi_1^{-1}(v)]^T$, **I** stands for the identity matrix, and

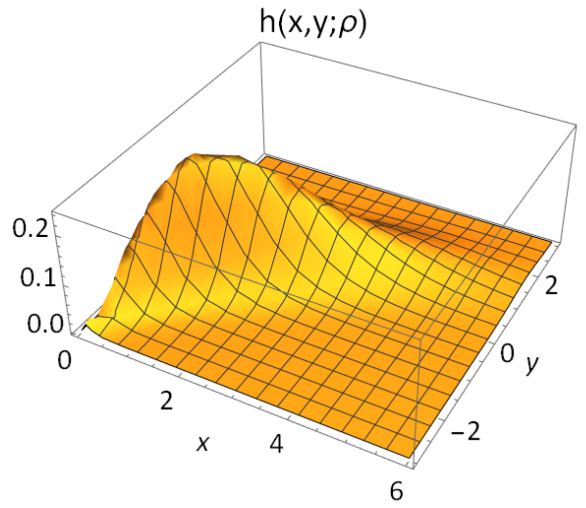
$$\boldsymbol{\Sigma}\equiv\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$$

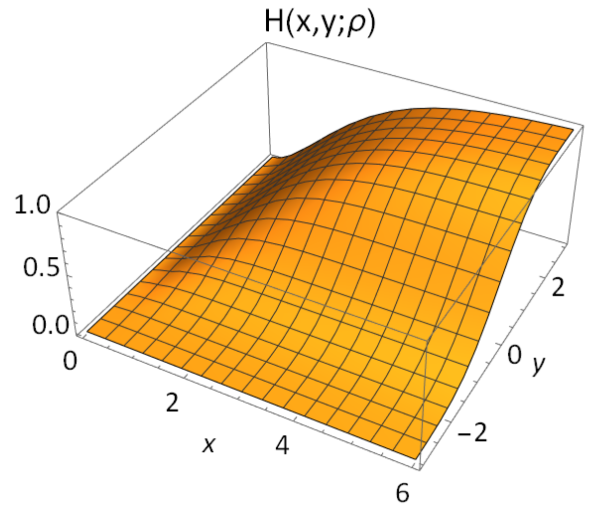
For concrete *f(x)* and *g(y)*, the PDF of joint distribution *H(x,y;ρ)* has the expression

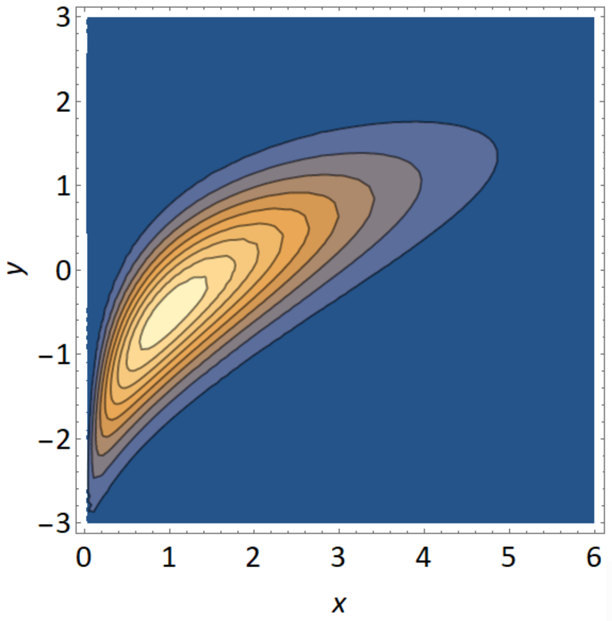
$$h(x,y;\rho)=\frac{x\exp(-x)}{\sqrt{2\pi}\sqrt{1-\rho^2}}\exp\left\{-\frac{\left[\sqrt{2}\rho\,\mathrm{erfc}^{-1}\left(2e^{-x}\left(-x+e^{x}-1\right)\right)+y\right]^2}{2\left(1-\rho^2\right)}\right\}.$$

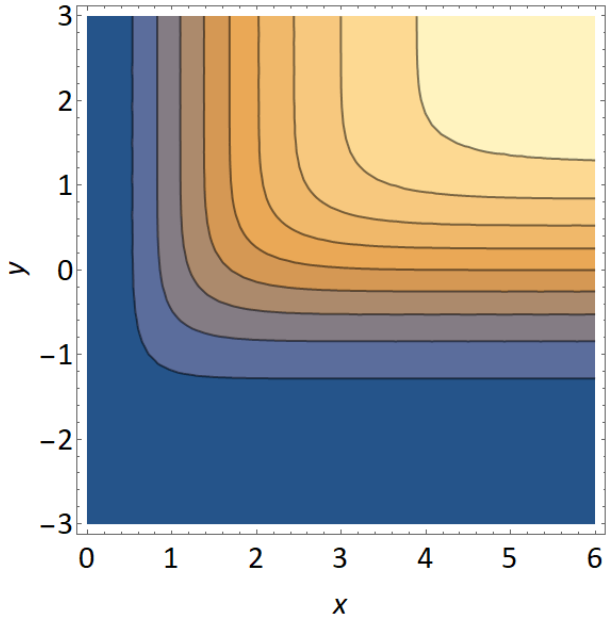
Copula

$\rho=0.8$









Copula

The conditional PDF of variable y is the probability of y when x is fixed, and it has the expression

$$g_y(y|x;\rho) \equiv \frac{h(x,y;\rho)}{f(x)} = c[F(x),G(y);\rho]g(y)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left\{-\frac{\left[\sqrt{2}\rho\,\mathrm{erfc}^{-1}\left(2e^{-x}\left(-x+e^x-1\right)\right)+y\right]^2}{2\left(1-\rho^2\right)}\right\}.$$

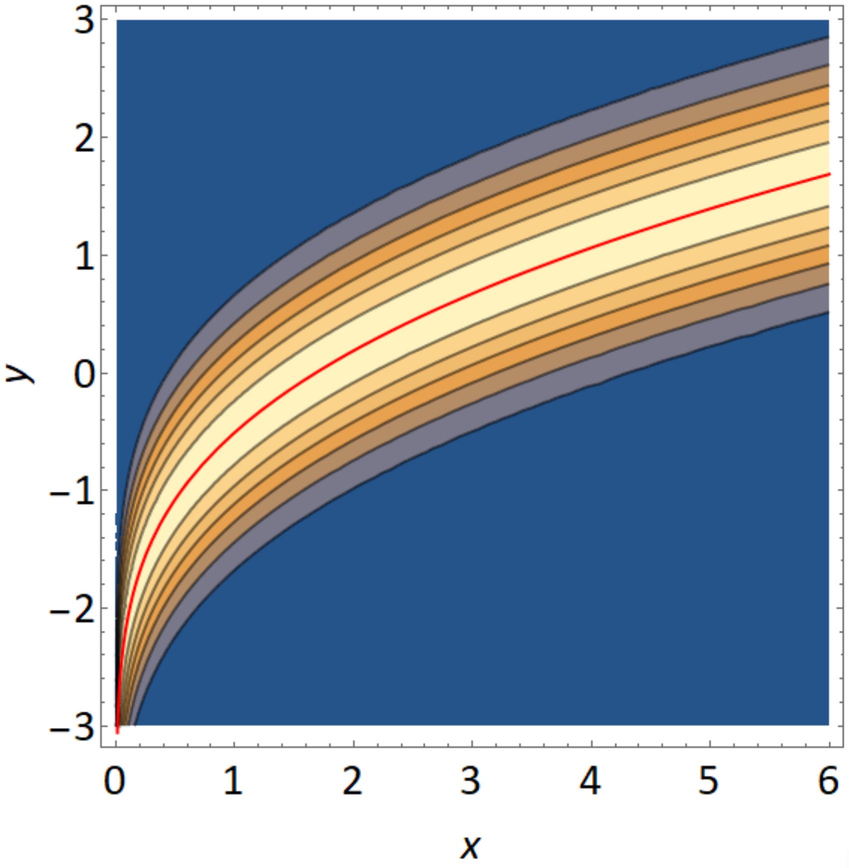
Since the variable y obeys a Gaussian distribution, the highest probability of y corresponds to

$$\left[\sqrt{2}\rho\,\mathrm{erfc}^{-1}\left(2e^{-x}(-x-1+e^x)\right)+y\right]^2 = 0$$

Copula

From the highest probability of *y*, we obtain

$$y = -\sqrt{2}\rho \,\mathrm{erfc}^{-1}\left(2e^{-x}\left(-x + e^{x} - 1\right)\right).$$



An improved Amati correlation

Two Gaussian distributions for $x = \log \frac{E_p}{300\text{keV}}$ and $y = \log \frac{E_{iso}}{1\text{erg}}$

$$f(x; \bar{a}_x, \sigma_x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(\bar{a}_x - x)^2}{2\sigma_x^2}} \qquad g(y; \bar{a}_y, \sigma_y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(\bar{a}_y - y)^2}{2\sigma_y^2}}$$

The PDF of redshift *z* of GBR data has the special form

$$w(z) = ze^{-z}$$

An improved Amati correlation

An improved Amati correlation:

$$y_{\text{copula}} = a + b\,x + c\,\text{erfc}^{-1}\left[2e^{-z}\left(e^{z} - z - 1\right)\right]$$

Here

$$a \equiv \bar{a}_y - \frac{\left(\rho_2\rho_3 - \rho_1\right)\bar{a}_x\sigma_y}{\left(\rho_2^2 - 1\right)\sigma_x},$$

$$b \equiv \frac{\left(\rho_2\rho_3 - \rho_1\right)\sigma_y}{\left(\rho_2^2 - 1\right)\sigma_x},$$

$$c \equiv \frac{\sqrt{2}\sigma_y\left(\rho_3 - \rho_1\rho_2\right)}{\rho_2^2 - 1}.$$

An improved Amati correlation

220 long GRB data points

79 low redshift GRBs

$z < 1.4$

141 high redshift GRBs

$z > 1.4$

Calibrating the Amati correlation

The distance modulus of GRBs

A fiducial cosmological model: LCDM

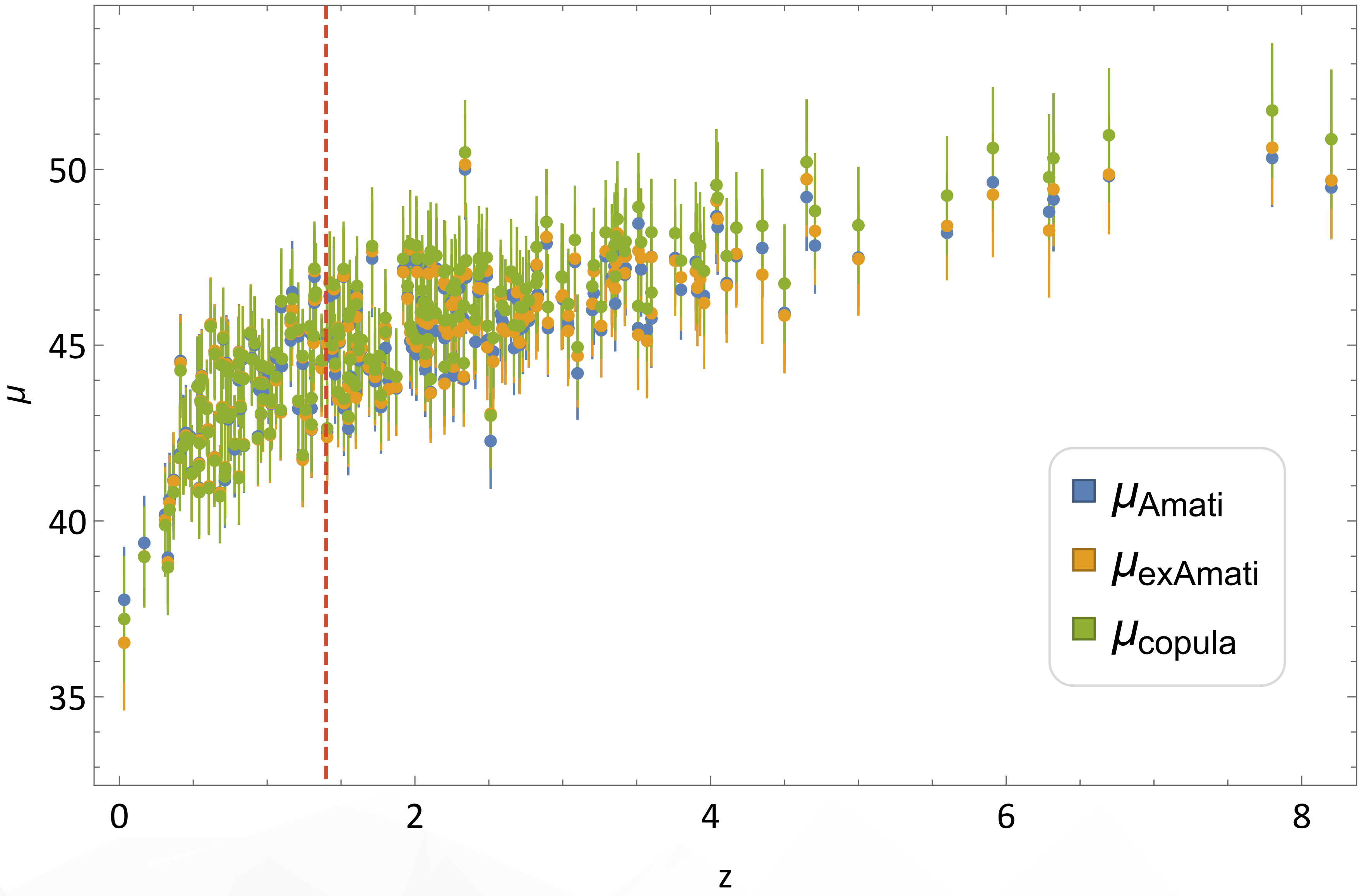
$$\Omega_{\rm m0} = 0.30$$

$$H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1}$$

An improved Amati correlation

	Amati method		extended Amati method		copula method	
	Best-fit(σ)	*0.68 CL*	*Best-fit(σ)*	*0.68 CL*	*Best-fit(σ)*	*0.68 CL*
σ_{int}	0.512(0.045)	$^{+0.054}_{-0.034}$	0.503(0.046)	$^{+0.063}_{-0.025}$	0.510(0.045)	$^{+0.054}_{-0.033}$
a	52.710(0.061)	$^{+0.06}_{-0.061}$	52.587(0.333)	$^{+0.324}_{-0.334}$	52.847(0.144)	$^{+0.145}_{-0.137}$
b	1.290(0.126)	$^{+0.126}_{-0.123}$	1.521(0.367)	$^{+0.422}_{-0.301}$	1.209(0.150)	$^{+0.145}_{-0.148}$
c	–	–	–	–	−0.217(0.207)	$^{+0.208}_{-0.198}$
α	–	–	0.319(0.729)	$^{+0.734}_{-0.702}$	–	–
β	–	–	−0.680(0.779)	$^{+0.624}_{-0.915}$	–	–
$-2\ln\mathcal{L}$	121.514		119.492		120.281	

An improved Amati correlation



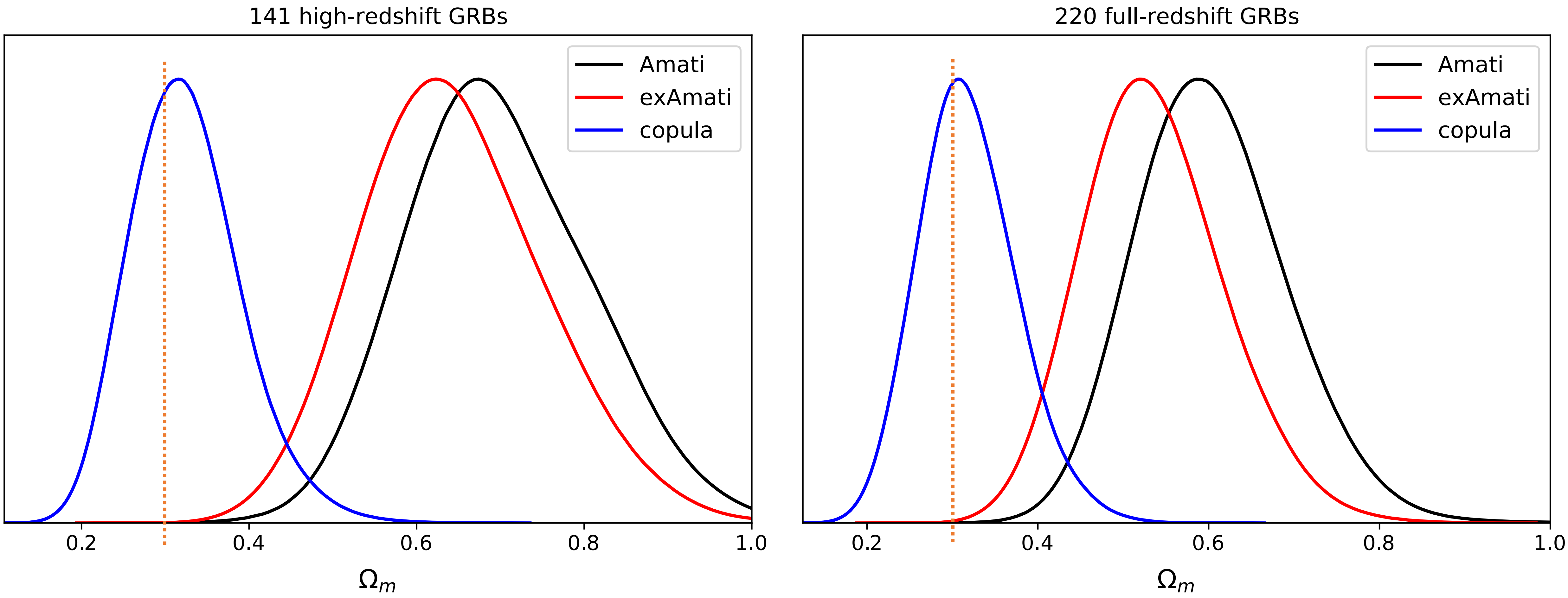
An improved Amati correlation

Constraint on LCDM

	high-redshift			full-redshift		
	141 $\Omega_{m0}(\sigma)$	68%CL	χ^2	220 $\Omega_{m0}(\sigma)$	68%CL	χ^2
Amati	0.677(0.108)	$^{+0.120}_{-0.100}$	98.519	0.589(0.088)	$^{+0.091}_{-0.082}$	177.657
extend Amati	0.622(0.109)	$^{+0.118}_{-0.100}$	91.668	0.519(0.083)	$^{+0.089}_{-0.075}$	168.671
copula	0.308(0.066)	$^{+0.072}_{-0.056}$	91.693	0.307(0.058)	$^{+0.063}_{-0.051}$	166.426

$$H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1}$$

An improved Amati correlation



$$H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1}$$

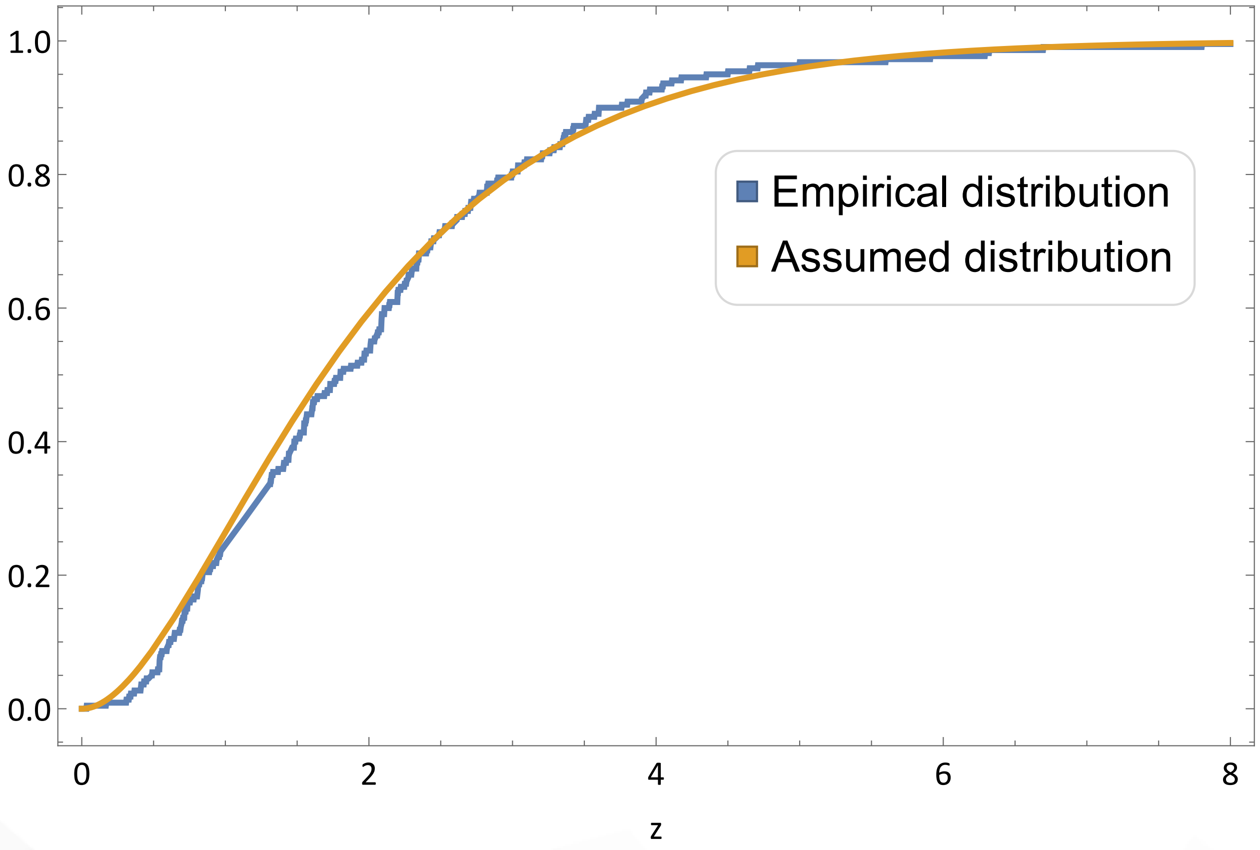
Conclusions

- Using the Gaussian copula, we obtain an improved Amati correlation.
- With the improved Amati correlation, the constraints on LCDM model from GRB data are very well consistent with the fiducial model.
- The GRBs can be used as an effective cosmological explorer.

END

THANK YOU FOR LISTENING

The Kolmogorov-Smirnov test (K-S test)



$$D = \max |W_N(z) - W(z)|,$$

$W_N(z)$ and $W(z)$ are the empirical and assumed CDFs of redshift distribution of GRBs respectively.

We set the significance level α to be α = 0.05, and obtain the critical value D = 0.09.

$$D = 0.06$$