

重庆大学
CHONGQING UNIVERSITY

Universal Bifurcation Threshold for Dynamical Gluon Mass Generation

Si-Xue Qin

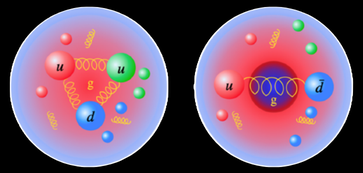
ICTS (USTC)

2026-11-07

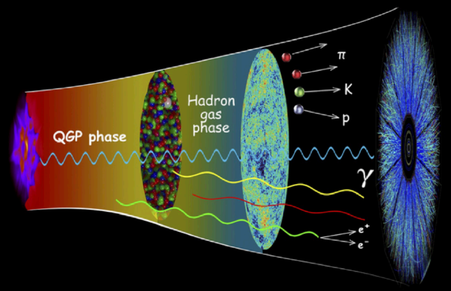
Introduction

Introduction: QCD frontiers

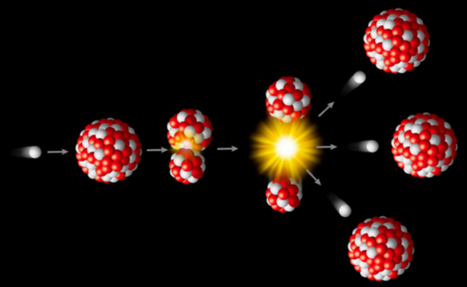
Hadron



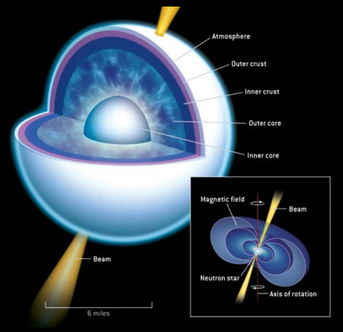
Hot Matter



Nucleus

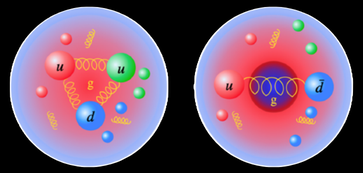


Dense Matter

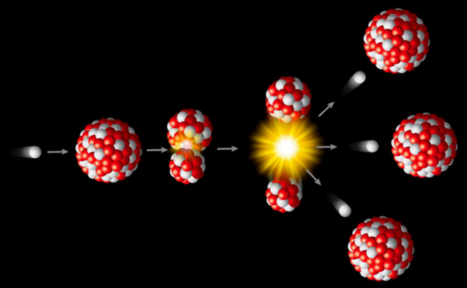


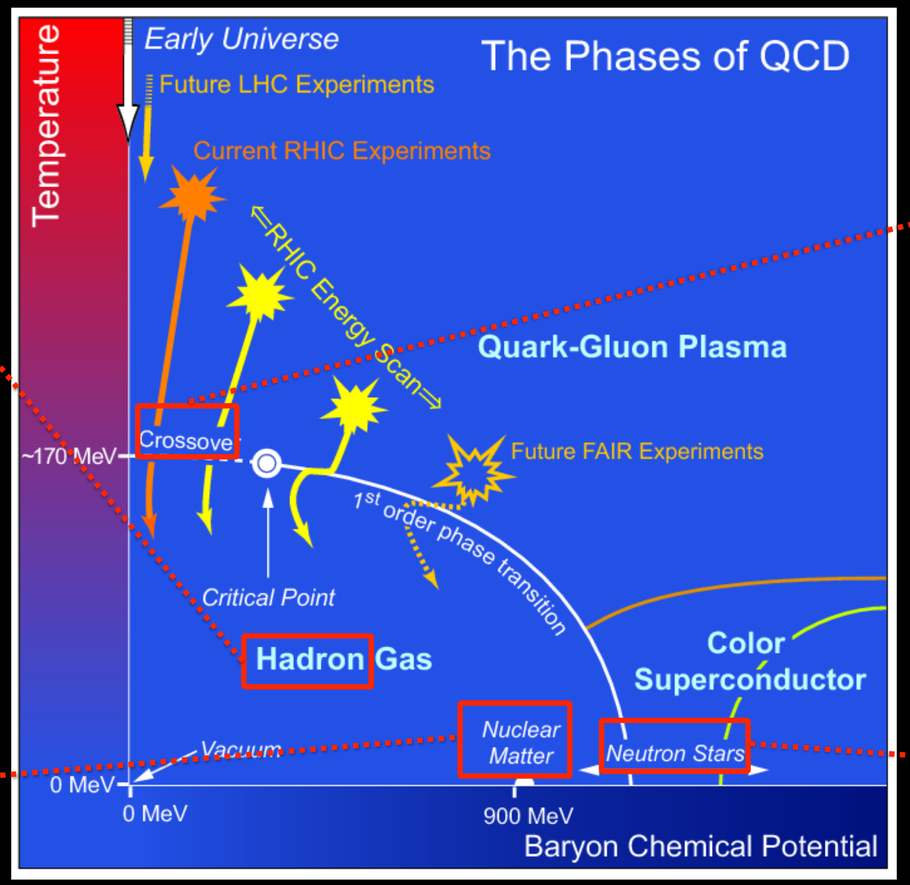
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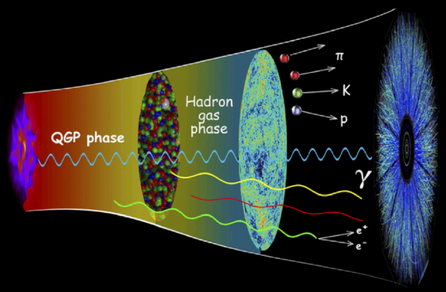


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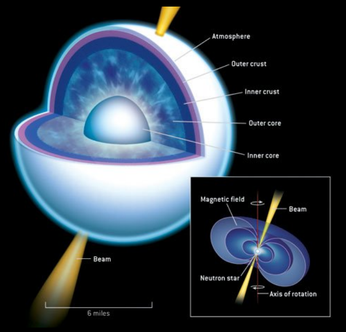




Hot Matter

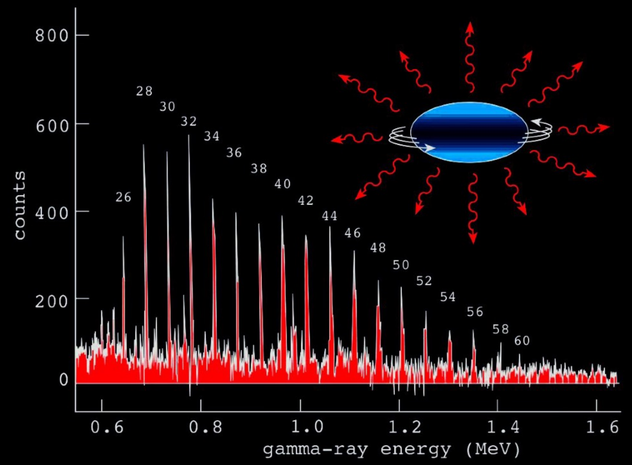


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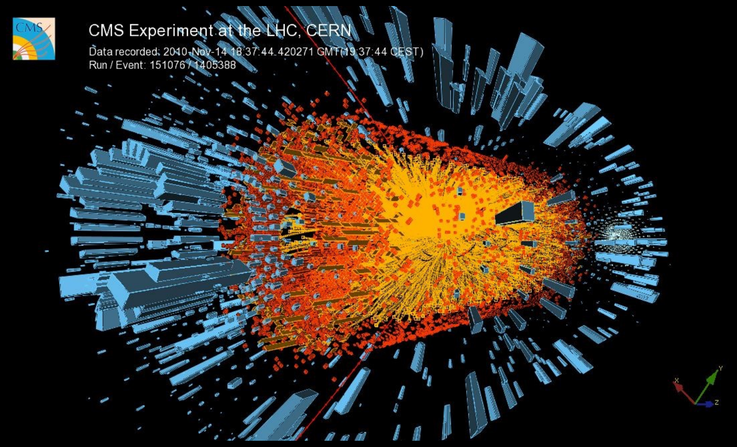


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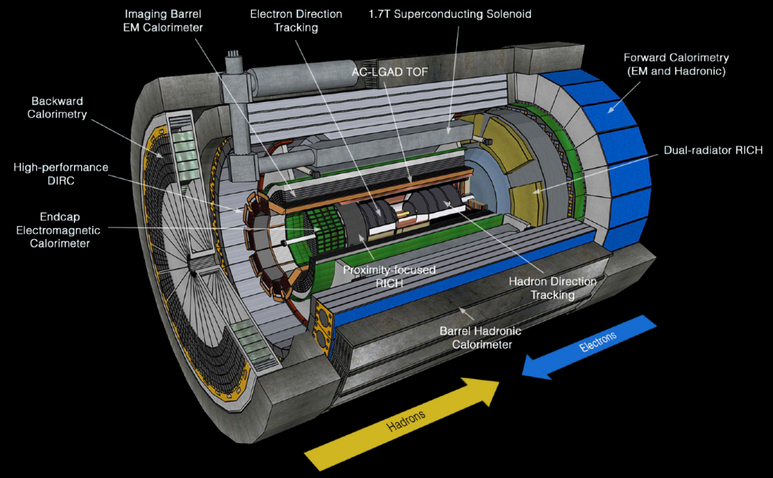
◉ Novel states of nuclei



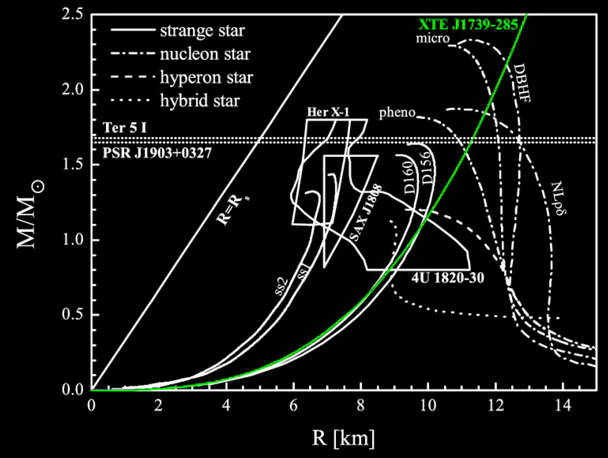
◉ Relativistic heavy-ion collision



◉ Electron-hadron collision



◉ Mass-radius relation of starts



Introduction: Core problem

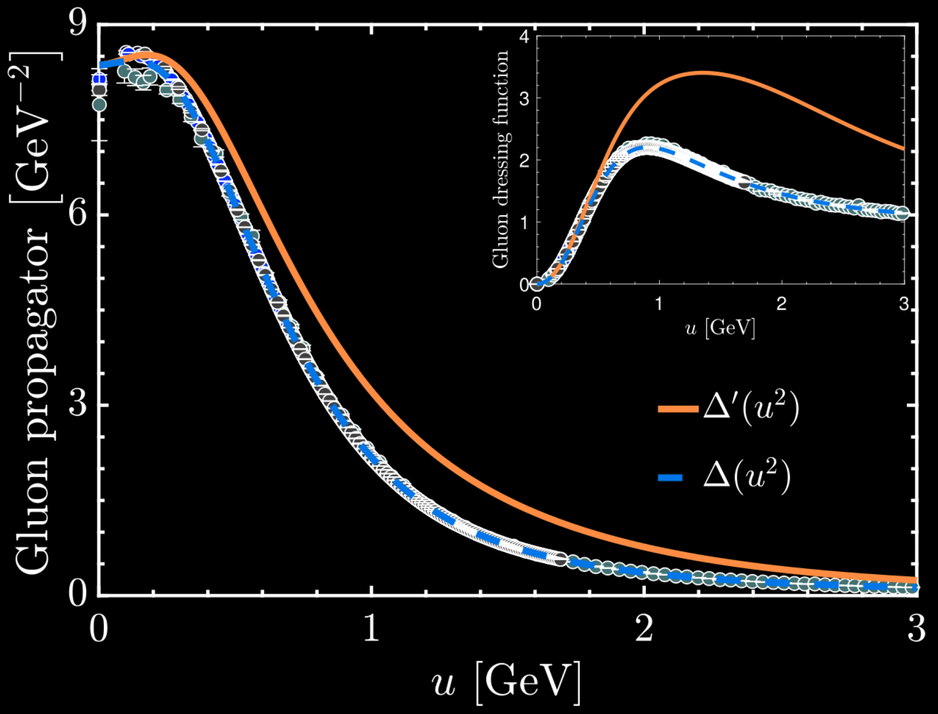


Wikipedia

Yang–Mills Existence and Mass Gap. Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those cited in Streater & Wightman (1964), Osterwalder & Schrader (1973) and Osterwalder & Schrader (1975).

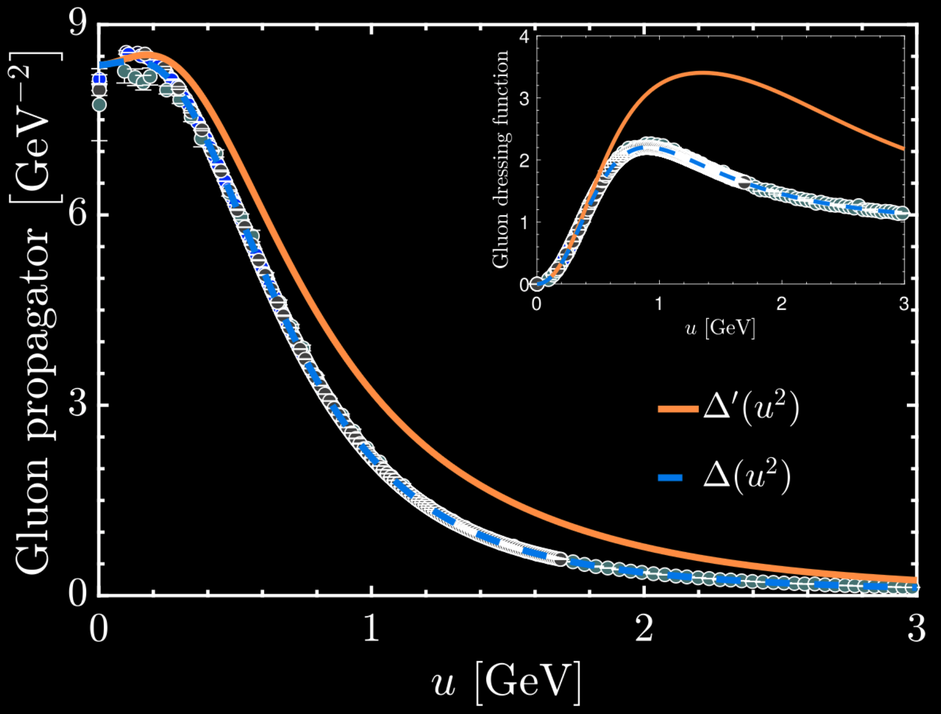
Introduction: Core problem

- **Gluon DSE:** 2006

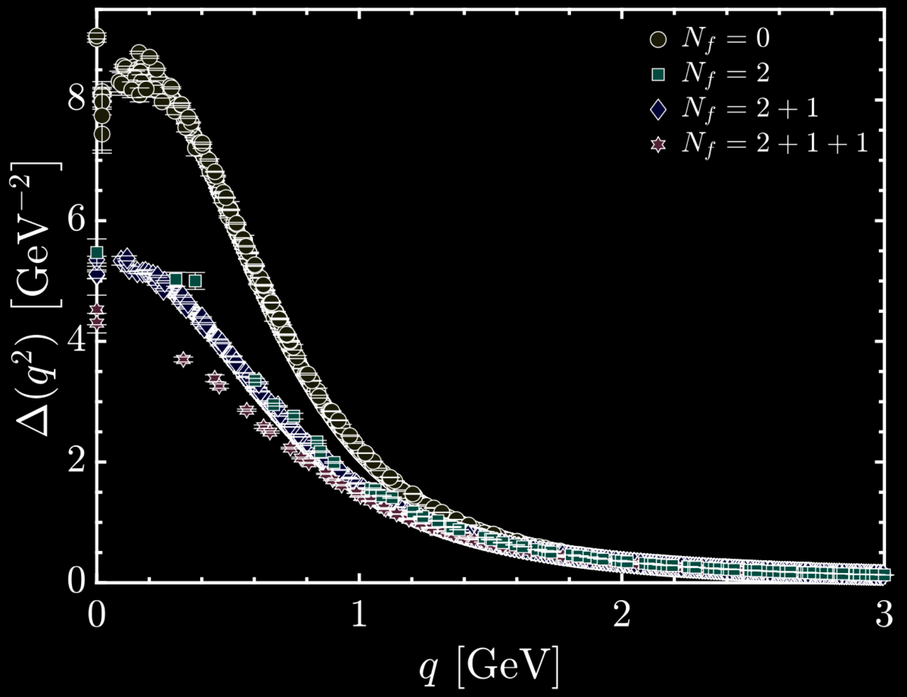


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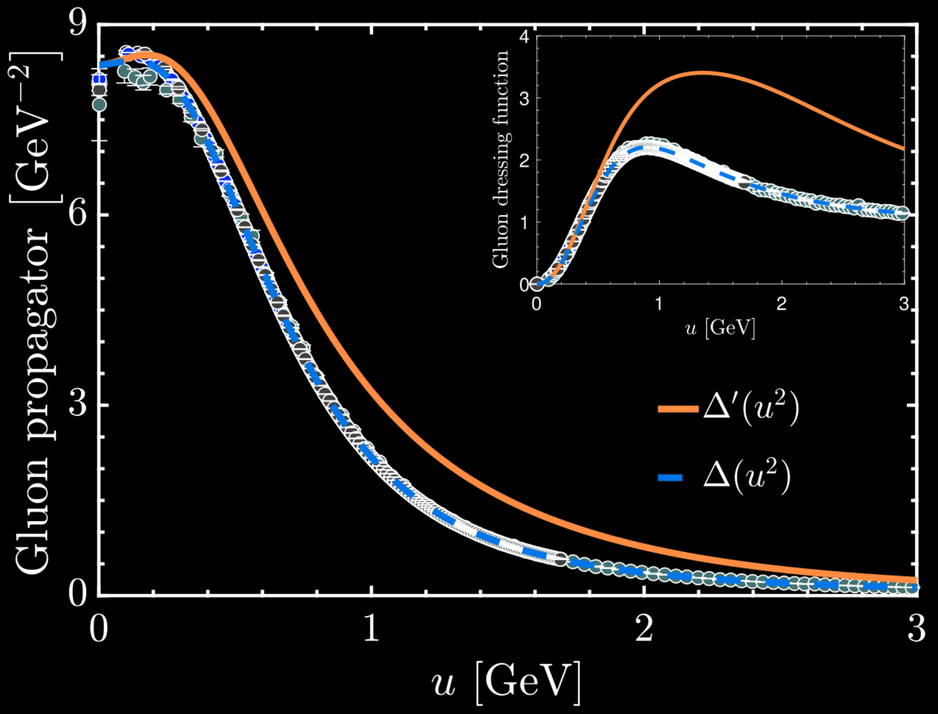


- **Lattice QCD:** 2009

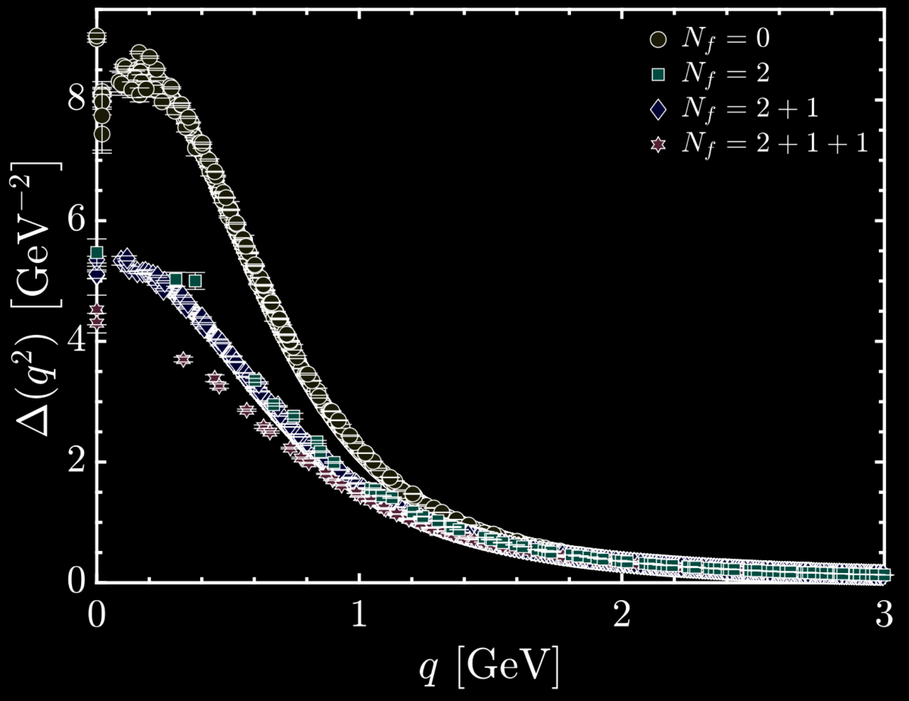


Introduction: Core problem

- **Gluon DSE:** 2006



- **Lattice QCD:** 2009



- **Gluon model:**

$$\mathcal{G}(k^2) \approx \frac{4\pi\alpha_s(k^2)}{k^2 + m_g{}^2(k^2)}$$

$$m_g^2(k^2) = \frac{M_g^4}{M_g{}^2 + k^2}$$

2011

Framework

Framework: DSEs

Classical Mechanics | Quantum Field Theory

Degrees of freedom

Generalized coord. | Fields on spacetime

Principle of Stationary Action

$$\frac{\delta S[q]}{\delta q} = 0$$

$$\left\langle \frac{\delta S[\phi(x)]}{\delta \phi(x)} \right\rangle = 0$$

Equations of Motion

Euler-Lagrange Equation | Dyson-Schwinger Equations

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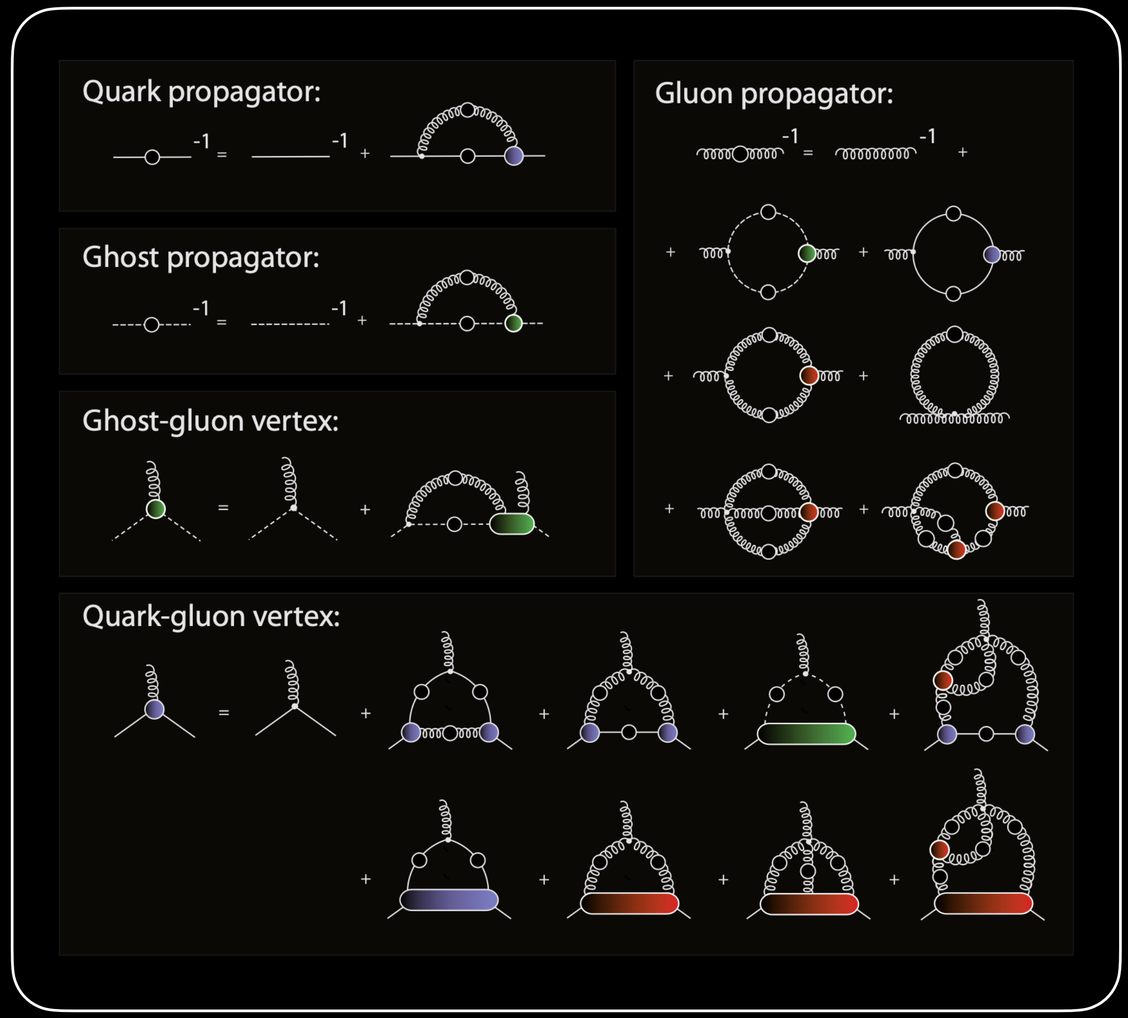
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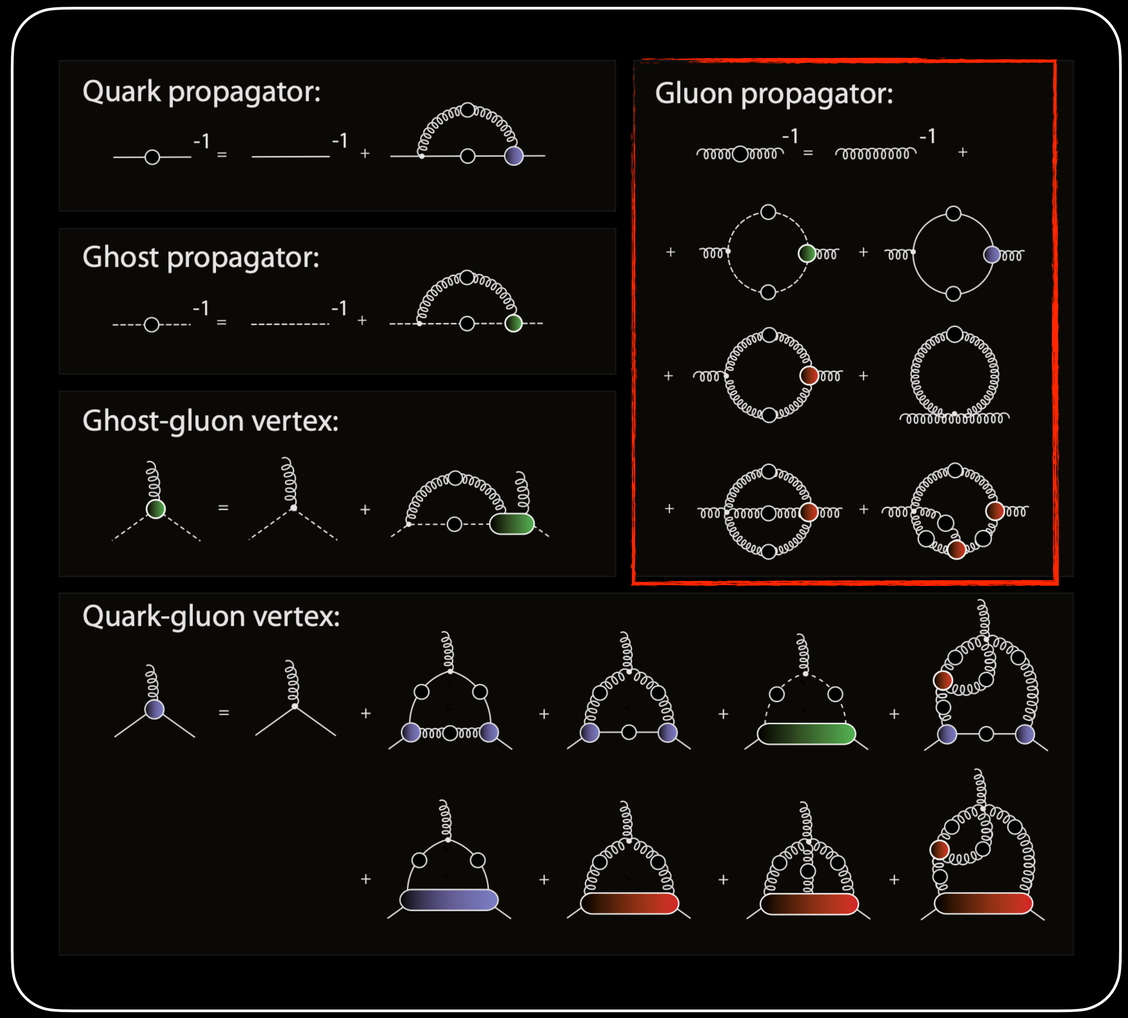
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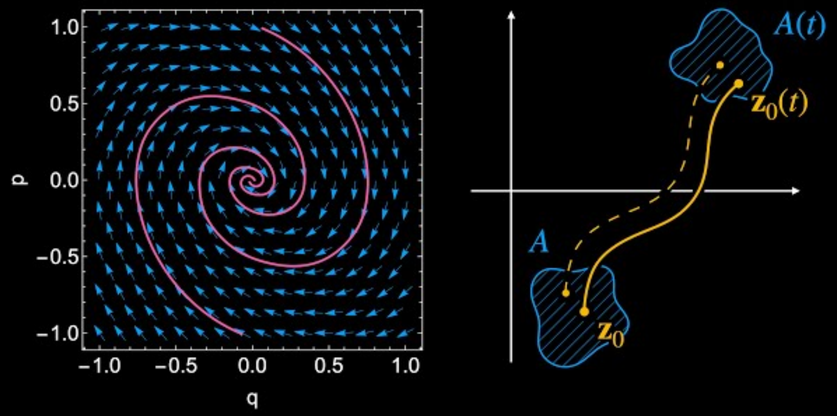
Dyson-Schwinger Equations



Framework: Statistical analogy

Liouville equation

$$\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0$$



Framework: Statistical analogy

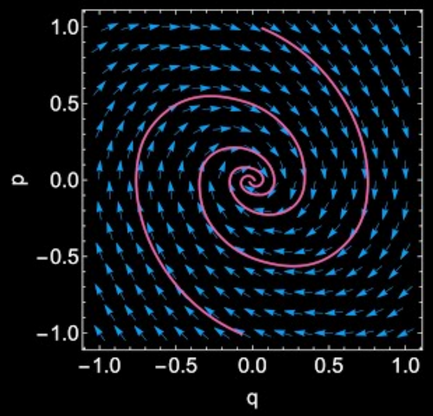
Liouville equation

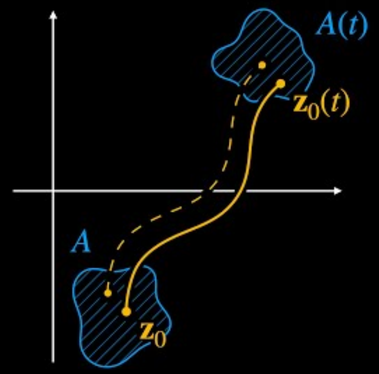
$$\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0$$

BBGKY hierarchy

$$\frac{\partial \rho_n}{\partial t} = \{H_n, \rho_n\} + (N-n)\sum_{i=1}^{n}\int \frac{\partial \rho_{n+1}}{\partial \mathbf{p}_i}\cdot\frac{\partial \phi_{i\,n+1}}{\partial \mathbf{r}_i} dz_{n+1}$$

DSE-like

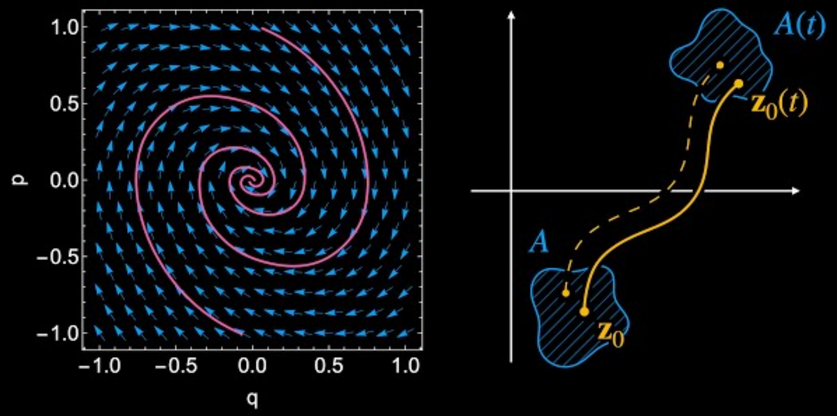




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DSE-like

One-body function

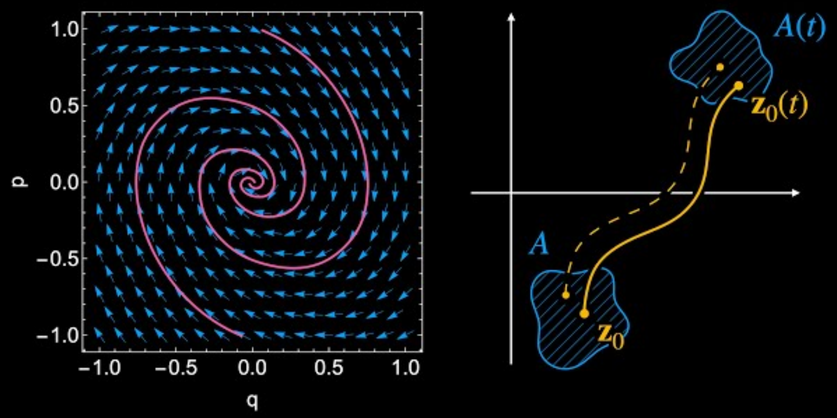
$$\rho_1(z_1) = \int_D \rho(Z,t)\, dz_2 \cdots dz_N$$

propagator-like

Framework: Statistical analogy

Liouville equation

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DSE-like

One-body function

$$\rho_1(z_1) = \int_D \rho(Z,t)\, dz_2 \cdots dz_N$$

propagator-like

Boltzmann equation

$$\frac{\partial \rho_1}{\partial t} = \{H_1, \rho_1\} + \int \left[\rho_1(\mathbf{p}_1')\rho_1(\mathbf{p}_2') - \rho_1(\mathbf{p}_1)\rho_1(\mathbf{p}_2)\right] N\nu\, d\sigma\, d^3\mathbf{p}_2$$

Framework: Constituent quarks

- **Quark propagator：**

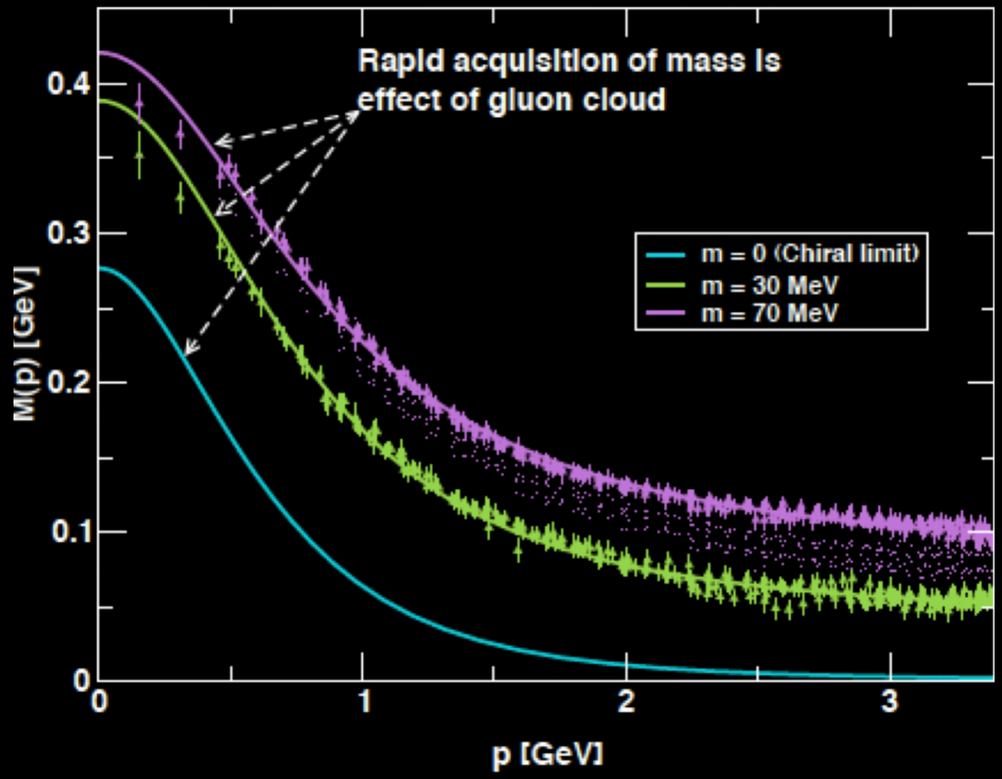
$$S(p) = \frac{1}{i\gamma \cdot pA(p^2) + B(p^2)} = \frac{Z(p^2)}{i\gamma \cdot p + M(p^2)}$$

Framework: Constituent quarks

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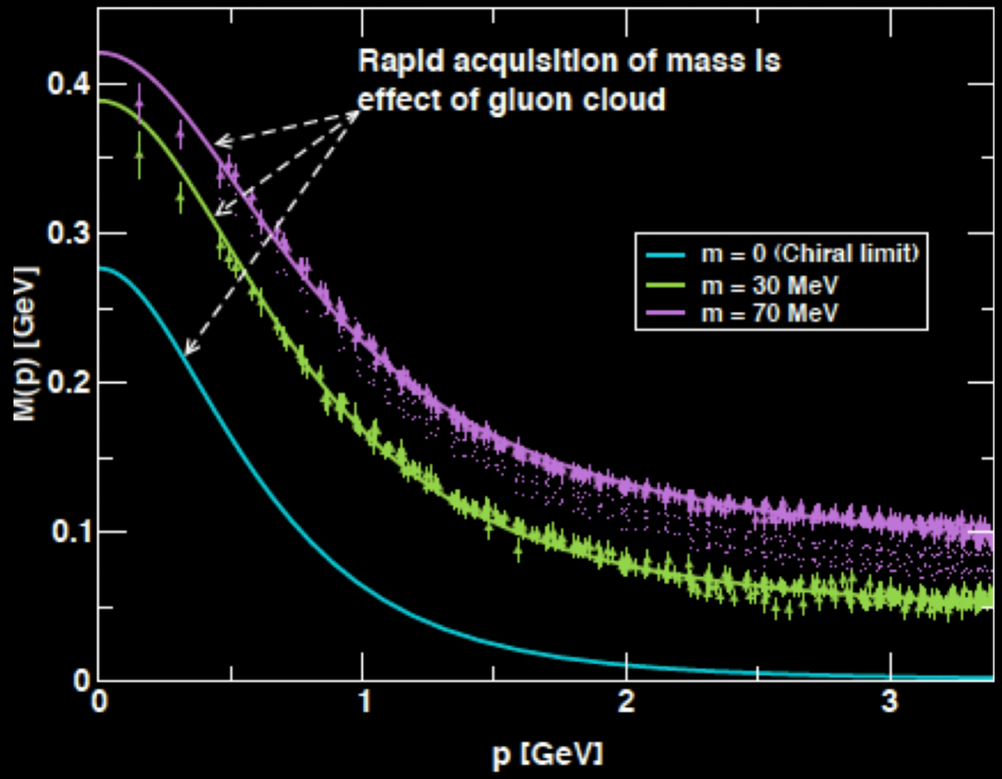


Framework: Constituent quarks

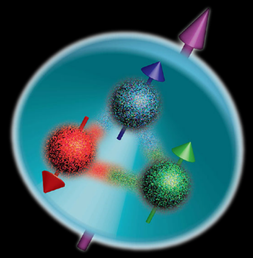
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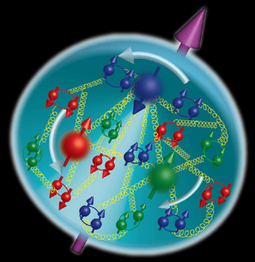
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- **Mass function:**



- **Quasi-quark:**

Low p 

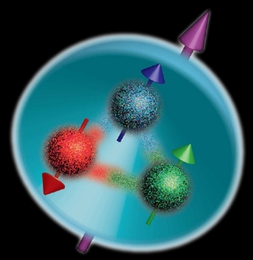
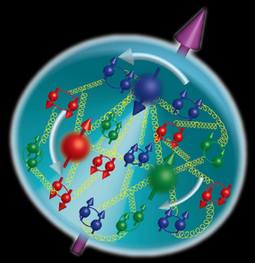
High p 

Framework: Constituent quarks

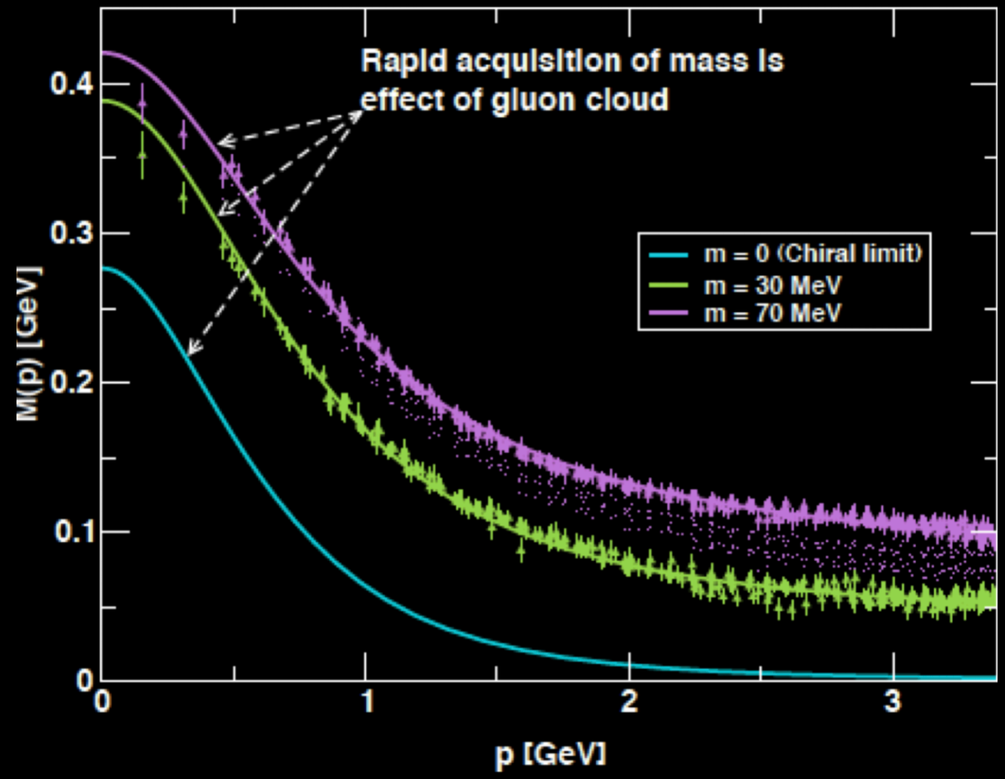
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- **Quasi-quark:**

Low p  High p 

- **Mass function:**



QCD vacuum — highly dispersive medium

Framework: Phase transitions

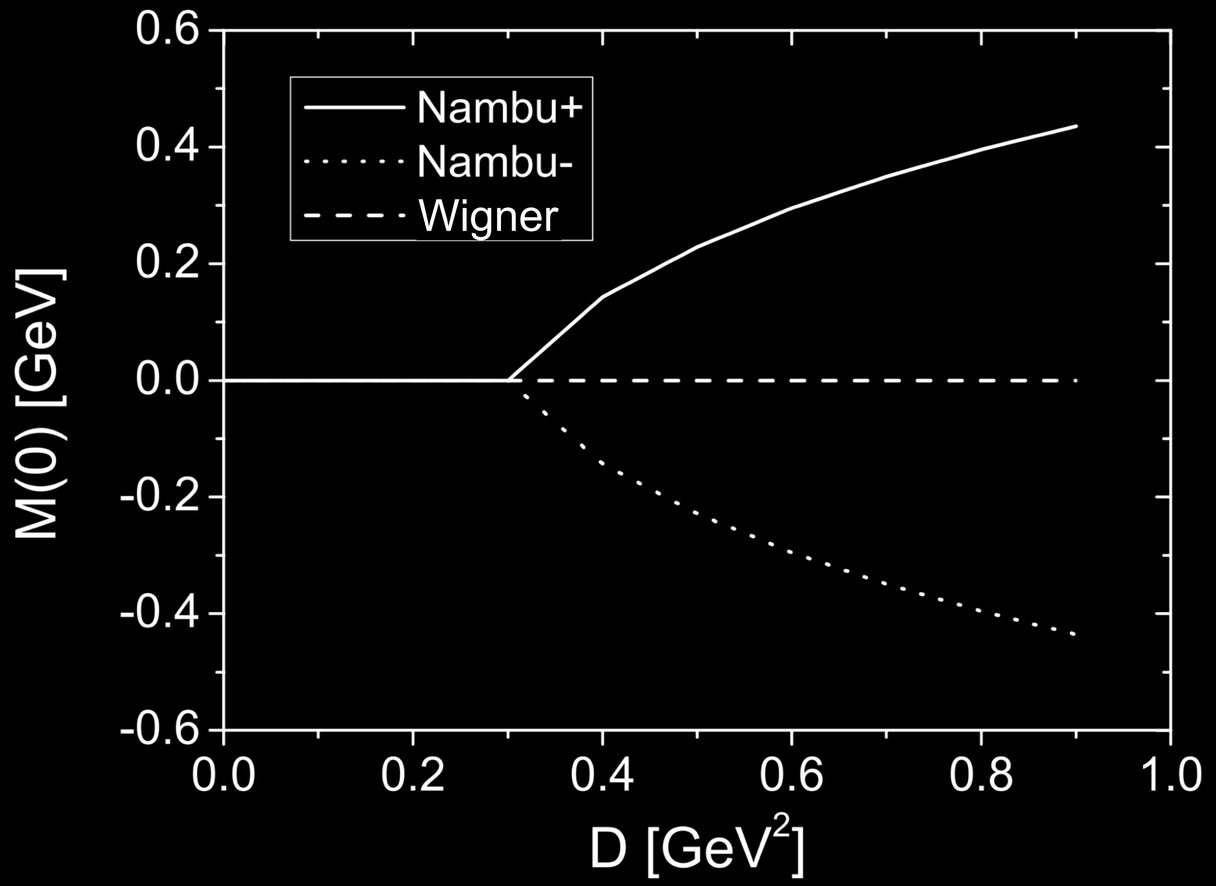
- **Fredholm-like integral equation:**

$$\phi(t) = f(t) + \lambda \int K(t, s; \phi)\, \phi(s)\, ds$$

Framework: Phase transitions

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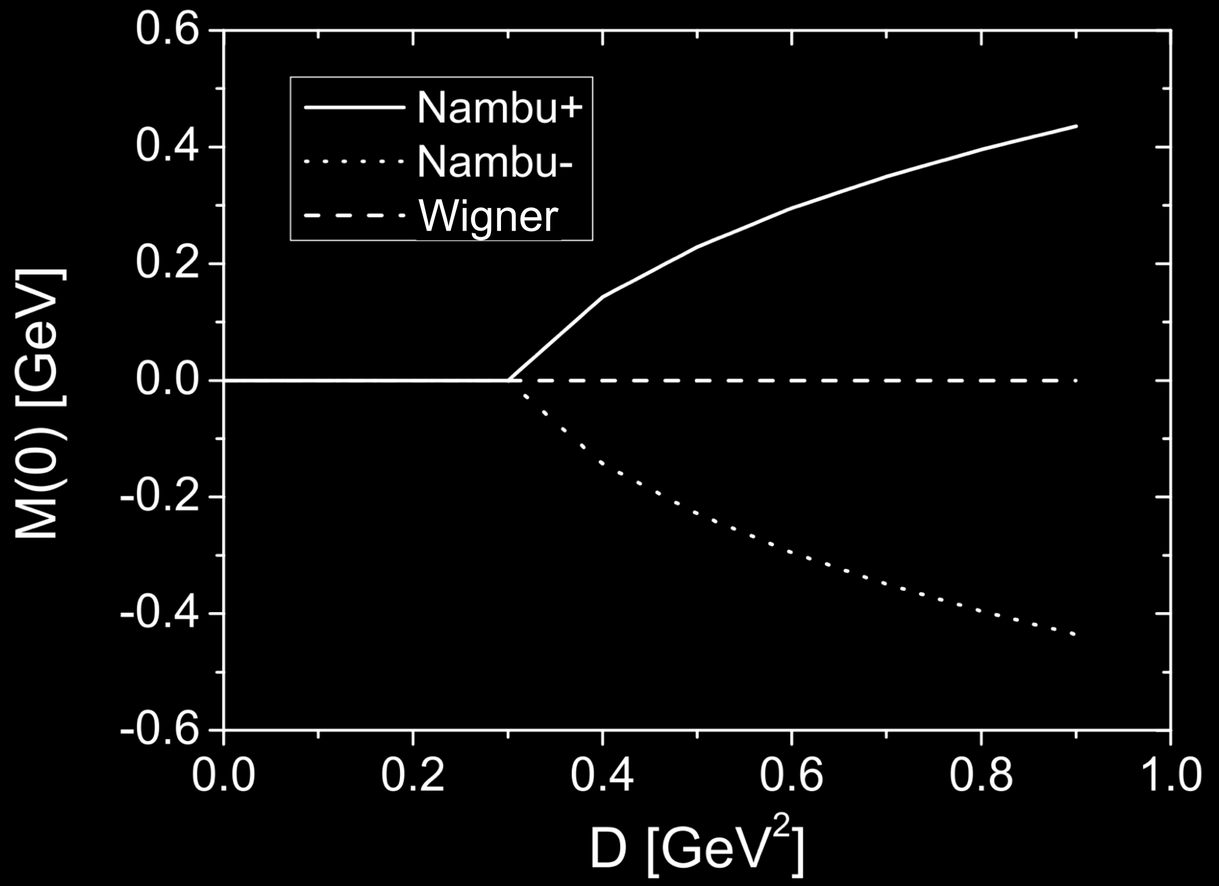


Framework: Phase transitions

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- Numerical solutions with hard cutoff.
- Integral operators are non-compact.

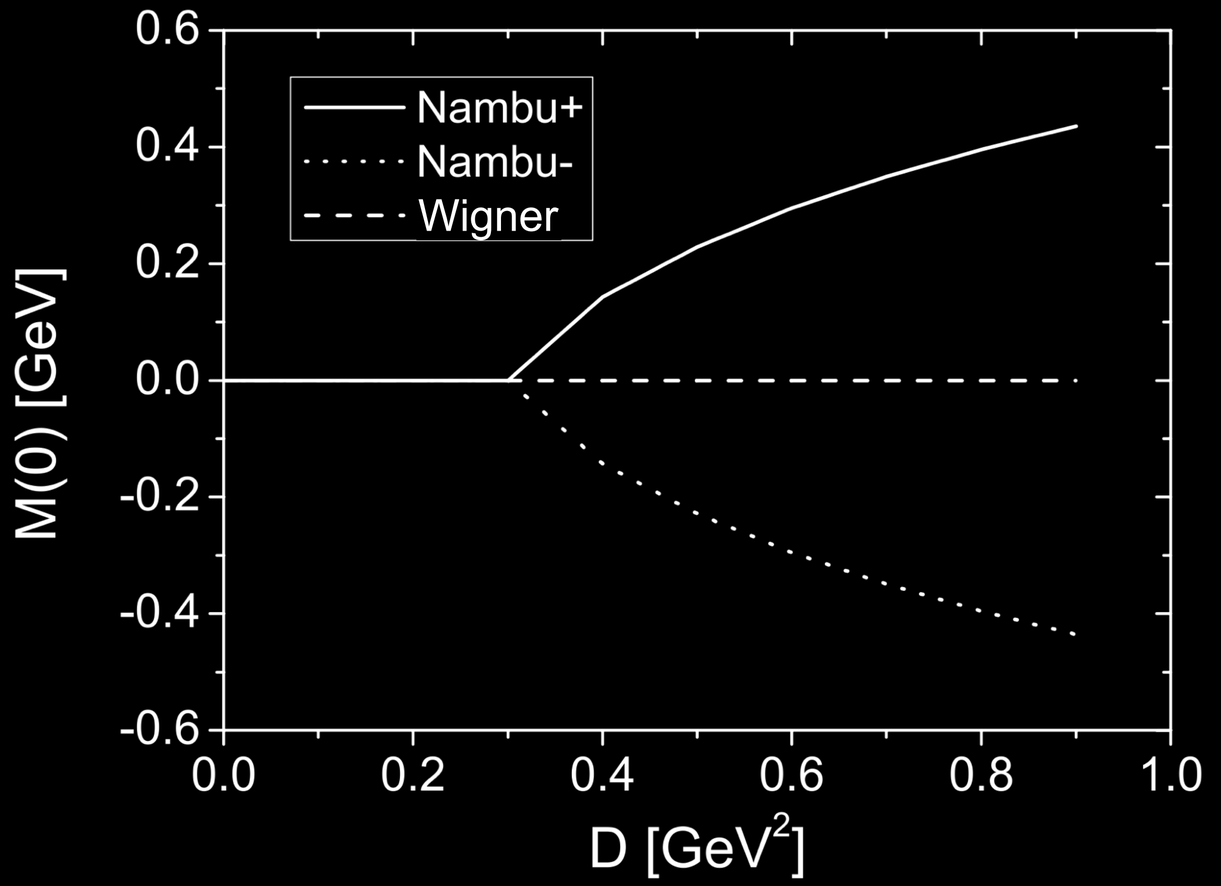


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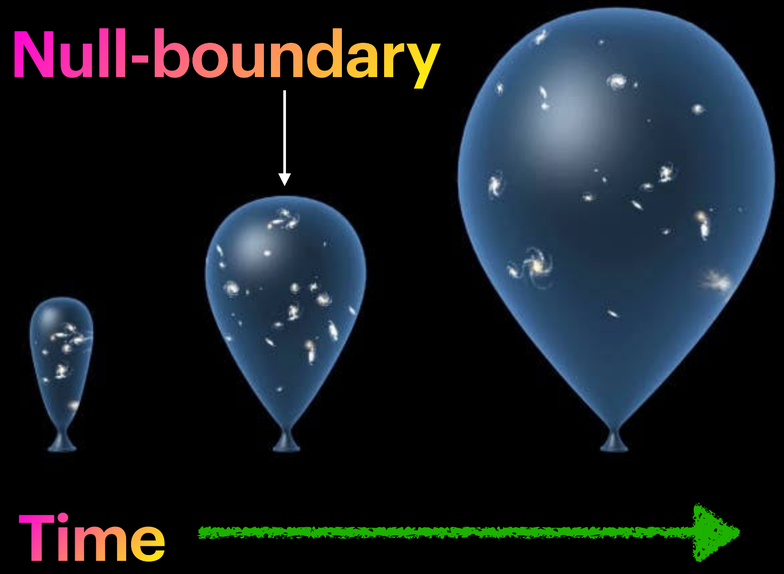
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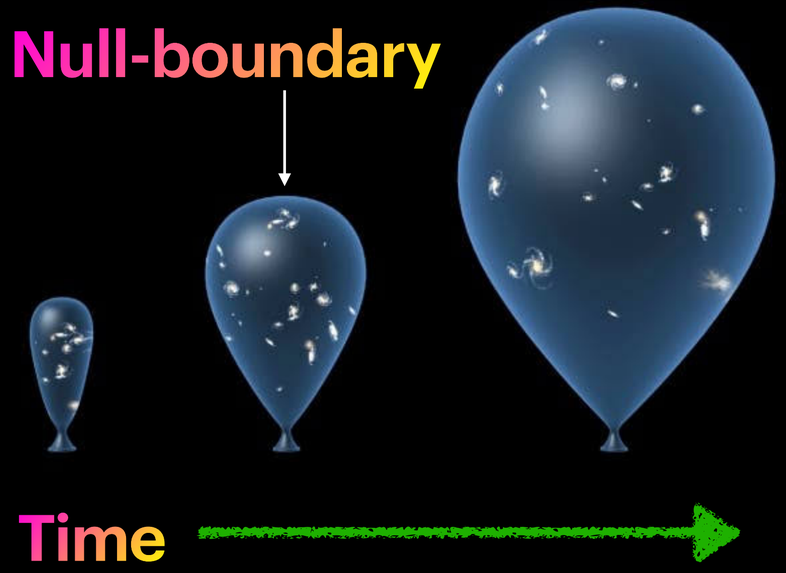
No admission ticket for bifurcation

Spacetime

Spacetime: Discrete dynamics



Spacetime: Discrete dynamics

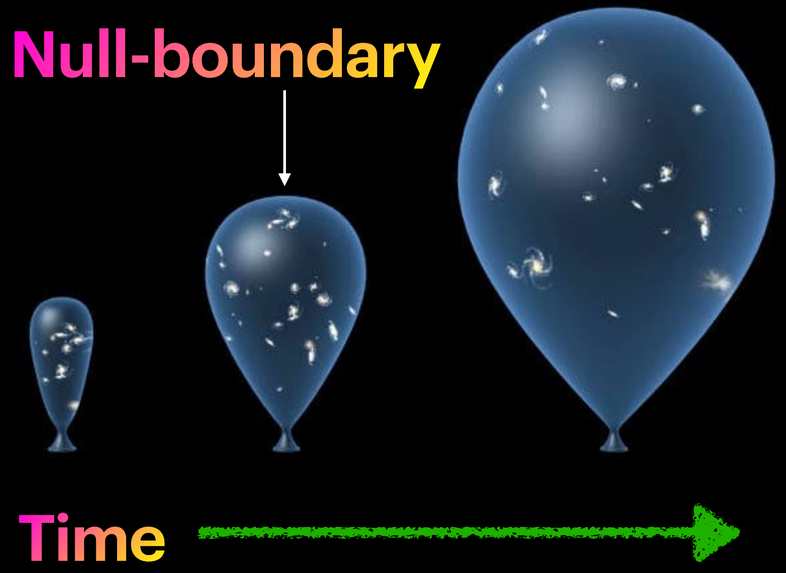


- **All in One:**

  The Universe manifold $\mathscr{M}$

  A composite scalar field $f(x)$

Spacetime: Discrete dynamics



- **Discrete Poisson Process:**

$$\mathcal{Z}[f] = \mathbb{E}\left[\exp\left(\int_{\mathscr{M}} f(x)\, N(dx)\right)\right]$$

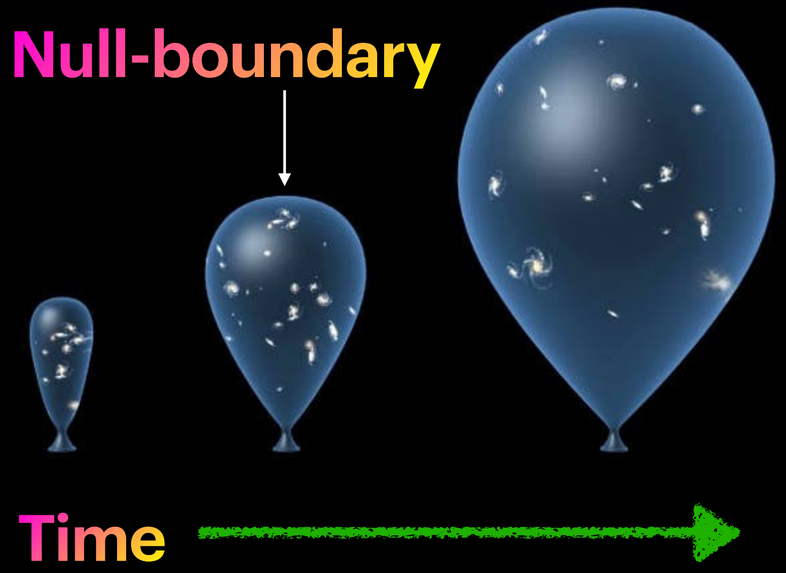
over all possible configurations

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Spacetime: Discrete dynamics



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- **Discrete Poisson Process:**

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over all possible configurations

- **Coarse-grained functional:**

$$\mathscr{Z}[f] = \exp\left\{\int_{\mathscr{M}} \lambda\left(e^{f(x)} - 1\right)\sqrt{g(x)}\, d^4x\right\}$$

by the Campbell's theorem

Spacetime: Emergent action

1. Planck basic parameters

$$\lambda \to \frac{\kappa}{\hbar} \sim \ell_P^{-4}$$

2. Lagrangian scalar field

$$f(x) \to \frac{\mathscr{L}_E(x)}{\kappa}$$

Spacetime: Emergent action

1. Planck basic parameters

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- **Macroscopic effective action:**

$$S_E = \int_{\mathscr{M}} \kappa \left[\exp\left(\frac{\mathscr{L}_E(x)}{\kappa} \right) - 1 \right] \sqrt{g(x)}\, d^4x$$

In low-energy regime $\mathscr{L}_E \ll \kappa$

$$S_E \sim \int \mathscr{L}_E(x)\, d^4x + \int \mathcal{O}\left(\frac{\mathscr{L}_E^2}{\kappa} \right) d^4x$$

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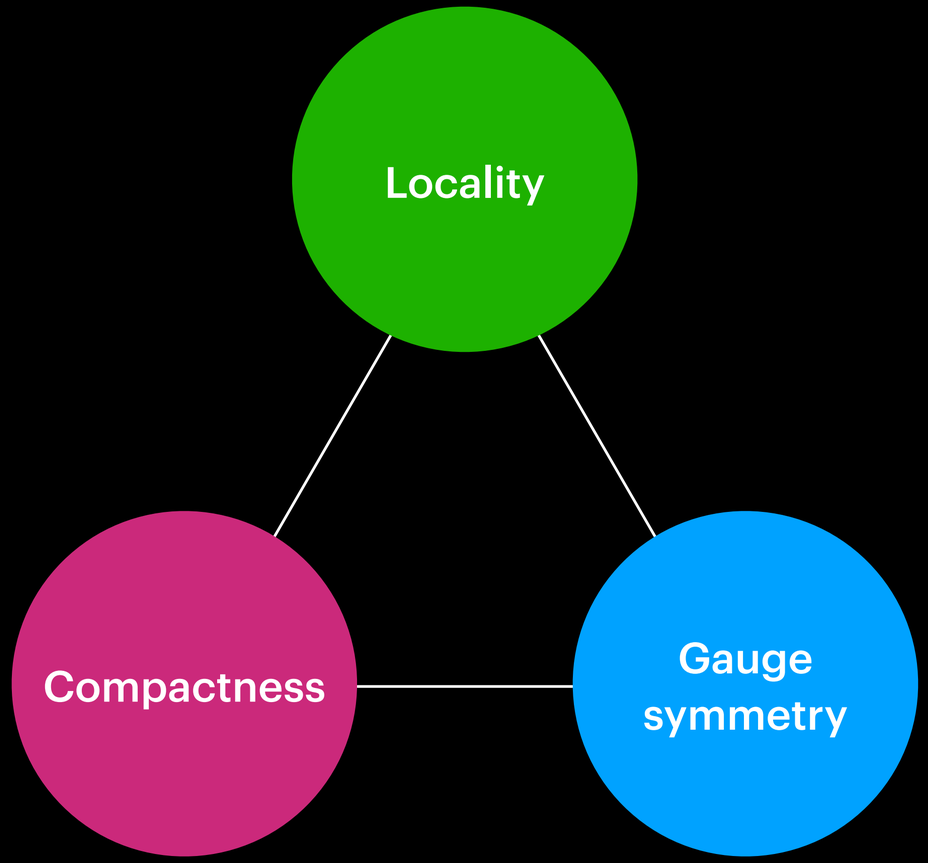
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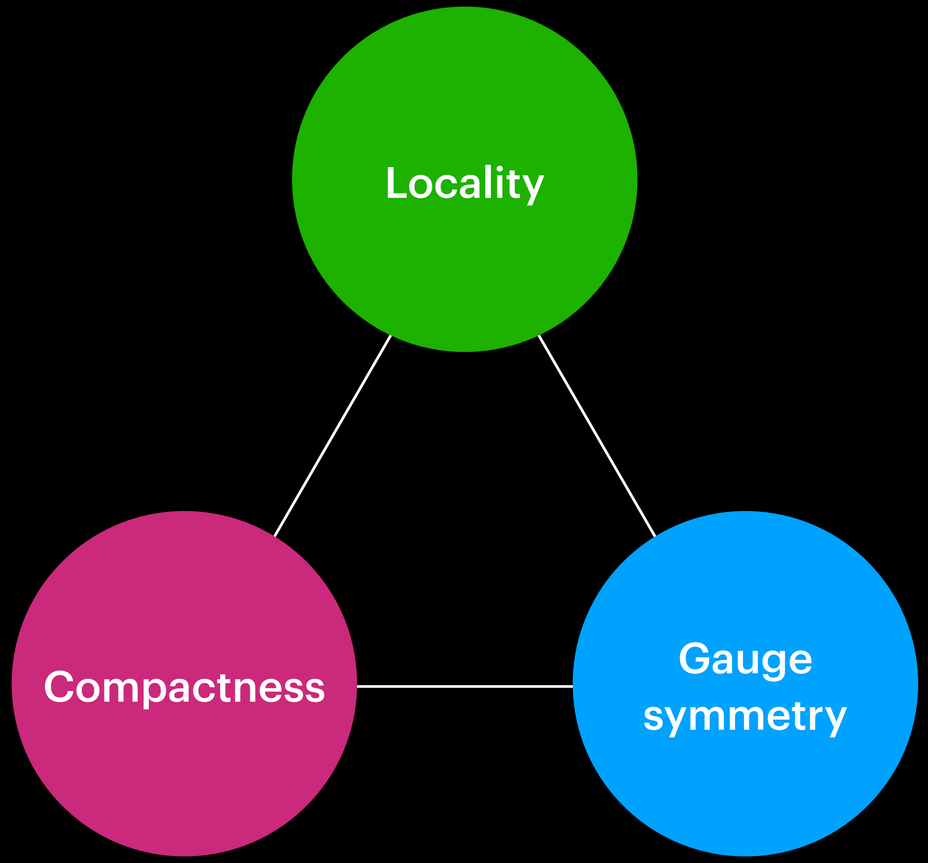
- **Double-Exponential Probability:**

$$P[\mathscr{L}_E] \sim \exp\left(-\frac{S_E}{\hbar} \right)$$ Euclidean EFT

Spacetime: Impossible triangle



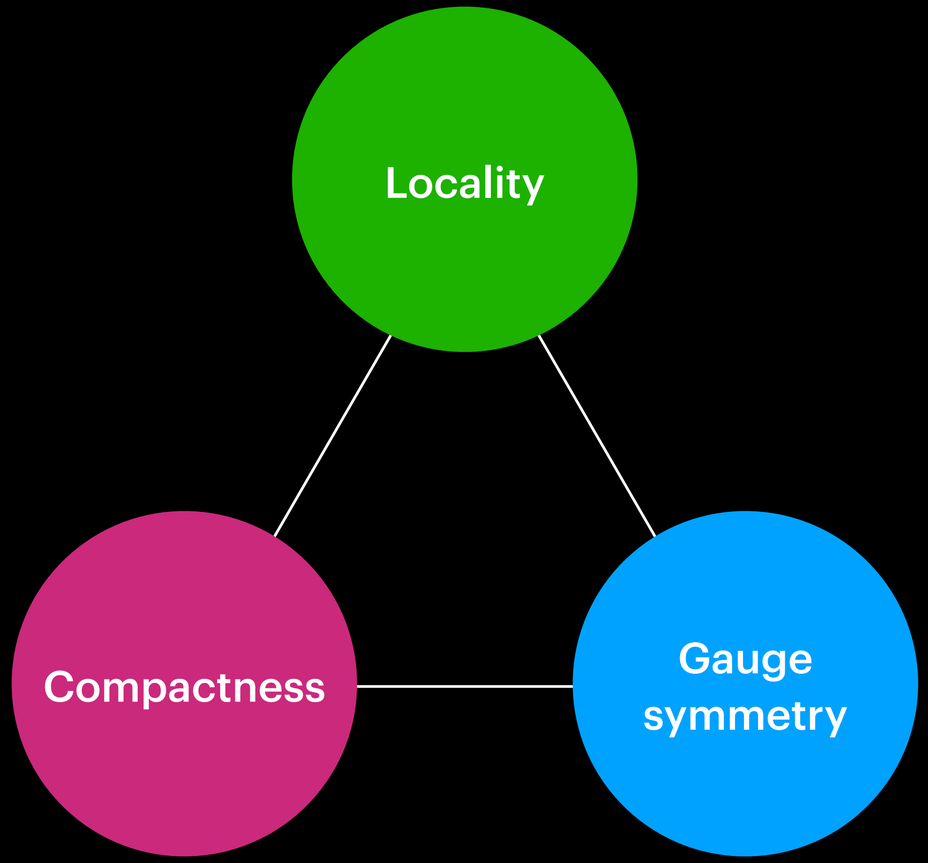
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- **Standard QFT:**

~~Compactness~~ →

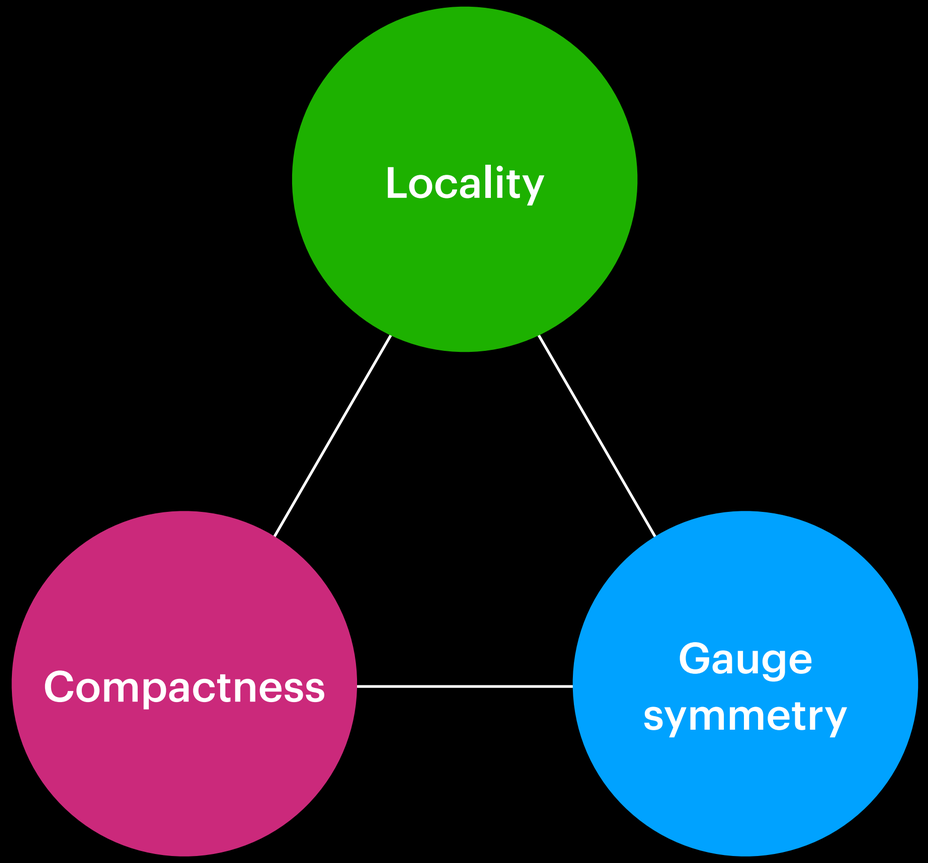
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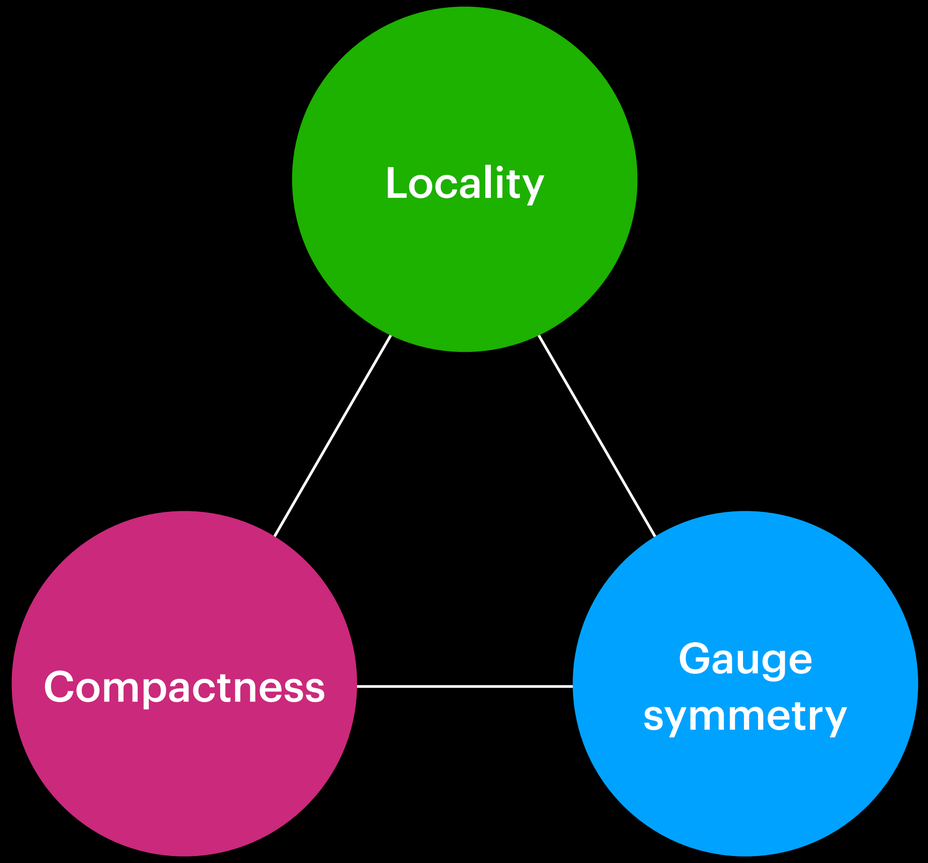
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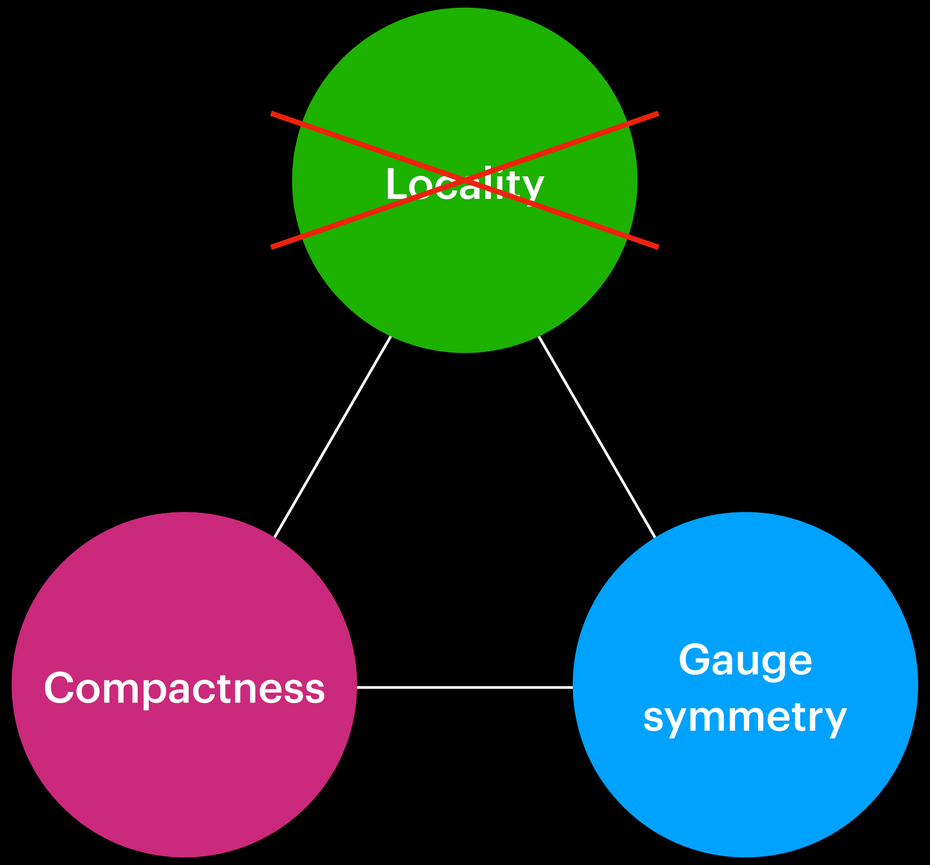
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Non-locality

Non-locality: Gauge theory

- **Non-local gauge theory:**

$$S_{\mathrm{NL}} = \int d^4x \, \mathrm{Tr} \left[\frac{1}{2} F_{\mu\nu} \exp\left(\tau_0 \hat{\mathscr{D}}^2 \right) F^{\mu\nu} \right] + \cdots$$

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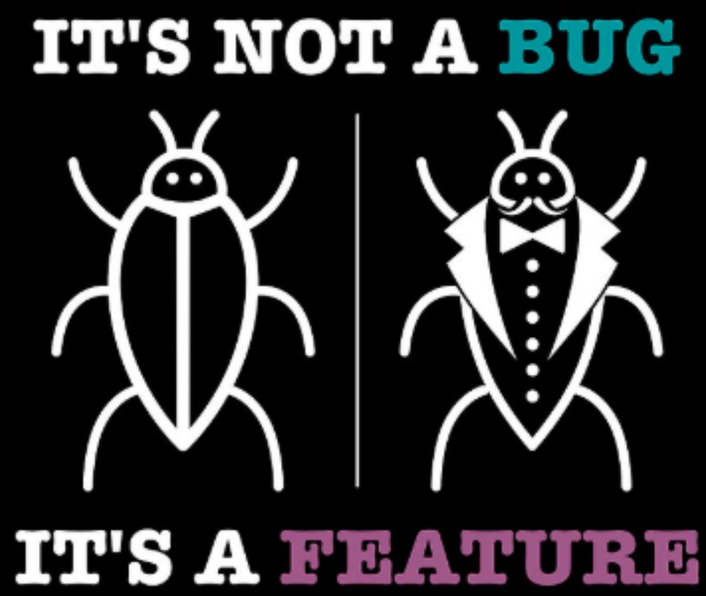
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Non-locality: Gauge theory

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$$\Delta(k) \propto k^{-2} \exp\left(-\tau_0 k^2\right)$$

suppression

$$\Gamma^{(n)}(k_i) \propto \exp\left(+\frac{\tau_0}{2} \sum k_i^2\right)$$

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- **Feynman diagrams:** ***L***-loops and ***I***-internal lines

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(e.g., Symanzik polynomials)

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- **One exception:** residual logarithmic divergence (one-loop)

$$\mathcal{K}_R(x,y) = \mathcal{K}_{\text{bare}}(x,y) - \mathcal{K}_{\text{bare}}(\tau_0^{-1}, y) \sim \mathcal{O}(y^{-2})$$

Gauge symmetry

Symmetry: PT-BFM framework

- **Abelian-like WTIs:** Effective Green's functions

Three-gluon vertex $q^{\mu}\hat{\Gamma}_{\mu\alpha\beta}(q,r,p) = \hat{\Delta}^{-1}_{\alpha\beta}(p) - \hat{\Delta}^{-1}_{\alpha\beta}(r)$ Gluon propagators

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$$\hat{\Delta}^{-1}_{\alpha\beta}(p) = [p^2 J(p^2) + m^2(p^2)]P_{\alpha\beta}(p) \qquad P_{\alpha\beta}(k) = \eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$$

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- **WTI splitting:** Schwinger mechanism

$$q^{\mu}\hat{\Gamma}_{\mu\alpha\beta}(q,r,p) = \underbrace{\left[J(p^2)(p^2 g_{\alpha\beta} - p_\alpha p_\beta) - J(r^2)(r^2 g_{\alpha\beta} - r_\alpha r_\beta)\right]}_{\text{kinetic term: analytic}} + \underbrace{\left[m^2(p^2)P_{\alpha\beta}(p) - m^2(r^2)P_{\alpha\beta}(r)\right]}_{\text{mass term: } 1/p^2 \ \& \ 1/r^2}$$

Symmetry: Eaten scalar pole

- **Massless pole in vertex:** Mass function as residues

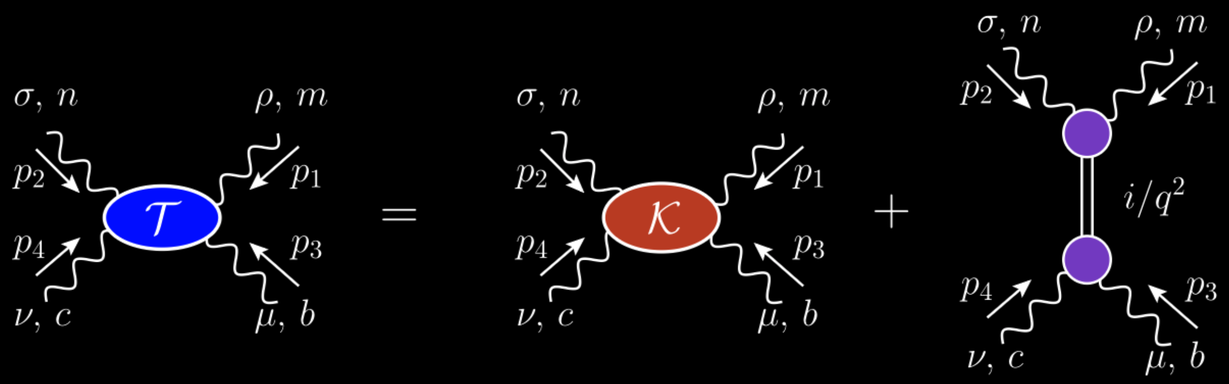
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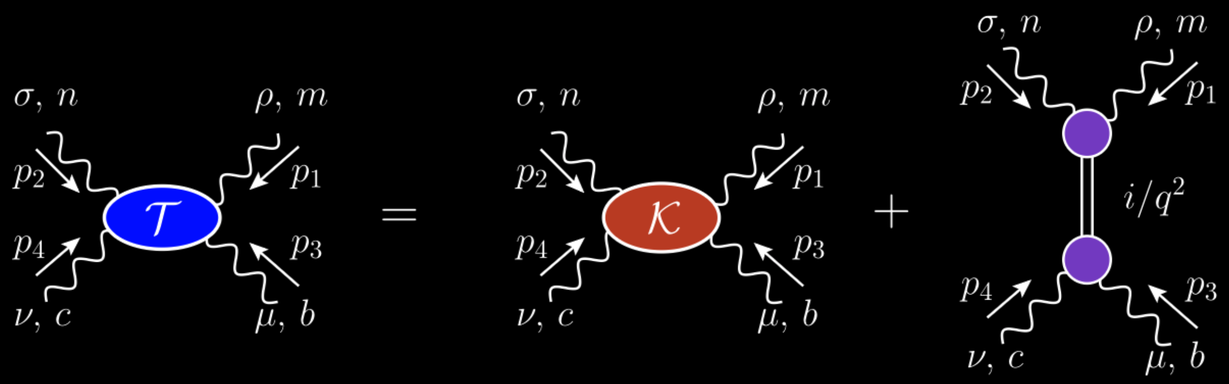


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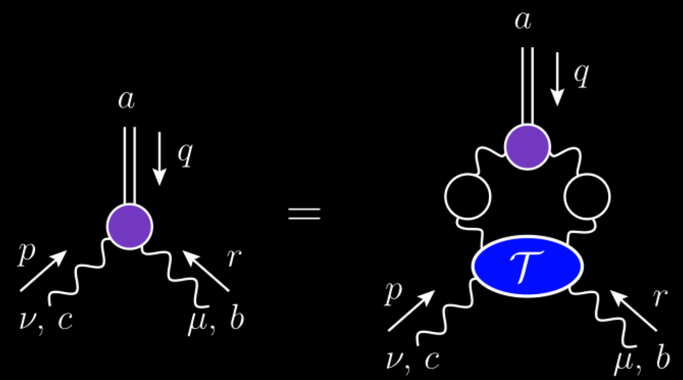
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- **BSE of the effective vertex:**



Symmetry: Non-linear BSE

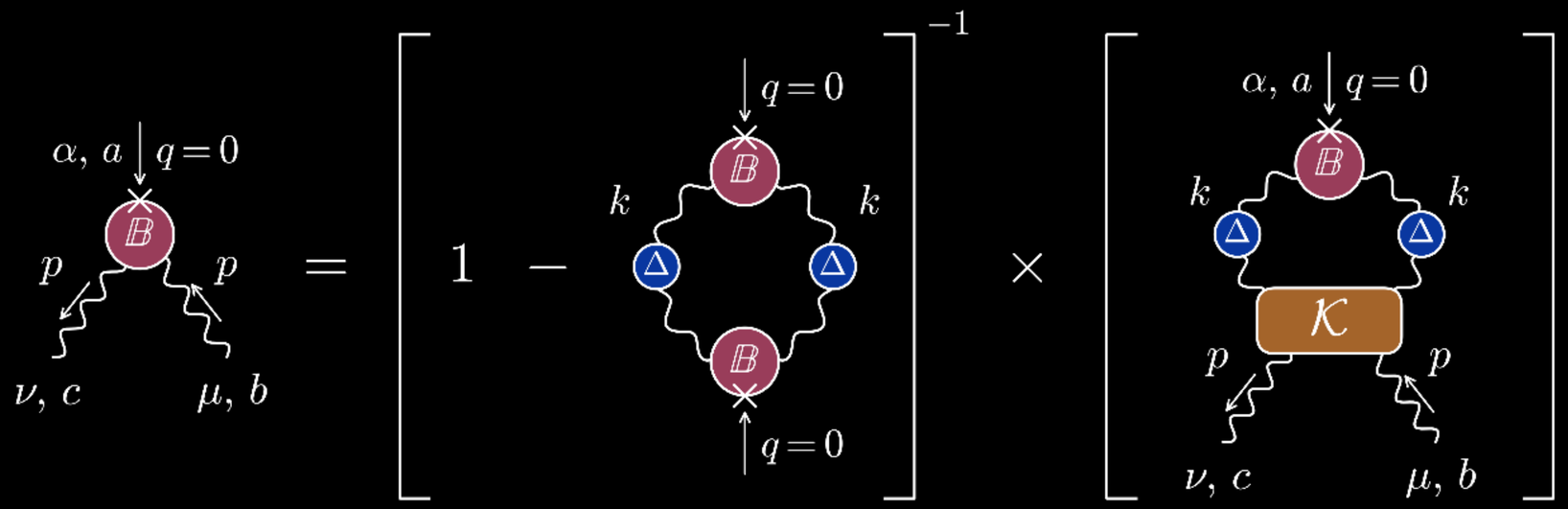
- **Residue non-linear equation:** Projecting in the soft-momentum limit $m \sim \mathbb{B}$

$$\mathbb{B}(p^2) = \frac{1}{1-\omega[\mathbb{B}]} \int_k k^2\, \Delta^2(k^2, m^2[\mathbb{B}])\, \mathscr{K}(p, k, m^2[\mathbb{B}])\, \mathbb{B}(k^2), \qquad \omega[\mathbb{B}] = \frac{3C_A}{2} \int_k k^2 \Delta^2(k^2) \mathbb{B}^2(k^2)$$

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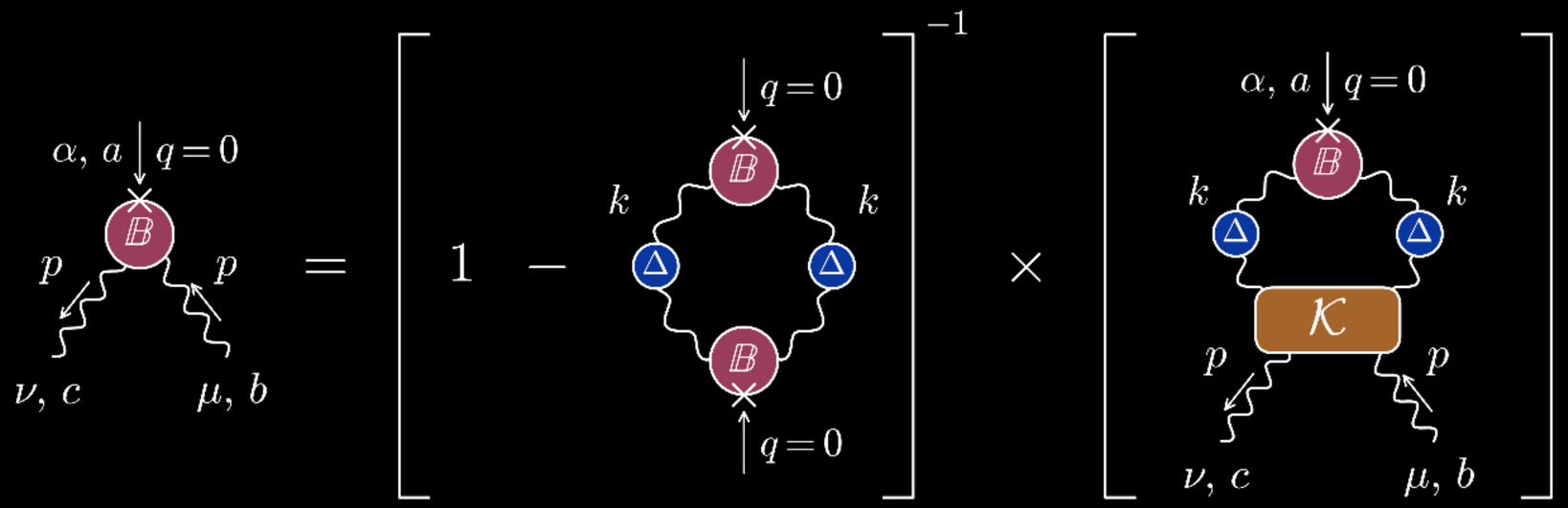
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renormalization group invariant (RGI)

Bifurcation

Bifurcation: Linearized BSE

$$\mathbb{B} = \mathscr{F}(\mathbb{B}) \xrightarrow{\text{Fréchet derivative}} \mathbb{B} = \left.\frac{\delta \mathscr{F}(\mathbb{B})}{\delta \mathbb{B}}\right|_{\mathbb{B}=0} \mathbb{B} \equiv \mathscr{L}\mathbb{B}$$

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Bifurcation: Convergent integral equation

- **Scale-invariant and power-law singularity:**

Propagator skeleton $\Delta_{\mathrm{pert}}(k^2) \to \dfrac{1}{k^2}$ Kernel skeleton $\mathscr{K}_{\mathrm{pert}}(p,k) \to \dfrac{12\pi C_A \alpha_s}{(p-k)^2}$

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Bifurcation: Euler-Cauchy equation

- **Local boundary-value differential problem:**

$$x^2 \frac{d^2\mathbb{B}}{dx^2} + 2x \frac{d\mathbb{B}}{dx} + \lambda \mathbb{B}(x) = 0$$

boundary condition $\mathbb{B}(1/\tau_0) = 0$

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Bifurcation: Dimensionless mode profile

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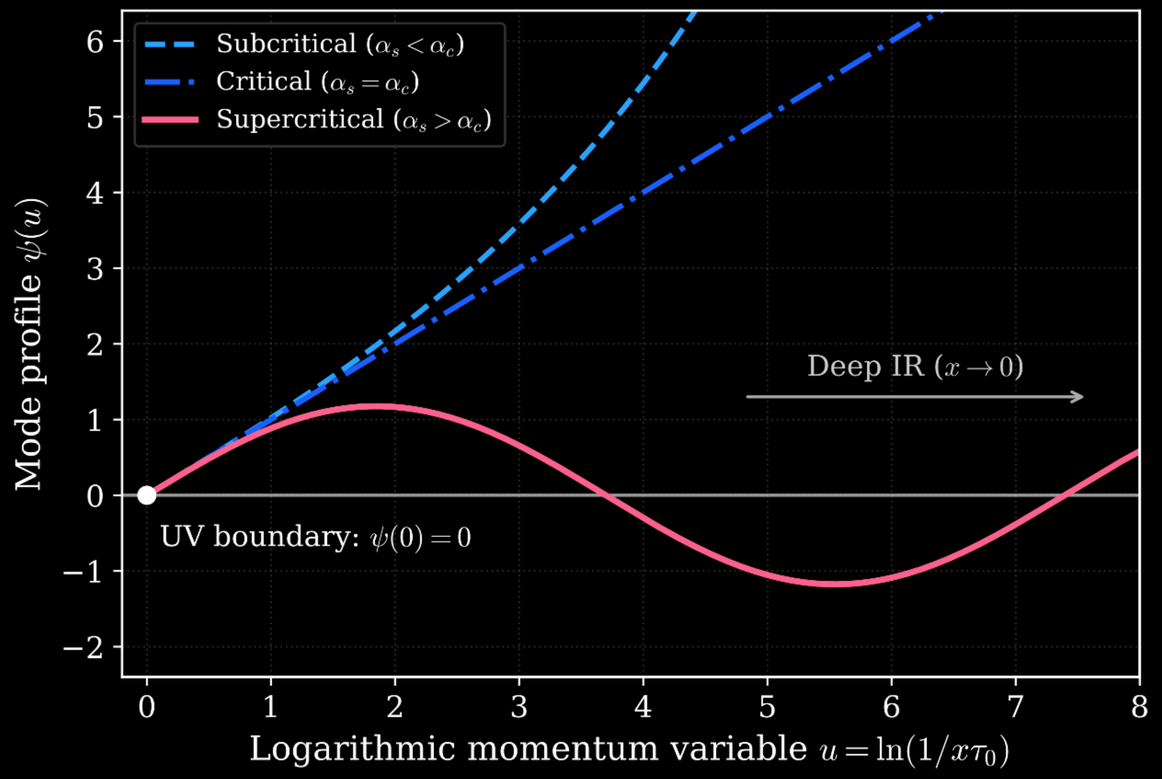
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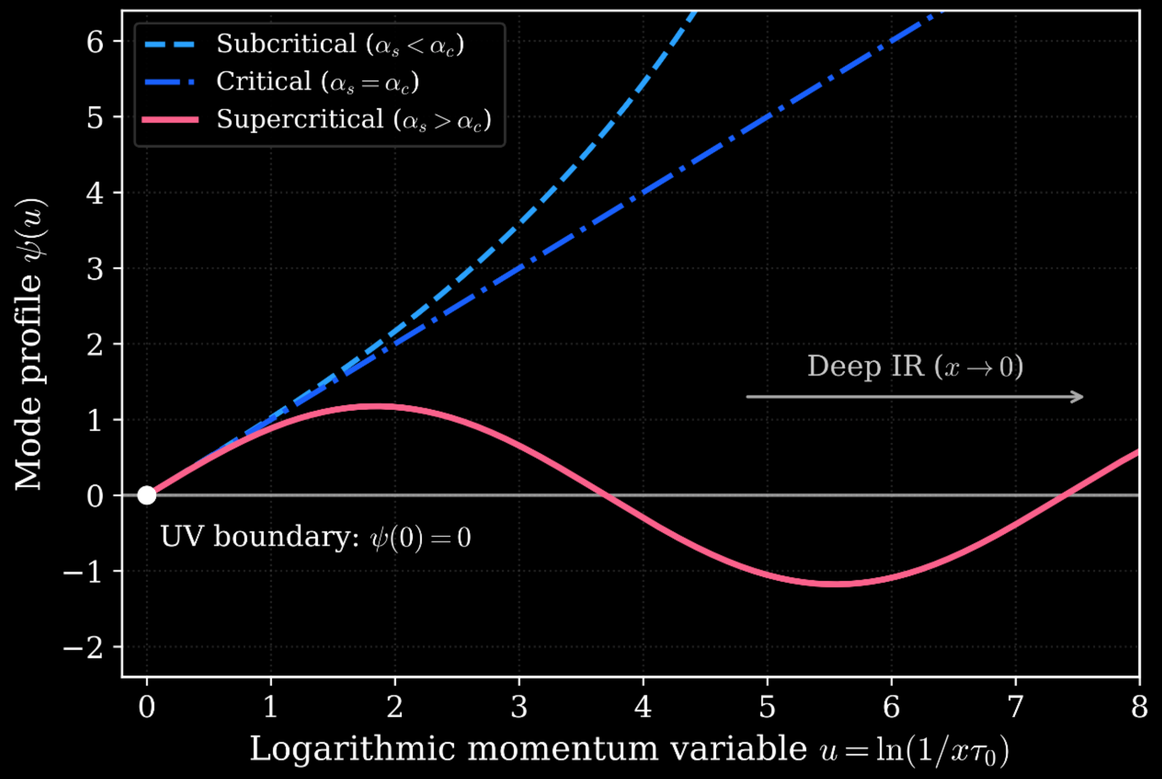
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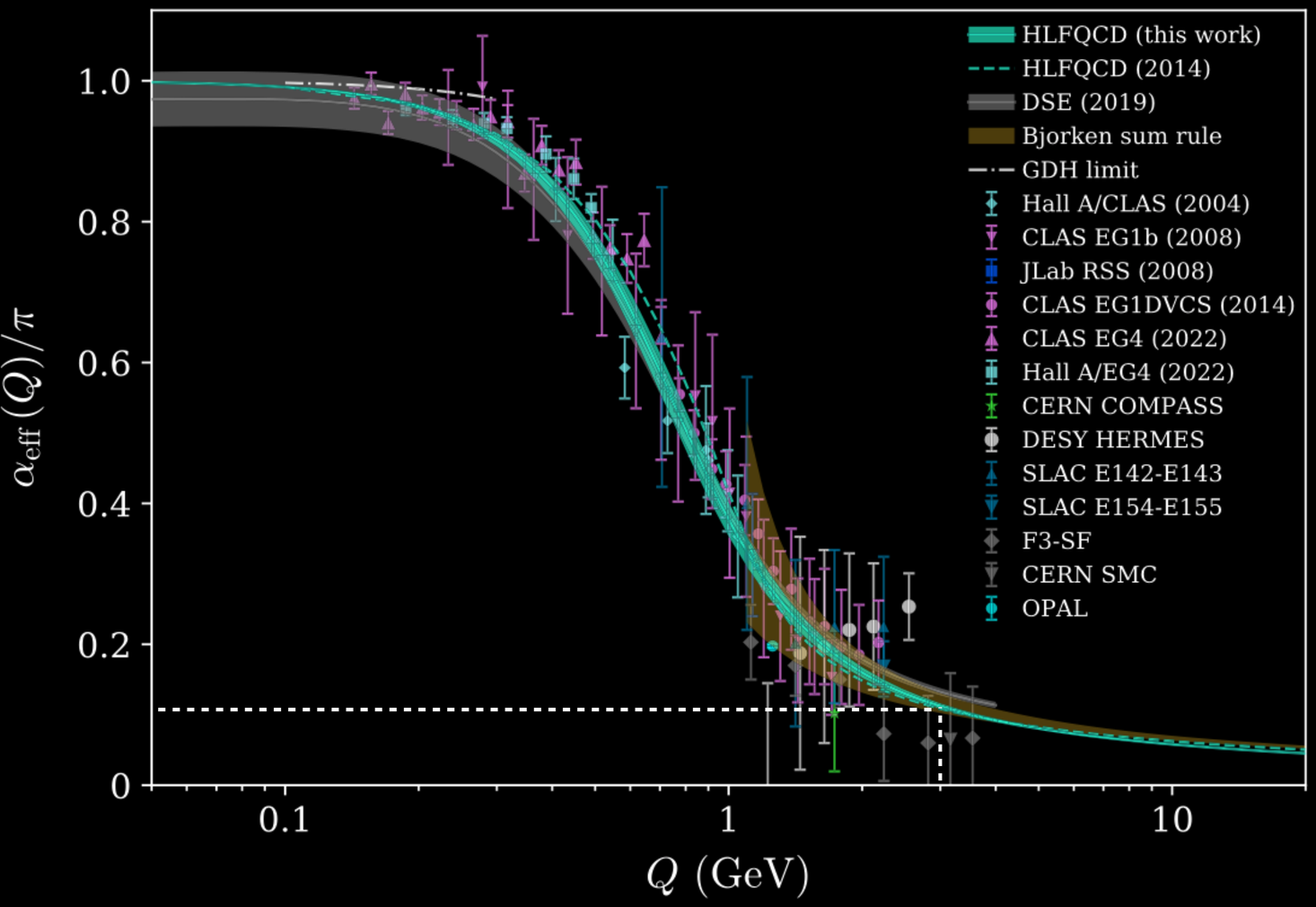
- **Infrared self-regularization:**

$$\omega[\mathbb{B}] \propto A^2 \int_0^\infty \psi^2(u)\, du$$

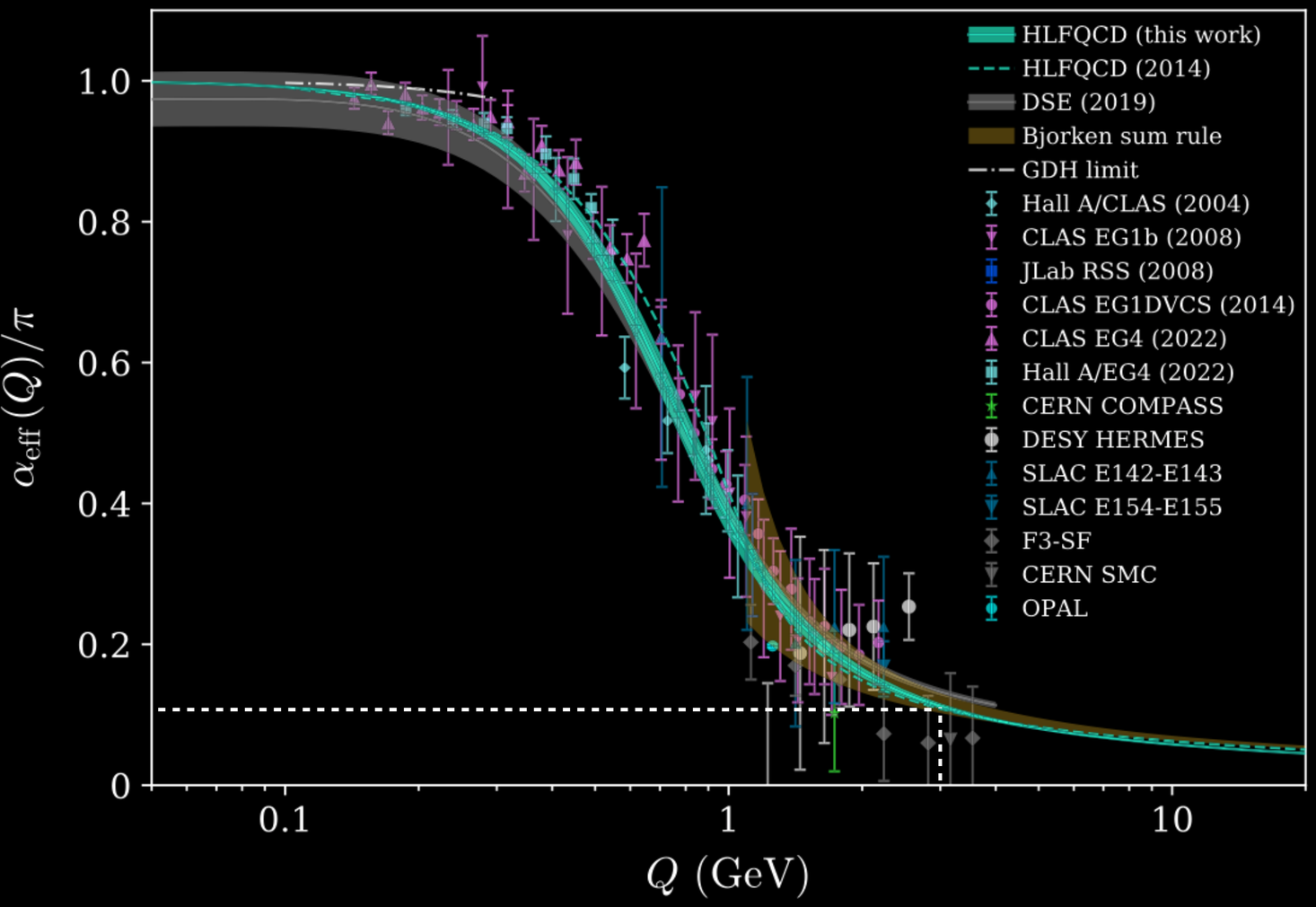
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Bifurcation: Physical massive vacuum



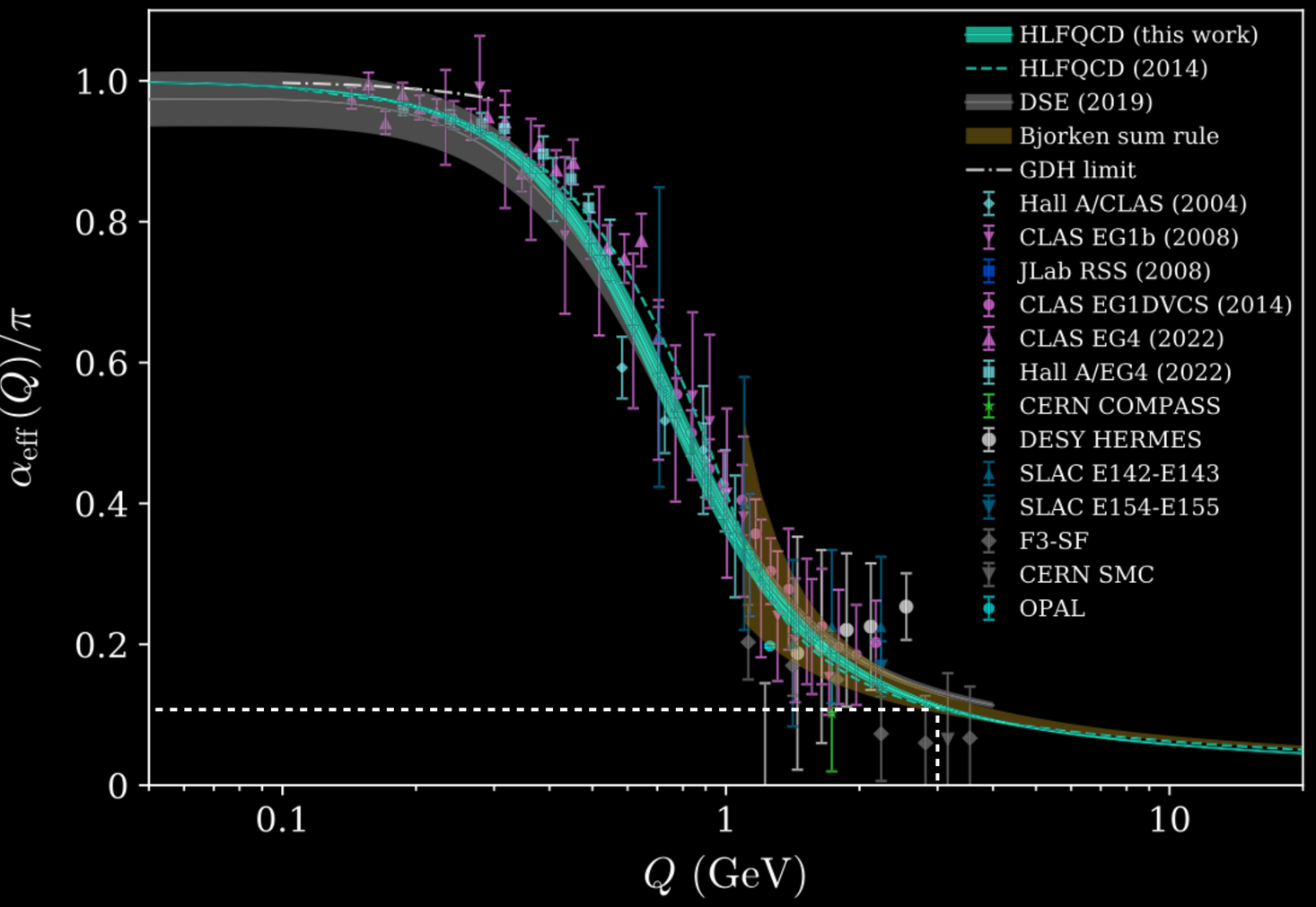
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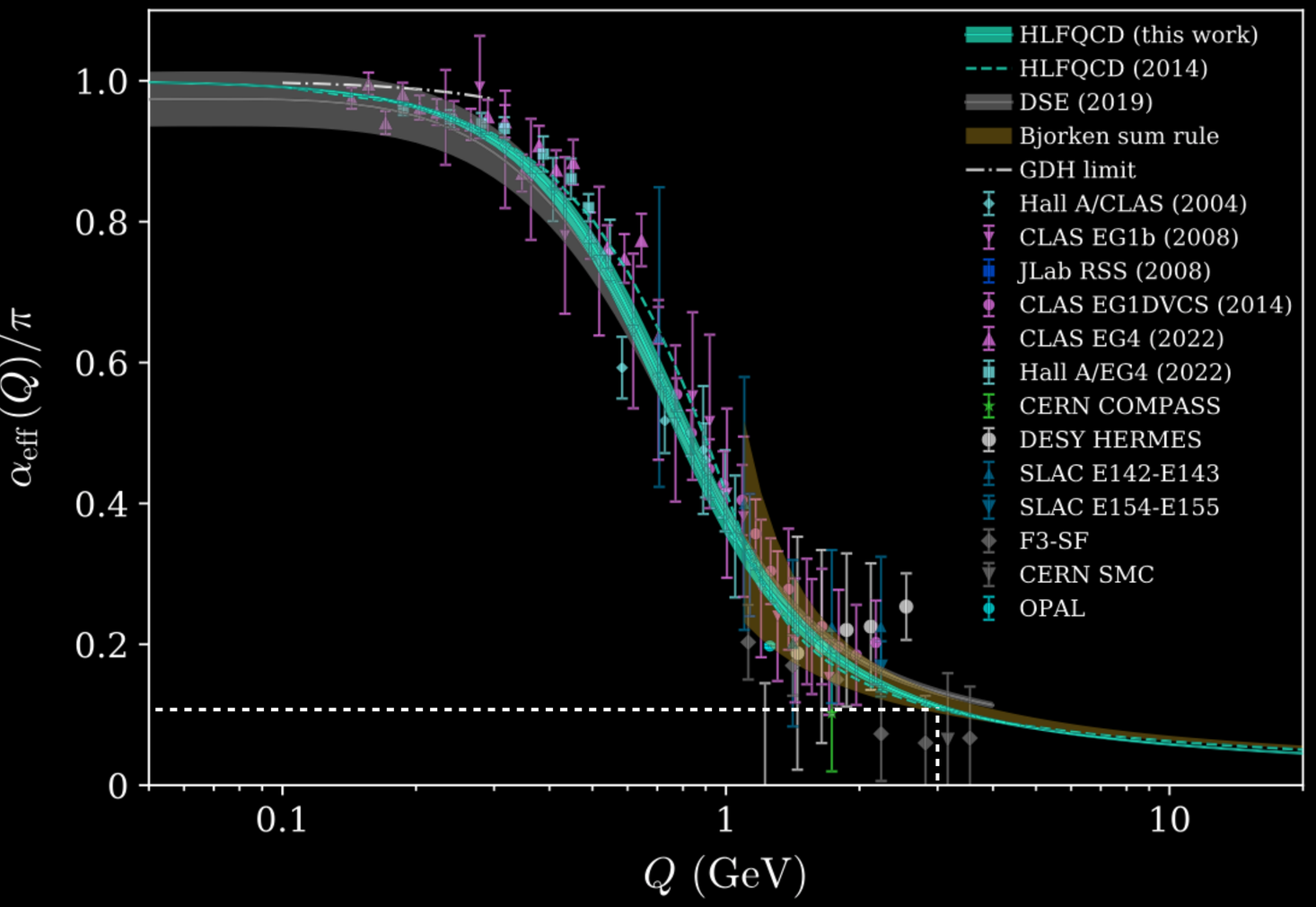
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QCD vacuum
broken massive phase

Summary

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- The fundamental problem of gluon mass generation has been revisited with the Schwinger mechanism and the bifurcation theory.

- The spacetime dynamics is far ***more*** than a scaffolding → new physics.
- The regularization itself is ***not*** limited to gauge sector → phase transition.

$$\alpha_c^{\text{conf}} = \frac{\pi}{3C_A} \qquad \alpha_c^{\text{chiral}} \sim \frac{\pi}{3C_F} A(0)$$