Effective-range expansion with a left-hand cut and its applications

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Workshop on Non-Perturbative Methods and Their Applications in High-Energy Physics
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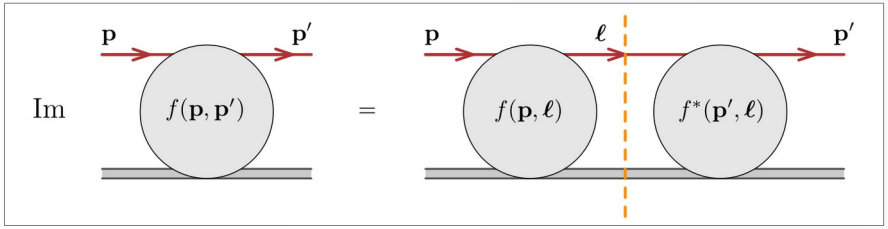
Collaborators: A. Fillin, V. Baru, X.-K. Dong, E. Epelbaum, J.-W. Fu, F.-K. Guo, C. Hanhart, B.-Y. Liu, A. Nefediev, J. Nieves, Q. Wang, W.-J. Wang, and B. Wu

Based on: PRL 131, 131903 (2023)
& PRL 135, 011903 (2025)
& PRD 114, 074030 (2026)
& arXiv: 2610.32139



Effective Range Expansion (ERE)

Unitarity: $SS^\dagger = 1, \quad \text{and} \quad S^\dagger S = 1.$ $\boxed{S = 1 + iT}$



$$-i(T - T^\dagger) = T^\dagger T$$

$$\boxed{\mathrm{Im}\frac{1}{T_\ell} = -\rho(s)}$$

$$2\mathrm{Im}\langle f|T|i\rangle = \sum_n \int \Big[\prod_{i=1}^{i_n} \frac{d^3p_i}{2p_i^0(2\pi)^3}\Big]\langle f|T^\dagger|n\rangle\langle n|T|i\rangle,$$

In general $\quad T_\ell = \dfrac{e^{i\delta_\ell}\sin\delta_\ell}{\rho} = \dfrac{1}{\rho\cot\delta_\ell - i\rho} = \dfrac{8\pi\sqrt{s}}{p\cot\delta_\ell - ip} \to \dfrac{1}{p\cot\delta - ip} = \dfrac{1}{f}$

ERE $\quad p\cot\delta_0 = -\dfrac{1}{a_0} + \dfrac{1}{2}r_0p^2 + \mathcal{O}(p^4)$ $\quad p^2 = 2\mu E, \quad p = \sqrt{2\mu E}$

K-matrix $\quad \dfrac{1}{T} = K^{-1} - i\rho$ $\quad K = \dfrac{\gamma}{s - s_0} + \alpha$

Dispersion relation

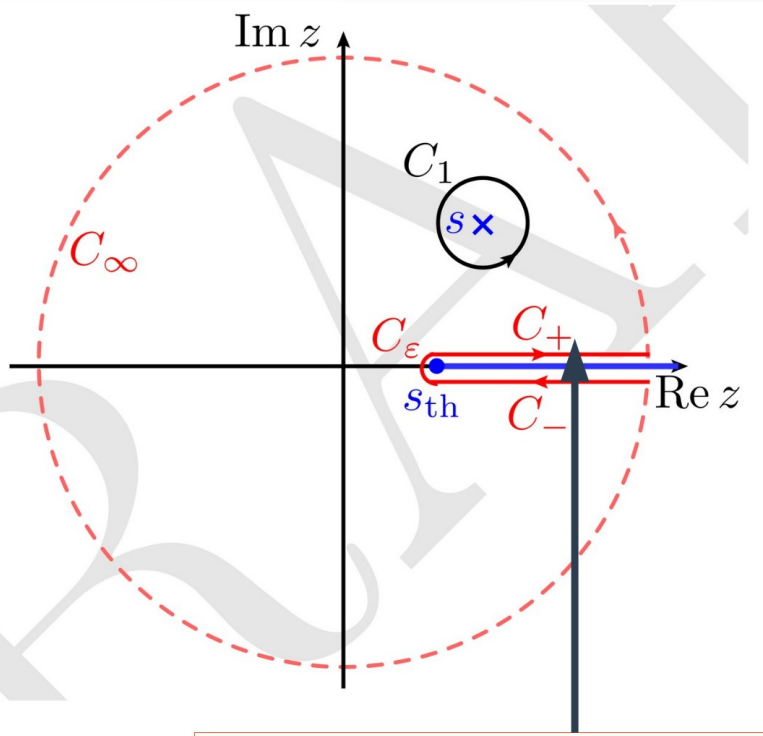
$$\mathrm{Im}\frac{1}{T_\ell} = -\rho(s)$$

$$\frac{1}{T_\ell(s)} = f(s) + \left[G(s_0) - \frac{s-s_0}{\pi}\int_{s_{\rm th}}^{\infty} dz \frac{\rho(z)}{(z-s)(z-s_0)} \right]$$

$$G(s) = i\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m_1^2+i\epsilon}\frac{1}{(q-p)^2-m_2^2+i\epsilon}$$

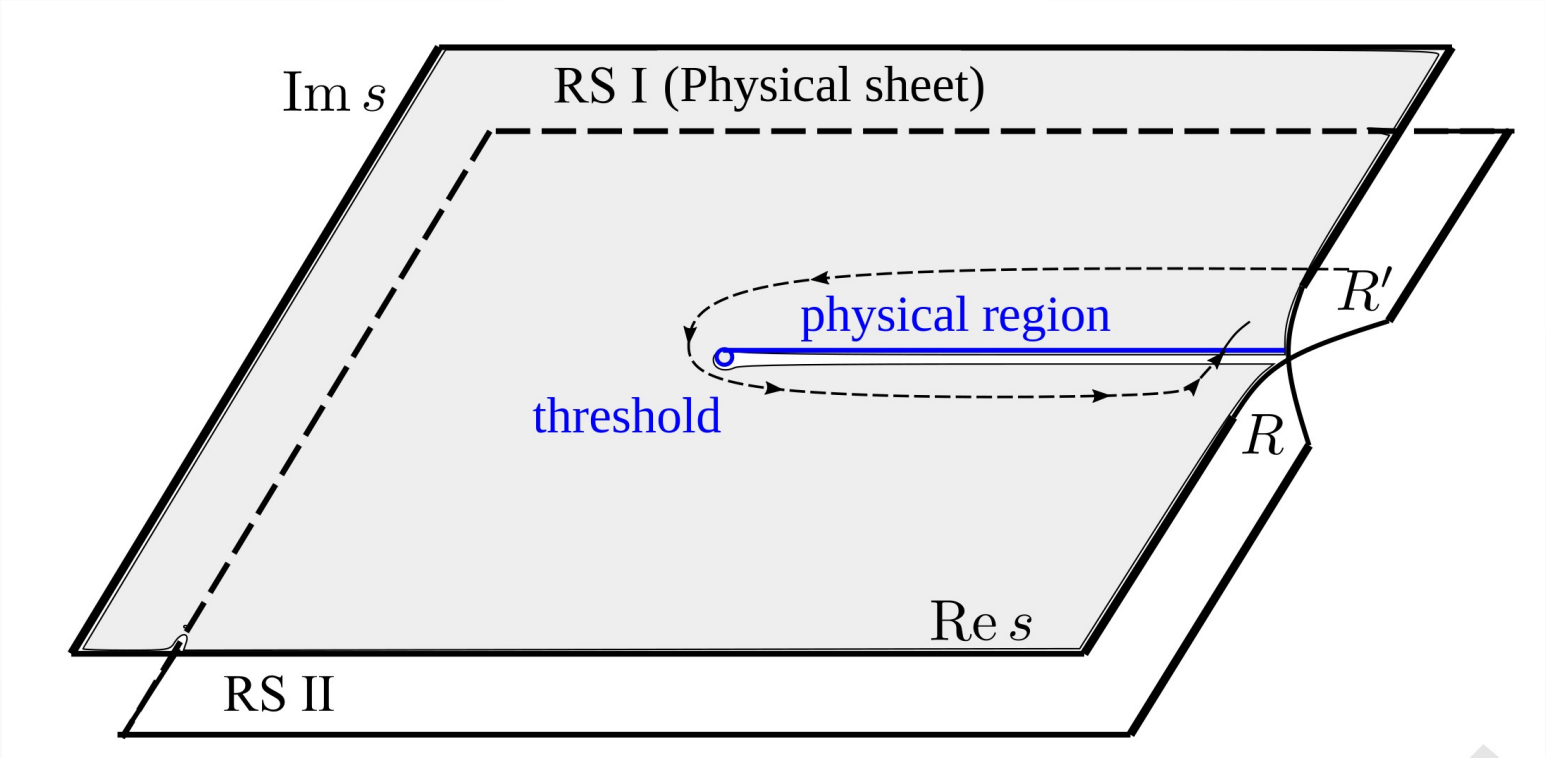
$$f(s) = -\frac{1}{V(s)}$$

$$-T(s) = \frac{V(s)}{1-V(s)G(s)}$$



$$T_\ell(s)_{\rm physical} = \lim_{\epsilon\to 0+} T_\ell(s+i\epsilon)$$

Riemann sheets

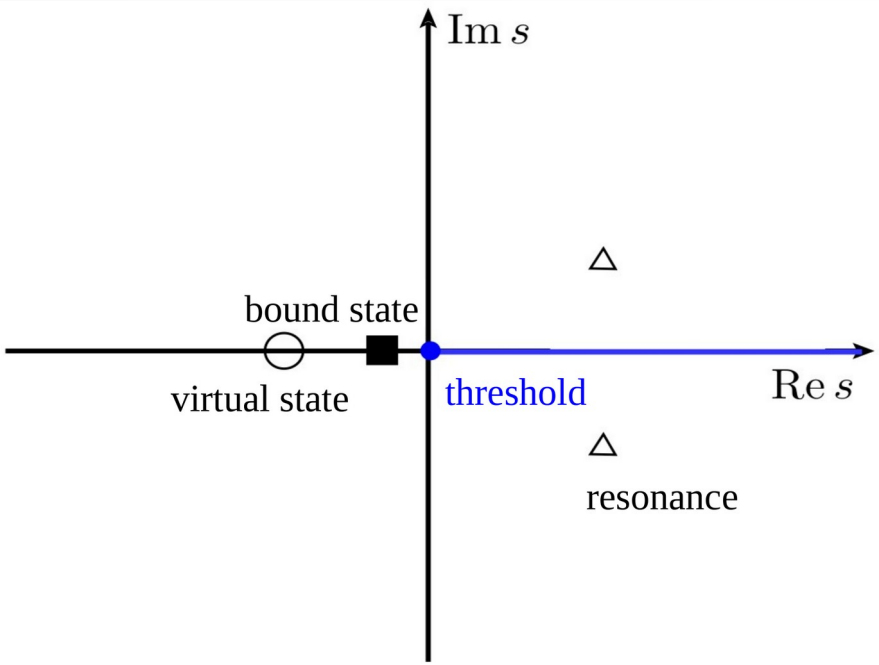


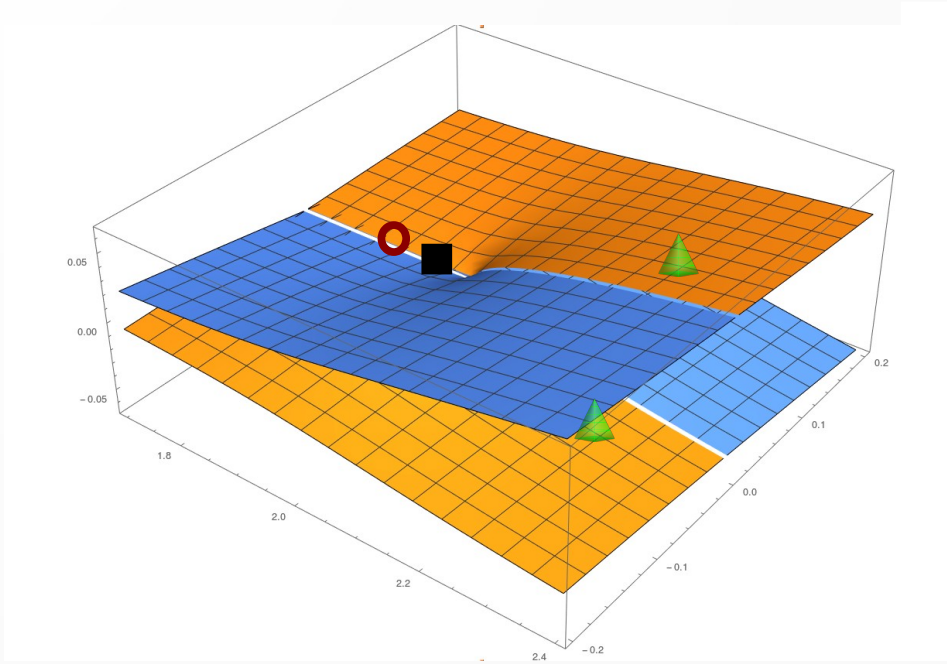
RS-I: $\mathrm{Im}\frac{1}{T_\ell} = -\rho(s)$

RS-II: $\frac{1}{T_\ell^{\mathrm{II}}} = \frac{1}{T_\ell} + 2i\rho$

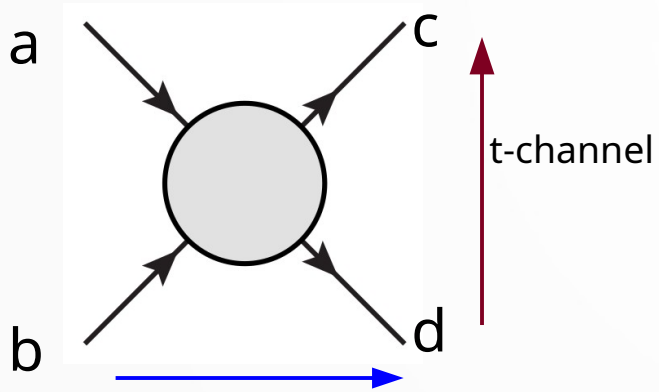
$$\rho(s) = \frac{|\mathbf{p}|}{8\pi\sqrt{s}}\theta(\sqrt{s} - 2m) = \frac{1}{16\pi}\sqrt{1 - 4m^2/s}$$

Pole structures





Crossing Symmetry & Left-Hand Cut (LHC)



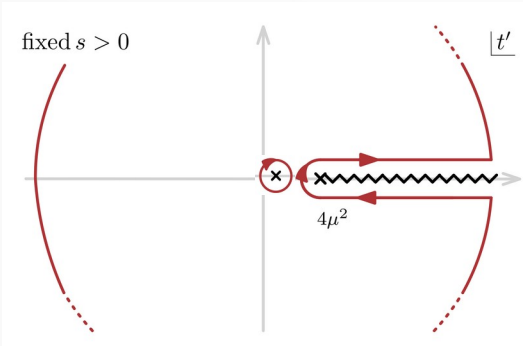
$T(s,t),\ T(s,\cos\theta)$

s-channel $\quad a(p_1)+b(p_2)\to c(p_3)+d(p_4),$

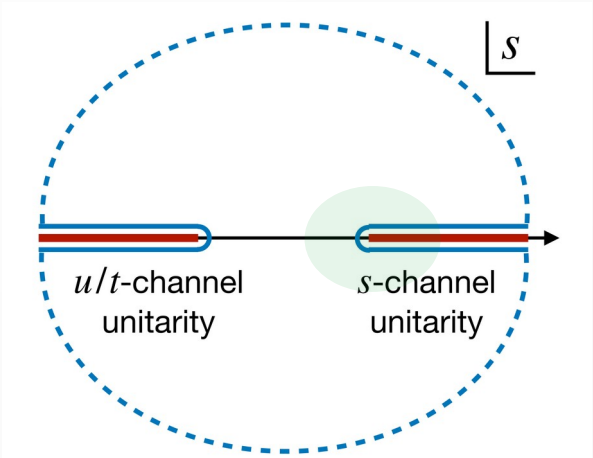
t-channel $\quad a(p_1)+\bar{c}(-p_3)\to \bar{b}(-p_2)+d(p_4),$

u-channel $\quad a(p_1)+\bar{d}(-p_4)\to c(p_3)+\bar{b}(-p_2).$

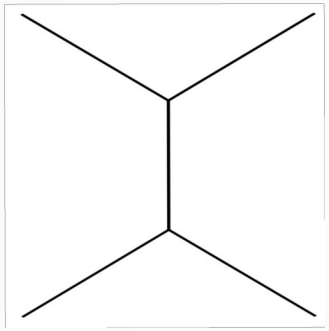
$$T_\ell(s)=\frac{1}{2}\int_{-1}^{1}P_\ell(\cos\theta)T(s,\cos\theta)d\cos\theta\propto\int dtP_\ell(\cos\theta(t))T(s,t)$$



t-channel unitarity



LHC from the one-particle exchange (OPE)

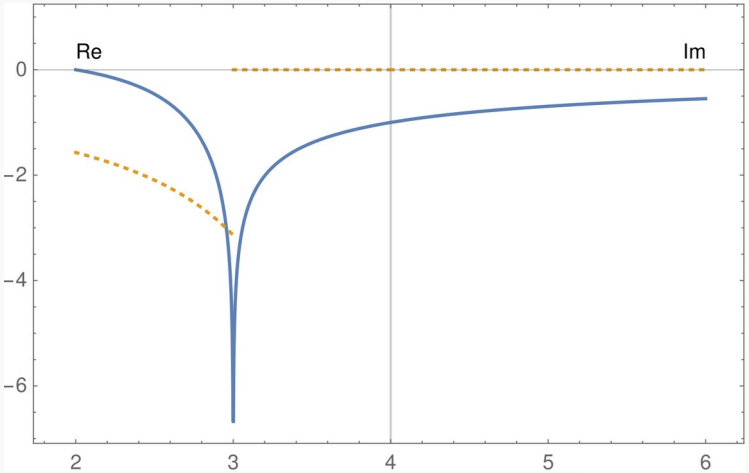


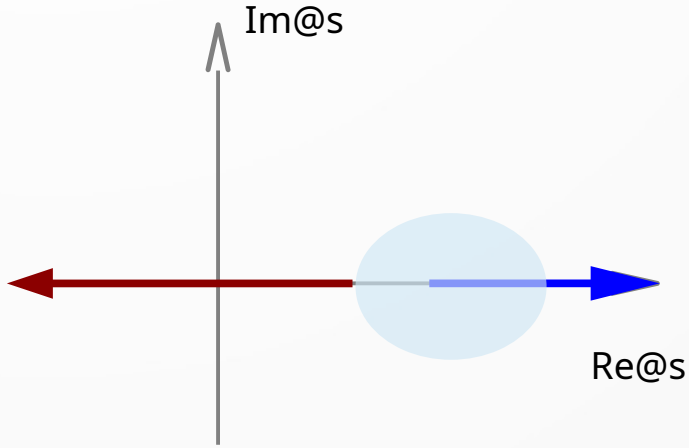
$$t = -2\mathbf{p}^2(1-\cos\theta)$$

$$\propto \frac{e^{-mr}}{r}$$

S-wave

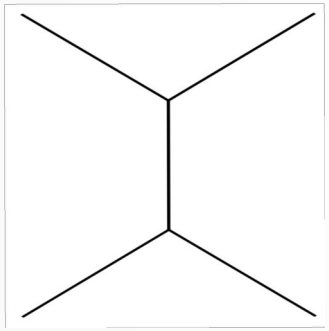
$$\frac{1}{2}\int_{-1}^{+1} d\cos\theta \frac{1}{-2\mathbf{p}^2(1-\cos\theta)-m_\rho^2+i\varepsilon} = -\frac{1}{4\mathbf{p}^2}\log\left(\frac{4\mathbf{p}^2+m_\rho^2}{m_\rho^2}+\frac{4\mathbf{p}^2}{m_\rho^4}i\varepsilon\right)$$





$$k\cot\delta = \frac{1}{a}+\frac{1}{2}rk^2+\mathcal{O}(k^4)$$

LHC from the one-particle exchange (OPE)

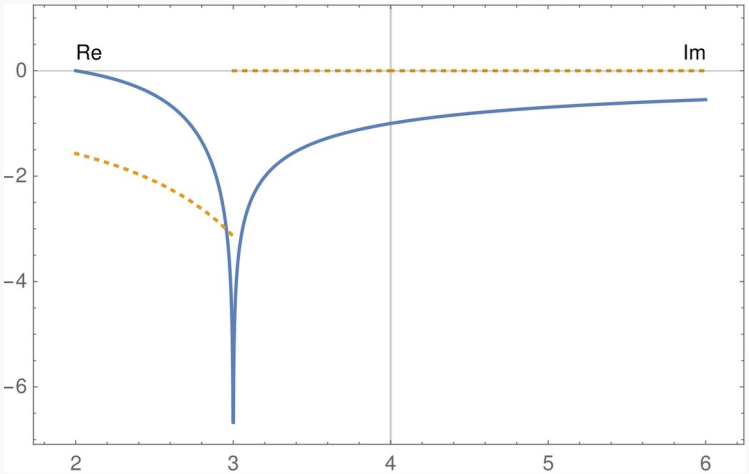


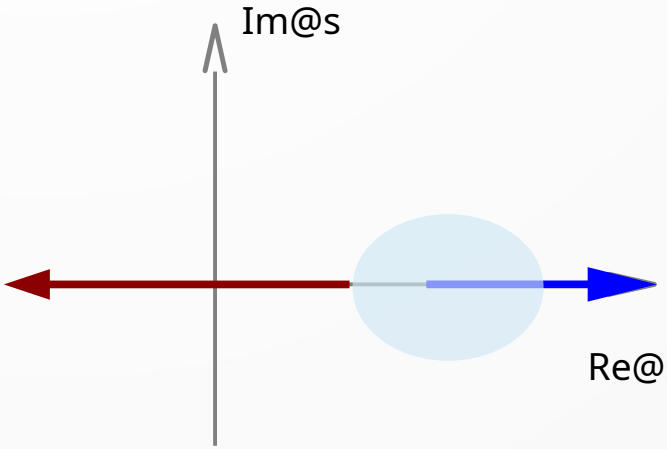
$$t = -2\mathbf{p}^2(1-\cos\theta)$$

$$\propto \frac{e^{-mr}}{r}$$

S-wave

$$\frac{1}{2}\int_{-1}^{+1} d\cos\theta \frac{1}{-2\mathbf{p}^2(1-\cos\theta)-m_\rho^2+i\varepsilon} = -\frac{1}{4\mathbf{p}^2}\log\left(\frac{4\mathbf{p}^2+m_\rho^2}{m_\rho^2}+\frac{4\mathbf{p}^2}{m_\rho^4}i\varepsilon\right)$$





$$k\cot\delta = \frac{1}{a}+\frac{1}{2}rk^2+\mathcal{O}(k^4)$$



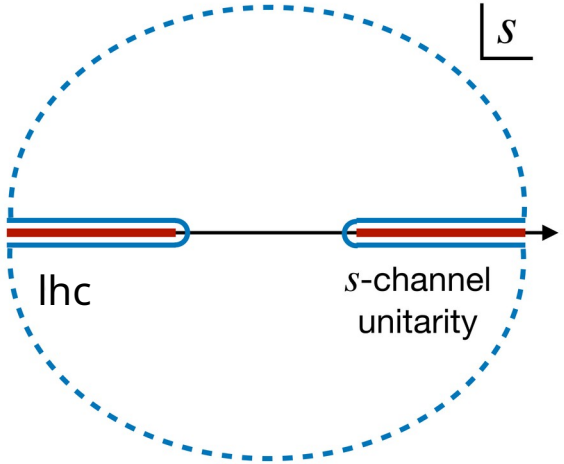
LSE ?

The N/D method

$$T(s) = \frac{N(s)}{D(s)}$$

$$\mathrm{Im}D = \mathrm{Im}\frac{N}{T} = N\mathrm{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\mathrm{thr}} \\ 0, & s < s_{\mathrm{thr}} \end{cases}$$

$$\mathrm{Im}N = \begin{cases} \mathrm{Im}TD, & s < s_{\mathrm{lhc}} \\ 0, & s > s_{\mathrm{lhc}} \end{cases}$$



$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s-s_0)^n}{\pi} \int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s-s_0)^{n-\ell}}{\pi} \int_{-\infty}^{s_{\mathrm{left}}} ds' \frac{\mathrm{Im}T(s')D(s')}{(s'-s_0)^{n-\ell}(s'-s)}.$$

N=1

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + P(s) + G(s)$$

$$T(s) = \frac{1}{D(s)}$$

$$d(k^2) = P(k^2) - ik = \frac{1}{a} + \frac{1}{2} r k^2 + \mathcal{O}(k^4) - ik$$

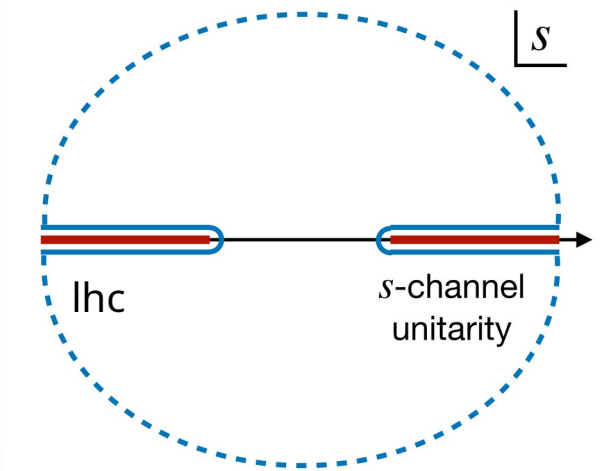
Non-relativistic
Neglect CDD

The N/D method

$$T(s) = \frac{N(s)}{D(s)}$$

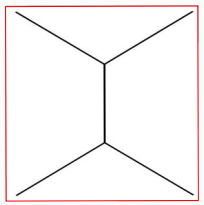
$$\mathrm{Im}D = \mathrm{Im}\frac{N}{T} = N\mathrm{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\mathrm{thr}} \\ 0, & s < s_{\mathrm{thr}} \end{cases}$$

$$\mathrm{Im}N = \begin{cases} \mathrm{Im}TD, & s < s_{\mathrm{lhc}} \\ 0, & s > s_{\mathrm{lhc}} \end{cases}$$



$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s-s_0)^n}{\pi} \int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s-s_0)^{n-\ell}}{\pi} \int_{-\infty}^{s_{\mathrm{left}}} ds' \frac{\mathrm{Im}T(s')D(s')}{(s'-s_0)^{n-\ell}(s'-s)}.$$

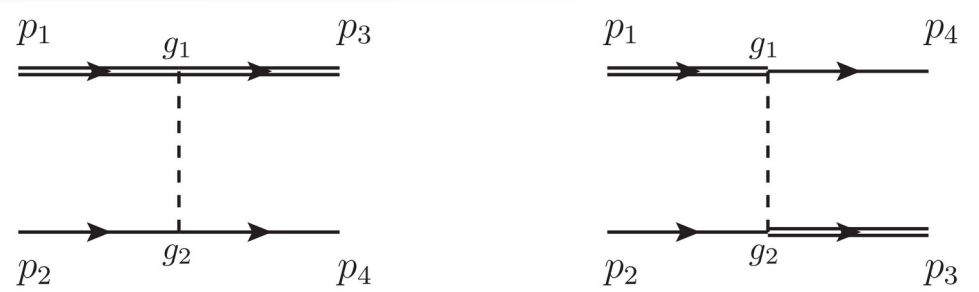


$$\frac{1}{T_\ell^{\mathrm{II}}} = \frac{1}{T_\ell} + 2i\rho \Rightarrow T^{\mathrm{II}} = \frac{1}{\frac{D}{N} + 2i\rho} = \frac{N}{D + 2i\rho N}$$

In general

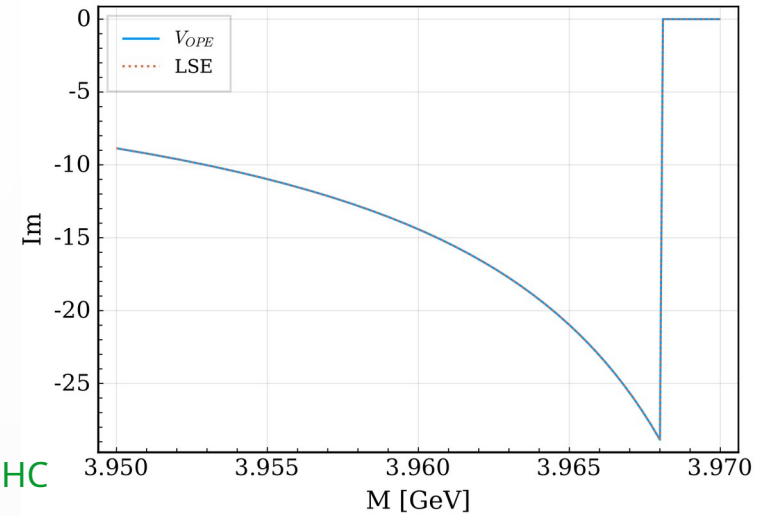
Along the lhc, $i\rho$ and D is real, N has imaginary part. ⇒ $D + 2i\rho N \neq 0$

The left-hand cut arising from OPE



$$T(E,k,k') = V(E,k,k') - \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} V(E,k,q)G(E,q)T(E,q,k')$$

free of the OPE LHC



$$\boxed{\mathrm{Im}\, f(k^2) = c\,\mathrm{Im}\, L(k^2) = -\frac{c}{4k^2}\pi, \quad \text{for } k^2 < k^2_{\mathrm{lhc}}}$$

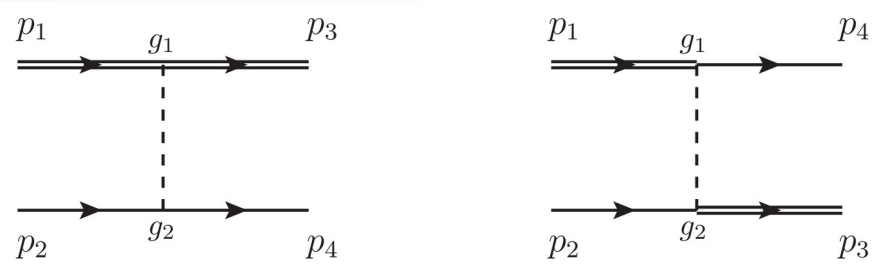
For a t-channel exchange at low-energies, an S-wave amplitude reads

$$L_t(s) = \frac{1}{2}\int \frac{1}{t-m_5^2}\mathrm{d}\cos\theta = -\frac{s}{\lambda(s,m_1^2,m_2^2)}\log\left(\frac{s-2(m_1^2+m_2^2)+m_5^2+\frac{(m_1^2-m_2^2)^2}{s}}{m_5^2}\right),$$

with m_5 the mass of changed particle. Likewise, the u-channel exchanged S-wave amplitude reads

$$L_u(s) = \frac{1}{2}\int \frac{1}{u-m_5^2}\mathrm{d}\cos\theta = -\frac{s}{\lambda(s,m_1^2,m_2^2)}\left(\log(s+m_5^2-2(m_1+m_2)^2) - \log(m_5^2-\frac{(m_1^2-m_2^2)^2}{s})\right).$$

The left-hand cut: nonrelativistic



$$\eta = |m_1 - m_2|/(m_1 + m_2)$$
$$\mu_{\rm ex}^2 = m_{\rm ex}^2 - (m_1 - m_2)^2$$
$$\mu_+^2 = 4\mu\mu_{\rm ex}^2/(m_1 + m_2)$$

Exchanged-particle: relativistic

$$L_t(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} = -\frac{1}{4k^2}\log\frac{m_{\rm ex}^2/4 + k^2}{m_{\rm ex}^2/4},$$
$$L_u(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{u - m_{\rm ex}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2},$$

$$\frac{1}{2}\int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_3)^2}{t - m_{\rm ex}^2}\mathrm{d}\cos\theta = -\frac{m_{\rm ex}^2}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} - 1$$
$$\frac{1}{2}\int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_4)^2}{u - m_{\rm ex}^2}\mathrm{d}\cos\theta \approx -\frac{\mu_{\rm ex}^2}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{u - m_{\rm ex}^2} - 1$$

$\mathcal{F}_\ell/2$

$$f(k^2) = \frac{n(k^2)}{d(k^2)}$$

$$\mathrm{Im}\, d(k^2) = -k\, n(k^2), \qquad \text{for } k^2 > 0,$$
$$\mathrm{Im}\, n(k^2) = d(k^2)\ \mathrm{Im}\, f(k^2), \qquad \text{for } k^2 < k_{\rm lhc}^2.$$

The N/D method: nonrelativistic

$$n(k^2) = n_m(k^2) + \frac{(k^2)^m}{\pi}\int_{-\infty}^{k_{\rm lhc}^2} \frac{d(k'^2)\,\mathrm{Im}\, f(k'^2)}{(k'^2-k^2)(k'^2)^m}\mathrm{d}k'^2$$

$\propto \mathrm{Im}L$

No singularity along lhc

$$n(k^2) = n'_m(k^2) + \frac{P(k^2)}{\pi}\int_{-\infty}^{k_{\rm lhc}^2} \frac{\mathrm{Im}f(k'^2)}{k'^2-k^2}\mathrm{d}k'^2 = n'_m(k^2) + P(k^2)\tilde{g}L(k^2)$$

$$\begin{aligned} n(k^2) &= n_0 + n_1k^2 + \frac{k^2}{\pi}\int_{-\infty}^{k_{\rm lhc}^2} \frac{(d_0+d_1k'^2)\mathrm{Im}f(k'^2)}{(k'^2-k^2)k'^2}\mathrm{d}k'^2 \\ &= n_0 + n_1k^2 - cL_0 + (d_0+d_1k^2)\frac{c}{\pi}\int_{-\infty}^{k_{\rm lhc}^2} \frac{\mathrm{Im}f(k'^2)}{k'^2-k^2}\mathrm{d}k'^2 \\ &= n'_0 + n_1k^2 + (d_0+d_1k^2)cL(k^2) \end{aligned}$$

$$n(k^2) = \tilde{n}(k^2) + \tilde{g}(L(k^2) - L_0)$$

$$L_0 = L(k^2=0) = -1/\mu_{\rm ex}^2$$

The N/D method: nonrelativistic

$$d(k^2) = d_n(k^2) - \frac{(k^2 - k_0^2)^n}{\pi} \int_0^\infty \frac{k' n(k'^2)\, \mathrm{d}k'^2}{(k'^2 - k^2)(k'^2 - k_0^2)^n}$$

$$d(k^2) = \tilde{d}(k^2) - ik(\tilde{n}(k^2) - \tilde{g}L_0) - \frac{\tilde{g}}{\pi} \int_0^\infty \frac{k' L(k'^2)}{k'^2 - k^2} \mathrm{d}k'^2$$

$$= \tilde{d}(k^2) - ik\, n(k^2) - \tilde{g} d^{\mathrm{R}}(k^2)$$

$$= \tilde{d}(k^2) - ik\tilde{n}(k^2) + ik\tilde{g}L_0 + \frac{i\tilde{g}}{2k} \log \frac{\mu_+/2 - ik}{\mu_+/2 - i\eta k}$$

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k} \left(\log \frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log \frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k} \right)$$

It is worth stressing that $d(k^2)$ is free of lhc, as the lhc associated with $n(k^2)$ below the threshold is counterbalanced by $d^R(k^2)$, which is crucial to ensure that $f(k^2)$ exhibits the correct lhc behavior. Along the rhc, both $n(k^2)$ and $d^{\mathrm{R}}(k^2)$ are real such that $\mathrm{Im} d(k^2) = -k\, n(k^2)$.

Effective range expansion with the left-hand cut

$$\frac{1}{f(k^2)} = \frac{\tilde{d}(k^2) - \tilde{g} d^{\mathrm{R}}(k^2)}{\tilde{n}(k^2) + \tilde{g}\,(L(k^2) - L_0)} - ik$$

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k}\left(\log\frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log\frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k}\right)$$

$$L(k^2) = -\frac{1}{4k^2}\log\frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2}$$

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i k^{2i} - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j k^{2j} + \tilde{g}(L(k^2) - L_0)} - ik$$

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik\right]^{-1}$$

$\tilde{g} \to 0$

$$\frac{1}{f(k^2)} = \frac{1}{a} + \frac{1}{2} r k^2 - ik$$

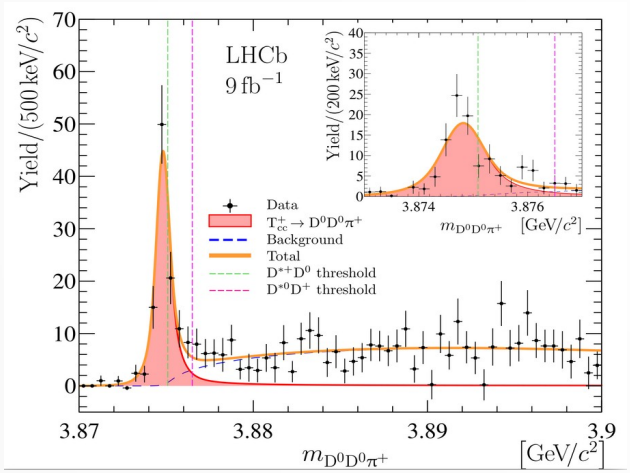
Scattering length

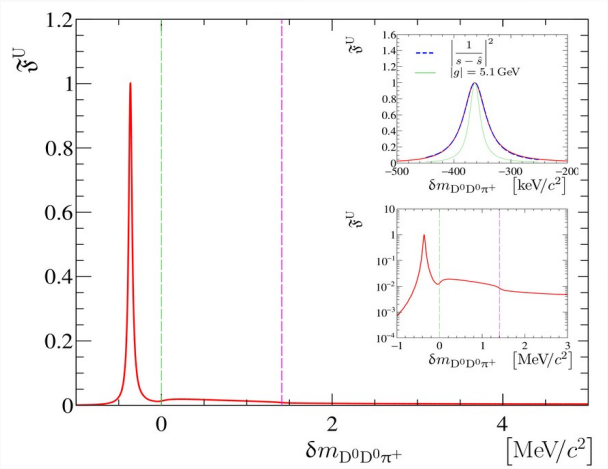
$$a = f(k^2 = 0) = \left[\tilde{d}_0 + \frac{\tilde{g}}{\mu_+}(1 - \eta)\right]^{-1}$$

Effective range

$$r = \left.\frac{\mathrm{d}^2(1/f + ik)}{\mathrm{d}k^2}\right|_{k=0} = 2\tilde{d}_1 - \frac{8\tilde{g}}{3\mu_+^3}(1 - \eta^3) - \frac{4\tilde{g}}{\mu_+^4 a_u}(1 - \eta^4)$$

Application I: Doubly Charm Tetraquark (Tcc)





Breit-Wigner fit

LHCb, Nature Phys. 18, (2022) 751

Parameter	Value
N	117 ± 16
$\delta m_{\rm BW}$	-273 ± 61 keV
$\Gamma_{\rm BW}$	410 ± 165 keV

☞ $\Re \sim 400$ keV.

Unitarized and analytical

LHCb, Nature Commun. 13 (2022), 3351

$\delta m = m_{T_{cc}^+} - m_{D^{*+}} - m_{D^0}$

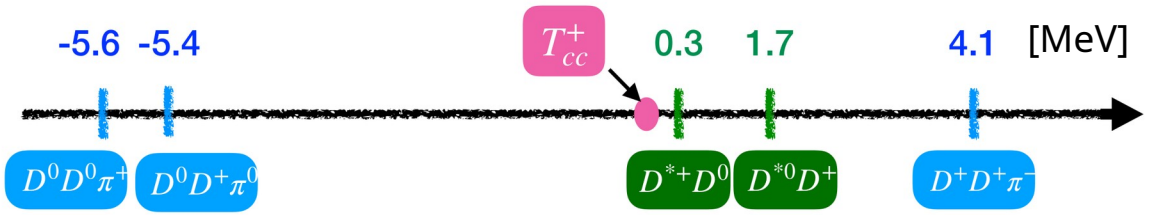
$\delta m_{\rm pole} = -360 \pm 40^{+4}_{-0}$ keV

$\Gamma_{\rm pole} = 48 \pm 2^{+0}_{-14}$ keV

☞ Three-body cuts



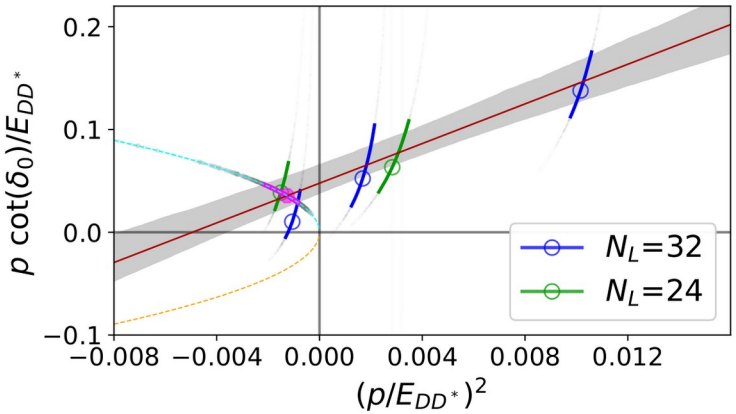
Du et al., PRD 105, 014024 (2022)

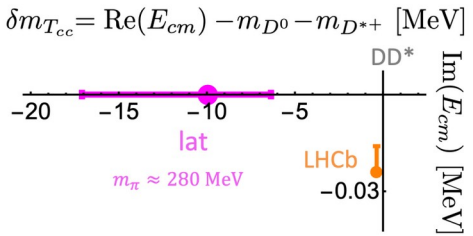


Doubly Charm Tetraquark on the Lattice

Padmanath *et al*, PRL129,032002(2022)

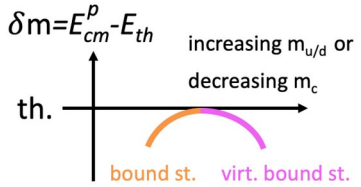
	m_D (MeV)	m_{D^*} (MeV)	$M_{\rm av}$ (MeV)	$a_{l=0}^{(J=1)}$ (fm)	$r_{l=0}^{(J=1)}$ (fm)	$\delta m_{T_{cc}}$ (MeV)	T_{cc}
Lattice ($m_\pi \simeq 280$ MeV, $m_c^{(h)}$)	1927(1)	2049(2)	3103(3)	1.04(29)	$0.96(^{+0.18}_{-0.20})$	$-9.9^{+3.6}_{-7.2}$	Virtual bound st.
Lattice ($m_\pi \simeq 280$ MeV, $m_c^{(l)}$)	1762(1)	1898(2)	2820(3)	0.86(0.22)	$0.92(^{+0.17}_{-0.19})$	$-15.0(^{+4.6}_{-9.3})$	Virtual bound st.
Experiment [2,41]	1864.85(5)	2010.26(5)	3068.6(1)	−7.15(51)	[−11.9(16.9), 0]	−0.36(4)	Bound st.



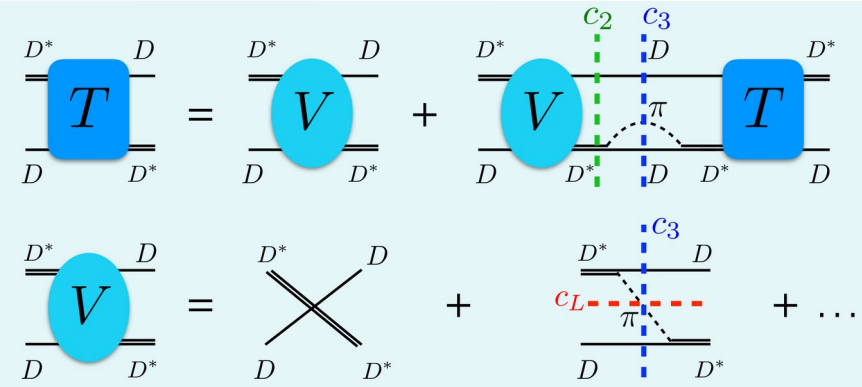


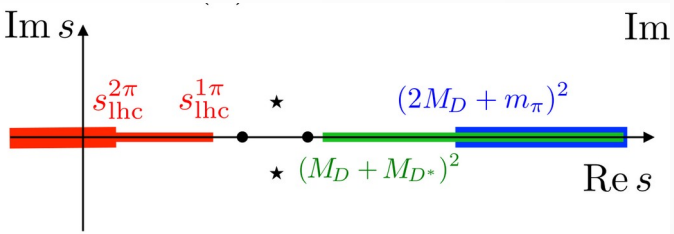
$$t = \frac{E_{\rm cm}}{2}\frac{1}{p\cot\delta - ip},$$

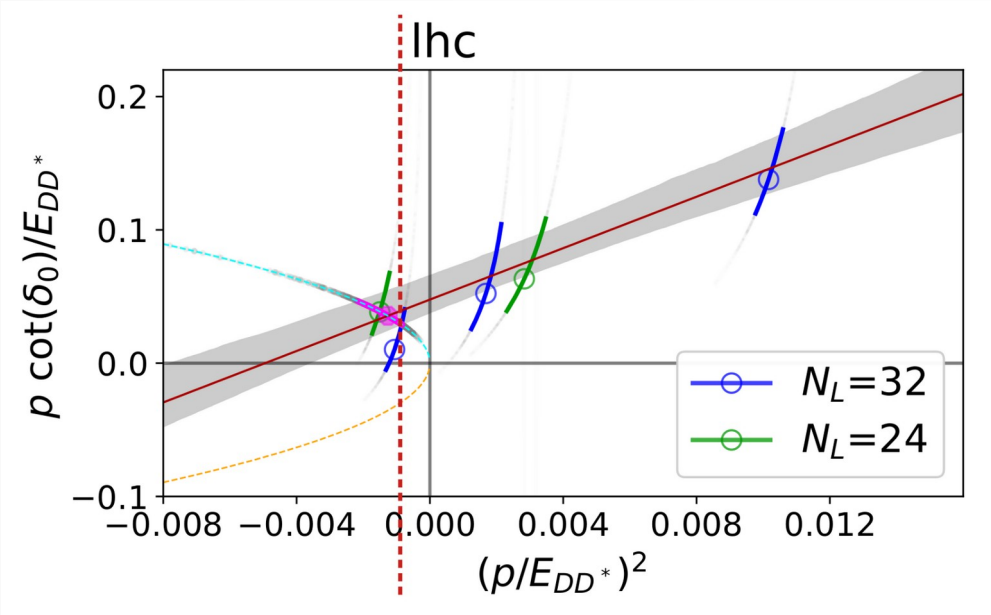
$$p\cot\delta = \frac{1}{a_0} + \frac{1}{2}r_0 p^2,$$



The left-hand cut







$m_\pi = 280$ MeV

☞ two-body branch point:

$E = M_D + M_{D^*}$

$\Longrightarrow p^2_{\rm rhc_2} = 0$

☞ three-body branch point:

$E = M_D + M_D + m_\pi$

$\Longrightarrow \left(\frac{p_{\rm rhc_3}}{E_{DD^*}}\right)^2 = +0.019$

☞ left-hand cut branch point:

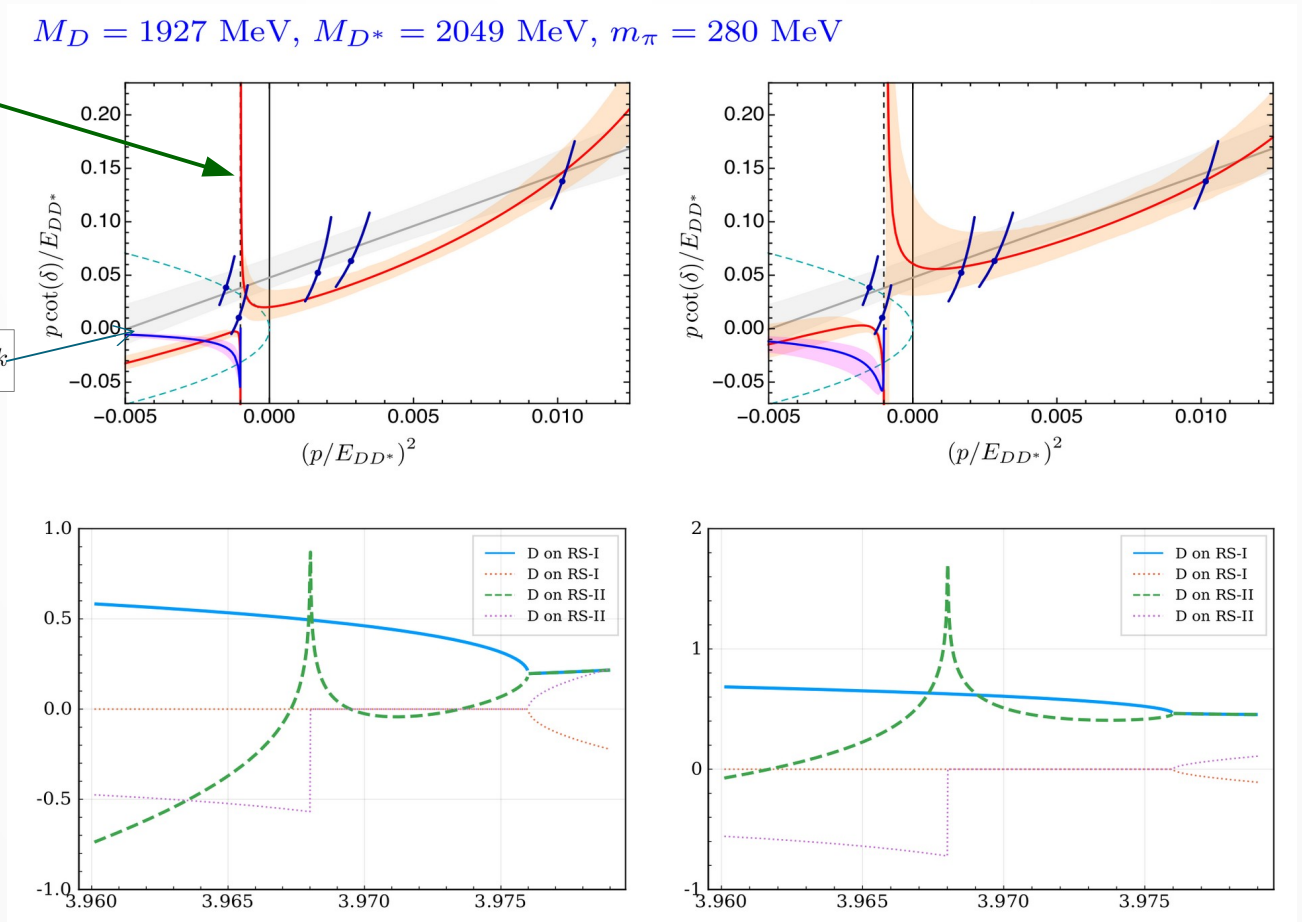
$\Longrightarrow \left(\frac{p^{1\pi}_{\rm lhc}}{E_{DD^*}}\right)^2 = -0.001$

$\left(\frac{\tilde{p}^{1\pi}_{\rm lhc}}{E_{DD^*}}\right)^2 = -0.190$

Phase shift with the LHC: Solving Lippmann-Schwinger Equation

Limit ERE

$$p\cot\delta = -\frac{2\pi}{\mu}\frac{1}{T} + ik$$

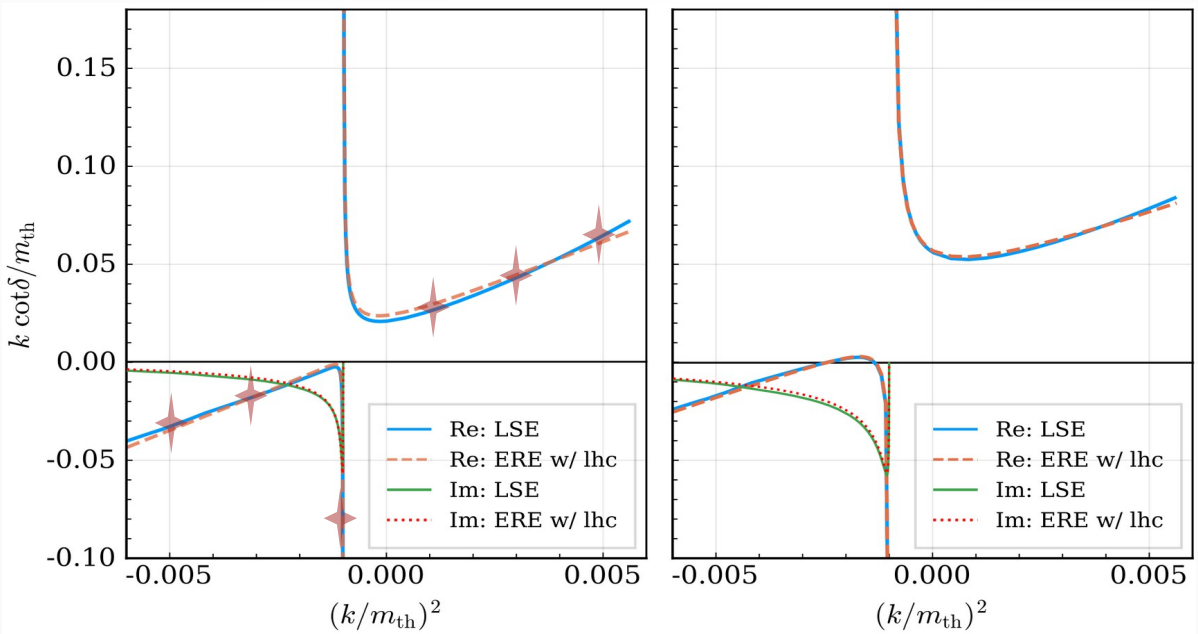


No virtual state along the lhc

Application to Tcc: on the Lattice [3 parameters]

Du et al., PRL 135, 011903 (2025)

$f_{[0,1]}$



ERE

$$f(k^2) = \frac{1}{k\cot\delta - ik}$$

$$k\cot\delta = \frac{1}{a} + \frac{1}{2}rk^2 + \mathcal{O}(k^4)$$

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g}d^{\rm R}(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik\right]^{-1}$$

Couplings to the exchanging-particle

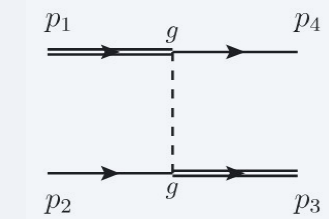
$$-\frac{2\pi}{\mu}\mathrm{Im}f = \mathrm{Im}T = \mathrm{Im}V_{\mathrm{OPE}}(k^2) = g_P\frac{-\pi}{4k^2}\mathcal{F}_\ell, \quad \text{for } k^2 < k^2_{\mathrm{lhc}}$$

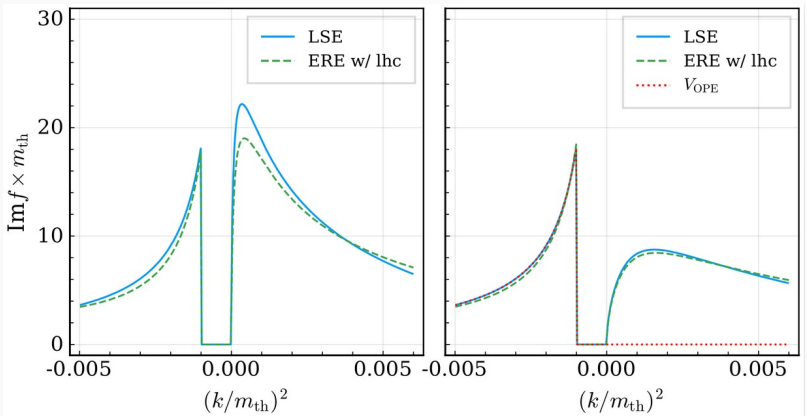
$$\mathrm{Im}\, n(k^2) = -\tilde{g}\frac{\pi}{4k^2}, \quad \text{for } k^2 < k^2_{\mathrm{lhc}}$$

$$d_u^{0,\mathrm{lhc}} = \tilde{d}_0 - \frac{\tilde{d}_1\mu_+^2}{4} + \frac{\mu_+}{2}\left(1+\frac{\tilde{g}}{\mu_{\mathrm{ex}}^2}\right) + \frac{\tilde{g}\log[2/(1+\eta)]}{\mu_+}$$

$$g_P = -\frac{2\pi\tilde{g}}{\mu d^{0,\mathrm{lhc}}\mathcal{F}_\ell}$$

$$g^2_{D^*D\pi}/(4F^2) = 9.2\ \mathrm{GeV}^{-2}$$





$9.0\ \mathrm{GeV}^{-2}$ $9.4\ \mathrm{GeV}^{-2}$

The amplitude zero

At leading order, i.e., $\tilde{n}(k^2) = 1$,

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g}d^{\mathrm{R}}(k^2)}{1+\tilde{g}(L(k^2)-L_0)} - ik\right]^{-1}$$

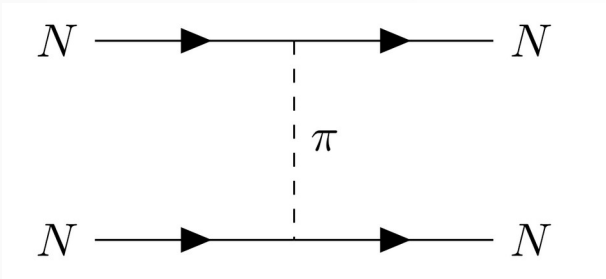
For a general u-channel exchange,

$$1 + \tilde{g}\left[L_u\left(k^2_{u,\mathrm{zero}}\right) + \frac{1}{\mu^2_{\mathrm{ex}}}\right] = 0,$$

$$k^2_{t,\mathrm{zero}} = -\frac{m^2_{\mathrm{ex}}}{4}\left[1 + \frac{1}{y}W(-e^{-y}y)\right]$$

$$y \equiv 1 + m^2_{\mathrm{ex}}/\tilde{g}$$

Application II: NN scattering



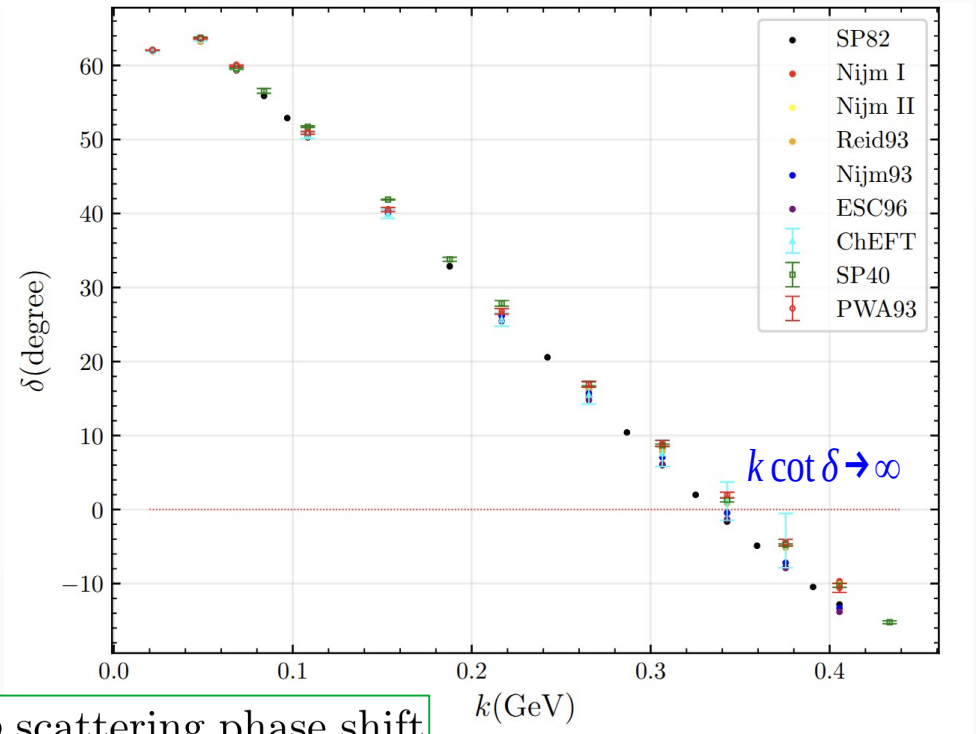
$$\mathcal{L}_{\pi NN} = i g_{\pi N} \bar{N} \gamma_5 \vec{\tau} \cdot \vec{\pi} N$$

$$\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi} \left(i \mathcal{D}_\mu \gamma^\mu - m_N + \frac{g_A}{2} \gamma^\mu \gamma_5 u_\mu \right) \Psi,$$

$$V(\vec{q}) = g^2_{\pi N} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{\sigma}_1 \cdot \vec{q}\, \vec{\sigma}_2 \cdot \vec{q}}{\vec{q}^{\,2} + m^2_\pi},$$

$$V_0(k^2) = -g^2_{\pi N} m^2_\pi L(k^2)$$

$$\frac{g^2_{\pi N}}{4\pi} = -\frac{\tilde{g}}{d^{0,\mathrm{lhc}}} \frac{4 m_N}{m^2_\pi}$$



$^1S_0$ np scattering phase shift

Application II: NN scattering: 1S0

PRD 114, 074030(2026)

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i k^{2i} - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j k^{2j} + \tilde{g}(L(k^2) - L_0)} - ik$$

ERE w/lhc	$g^2_{\pi NN}/4\pi$	$E_{\mathrm{pole}} - m_{\mathrm{thr}}$
d_0, d_1, d_2, n_1, g_0	$12.4^{+2.2}_{-2.3}$	-66.4 keV
d_0, d_1, n_1, n_2, g_0	$11.0^{+2.6}_{-2.8}$	-66.6 keV
$d_0, d_1, d_2, n_1, n_2, g_0$	$13.0^{+3.0}_{-4.1}$	-66.3 keV

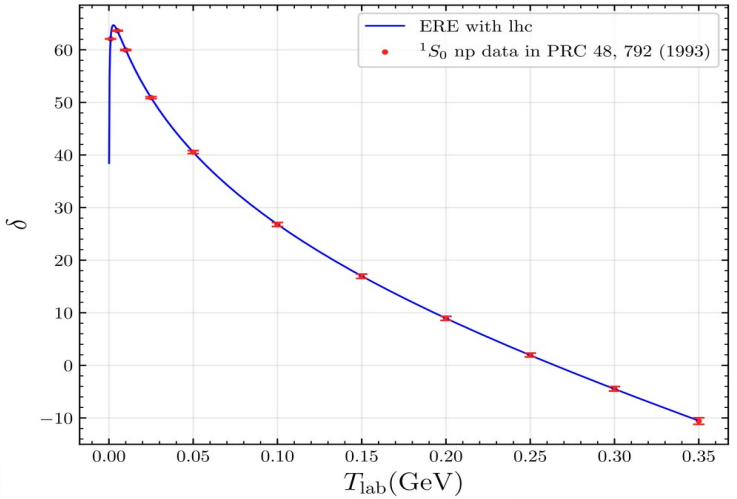
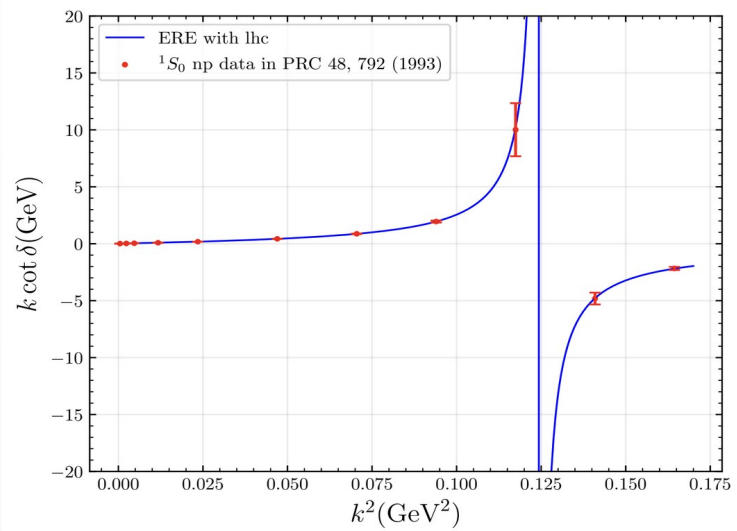


TABLE I. Some deduced values for the πNN coupling constant. The quoted uncertainty are those quoted by the authors and usually do not include systematic uncertainties.

Source	Year	System	$g^2_{\pi NN}/4\pi$
Karlsruhe-Helsinki [3]	1980	πp	14.28(18)[a]
Kroll [4]	1981	pp	14.52(40)[a]
Nijmegen [6]	1993	pp,np	13.58 (5)[a]
VPI [8]	1994	pp,np	13.70
Nijmegen [7]	1997	pp,np	13.54 (5)[a]
Timmermans [12]	1997	$\pi^+ p$	13.45(14)[a]
VPI [9]	1994	GMO, πp	13.75(15)[a]
Uppsala [2]	1998	$np \to pn$	14.52(26)
Pavan *et al.* [11]	1999	πp	13.73 (9)
Schröder *et al.*, corrected [14,10]	1999	GMO, $\pi^{\pm} p$	13.77(18)
Present work	2001	GMO, $\pi^{\pm} p$	14.11(20)

[a]Statistical uncertainty only.

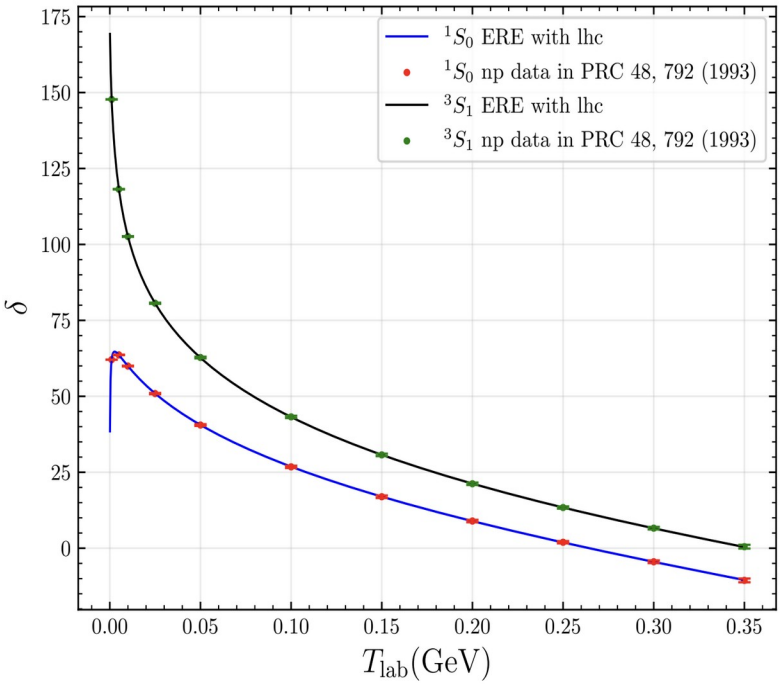
PRC66(2002)014005



Application II: NN scattering: 1S0 + 3S1

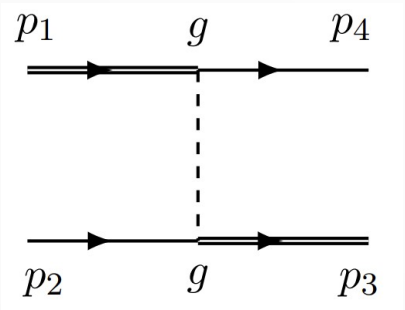
PRD 114, 074030(2026)

ERE w/ lhc		$E_{\rm pole}$ (keV)		a (fm)		r (fm)		$\frac{g_{\pi N}^2}{4\pi}$	$\frac{\chi^2}{\rm d.o.f}$
$\|^1S_0\rangle$	$\|^3S_1\rangle$	$\|^1S_0\rangle$	$\|^3S_1\rangle$	$\|^1S_0\rangle$	$\|^3S_1\rangle$	$\|^1S_0\rangle$	$\|^3S_1\rangle$		
$f_{[2,1]}$	$f_{[1,2]}$	$-66.33(4)$	$-2208(4)$	$23.73(1)$	$-5.418(2)$	$2.68(1)$	$1.741(3)$	$12.2(1.4)$	0.18
$f_{[1,2]}$	$f_{[1,2]}$	$-66.33(4)$	$-2209(4)$	$23.73(1)$	$-5.418(2)$	$2.68(1)$	$1.742(3)$	$12.8(1.2)$	0.13
$f_{[2,1]}$	$f_{[2,1]}$	$-66.34(4)$	$-2211(5)$	$23.73(1)$	$-5.418(2)$	$2.68(1)$	$1.743(3)$	$11.9^{+1.4}_{-1.5}$	0.10
$f_{[1,2]}$	$f_{[2,1]}$	$-66.34(4)$	$-2212(4)$	$23.73(1)$	$-5.418(2)$	$2.68(1)$	$1.744(3)$	$12.7(1.2)$	0.05
$f_{[1,2]}$	-	$-66.4^{+0.6}_{-0.5}$	-	$23.72^{+0.10}_{-0.08}$	-	$2.68^{+0.02}_{-0.03}$	-	$12.4^{+2.2}_{-2.3}$	0.008
-	$f_{[2,1]}$	-	-2222^{+14}_{-12}	-	$-5.416^{+0.002}_{-0.003}$	-	$1.749(7)$	$18.3^{+8.4}_{-8.6}$	0.04
[37]		-66.5	-2237	23.719	-5.403	2.626	1.749	-	-
[38]		$-66.08(13)$	$-2223(6)$	$23.748(10)$	$-5.424(4)$	$2.75(5)$	$1.759(5)$	-	-
[39]		$-66.08(16)$	$-2223(10)$	$23.740(20)$	$-5.419(7)$	$2.77(5)$	$1.753(8)$	-	-
[21]		-	$-2224.7(35)$	-	-	-	-	-	-
[47]		-	$-2224.575(9)$	$23.74(2)$	-	-	-	-	-
[42]		-	-	-	-	-	-	$13.76(8)$	-
[43, 44]		-	-	-	-	-	-	$13.69(27)$	-
[41]		-	-	-	-	-	-	$14.23(11)$	-
[45]		-	-	-	-	-	-	$13.14^{+0.07}_{-0.04}$	-
[46]		-	-	-	-	-	-	$13.6(3)$	-



Extension to higher orders

PRD 114, 014003(2025)



$$g^2 L_u(k^2) = \frac{g^2}{2}\int \frac{\mathrm{d}\cos\theta}{u-m_{\rm ex}^2} = \frac{g^2}{2}\int \frac{\mathrm{d}\cos\theta}{\left(\sqrt{k^2+m_1^2}-\sqrt{k^2+m_2^2}\right)^2 - 2k^2(1-\cos\theta) - m_{\rm ex}^2}$$
$$= -\frac{g^2}{4k^2}\log\frac{m_1^2+m_2^2-m_{\rm ex}^2-2\sqrt{(k^2+m_1^2)(k^2+m_2^2)}-2k^2}{m_1^2+m_2^2-m_{\rm ex}^2-2\sqrt{(k^2+m_1^2)(k^2+m_2^2)}+2k^2},$$

$$u - m_{\rm ex}^2 = \Delta^2 - m_{\rm ex}^2 - \left[2(1-\cos\theta)+\frac{\Delta^2}{m_1 m_2}\right]k^2 + \frac{\Delta^2 m_{\rm th}^2}{4m_1^3 m_2^3}k^4 - \frac{\Delta^2 m_{\rm th}^2(m_1^2+m_2^2)}{8m_1^5 m_2^5}k^6 + \ldots,$$

$$L_u^{(4)}(k^2) = -\frac{1}{4k^2}\log\frac{\left(2c_4k^2+\sqrt{\mu_{(2)}^2c_4+1}-1\right)\left(2c_4k^2-\sqrt{\mu_{(2)}^2c_4+1}-1\right)}{\left(2c_4k^2+\sqrt{\mu_{(2)}^2c_4+\eta^4}-\eta^2\right)\left(2c_4k^2-\sqrt{\mu_{(2)}^2c_4+\eta^4}-\eta^2\right)} = -\frac{1}{4k^2}\log\frac{k^2+\mu_{(4)}^2/4}{\eta_{(4)}^2k^2+\mu_{(4)}^2/4} - \frac{1}{4k^2}\log\frac{2c_4\eta_{(4)}^2k^2-\sqrt{\mu_{(2)}^2c_4+\eta^4}-\eta^2}{2c_4k^2-\sqrt{\mu_{(2)}^2c_4+\eta^4}-\eta^2},$$

$$\mu_{(4)}^2 = 2\frac{\sqrt{\mu_{(2)}^2c_4+1}-1}{c_4},\ \eta_{(4)}^2 = \frac{\sqrt{\mu_{(2)}^2c_4+1}-1}{\sqrt{\mu_{(2)}^2c_4+\eta^4}-\eta^2},\ \text{with } c_4 = \frac{\Delta^2 m_{\rm th}^2}{4m_1^3m_2^3}.$$

A relativistic version

PRD 114, 014003(2025)

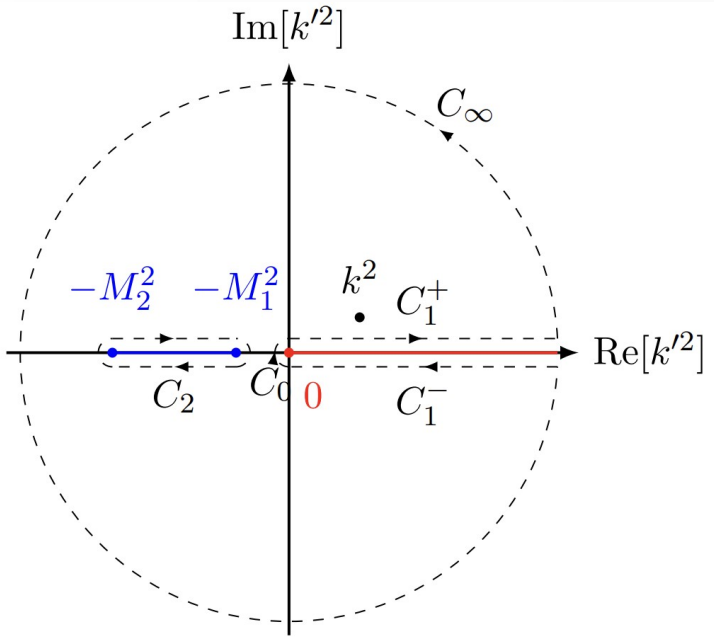
$$S = 1 + i8\pi\sqrt{s}f$$

$$n(k^2) = \bar{n}(k^2) + \bar{g}(k^2)(L(k^2) - L_0)\ ,$$

$$d(k^2) = \bar{d}(k^2) - ik\big(\bar{n}(k^2) - \bar{g}(k^2)L_0\big) - \frac{\bar{g}(k^2)}{\pi}\int_0^\infty \frac{k' L(k'^2)}{k'^2 - k^2}\mathrm{d}k'^2$$

$$\begin{aligned}\int_0^\infty \frac{k' L_u(k'^2)}{k'^2 - k^2}\mathrm{d}k'^2 &= \frac{1}{2}\int_{-M_1^2}^{-M_2^2} \mathrm{Disc}\left[\frac{k' L_u(k'^2)}{k'^2 - k^2}\right]\mathrm{d}k'^2 + i\pi k L_u(k^2)\\ &= -\frac{i\pi}{4}\int_{-M_1^2}^{-M_2^2}\frac{1}{k'(k'^2 - k^2)}\mathrm{d}k'^2 + i\pi k L_u(k^2)\\ &= \frac{i\pi}{4k}\left[\log\left(\frac{M_1 + ik}{M_1 - ik}\right) - \log\left(\frac{M_2 + ik}{M_2 - ik}\right)\right] + i\pi k L_u(k^2).\end{aligned}$$

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k}\left[\log\left(\frac{M_1 + ik}{M_1 - ik}\right) - \log\left(\frac{M_2 + ik}{M_2 - ik}\right)\right]$$



$$\int_{C_1^- + C_0 + C_1^+ + C_\infty + C_2} \frac{k' L_u(k'^2)}{k'^2 - k^2}\mathrm{d}k'^2 = 2\pi i k L_u(k^2)$$

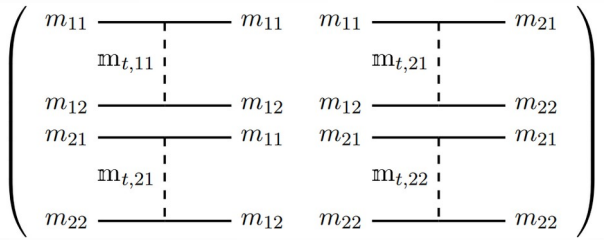
$$M_1 = \frac{1}{2}\sqrt{\frac{\lambda(m_1^2, m_2^2, m_{\mathrm{ex}}^2)}{m_{\mathrm{ex}}^2 - 2(m_1^2 + m_2^2)}}, \qquad M_2 = \frac{\sqrt{-\lambda(m_1^2, m_2^2, m_{\mathrm{ex}}^2)}}{2m_{\mathrm{ex}}}$$

$$m_1^2 + m_2^2 - m_{\mathrm{ex}}^2 - 2\sqrt{(k^2 + m_1^2)(k^2 + m_2^2)} \mp 2k^2 = 0.$$

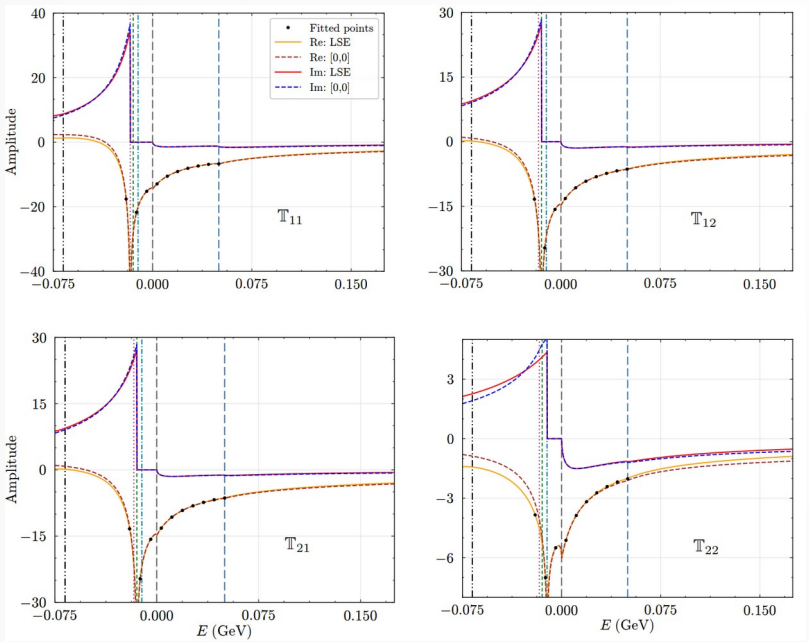
Extension to Coupled-channels

arXiv: 2610.32139

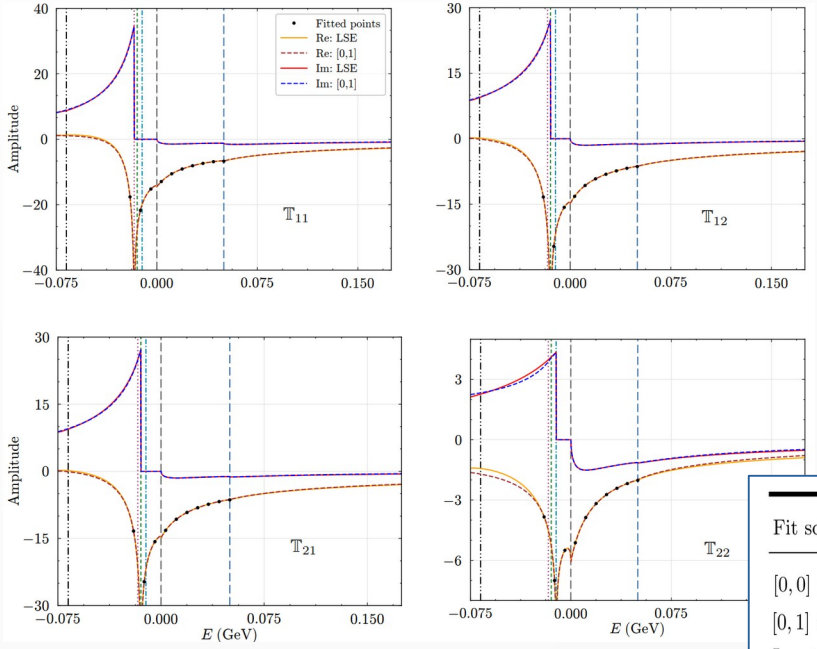
$$\mathbb{L}^{(2)} = \begin{pmatrix} c_{11}L_{t,11}(k_1^2) & c_{12}L_{t,12}(k_2^2) \\ c_{21}L_{t,21}(k_1^2) & c_{22}L_{t,22}(k_2^2) \end{pmatrix} =$$



$m_{11} = 2$ GeV, $m_{12} = 1$ GeV,
$m_{21} = 1.8$ GeV, $m_{22} = 1.25$ GeV,
$\mathbb{m}_{t,11} = 0.3$ GeV, $\mathbb{m}_{t,12} = 0.5$ GeV,
$\mathbb{m}_{t,22} = 0.6$ GeV.



$[0,0]$



$[0,1]$

Fit scheme	c_{11}	c_{21}	c_{22}
$[0,0]$ for Re[$\mathbb{T}$]	1.06	1.56	0.58
$[0,1]$ for Re[$\mathbb{T}$]	0.99	1.49	0.50
Input values in LSE	1.00	1.50	0.50

Summary & Outlook

★ The ERE with the left-hand cut

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^R(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik \right]^{-1}$$

↪ correct behavior of the left-hand cut

↪ can be used to extract the couplings of the exchanged particle to the scattering particles

↪ amplitude zeros caused by the interplay between the short- and long-range interactions

⇨ Applications to Tcc & NN, extracted coupling constants!

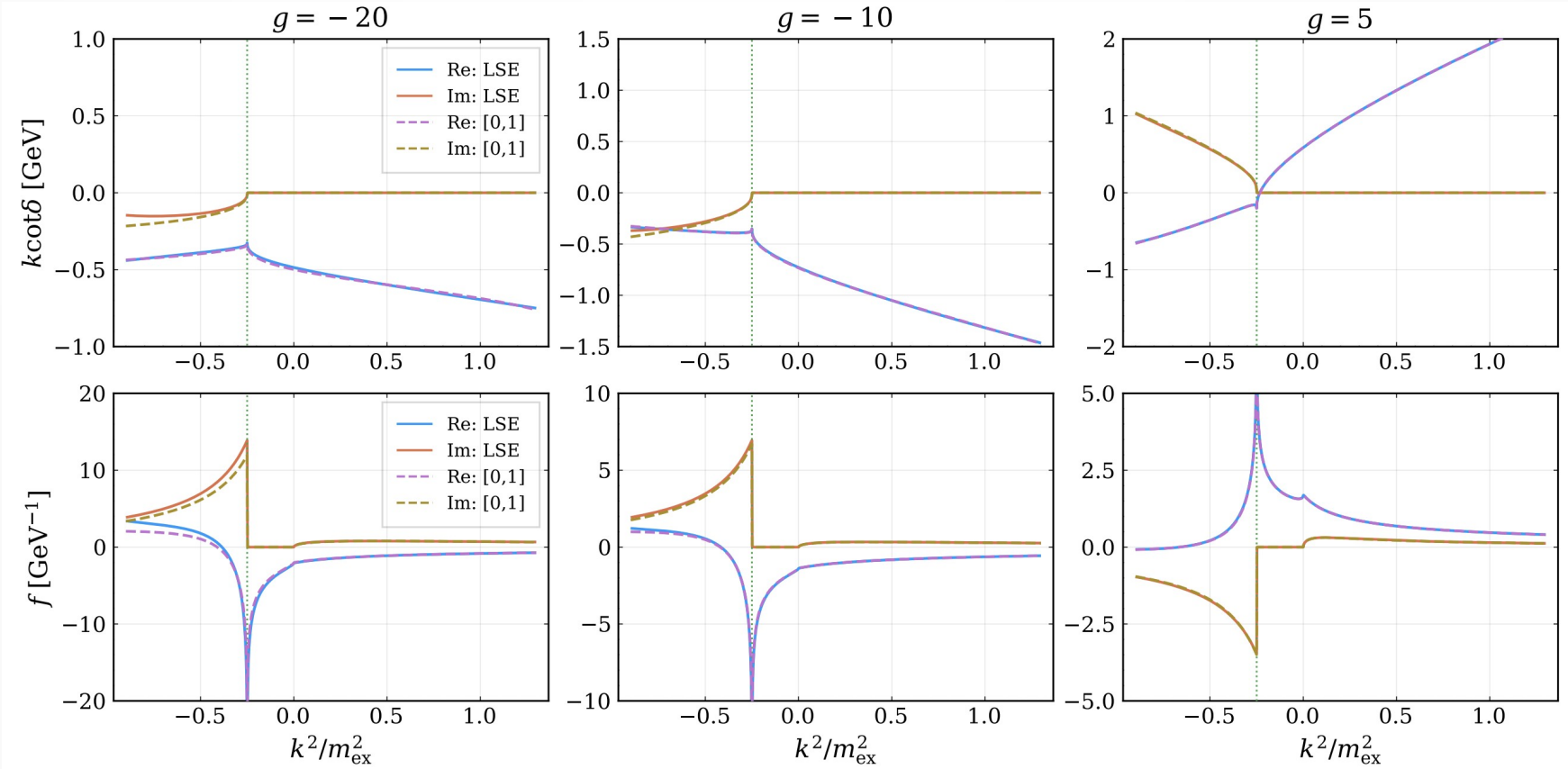
Put the new parameterization on LATTICE!

Thank you very much for your attention!

Application III: Yukawa potential

$$Y_{\rm OPE}(p,q) = -\frac{g}{4pq}\log\frac{(p+q)^2+m_{\rm ex}^2}{(p-q)^2+m_{\rm ex}^2},$$

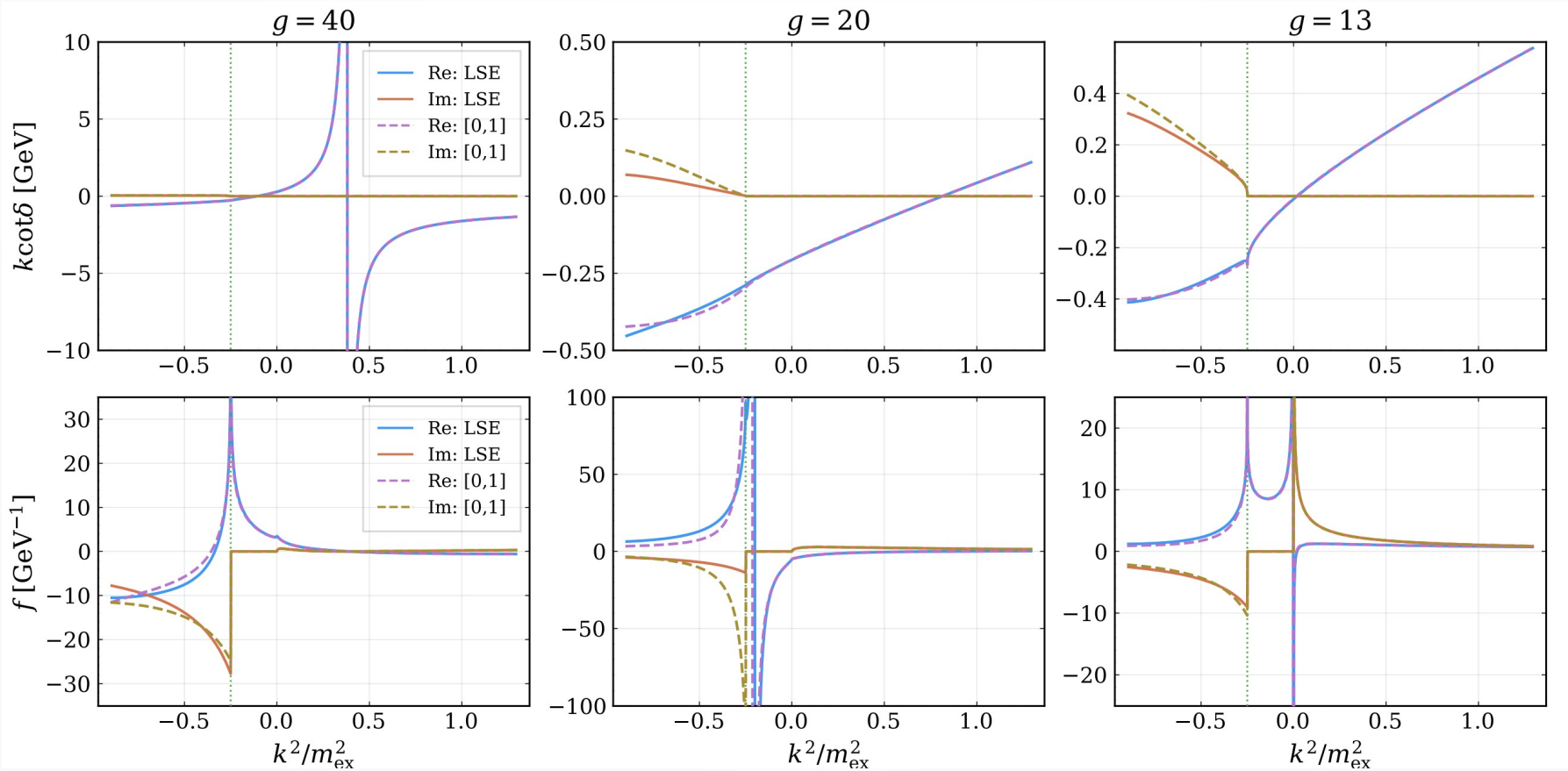
$$V = -ge^{-m_{\rm ex}r}/r$$



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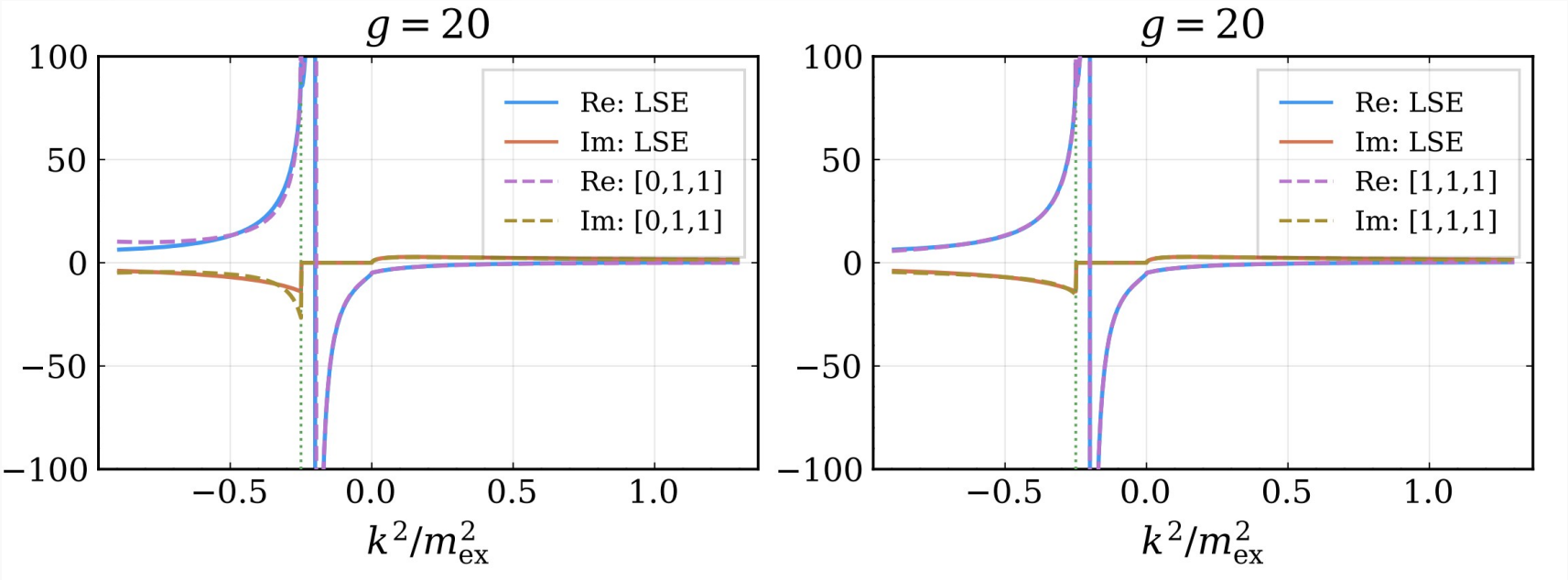


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$$\frac{1}{f_{[m,n,\ell]}(k^2)} = \frac{\sum_{i=0}^{n} d_i k^{2i} - d^{\mathrm{R}}(k^2)\left(\sum_{i=0}^{\ell} g_i k^{2\ell}\right)}{1+\sum_{i=1}^{m} n_i k^{2i} + (L(k^2)-L_0)\left(\sum_{i=0}^{\ell} g_i k^{2\ell}\right)} - ik,$$



Without $d^R(k^2)$

$$d(k^2) = \tilde{d}(k^2) - ik(\tilde{n}(k^2) - \tilde{g}L_0) - \frac{\tilde{g}}{\pi}\int_0^\infty \frac{k' L(k'^2)}{k'^2 - k^2}\mathrm{d}k'^2$$
$$= \tilde{d}(k^2) - ik\, n(k^2) - \tilde{g} d^{\mathrm{R}}(k^2)$$

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k}\left(\log\frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log\frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k}\right)$$

$$f^{-1} = \frac{\frac{1}{a} + \frac{1}{2}rk^2}{1 + \tilde{g}(L(k^2) - L_0)} - ik$$

$$d(k^2) = \frac{1}{a} + \frac{1}{2}rk^2 - ik - ik\tilde{g}(L(k^2) - L_0)$$

lhc

