Tame multi-leg Feynman integrals beyond one loop

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Based on works: L.H. Huang, R.J. Huang, Y.Q. Ma, arXiv: 2412.21053
R.J. Huang, D.S. Jian, Y.Q. Ma, D.M. Mu, W.H. Wu, arXiv: 2412.21054

北京大学

Outline

I. Introduction

II. Feynman integrals

III. A new representation

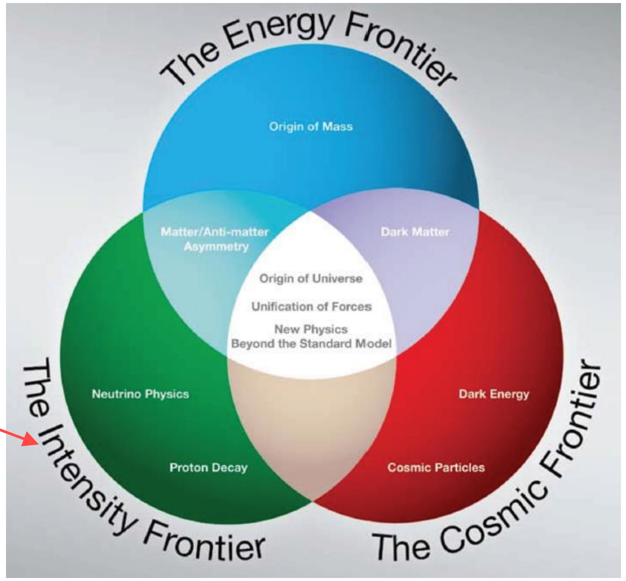
IV. Examples

V. Summary and outlook

Precision: gateway to discovery

➢ **New particles/physics have not been discovered yet at LHC**

- Currently main strategy: search anomalous deviations from theory
- Interplay between exp. and th.



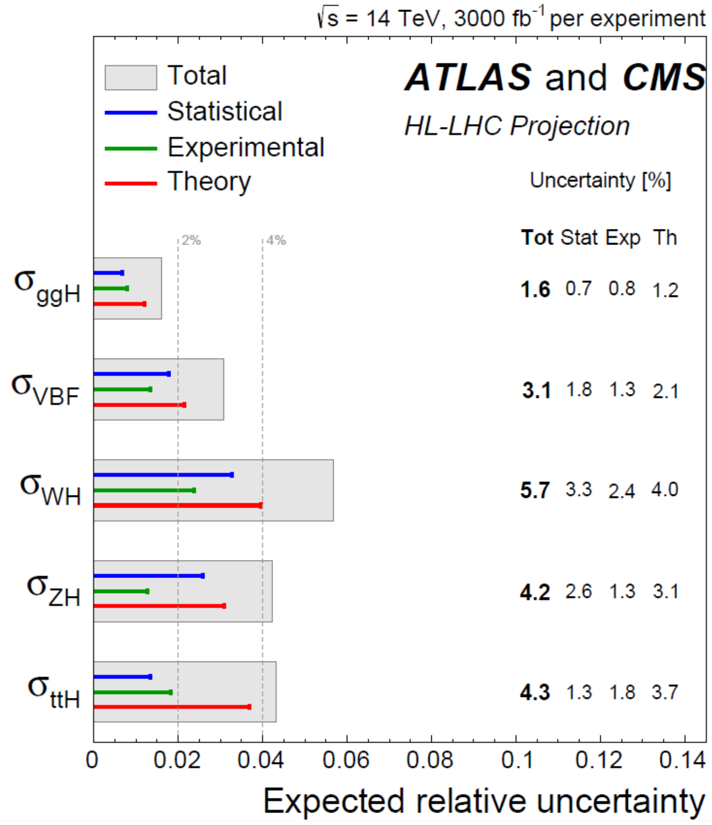
To make full use of data: theoretical errors should be much smaller than experimental errors, ideally:

$$\boxed{Error_{th} < \frac{1}{3} Error_{exp}}$$

Higgs production

➢ Looking ahead

- **Run III of LHC (22-25)**
- **HL-LHC (29-40): expect $O(1\%)$ uncertainty**
- **Requirement: reducing theoretical uncertainties by at least a factor of 5-10**



CERN2019

Assuming theoretical errors halved

A big challenge

➢ Requirement

- Reducing theoretical uncertainties by a factor of 5-10： 1-2 higher orders in α_s !

➢ How difficult to compute one more α_s order?

- RGE of PDFs (AP kernel): important for all LHC processes

3-loop: 2004 Moch, Vermaseren, Vogt, NPB2004 4-loop: not available yet

One order takes more than 20 years!

Can theorists keep up experimental requirement (α_s^2 in 16 years)?

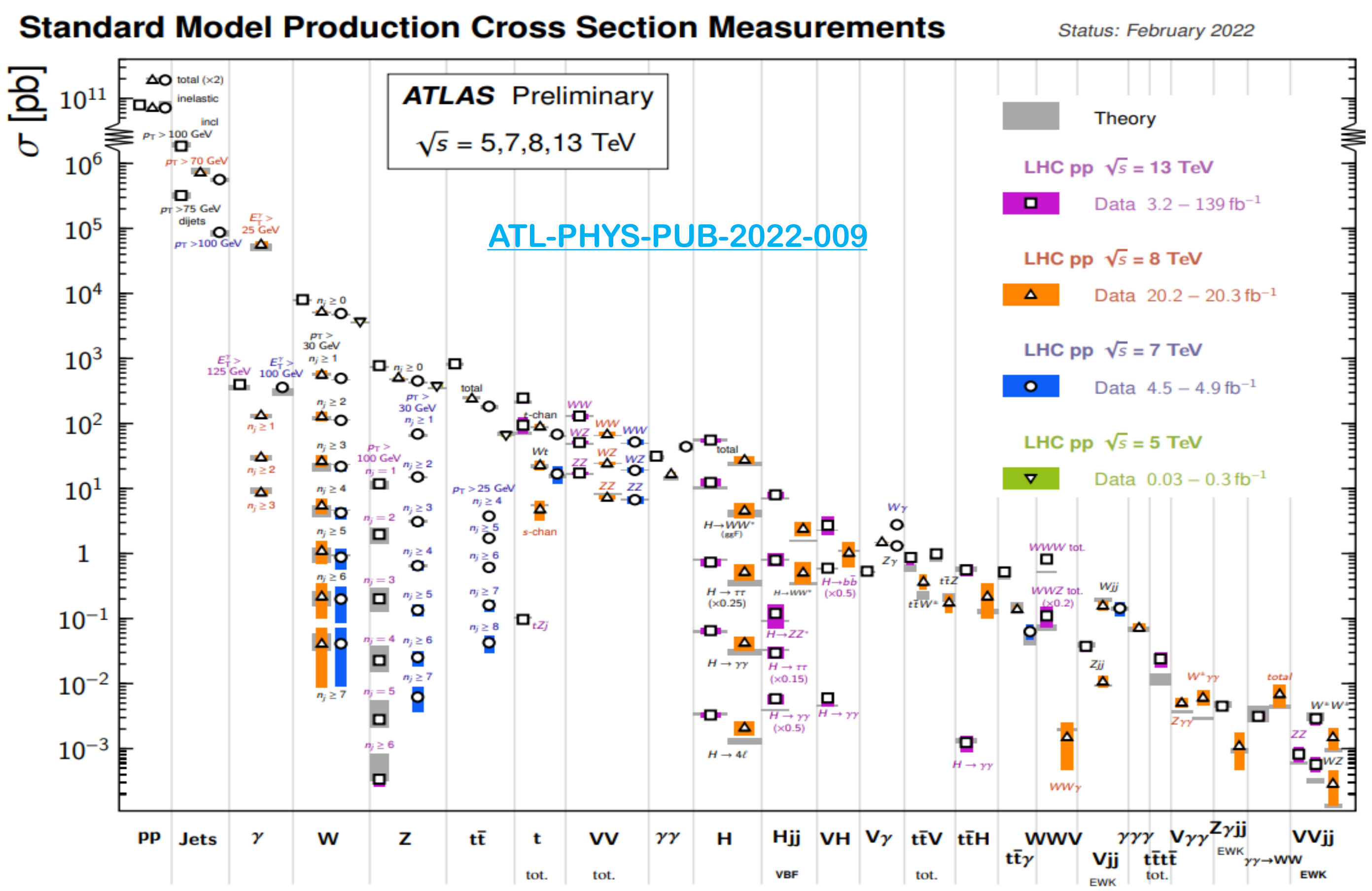
Era of precision physics

➢ High-precision data

- Many observables probed at precent level precision

➢ QCD cor. requirement: ideally

- Most processes: N2LO
- Many processes: N3LO
- Some processes: N4LO
- A few processes: N5LO



Current status of perturbative calcualtion

➢ Accomplished processes

- NLO solved, automatic codes exist: MadGraph, Helac, etc

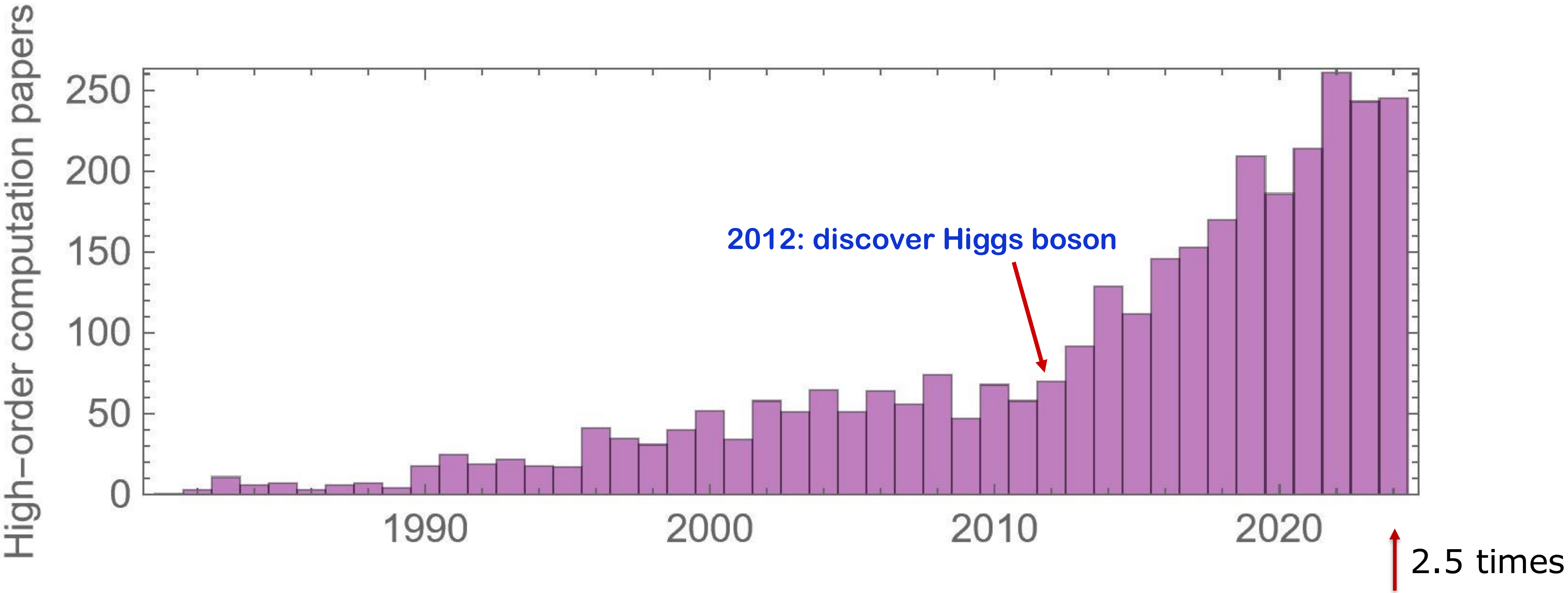
Order \ Legs	2→1	2→2	2→3	2→4	2→5	2→6
NLO	★★★	★★★	★★★	★★★	★★★	★★★
N2LO	★★★	★★	★	?	?	
N3LO	★★	★	?			
N4LO	★	?				
N5LO	?					

Novel methods for high-order computation are highly demanded!!!

➢ A "billion-dollar project"

- LHC cost about 10 billion dollars
- It is waste of money unless having high-precision computation

High-order community



- In 10 years: high-order papers increased by 2.5 times
- The number of papers is almost unchanged for the whole hep-ph field

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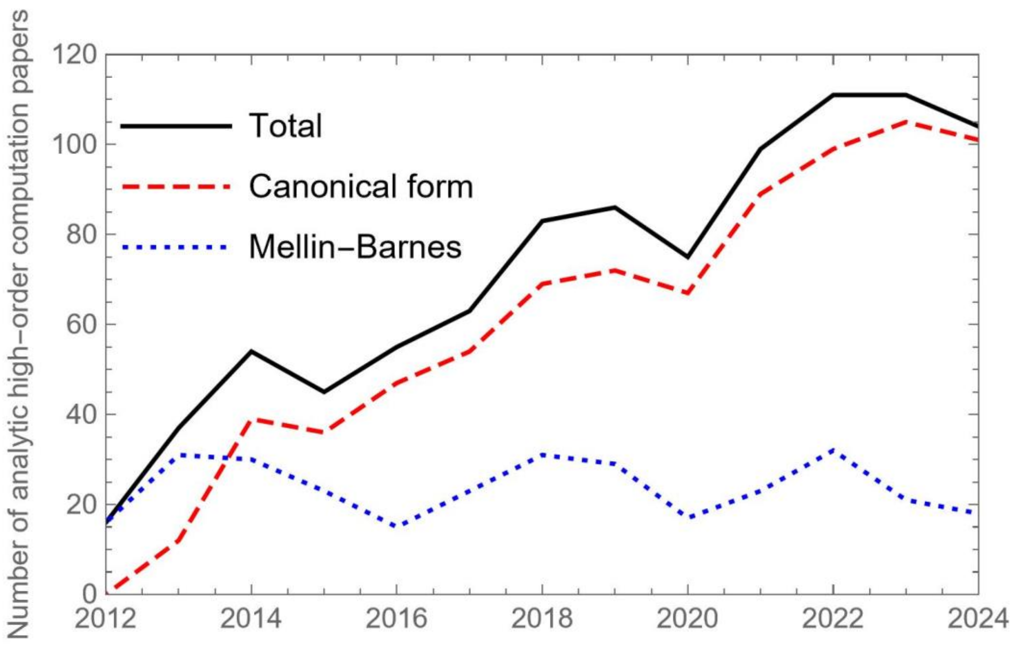
Feynman integrals: a key obstacle in high-order computation

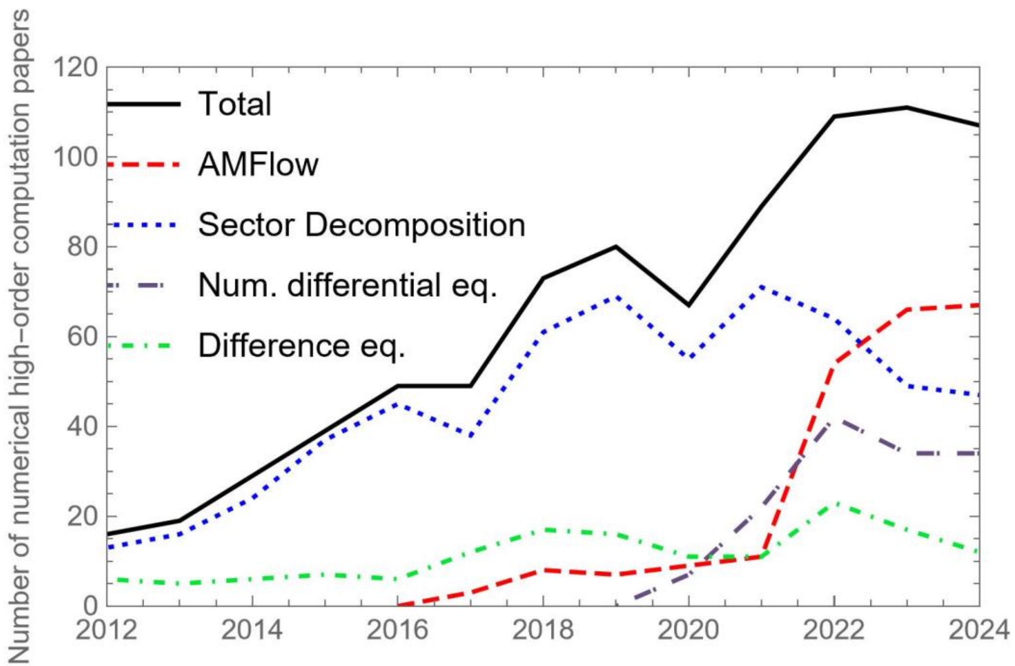
$$\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D,\vec{s}\,)I_{\vec{\nu}'}(D,\vec{s}\,) = 0$$

1) Reduce loop integrals to basis (Master Integrals)

2) Calculate MIs

Computation of MIs





- ✓ Efficient for numerical evaluation
- ✓ High precision
- × High manpower cost
- × Only applicable for some processes

- ✓ Applicable for any process
- ✓ Low manpower cost for AMF/SD
- ✓ High precision except for SD
- ✓ Efficient by combining AMF+differential eq.

But, all depend on reduction!!!

Integration-by-parts reduction: the bottleneck!

➢ The state-of-the-art IBP method: very challenging

- 4-loop DGLAP kernel cannot be obtained
- $H+2j$ production: exact two-loop contribution is missing — Chen, et al., JHEP2022
- $H+t\bar{t}$ production: exact two-loop contribution is missing — Catani, et al., PRL2023

➢ Improvements for IBPs

- Syzygy equations: trimming IBP system
  - Gluza, Kajda, Kosower, PRD2011
  - Böhm, Georgoudis, Larsen, Schulze, Zhang, PRD2018
  - NeatIBP: Wu, et al. CPC2024
- Block-triangular form: search simple IBP system
  - Liu, YQM, PRD2019
  - Guan, Liu, YQM, CPC2020
  - Blade: Guan, Liu YQM, Wu, 2405.14621

Improve efficiency by a hundredfold $\approx$ half order in α_s

Need to calculate two more orders in α_s! How?

Ways to bypass IBP

My lessons after 7-year study

- Reduction is very hard, no matter using any method, the reason: too many integration variables

  - IBP
  - Intersection number
  - Asymptotic expansion
  - Iterative reduction
  - …

Conservation of suffering!

Unless a magic: deeper understanding of FIs!

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A possible simplification?

➢ Feynman parametrization

$$J(\vec{\nu};D) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\Gamma(\nu_1)\cdots\Gamma(\nu_K)} \int \prod_{i=1}^{K} (x_i^{\nu_i - 1} \mathrm{d}x_i) \delta(1-X) \frac{\mathcal{U}^{N_\nu-(L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$

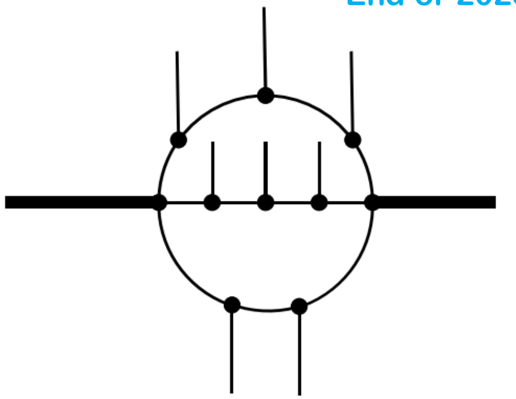
- U: degree L in the Feynman parameters x_i
- F: degree $L+1$

$$\mathcal{U} = \sum_{T\in T(G)} \prod_{e_i \notin T} x_i$$

➢ Will things be simpler if we fix U unintegrated?

End of 2023

$$J(\vec{\nu};D) = \int [\mathrm{d}\mathbf{X}] \prod_{a=1}^{B} X_a^{\nu_a - 1} \mathcal{U}^{\nu - \frac{(L+1)D}{2}} I_{\vec{\nu}}^{\frac{LD}{2}}(\vec{X})$$



X_a: the summation of Feynman parameter for the a-th branch

A surprised observation!

L.H. Huang, R.J. Huang, YQM, 2412.21053

- **The integrands are as simple as one-loop FIs!**

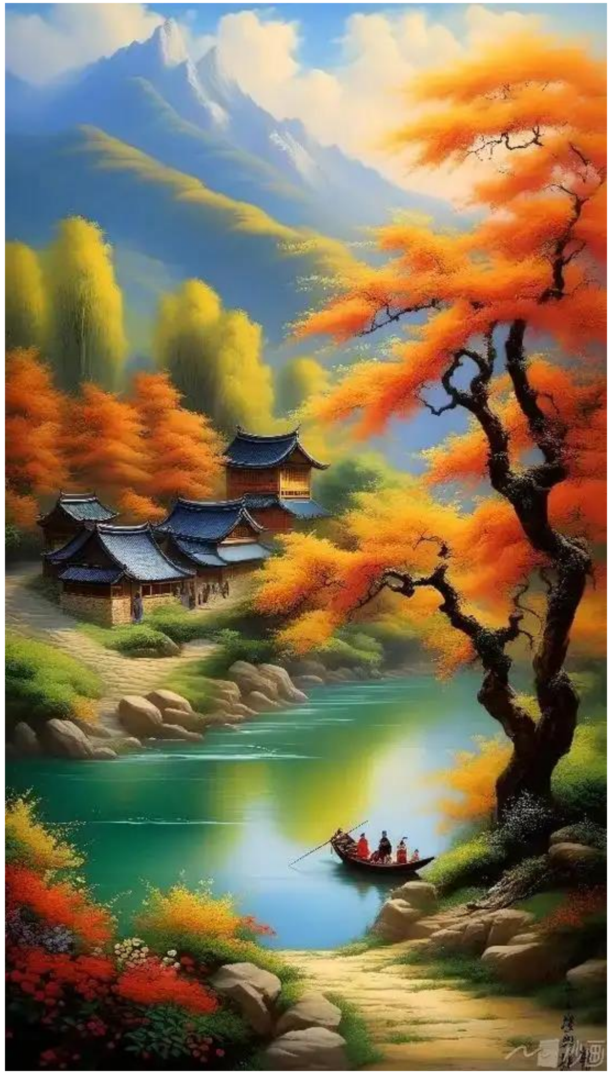
$$J(\vec{\nu};D)=\int[\mathrm{d}\mathbf{X}]\prod_{a=1}^{B}X_a^{\nu_a-1}\mathcal{U}^{\nu-\frac{(L+1)D}{2}}I_{\vec{\nu}}^{\frac{LD}{2}}(\vec{X})$$

A new representation

  - Because F is then degree 2
  - Integrand can be computed easily

- **Much less unintegrated parameters!**

  - 2 loops: $B-1=2$
  - 3 loops: $B-1=5$



Definition

➢ An L-loop amplitude

$$\mathcal{M} \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D l_i}{\mathrm{i}\pi^{D/2}} \frac{P(l)}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_N^{\nu_N}} \,,$$

- **With**

$$\mathcal{D}_\alpha = \sum_{i,j=1}^{L} \hat{\mathcal{A}}_{ij}^{\alpha}\, l_i \cdot l_j + 2 \sum_{i=1}^{L} \hat{\mathcal{B}}_i^{\alpha} \cdot l_i + \hat{\mathcal{C}}^{\alpha}$$

- **Two propagators are in the same branches if they have identical:** $\hat{\mathcal{A}}_{i,j}^{\alpha}$ and $\hat{\mathcal{A}}_{i,j}^{\beta}$
- **B: number of branches**
- $n_1, \cdots, n_b, \cdots, n_B$: **number of propagators in each branch**
- **Corresponding between** α **and** (b, i)

Feynman parametrization

- **First combine denominators in each branch, then combine them**

$$\frac{1}{\mathcal{D}_1^{\nu_1}\cdots\mathcal{D}_N^{\nu_N}} \equiv \prod_{b=1}^{B}\prod_{i=1}^{n_b}\frac{1}{\mathcal{D}_{(b,i)}^{\nu_{(b,i)}}} = \frac{\Gamma(\nu)}{\prod_{\alpha=1}^{N}\Gamma\left(\nu_\alpha\right)}\int_0^\infty [\mathrm{d}\mathbf{X}]\,[\mathrm{d}\mathbf{y}]\,\frac{\prod_{b=1}^{B} X_b^{\nu_b-1}\prod_{\alpha=1}^{N} y_\alpha^{\nu_\alpha-1}}{\left(\sum_{b=1}^{B}\sum_{i=1}^{n_b} X_b y_{(b,i)}\mathcal{D}_{(b,i)}\right)^{\nu}}$$

- **With:** $\nu_b = \sum_{i=1}^{n_b}\nu_{(b,i)},\ \nu = \sum_{\alpha=1}^{N}\nu_\alpha$

$$[\mathrm{d}\mathbf{X}] = \prod_{b=1}^{B}\mathrm{d}X_b\delta\left(1-\sum_{b=1}^{B}X_b\right), \quad [\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N}\mathrm{d}y_\alpha\prod_{b=1}^{B}\delta\left(1-\sum_{i=1}^{n_b}y_{(b,i)}\right)$$

Feynman parametrization(cont.)

$$[\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N} \mathrm{d}y_\alpha \prod_{b=1}^{B} \delta\left(1 - \sum_{i=1}^{n_b} y_{(b,i)}\right)$$

➢ The denominator

$$\sum_{b=1}^{B}\sum_{i=1}^{n_b} X_b y_{(b,i)} \mathcal{D}_{(b,i)} = \sum_{i,j=1}^{L} \mathcal{A}_{ij}\, l_i \cdot l_j + 2\sum_{i=1}^{L} \mathcal{B}_i \cdot l_i + \mathcal{C}$$

- A is independent of $\boldsymbol{y}$! B and C are linear in $\boldsymbol{y}$
- Define:

$$\mathcal{U} = \det(\mathcal{A})\,, \quad \text{independent of } y$$

$$\mathcal{F} = (\mathcal{B}_\mu)^T \mathcal{A}^{adj} \mathcal{B}^\mu - \mathcal{C} \det(\mathcal{A}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{N} R_{\alpha\beta}\, y_\alpha y_\beta = \frac{1}{2}\mathbf{y}^T \cdot R \cdot \mathbf{y}$$

$$y_{(b,i)} \to y_{(b,i)} \times 1 = y_{(b,i)} \sum_j y_{(b,j)}$$

A new representation

➢ **Formula after straightforwardly integrated out loop momenta**

$$\mathcal{M} = \int [\mathrm{d}\mathbf{X}]\, \hat{\mathcal{M}}(\mathbf{X}) \qquad \hat{\mathcal{M}}(\mathbf{X}) = \mathcal{U}^{-\frac{(L+1)D}{2}} \sum_{\Delta,\vec{\nu}'} K^{\Delta}_{\vec{\nu}'}(\mathbf{X})\, I^{\Delta}_{\vec{\nu}'}(\mathbf{X})$$

- $\Delta = \frac{\mathrm{LD}}{2}$, K's are rational in $\boldsymbol{X}$
- Fixed-Branch Integrals (FBIs) defined as

$$I^{\Delta}_{\vec{\nu}}(\mathbf{X}) = \frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{\alpha=1}^{N}\Gamma(\nu_\alpha)} \int [\mathrm{d}\mathbf{y}]\, \frac{\prod_{\alpha=1}^{N} y_\alpha^{\nu_\alpha-1}}{\left(\frac{1}{2}\mathbf{y}^T\!\cdot\! R\!\cdot\!\mathbf{y} - \mathrm{i}0^+\right)^{\nu-\Delta}}$$

- The same as one-loop integrals, except for more delta functions $[\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N} \mathrm{d}y_\alpha \prod_{b=1}^{B} \delta\left(1 - \sum_{i=1}^{n_b} y_{(b,i)}\right)$

Compute FBIs: from matrix *R* to matrix *S*

➤ **Add a line for each branch; number of 1's equals to n_b**

- E.g., if $B = 3$ and $(n_1, n_2, n_3) = (2, 1, 1)$

$$S = \begin{pmatrix} & & & 1 & 1 & 0 & 0 \\ & 0_{3\times 3} & & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & R & \\ 0 & 0 & 1 & & & & \end{pmatrix}$$

Generalized Gram matrix

Reduction relations for FBIs

➢ Recursion relation

$$S\cdot(t_1,\cdots,t_B,\nu_1 I^{\Delta}_{\vec{\nu}+\vec{e}_1},\cdots,\nu_N I^{\Delta}_{\vec{\nu}+\vec{e}_N})^T=(-I^{\Delta-1}_{\vec{\nu}},\cdots,-I^{\Delta-1}_{\vec{\nu}},I^{\Delta-1}_{\vec{\nu}-\vec{e}_1},\cdots,I^{\Delta-1}_{\vec{\nu}-\vec{e}_N})^T$$

- With t_b determined by the equation itself

➢ Dimension-shift relation

$$CI^{\Delta-1}_{\vec{\nu}}=(2\Delta-\nu-B)\,z_0 I^{\Delta}_{\vec{\nu}}+\sum_{\alpha=1}^{N} z_\alpha I^{\Delta-1}_{\vec{\nu}-\vec{e}_\alpha}$$

- With $z_0=0$ or 1 depending on generalized Gram determinant $\det S=0$ or not
- Other parameters determined by

$$S\cdot(C_1,\cdots,C_B,z_1,\cdots,z_N)^T=(z_0,\cdots,z_0,0,\cdots,0)^T$$

- Choose $C=\sum_{b=1}^{B} C_b$ as nonzero as possible

Reduction: 4 different cases

- FBIs have at most one master integral in each sector

1. $\det(S) \neq 0$ and $C \neq 0$: using recursion relation, leaving one master integral

2. $\det(S) \neq 0$ and $C = 0$: $(2\Delta - \nu - B)\, I_{\vec{\nu}}^{\Delta} = -\sum_{\alpha=1}^{N} z_\alpha I_{\vec{\nu}-\vec{e}_\alpha}^{\Delta-1}$

3. $\det(S) = 0$ and $C \neq 0$: $C I_{\vec{\nu}}^{\Delta-1} = \sum_{\alpha=1}^{N} z_\alpha I_{\vec{\nu}-\vec{e}_\alpha}^{\Delta-1}$

4. $\det(S) = 0$ and $C = 0$: $I_{\vec{\nu}}^{\Delta} = -\sum_{\alpha\neq\beta} \frac{z_\alpha}{z_\beta} I_{\vec{\nu}+\vec{e}_\beta-\vec{e}_\alpha}^{\Delta}$

2-4: no master integral

Compute master integrals of FBIs

➢ Using auxiliary mass flow method:

$$\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)=\frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{\alpha=1}^{N}\Gamma\left(\nu_{\alpha}\right)}\int[\mathrm{d}\mathbf{y}]\,\frac{\prod_{\alpha=1}^{N}y_{\alpha}^{\nu_{\alpha}-1}}{\left(\frac{1}{2}\mathbf{y}^{T}\cdot R\cdot\mathbf{y}+\eta\right)^{\nu-\Delta}}$$

- **Equivalent to $R_{\alpha\beta}\to R_{\alpha\beta}+2\eta/B^2$, thus have**

$$(2z_0\eta-C)\frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)=(2\Delta-\nu-B)\,z_0\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)+\sum_{\alpha=1}^{N}z_{\alpha}\mathcal{I}_{\vec{\nu}-\vec{e}_{\alpha}}^{\Delta-1}(\eta)$$

- **Solve it with $\eta\to\infty$ as boundary condition**
- **Using Dimension-Change Transformation to obtain desired FBIs** Huang, Jian, YQM, Mu, Wu, 2412.21054

$$I_{\vec{\nu}}^{\Delta+\delta}=\frac{1}{\Gamma(\delta)}\int_{-\mathrm{i}0^{+}}^{-\mathrm{i}\infty}\mathrm{d}\eta\;\eta^{\delta-1}\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)$$

Comparison

- One-loop FIs: a special case of FBIs, with $B = 1$

$$[\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N} \mathrm{d}y_\alpha \prod_{b=1}^{B} \delta\left(1 - \sum_{i=1}^{n_b} y_{(b,i)}\right)$$

- B is an unimportant parameter in the computation of FBIs

$$C I_{\vec{\nu}}^{\Delta-1} = (2\Delta - \nu - B)\, z_0 I_{\vec{\nu}}^{\Delta} + \sum_{\alpha=1}^{N} z_\alpha I_{\vec{\nu}-\vec{e}_\alpha}^{\Delta-1}$$

- FBIs are as simple as one-loop FIs, thus a solved problem

Comment on remained integration

- **Integral with known integrand**

$$\mathcal{M} = \int [\mathrm{d}\mathbf{X}]\, \hat{\mathcal{M}}\,(\mathbf{X})$$

- **Contour deformation to avoid divergences**

$$\tilde{X}_b = X_b + \mathrm{i}X_b(1 - X_b)G_b(\mathbf{X})$$

$$G_b(\mathbf{X}) = \kappa \sum_j \lambda k_j \frac{\partial_{X_b} P_j}{P_j^2 + (\partial_{X_b} P_j)^2} \exp(-\frac{P_j^2}{\lambda^2 k_j^2})$$

- Adjust parameters
- Subtract out divergences
- Then use existed techniques to perform integration

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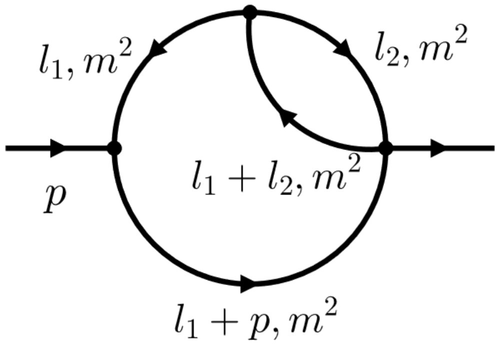
Example

➤ **A two-loop two-leg amplitude**

$$\mathcal{M} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D l_i}{\mathrm{i}\pi^{D/2}} \frac{(l_2 \cdot p)^2 + 2m^2 (l_1 \cdot p)}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4}$$



$$\mathcal{D}_1 = l_1^2 - m^2, \; \mathcal{D}_2 = (l_1 + p)^2 - m^2,$$

$$\mathcal{D}_3 = l_2^2 - m^2, \; \mathcal{D}_4 = (l_1 + l_2)^2 - m^2.$$

$$\mathcal{U} = X_1 X_2 + X_1 X_3 + X_2 X_3$$

$$R = 2m^2 \mathcal{U} \begin{pmatrix} X_1 & X_1 - \frac{sX_1}{2m^2} & 0 & 0 \\ X_1 - \frac{sX_1}{2m^2} & X_1 - \frac{sX_1 X_2 X_3}{m^2 \mathcal{U}} & 0 & 0 \\ 0 & 0 & X_2 & 0 \\ 0 & 0 & 0 & X_3 \end{pmatrix}$$

$$S = \left(\begin{array}{ccc|cccc} & & & 1 & 1 & 0 & 0 \\ & 0_{3\times 3} & & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & & & R & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \end{array} \right)$$

Example

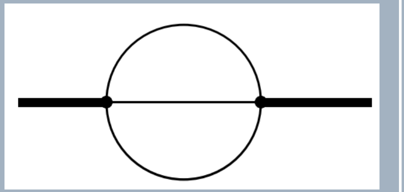
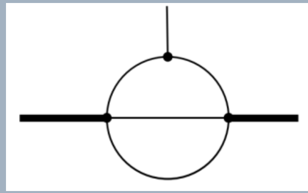
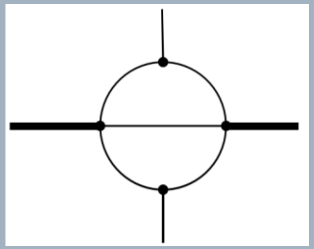
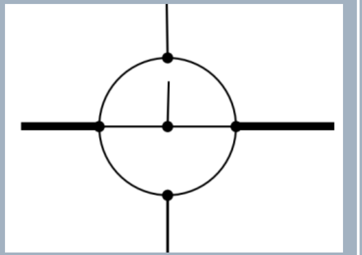
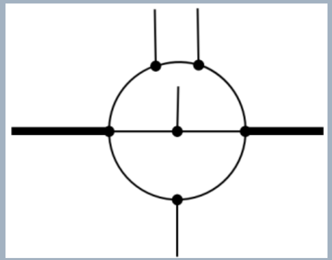
➢ **Reduction of integrand**

$$\mathcal{M} = \int [\mathrm{d}\mathbf{X}]\, \hat{\mathcal{M}}\,(\mathbf{X})$$

$$\begin{aligned}
\hat{\mathcal{M}}\,(\mathbf{X}) = & -\frac{s}{2} X_1(X_1+X_3)\mathcal{U}^{2-\frac{3D}{2}}\, I^{D+1}_{(1,1,1,1)}(\mathbf{X}) \\
& + 2s^2 X_1^3 X_3^2 \mathcal{U}^{2-\frac{3D}{2}}\, I^{D+2}_{(1,3,1,1)}(\mathbf{X}) \\
& - s m^2 X_1 \mathcal{U}^{4-\frac{3D}{2}}\, I^{D}_{(1,1,1,1)}(\mathbf{X}) \\
& + m^2 \mathcal{U}^{3-\frac{3D}{2}}\, I^{D}_{(1,0,1,1)}(\mathbf{X}) \\
& - m^2 \mathcal{U}^{3-\frac{3D}{2}}\, I^{D}_{(0,1,1,1)}(\mathbf{X}),
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{M}}\,(\mathbf{X}) = & \frac{\mathcal{U}^{2-\frac{3D}{2}}}{4(2D-5)(X_2+X_3)^2} \Big(-sX_1\mathcal{U}^2 \\
& \big(-4m^2(X_2+X_3)(X_1-2(D-3)(X_2+X_3))+ \\
& s(\mathcal{U}+(5-2D)X_3^2)\big)\, \boxed{I^{D}_{(1,1,1,1)}(\mathbf{X})} \\
& -\mathcal{U}\,\big(-4m^2(2D-5)(X_2+X_3)^2 \\
& -s(\mathcal{U}+(5-2D)X_3^2)\big)\, \boxed{I^{D}_{(1,0,1,1)}(\mathbf{X})} \\
& +\big(-4m^2(2D-5)(X_2+X_3)^2\mathcal{U} \\
& + s((2D-5)X_3^2(3X_1X_2+3X_1X_3-X_2X_3) \\
& +(X_1X_2+X_1X_3-X_2X_3)\mathcal{U})\big)\, \boxed{I^{D}_{(0,1,1,1)}(\mathbf{X})}\Big)
\end{aligned}$$

Example: compute FBIs

					
DCT/pt (ms)	0.14	0.19	0.34	0.76	2.91
#points	726 =121*6	726	726	726	12826 =121*106

- To obtain 6-digit precision using Adaptive Gausian-Kronrod Rule with degree 5 (11*11=121 points)
- A very preliminary implementation
- Much more to improve

Summary and outlook

- Reveal a deep structure of FIs: simple integrand followed by integration over a few variables:

  2 for two-loop, and 5 for three-loop: independent of number of external legs!

- The integrand (FBIs) can be fully solved, similar to one-loop FIs

- All previous FIs techniques can be applied to resolve the remained integration

  Either fully numerically, or via reduction + computing MIs

- Optimistic to overcome multi-leg FIs computation beyond one-loop, and to meet the requirement of high-precision LHC data

Thank you !