Inverse Problem Approach

——A novel non-perturbative QCD method

Fu-Sheng Yu

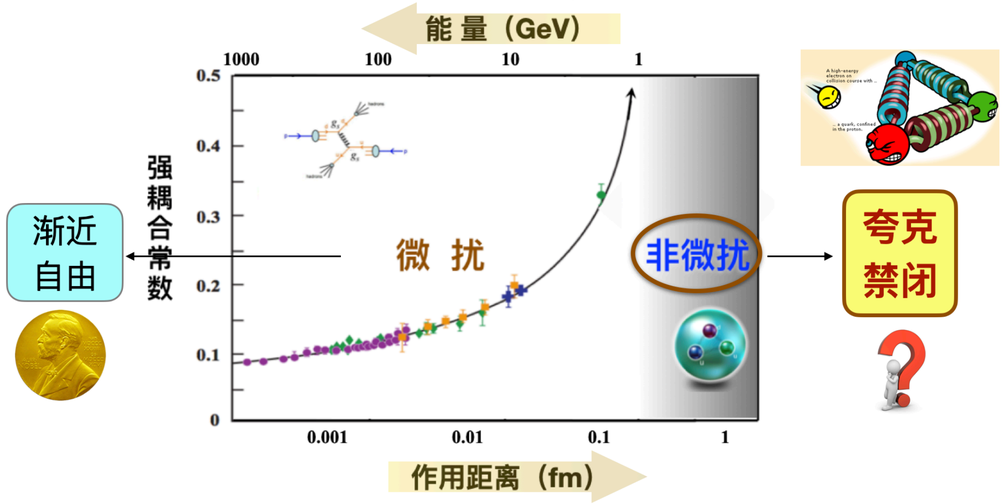
Lanzhou University

Based on [A.-S. Xiong, F.-S. Yu, Y. Zheng, T. Wei, arXiv:2211.13753, arXiv:2610.40111 and more in preparations]

2026 Workshop on Non-Perturbative Methods and Their Applications in High-Energy Physics

@ Hefei, 2026.11.7

Problem of Non-Perturbation QCD



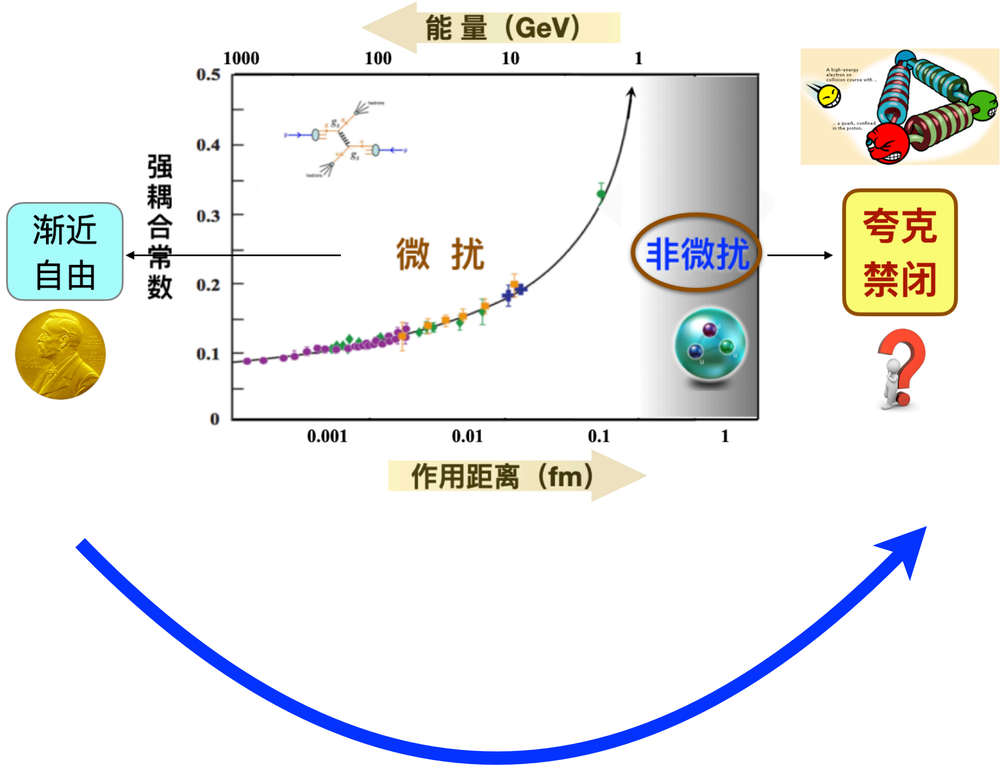
- **Particle physics:** color confinement
- **New physics**: muon g-2, B-meson anomalies
- **Parton physics**: mass and spin of nucleon, PDF, GPD, TMD, LCDA
- **Hadron physics**: tetraquarks, pentaquark, glueballs
- **High energy nuclear physics**: QCD phase transition, critical point
- **Low energy nuclear physics**: nuclear force

Problem of Non-Perturbation QCD

微扰论大势已成

Perturbation theory is now firmly established

X.G.Wu's talk



Fundamental Science Questions for 21st Century

Can't build a Theory of Everything before status of QCD is decided

C.Roberts's talk

Can we calculate non-perturbative QCD from perturbative inputs?

Inverse Problem Approach

Illustration of Inverse Problem Approach

i) Basic Idea: Inverse Problem

$$\int_{t_{\min}}^{\Lambda} ds \frac{\mathrm{Im}\,\Pi(s)}{s-q^2} = \pi\Pi(q^2) - \int_{\Lambda}^{\infty} ds \frac{\mathrm{Im}\,\Pi(s)}{s-q^2}$$

Non-perturbative objective

Perturbative input

$q^2 \ll 0$

ii) Challenge: Ill-posedness

$$\int_a^b \frac{f(x)}{x-y}\,dx = g(y)$$

$$\Rightarrow \quad Kf = g$$

Solutions:

- existence ✓
- uniqueness ✓
- stability ✗

iii) Strategy: Regularization

- Well-posed approximation

$$Kf = g \quad \Rightarrow \quad \text{bounded } R_\alpha \sim K^{-1}$$

- Regularization operator

$$R_\alpha = (K^*K + \alpha I)^{-1}K^*$$

iv) Rigor: Error & Convergence

- Uncertainty analysis

$$\|f_\alpha^\delta - f\| \leqslant \delta\|R_\alpha\| + \|R_\alpha Kf - f\|$$

- Convergence of solution

$$f_\alpha^\delta \to f, \text{ as } \delta \to 0$$

1. Basic Idea: Dispersion Relation and Inverse Problem

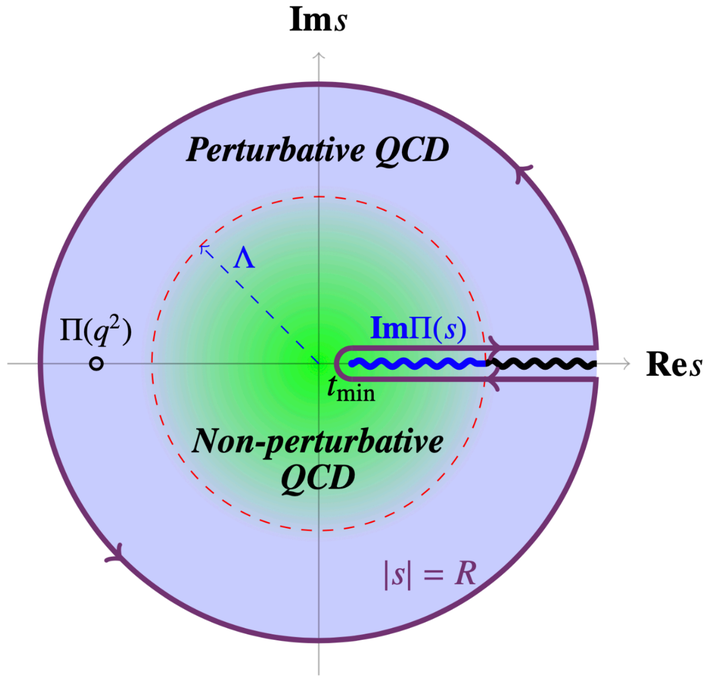
- Based on Quantum Field Theory and correlation functions

$$\Pi(q^2) = i \int \mathrm{d}^4x\, e^{iqx} \langle \Omega \,|\, T\{\mathcal{O}_1(x)\mathcal{O}_2(0)\} \,|\, \Omega \rangle$$

- Analyticity of QFT $\Longrightarrow$ Dispersion Relation

$$\Pi(q^2) = \frac{1}{\pi} \int_{t_{min}}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\,\Pi(s)}{s - q^2}$$

- Any dispersion relation would be studied similarly.



- **Inverse Problem:**

$$\int_{t_{\min}}^{\Lambda} \mathrm{d}s\, \frac{\mathrm{Im}\,\Pi(s)}{s - q^2} = \pi \Pi(q^2) - \int_{\Lambda}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\,\Pi(s)}{s - q^2}$$

Non-perturbative objective

Perturbative input

$$q^2 \ll 0$$

1. Basic Idea : Inverse Problem

- Inverse problem of dispersion relation:

$$\int_{t_{\min}}^{\Lambda} \mathrm{d}s \, \frac{\mathrm{Im}\,\Pi(s)}{s - q^2} = \pi \Pi(q^2) - \int_{\Lambda}^{\infty} \mathrm{d}s \, \frac{\mathrm{Im}\,\Pi(s)}{s - q^2}$$

$$\int_{t_{\min}}^{\Lambda} \frac{f(x)}{x - y} \, \mathrm{d}x = g(y)\,, \quad \text{with } y \in [q^2_{\mathrm{low}}, q^2_{\mathrm{up}}]\,, \qquad \Lambda > t_{\min} \geqslant 0 \text{ and } q^2_{\mathrm{low}} < q^2_{\mathrm{up}} \ll 0$$

$$Kf = g\,, \quad \text{with } f \in F\,,\ g \in G\,, \qquad K : F \to G \text{ is the linear integral operator,}$$

$$F = L^2(t_{\min}, \Lambda) \qquad G = L^2(q^2_{\mathrm{low}}, q^2_{\mathrm{up}}) \qquad \|f\|_{L^2(a,b)} = \left(\int_a^b |f(x)|^2 \, \mathrm{d}x \right)^{1/2}$$

2. Property & Challenge : ill-posedness

The inverse problem of dispersion relation is ill-posed

Proof seen in arXiv:2211.13753

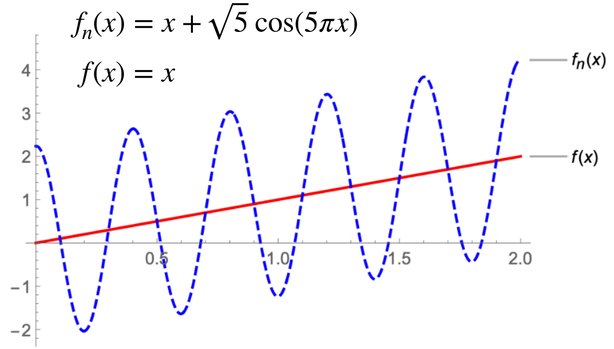
1) Existence ✓

2) Uniqueness ✓

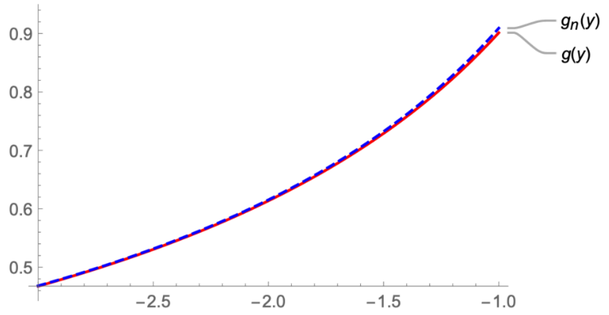
3) Stability ✗

$$\int_{t_{\min}}^{\Lambda} \frac{f(x)}{x-y}\,\mathrm{d}x = g(y)\,, \quad \text{with } y \in [q^2_{\text{low}}, q^2_{\text{up}}]\,,$$

$f(x)$ $g(y)$



Instability



uniqueness

2. Property & Challenge : Instability

- K is a compact operator, which cannot have a bounded inverse in an infinite-dimensional space

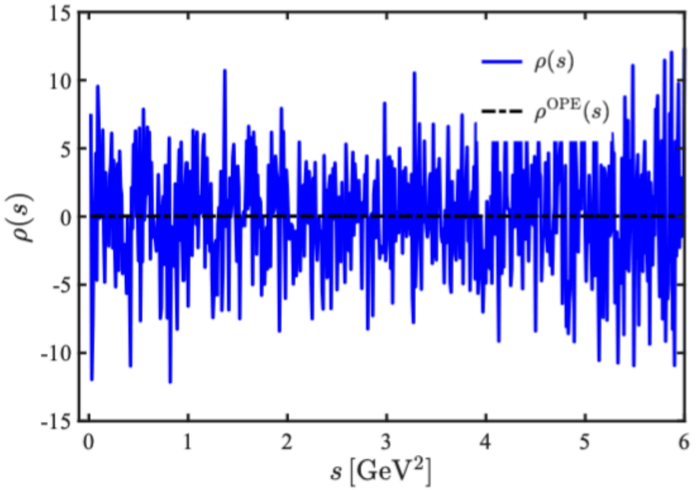
- Discretization: $K_{ij} f_j = g_i \ , \quad f_j = (K^{-1})_{ji} g_i \ .$ $\quad K^{-1} = \dfrac{1}{\det(K)} \mathrm{adj}(K),$

$\det(K) \to 0$, since K is a continuous operator

discretization dimensions:

$n = 10 \ , \ \det(K) = 10^{-88} \ ; \quad n = 20 \ , \ \det(K) = 10^{-244}$

- K^{-1} diverges $\Longrightarrow$ A small error of g^{δ} would induce infinitely large error of solution



3. Strategy : Regularization

- Basic idea: Approximate an ill-posed problem by a neighboring well-posed problem
- **Definition**: A regularization strategy is a family of linear and bounded operator

$R_\alpha : G \to F, \alpha > 0$, such that $\lim_{\alpha \to 0} R_\alpha K f = f$

- $f_\alpha^\delta = R_\alpha g^\delta$ is an approximation of the solution of $Kf = g$
- As $\alpha \to 0, R_\alpha \to K^{-1}$
- **Tikhonov regularization:** $R_\alpha = (K^*K + \alpha I)^{-1}K^*, \quad f_\alpha^\delta = (K^*K + \alpha I)^{-1}K^* g^\delta$

Adjoint operator: $(Kf, g) = (f, K^*g)$. $\qquad K : F \to G, \quad K^* : G \to F$

3. Strategy : Tikhonov Regularization

• Three equivalent understandings

1. Operator : $R_\alpha = (K^*K + \alpha I)^{-1} K^*.$

Tikhonov, 1943, 1963

2. Minimization of functional : $J_\alpha(f) := \|Kf - g^\delta\|_G^2 + \alpha \|f\|_F^2$

$$f_\alpha^\delta = \arg\min_{f \in F} J_\alpha(f)$$

Residue of data, χ^2 fit

Penalty, suppress high-frequency oscillation

3. Singular Value Decomposition : $Kf_j = \mu_j g_j$, $K^* g_j = \mu_j f_j$, $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots > 0$

$$f_\alpha = \sum \frac{1}{\mu_j} (g, g_j) f_j \quad \longrightarrow \quad f_\alpha = \sum \frac{\mu_j}{\mu_j^2 + \alpha} (g, g_j) f_j$$

Divergent if $\mu_j \to 0$

- $\mu_j^2 \gg \alpha$ terms retained
- $\mu_j^2 \ll \alpha$ terms neglected

3. Strategy : Determination of Regulator

- Two steps of regularization: Construct regularization operator, and Determine the value of regularization parameter.

- $\delta \to 0$ is not satisfied in practice. We have to determine α in case of finite input errors.

- In physics, the results should not depend on the non-physical parameters. Therefore, there should exist a plateau of α.

✓ Quasi-Optimality Criterion: $\alpha_{\text{opt}} = \arg\min_{\alpha>0} \left\{ \left\| \alpha \frac{\mathrm{d}f_\alpha}{\mathrm{d}\alpha} \right\| \right\}$ A.Kirsch, 2011

- Two criterions: mathematical method + physical stability plateau.
- Robustness of α

Numerical Results under Tikhonov Regularization

f_π, f_{a_1}, and the continuum

Integral Equation:

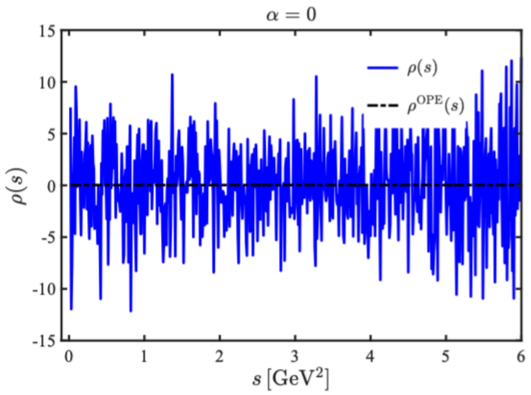
$$\frac{f_\pi^2}{(m_\pi^2-\sigma_0)(m_\pi^2-q^2)}+\int_{s_{\text{th}}}^{\Lambda} ds\frac{\rho^c(s)}{(s-\sigma_0)(s-q^2)}=\frac{\Pi_q(q^2)-\Pi_q(\sigma_0)}{(q^2-\sigma_0)}-\int_{\Lambda}^{\infty} ds\frac{\rho^{\text{OPE}}(s)}{(s-\sigma_0)(s-q^2)}$$

OPE Input:

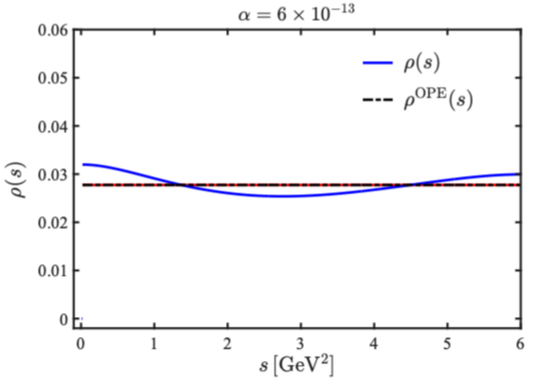
$$\Pi_q(q^2)=-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s(\mu)}{\pi}\right)\ln\left(\frac{-q^2}{\mu^2}\right)+\frac{(m_u+m_d)\langle\bar{q}q\rangle}{(q^2)^2}+\frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2}-\frac{352\pi}{81}\frac{\alpha_s(\mu)\langle\bar{q}q\rangle^2}{(q^2)^3}$$

Prior Information:

$$\rho(s)=f_\pi^2\delta(s-m_\pi^2)+f_{a_1(1260)}^2\mathbb{BW}\left(s,m_{a_1(1260)},\Gamma_{a_1(1260)}\right)+\rho^c(s)$$

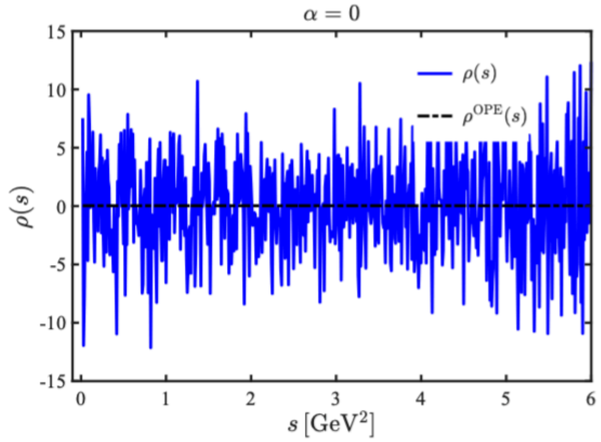


(a) Without regularization

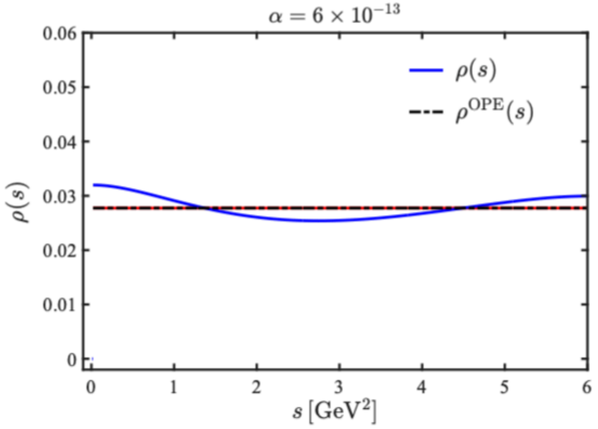


(b) Under regularization

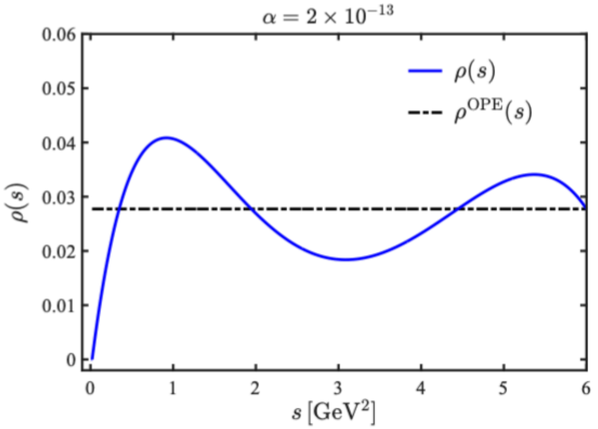
f_π, f_{a_1}, and the continuum



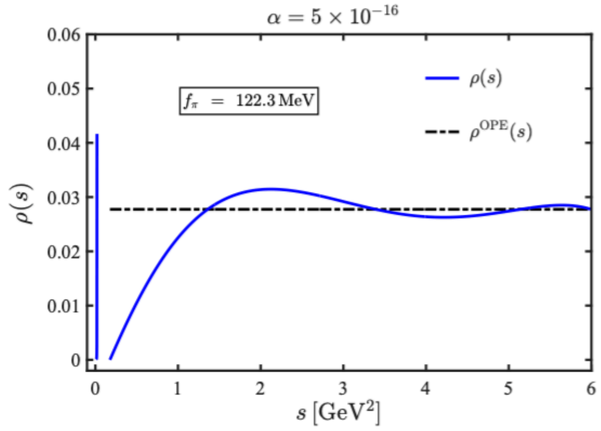
(a) Without regularization



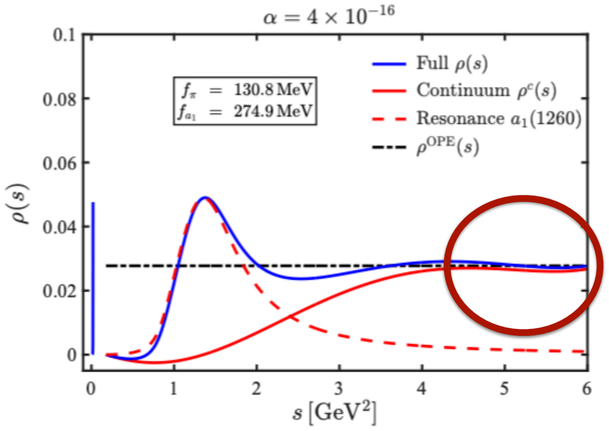
(b) Under regularization



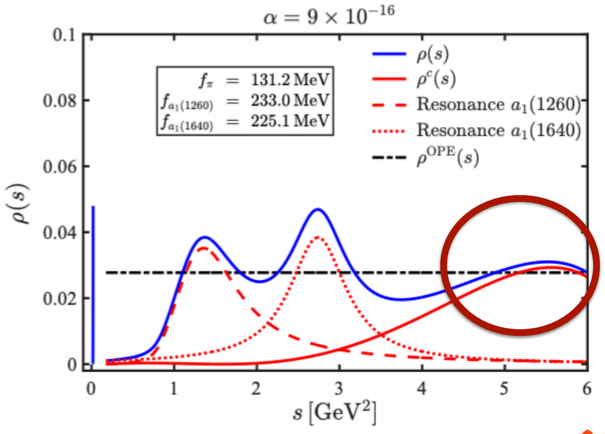
(c) + boundary conditions



(d) + pion pole



(e) + $a_1(1260)$ resonance



(f) + $a_1(1640)$ resonance

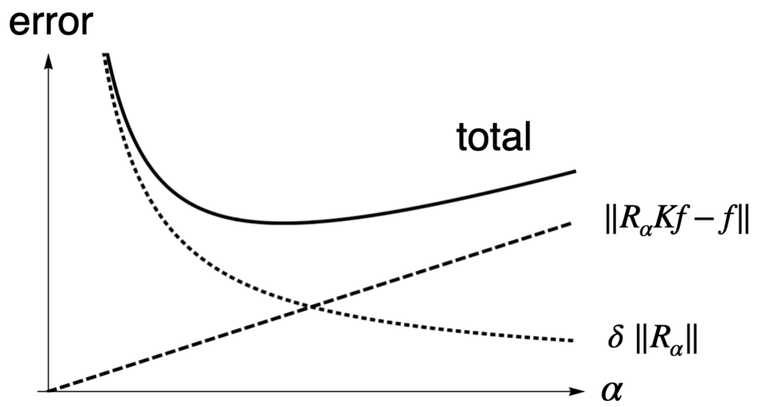
4. Rigorousness : Error & Convergence

- Given $\|g^{\delta} - g\| \leqslant \delta$, we consider the error of the solution $\|f_{\alpha}^{\delta} - f\|$

$$\|f_{\alpha}^{\delta} - f\| \leqslant \|f_{\alpha}^{\delta} - f_{\alpha}\| + \|f_{\alpha} - f\|$$

$$\leqslant \|R_{\alpha}\|\|g^{\delta} - g\| + \|R_{\alpha}Kf - f\|$$

$$\leqslant \delta \, \|R_{\alpha}\| + \|R_{\alpha}Kf - f\|$$



- α could be neither too large nor too small, but within an appropriate range.

- Error of solution: $\|f_{\alpha}^{\delta} - f\| \leqslant \delta \, \|R_{\alpha}\| + \|R_{\alpha}Kf - f\|$

  $\|R_{\alpha}\|$ → Input-induced uncertainty

  $\|R_{\alpha}Kf - f\|$ → Method-intrinsic uncertainty

4. Rigorousness : Error & Convergence

- Error of solution: $\|f_\alpha^\delta - f\| \leqslant \delta \, \|R_\alpha\| + \|R_\alpha K f - f\|$

  (1)Tikhonov regularization: $\|R_\alpha\| \leqslant 1/\left(2\sqrt{\alpha}\right)$

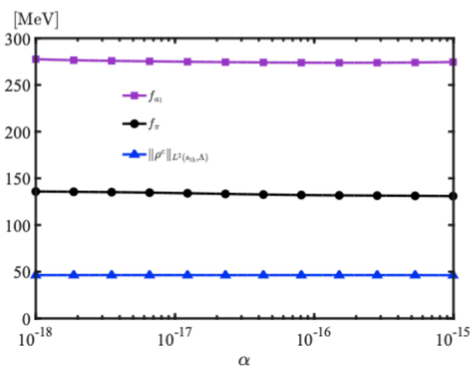
  (2)Under a priori condition $f = K^*Kz \in K^*K(F), \quad \|z\| \leqslant E, \ \|R_\alpha Kf - f\| \leqslant \alpha E$

- $\|f_\alpha^\delta - f\| \leqslant \dfrac{\delta}{2\sqrt{\alpha}} + \alpha E$, Chose $\alpha = \left(\dfrac{\delta}{E}\right)^{\frac{2}{3}}$,

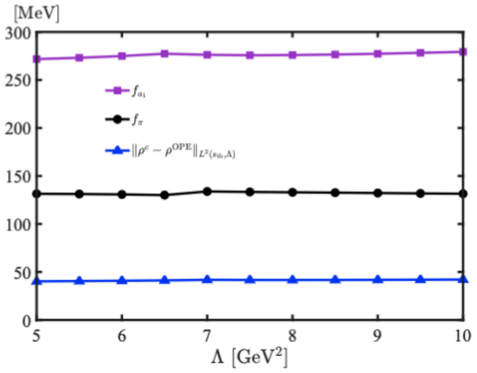
  $\|f_\alpha^\delta - f\| \leqslant \dfrac{3}{2}\,\delta^{\frac{2}{3}}\,E^{\frac{1}{3}} \to 0, \quad \delta \to 0$

- The solution is convergent to the true solution as $\delta \to 0$!! This is the most important !
- The precision can be systematically improved !

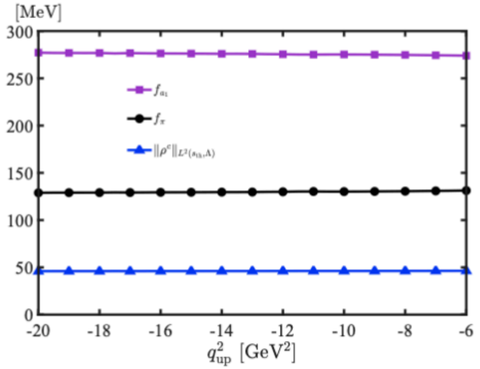
Stability Plateaus of parameters



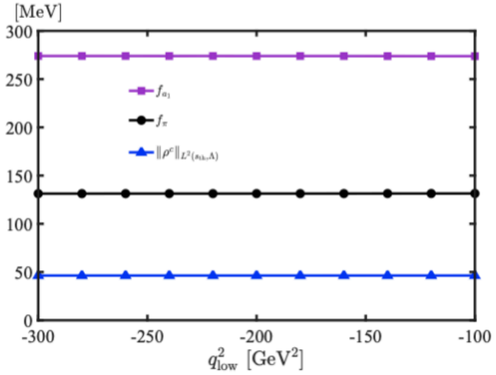
(a) α-dependence



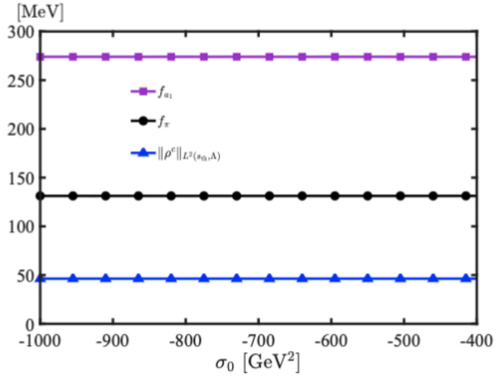
(b) Λ-dependence



(c) q^2_{up}-dependence



(d) q^2_{low}-dependence



(e) σ_0-dependence

Error Budget

$$\|f_\alpha^\delta - f\| \leqslant$$

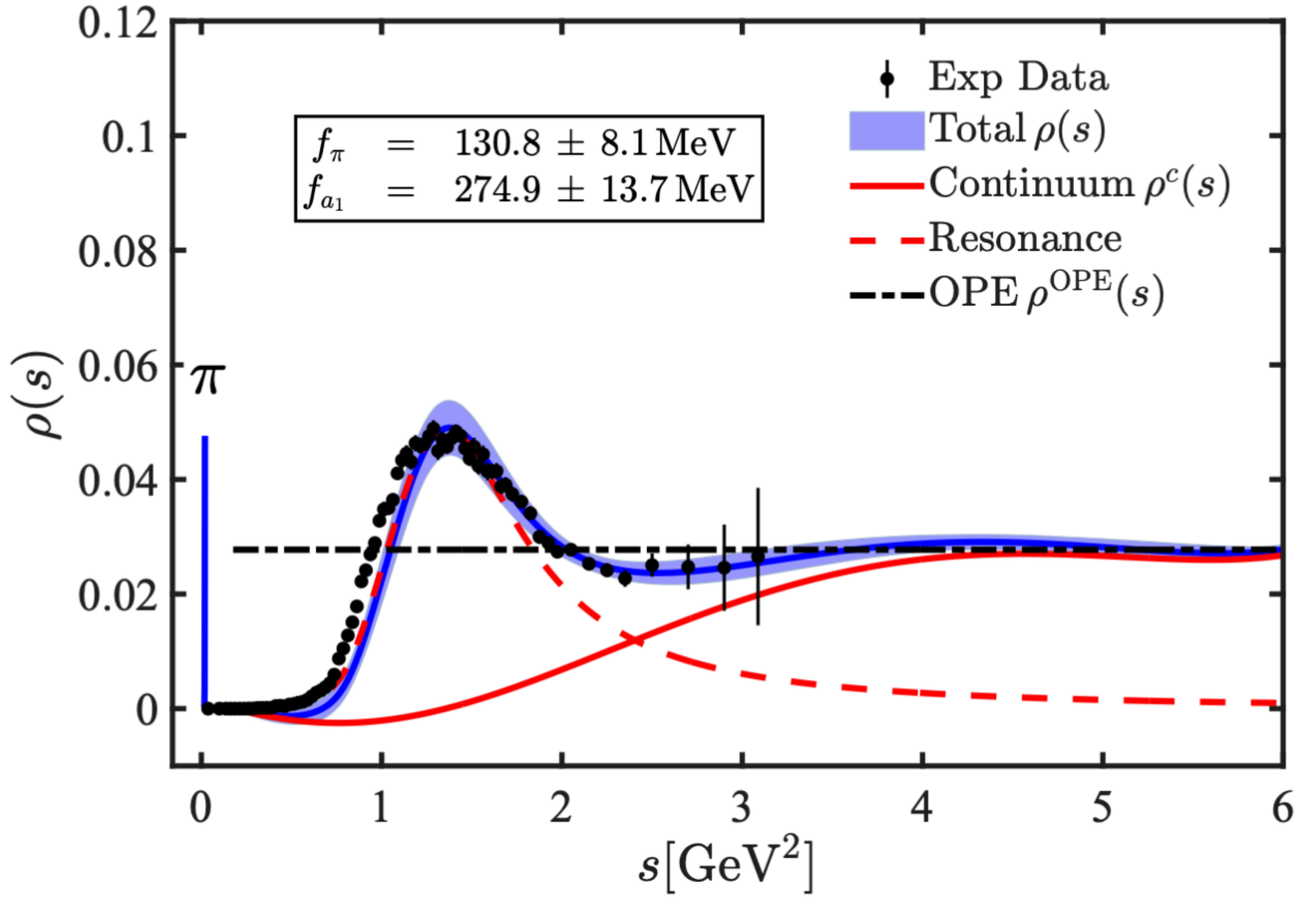
$$\delta \, \|R_\alpha\|$$

$$+$$

$$\|R_\alpha K f - f\|$$

	Source	Prescription	Δf_π [MeV]	Δf_{a_1} [MeV]
	$m_{u(d)}$	$2.16 \pm 0.04\,(4.70 \pm 0.04)$ [MeV]	0.01	0.01
	$\langle \bar{q}q \rangle$	$-(0.272 \pm 0.005)^3$ [GeV]^3	0.57	1.03
	$\langle \alpha_s GG \rangle$	0.07 ± 0.02 [GeV^4]	3.51	4.57
Input	$M_{a_1(1260)}$	1202 ± 15 [MeV]	0.78	3.52
Errors	$\Gamma_{a_1(1260)}$	422 ± 12 [MeV]	0.08	2.05
	μ	[1, 3] GeV	0.77	2.41
	OPE truncation	dim-6 operators	6.73	11.30
	Total input-induced uncertainty		**7.69**	**13.12**
	α	4×10^{-16}	0.34	0.40
	Λ	[5, 10] GeV^2	2.11	3.77
Method	q^2_{up}	$[-20, -6]$ GeV^2	1.19	1.49
Errors	q^2_{low}	$[-300, -100]$ GeV^2	0.08	0.10
	σ_0	$[-1000, -400]$ GeV^2	0.02	0.03
	Total method-intrinsic uncertainty		**2.45**	**4.11**
Total uncertainty			**8.07**	**13.74**

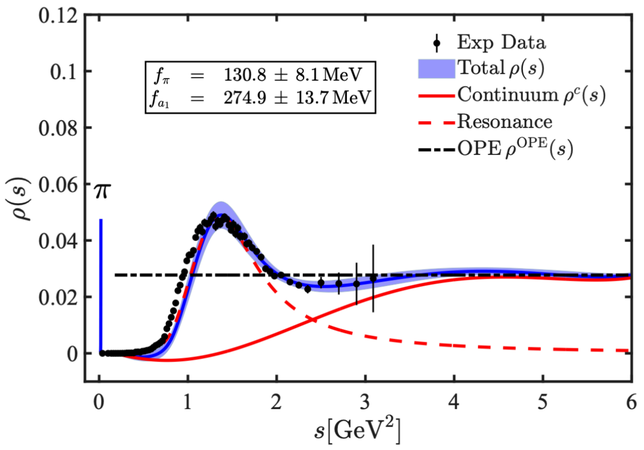
Final Result

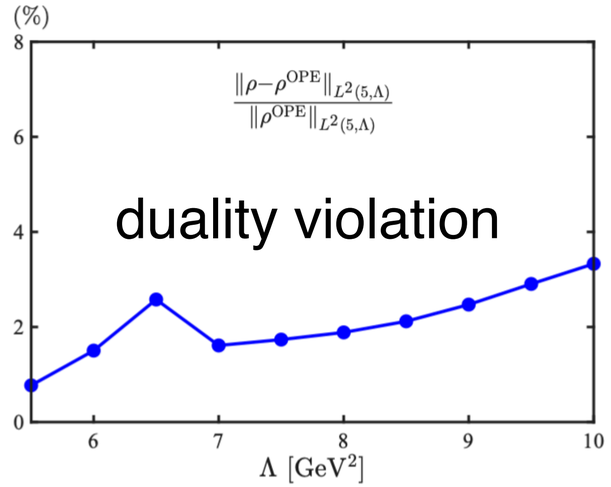


Method	f_π [MeV]	$f_{a_1(1260)}$ [MeV]
This work	130.8 ± 8.1	274.9 ± 13.7
Experiment	130.57 ± 0.14 [17]	
Lattice QCD	130.20 ± 0.80 [28]	240.0 ± 28.5 [29]
QCDSR (Π_q)	127 ± 15 [15], 135 ± 1 [21]	151 ± 20 [21]
QCDSR (Π_g)		238 ± 10 [30]

5. Physical Discussions: Quark-Hadron Duality

- Inverse Problem Approach:
  - Quark-hadron duality is automatically obtained at high-energy scale, $s \gtrsim 5\ \mathrm{GeV}^2$
  - But violated at low-energy scale
  - Duality violation: $(1-4)\,\%$



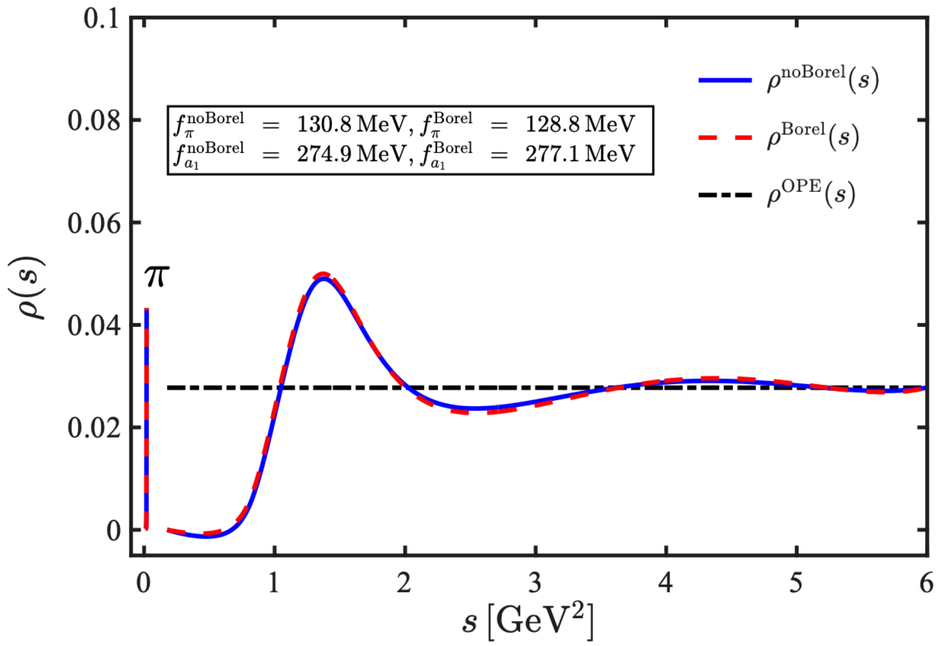


- QCD sum rules:
  - Quark-hadron duality at low-energy scale
  - Free parameter s_0 , channel dependent, large error
  - $s_0^\pi \simeq 0.7\ \mathrm{GeV}^2$, $s_0^{a_1} \simeq (1.3-1.7)\ \mathrm{GeV}^2$

$$\int_{s_{\mathrm{th}}^H}^{\infty} ds\, \frac{\rho^c(s)}{s-q^2} \simeq \int_{s_0}^{\infty} ds\, \frac{\frac{1}{\pi}\,\mathrm{Im}\,\Pi^{\mathrm{pert}}(s)}{s-q^2}$$

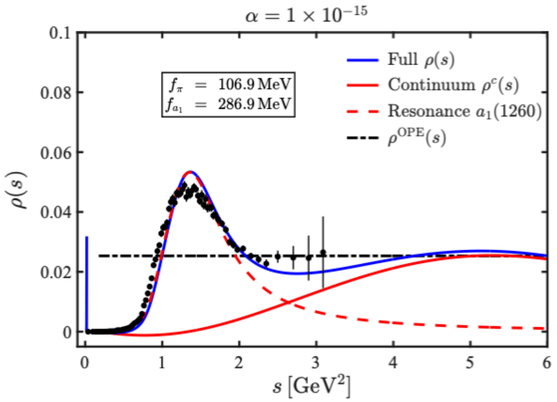
5. Physical Discussions: Borel Transformation

- Inverse Problem Approach:
  - Equivalent results with and without Borel transformation.
  - Don't need Borel transformation

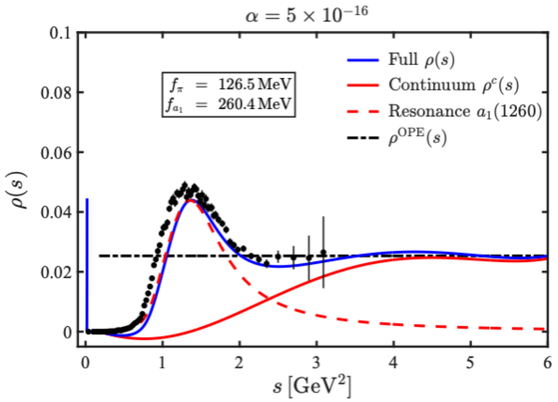


- QCD sum rules:
  - Borel transformation is required, to reduce sensitivity to the quark-hadron duality approximation
  - Borel window is determined by OPE convergence and pole dominance, introducing some arbitrariness and bias

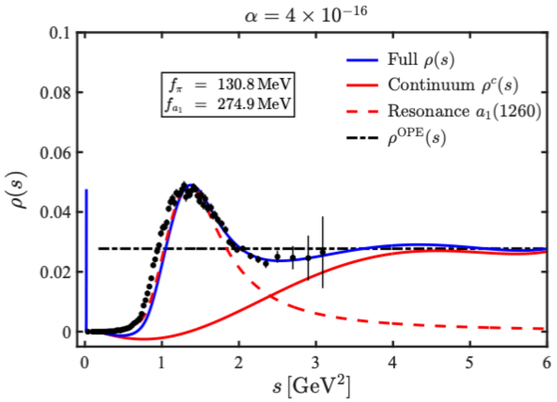
5. Physical Discussions: Convergence of Solution



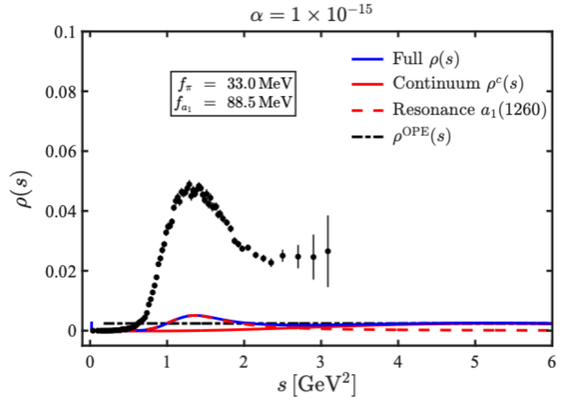
(a) pure LO



(b) LO+Condensates



(c) LO + NLO + Condensates



(d) pure NLO

5. Physical Discussions: Decomposition of Spectrum

	f_π^2	$f_{a_1(1260)}^2 \eta$	$\int_{s_{th}}^{\Lambda} ds\, \rho_i^c(s)$
pure LO	0.011 (66.6%)	0.065 (108.6%)	0.075 (84.3%)
pure NLO	0.001 (6.3%)	0.006 (10.3%)	0.007 (8.0%)
$m_q \langle \bar{q}q \rangle$	−0.0002 (−1.3%)	0.0005 (0.8%)	−0.0003 (−0.3%)
$\langle \alpha_s G^2 \rangle$	0.003 (18.2%)	−0.007 (−11.5%)	0.004 (4.2%)
$\langle \bar{q}q\bar{q}q \rangle$	0.002 (9.9%)	−0.005 (−8.3%)	0.003 (3.7%)
Sum	0.017 (100.0%)	0.060 (100.0%)	0.089 (100.0%)

$$\eta \equiv \int_{s_{\rm th}}^{\Lambda} ds\, \mathbb{BW}\left(s, M_{a_1(1260)}, \Gamma_{a_1(1260)}\right) = 0.80$$

Advantages

1. **The framework is based on rigorous mathematical principles.** The dispersion-relation inverse problem is proved to admit a unique but unstable solution. The well-established Tikhonov regularization is employed, ensuring that the regularized solution converges to the true solution as the input error decreases.

2. **It enables one to analyze uncertainties.** In particular for the method-intrinsic errors. The precision of solution can be systematically improved by reducing the input errors and by refining prior information and regularization methods.

Advantages

3. **The framework treats the entire non-perturbative region simultaneously.** It is beneficial for studying excited states and the continuum spectrum.

4. **Quark-hadron duality emerges naturally at high energy scales**. This avoids the low-energy duality assumption and the Borel transformation, thereby eliminating the uncertainties associated with them.

5. **Numerical calculations can be completed with only modest computational resources.** Provided that the perturbative input is available, the numerical solution of the inverse problem can be obtained very efficiently.

Summary

- We propose a novel method to calculate the non-perturbative quantities.
- With the **dispersion relation** of QFT, the non-perturbative quantities are obtained by solving the **inverse problem** with the perturbative calculations as inputs.
- **The precision of the predictions can be systematically improved.**
- The mathematical basis has been provided.
- Physical applications are expected.

Thank you!

Backups

Numerical Method of Tikhonov Regularization

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h}, x \in [x_{i-1}, x_i], \\ -\frac{x-x_{i+1}}{h}, x \in [x_i, x_{i+1}], \\ 0, otherwise, \end{cases}$$

$$\varphi_0(x) = \begin{cases} -\frac{x-x_1}{h}, x \in [x_0, x_1], \\ 0, otherwise, \end{cases}$$

$$\varphi_n(x) = \begin{cases} \frac{x-x_{n-1}}{h}, x \in [x_{n-1}, x_n], \\ 0, otherwise. \end{cases}$$

$$f_\alpha^\delta = \arg\min_{f \in L^2(a,b)} J(f) = \arg\min_{f \in L^2(a,b)} \left(\frac{1}{2} \|Kf - g^\delta\|_{L^2(c,d)}^2 + \frac{\alpha}{2} \|f\|_{L^2(a,b)}^2 \right)$$

$$f_{\alpha,n}^\delta(x) = \sum_{i=0}^{n} c_i \varphi_i(x)$$

$$J(f_{\alpha,n}^\delta) = \frac{1}{2} \left\| \sum_{i=0}^{n} c_i K\varphi_i - g^\delta \right\|_{L^2(c,d)}^2 + \frac{\alpha}{2} \left\| \sum_{i=0}^{n} c_i \varphi_i \right\|_{L^2(a,b)}^2$$

$$= \frac{1}{2} \sum_{i,j=0}^{n} c_i c_j (K\varphi_i, K\varphi_j)_{L^2(c,d)} - \sum_{i=0}^{n} c_i (K\varphi_i, g^\delta)_{L^2(c,d)} + \frac{1}{2} (g^\delta, g^\delta)_{L^2(c,d)} + \frac{\alpha}{2} \sum_{i,j=0}^{n} c_i c_j (\varphi_i, \varphi_j)_{L^2(a,b)}$$

$$X_n = \text{span}\{\varphi_0, \varphi_1, \cdots, \varphi_n\}$$

$$f_{\alpha,n}^\delta(x) = \sum_{i=0}^{n} c_i \varphi_i(x)$$

$$A_{ij} = (K\varphi_i, K\varphi_j)_{L^2(c,d)} \qquad B_{ij} = (\varphi_i, \varphi_j)_{L^2(a,b)} \qquad C = (c_0, c_1, \cdots, c_n)^T$$

$$(A + \alpha B)C = D \qquad D_i = (K\varphi_i, g^\delta)_{L^2(c,d)}$$

Theorem 4.3. *If the noise δ and the regularization parameter α are fixed, we have $\|f_{\alpha,n}^\delta - f_\alpha^\delta\|_{L^2(a,b)} \to 0$, as $n \to \infty$.*

反问题方法应用举例

积分问题：已知积分结果，求被积函数 $\int_a^b K(x,y) f(x)\,\mathrm{d}x = g(y)$

1. 色散积分（谱分析）（连续）：$\int_{t_{\min}}^{\Lambda} \mathrm{d}s\,\frac{\mathrm{Im}\,\Pi(s)}{s-q^2} = \pi\,\Pi(q^2) - \int_{\Lambda}^{\infty} \mathrm{d}s\,\frac{\mathrm{Im}\,\Pi(s)}{s-q^2}$

2. 矩（LCDA、PDF）（离散）：$\int_0^1 x^n\,\varphi(x)\,\mathrm{d}x = \langle x^n \rangle \qquad \int_0^1 (2u-1)^n\,\varphi(u)\,\mathrm{d}u = \langle \xi^n \rangle, \quad \xi = 2u-1$

3. 傅立叶变换（LaMET）（离散）：$\int e^{-i\lambda x} f(x)\,\mathrm{d}x = g(\lambda)$

4. 动量关联函数（强子强子相互作用）（离散）：$\int \mathrm{d}^3 r\, S_{12}(r)\,|\,\Psi(r,k)\,|^2 = C(k)$

General Regularization Theory : Matrix Example

- Suppose A is a positive semidefinite square matrix. Eigenvalues $\lambda_i \geqslant 0$
- A simple approximation is $A + \alpha I$

  Convergence: $A + \alpha I \rightarrow A$, as $\alpha \rightarrow 0$

- A is non-invertible $\Leftrightarrow$ A is singular $\Leftrightarrow \det(A) = 0$

  $\det(A) = \Pi_i \lambda_i$, $\det(A) \rightarrow 0 \Rightarrow \exists\, \lambda_i \rightarrow 0$

- Characteristic equations: $Ax = \lambda x \;\Rightarrow\; (A + \alpha I)x = (\lambda + \alpha)x$

  $(\lambda + \alpha)$ is the eigenvalue of $A + \alpha I$

  $\det(A + \alpha I) = \Pi_i(\lambda_i + \alpha) \neq 0, \qquad \alpha > 0$

- $(A + \alpha I)^{-1}$ is a bounded and convergent approximation of A^{-1}

Tikhonov Regularization : Operator

- For $Kf = g$, $K : F \to G$ is not square in general

  Adjoint operator: $(Kf, g) = (f, K^*g)$. $\qquad K : F \to G, \quad K^* : G \to F$

  $$Kf = g \quad \Rightarrow \quad K^*Kf = K^*g \quad \Rightarrow \quad f = (K^*K)^{-1}K^*g$$

- K^*K is square, which can have a regularized term: $f_\alpha = (K^*K + \alpha I)^{-1}K^*g$

- Tikhonov regularization: $R_\alpha = (K^*K + \alpha I)^{-1}K^*, \quad f_\alpha^\delta = R_\alpha g^\delta$

  i.e., $f_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*g^\delta$

- R_α is the most simplest bounded approximation of K^{-1}. As $\alpha \to 0$, $R_\alpha \to K^{-1}$

- R_α is bounded and convergent. It is a regularized operator