Extensive properties in black hole thermodynamics

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Contents

1.Introduction/Motivation

Northwest University

Black hole thermodynamics:

- Application thermodynamics to BH
- Combination of GR and QM.

$$T = \frac{\hbar \kappa}{2\pi k_B c} \qquad S = \frac{k_B c^3 A_H}{4G\hbar}$$

Bekenstein, PRD(1973), Hawking, CMP(1975)

	Thermodynamics	Black hole
Zero law	$T = const$ at equilibrium	$\kappa = const$ along horizon
First law	$dE = Tds - pdV + \mu dN$	$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega \delta J + \Phi \delta Q$
Second law	$\delta S \geq 0$	$\delta A \geq 0$
Third law	Absolute zero cannot be reached	Extreme black hole cannot reach

Bardeen et al, PRL(1973), Smarr, PRL(1973)

Thermodynamics in ordinary system

- **Boltzman-Gibbs thermodynamics**

Extensive →
- Proportional to the size of the system
- Parameters are additive, $E = E_1 + E_2$

Intensive →
- Does not change with the size of the system
- Parameters are not additive, $T = T_1 = T_2$

Local equilibrium
- Locally define all physical quantities

Homogeneous Function

IF $f(x_1,\cdots,x_n)$ → $f(\lambda x_1,\cdots,\lambda x_n) = \lambda^m f(x_1,\cdots,x_n)$

The Euler throrem: $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = mf$ *P-V-T* system: $E = TS - PV + \mu N$

$m = 1$ is required $E(\lambda S, \lambda V, \lambda N) = \lambda E(S, V, N)$

Non-Extensive thermodynamics

- **Tsallis thermodynamics**

  - Non-Extensive
  - Long-rang interaction or few body system

- **Whether BH thermodynamics be extensive or not?**

  - Black hole thermodynamics is **special** , i.e., thermodynamics on its horizon!
  - In some seems, horizon is a null surface with equal gravitational potential.
  - BH thermodynamics is reasonable to considered as **extensive.**
  - **Why** BH thermodynamics is non-extensive.

1.Introduction/Motivation

The Smarr relation:

$$M=\frac{d-2}{d-3}\frac{\kappa A}{8\pi G}-\frac{1}{d-3}\frac{\Theta\Lambda}{4\pi G}$$

Classical and Quantum Gravity

Enthalpy and the mechanics of AdS black holes

David Kastor[1], Sourya Ray[2] and Jennie Traschen[1]

Published 22 September 2009 • 2009 IOP Publishing Ltd

Classical and Quantum Gravity, Volume 26, Number 19

Citation David Kastor *et al* 2009 *Class. Quantum Grav.* **26** 195011

Einstein field equation with energy-momentum tensor

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=-\Lambda g_{\mu\nu},$$

$$T^{\mu\nu}=(\rho+p)u^{\mu}u^{\nu}+p\eta^{\mu\nu}$$

$$p=-\frac{\Lambda}{8\pi G},\quad V=-\Theta$$

◆ Kerr–Newman–AdS black holes in the extended phase space

Scaling argument:

$M \propto l^{D-3}$ $\Lambda \propto l^{-2}$ $A \propto l^{D-2}$

$$(D-3)M = (D-2)\left(\frac{\partial M}{\partial A}\right)A - 2\left(\frac{\partial M}{\partial \Lambda}\right)\Lambda$$

First law and Smarr relation:

$$dM = TdS + VdP + \Omega dJ + \Phi dQ$$

$$M = 2TS - 2PV + 2\Omega J + \Phi Q$$

The thermodynamic variables

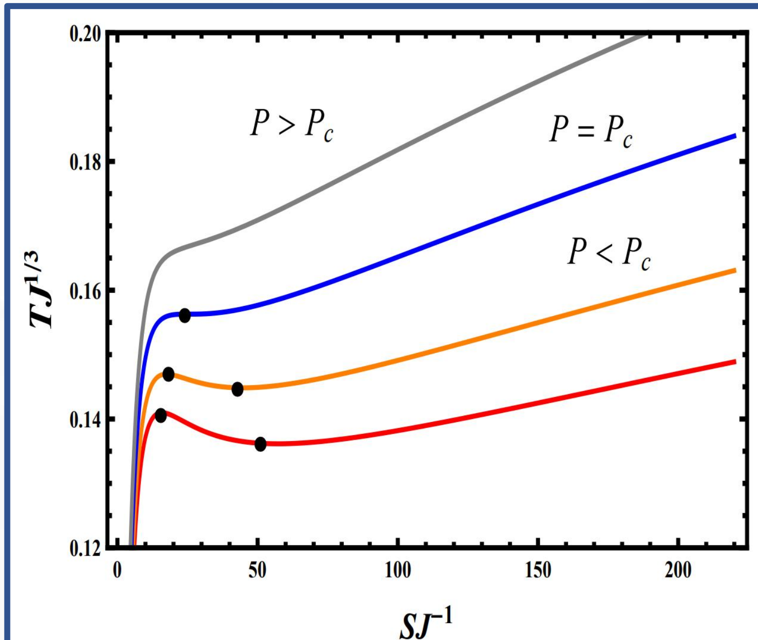
$$M = \frac{(r_0^2+a^2)(r_0^2+l^2)+q^2l^2}{2r_0l^2\Xi^2}$$

$$T = r_0\left(1+\frac{a^2}{l^2}+3\frac{r_0^2}{l^2}-\frac{a^2+q^2}{r_0^2}\right)/4\pi(r_0^2+a^2)$$

$$V = \frac{2\pi}{3}\frac{(r_0^2+a^2)(2r_0^2l^2+a^2l^2-r_0^2a^2)}{r_0l^2\Xi^2}$$

$$S = \frac{\pi(r_0^2+a^2)}{\Xi} \qquad \Phi = \frac{q\,r_0}{r_0^2+a^2}$$

The phase transition



More issues:

- Ensemble theory or ensemble of the theories
- Mass ***M*** regard as enthalpy ***H***
- The first law and Smarr relation of the rotating, charged AdS black holes

$$dM = TdS + \Omega dJ + \Phi dQ + \boxed{VdP,}$$

$$(d-3)M = (d-2)(TS + \Omega J) + (d-3)\Phi Q - 2PV$$

The extensive of the relation is absent.

The Smarr relation **is not consistent with** the Euler relation

Extensive:

- Parameters are additive
- First-order homogeneouty

1.Introduction/Motivation

Northwest University

Holographic interpretation of cosmological constants

Classical and Quantum Gravity

PAPER

The extended thermodynamic phase structure of Taub–NUT and Taub–Bolt

Clifford V Johnson[1]

Published 28 October 2014 • © 2014 IOP Publishing Ltd

Classical and Quantum Gravity, Volume 31, Number 22

Citation Clifford V Johnson 2014 *Class. Quantum Grav.* **31** 225005

Regular Article - Theoretical Physics | Open Access | Published: 14 December 2015

Holographic black hole chemistry

Andreas Karch & Brandon Robinson

Journal of High Energy Physics **2015**, 1–15 (2015) | Cite this article

398 Accesses | **73** Citations | **2** Altmetric | Metrics

l is related to *N*, N^2 the number of branes

Thermodynamic in frame of Holograhpy

Open Access

Holographic thermodynamics requires a chemical potential for color

Manus R. Visser

Phys. Rev. D **105**, 106014 – Published 16 May 2022

$$dE = TdS + \Omega dJ + \tilde{\Phi} d\tilde{Q} - pdV + \mu dC$$

$$E = TS + \Omega J + \tilde{\Phi}\tilde{Q} + \mu C$$

$$dM = \frac{\kappa}{8\pi G_N} dA + \Phi dQ + \frac{\Theta}{8\pi G_N} d\Lambda - (M - \Phi Q)\frac{dG_N}{G_N}$$

$$M = \frac{d-1}{d-2}\frac{\kappa A}{8\pi G_N} + \Phi Q - \frac{1}{d-2}\frac{\Theta\Lambda}{4\pi G_N}$$

Regular Article - Theoretical Physics | Open Access | Published: 18 August 2022

Holographic CFT phase transitions and criticality for charged AdS black holes

Wan Cong, David Kubizňák, Robert B. Mann & Manus R. Visser

Journal of High Energy Physics **2022**, Article number: 174 (2022) | Cite this article

168 Accesses | **8** Citations | **1** Altmetric | Metrics

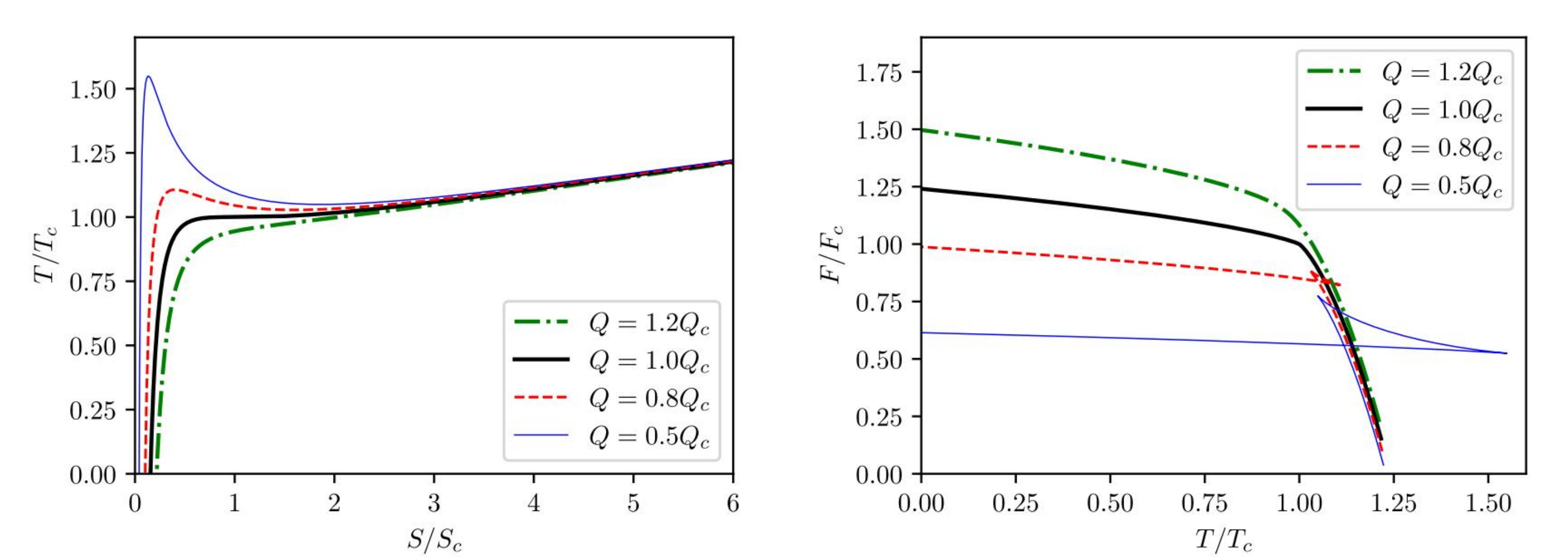
1.Introduction/Motivation

Holographic interpretation of cosmological constants

$C = L^{d-2}/G$ fixing L

$$dM = TdS + \widetilde{\Phi}\widetilde{Q} + \mu dC$$
$$M = TS + \widetilde{\Phi}\widetilde{Q} + \mu C$$

Phase Transition



ZYG, LZ, CQG(2022)

西北大学 Northwest University

Motivition

- **The Smarr relation of the charged rotating AdS black hole in three dimensional spacetime:**

$$(d-3)M=(d-2)(TS+\Omega J)+(d-3)\Phi Q-2PV$$

- No mass term, Absent of the electric field
- Whether BTZ black holes thermodynamics become additive again

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- **The Smarr relation of the modified gravity model.**

$$(d-3)M=(d-2)TS+(d-3)\Phi Q-2PV+(\alpha,\beta)+\bullet\bullet\bullet\bullet\bullet\bullet$$

- The mass *M* is not a first order homogeneous function
- Whether coupling parameters can be treated as thermodynamic variables

2. Holographic thermodynamics of the BTZ black hole

➢ **Introduce the central charge C**

Holographic Dictionary : $$C = \frac{V_{d-2} l^{d-2}}{16\pi G} \propto \frac{l^2}{G}$$

➢ **Definition of μ (on-shell Euclidean action)**

- Thermal partition function of *CFT*: $W = -T \ln Z_{CFT}$
- On-shell Euclidean action: $I_E = -\ln Z_{AdS}$
- *AdS/CFT* correspondence: $Z_{CFT} = Z_{AdS}$
- The periodicity of Killing time : $\beta = 1/T$

$$\Rightarrow \quad \mu C = -T \ln Z_{CFT} = -T \ln Z_{AdS} = T I_E$$

2. Holographic thermodynamics of the BTZ black hole

Additive thermodynamics of the BTZ BH $J=0,\ Q\neq 0$

➢ **The metric:**

$$f(r) = -8GM + \frac{r^2}{l^2} + \frac{\kappa\mu_0 Q^2}{4\pi^2}\ln\left(\frac{r}{l}\right) \qquad \varphi(r) = Q\ln\left(\frac{r}{l}\right)$$

➢ **The thermodynamic variables**

$$\mu_0 = 2\pi,\ \kappa = 8\pi G,\ c = 1$$

$$M = \frac{r_0^2}{8Gl^2} - \frac{Q^2}{2}\ln\left(\frac{r_0}{l}\right) \qquad T = \frac{f'(r_0)}{4\pi} = \frac{r_0}{2\pi l^2} - \frac{GQ^2}{\pi r_0}$$

$$S = \frac{A_1}{4G} = \frac{\pi r_0}{2G} \qquad \Phi = \left(\frac{\partial M}{\partial Q}\right)_{S,l} = -Q\ln\left(\frac{r_0}{l}\right)$$

➢ **The heat capacity :**

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q \geq 0$$

Stable

2. Holographic thermodynamics of the BTZ black hole

On-shell Euclidean action

- Einstein-Hilbert action :

$$I_{EH} = -\frac{1}{2\kappa}\int dx^3\sqrt{g}(R-2\Lambda)+\frac{1}{4\mu_0}\int d^3xF_{\mu\nu}F^{\mu\nu}$$

- Gibbons-Hawking action :

$$I_{GHY} = -\frac{1}{\kappa}\int_{\partial M} d^2x\sqrt{h}K-\frac{1}{\mu_0}\int_{\partial M} d^2xn_rF^{rt}A_t$$

- The counterterm action :

$$I_{ct} = \frac{1}{\kappa}\int_{\partial M} dx^2\sqrt{h}\left(\frac{1}{l}\right)$$

$$TI_E = \frac{1}{\beta}(I_{EH}+I_{GHY}+I_{ct}) = \mu C$$

$$\mu_0 = 2\pi, \kappa = 8\pi G, c = 1$$

➤ The total action central charge:

$$I_E = I_{EH}+I_{GHY}+I_{ct} = \beta\left(\frac{Q^2}{2}\ln\left(\frac{r_0}{l}\right)+\frac{Q^2}{2}-\frac{r_0^2}{8Gl^2}\right)$$

2. Holographic thermodynamics of the BTZ black hole

Additive thermodynamics of the BTZ BH $J=0,\ Q\neq 0$

- **Order homogeneity**

The thermodynamical variables with central charge

$$\tilde{Q}=\frac{Ql}{\sqrt{G}} \qquad \tilde{\Phi}=\frac{\Phi\sqrt{G}}{l}$$

$$M=\frac{S^2}{16\pi^2 lC}-\frac{\tilde{Q}\ln[S/4\pi C]}{16lC} \qquad T=\left(\frac{\partial M}{\partial S}\right)_{\tilde{Q},C}=\frac{S}{8\pi^2 lC}-\frac{\tilde{Q}^2}{16lSC}$$

$$\tilde{\Phi}=\left(\frac{\partial M}{\partial \tilde{Q}}\right)_{S,C}=-\frac{\tilde{Q}\ln[S/4\pi C]}{8lC} \qquad \mu=\left(\frac{\partial M}{\partial C}\right)_{S,\tilde{Q}}=-\frac{S^2}{16\pi^2 lC^2}+\frac{\tilde{Q}^2}{16lC^2}+\frac{\tilde{Q}^2\ln[S/4\pi C]}{16lC^2}$$

- **Thermodynamic behavior**

$$dM=TdS+\tilde{\Phi}d\tilde{Q}+\mu dC,\ M=TS+\tilde{\Phi}\tilde{Q}+\mu C$$

Neither a extremal point nor an inflection point, there are ***no*** phase transition.

2. Holographic thermodynamics of the BTZ black hole

Additive thermodynamics of the BTZ BH $J \neq 0,\ Q = 0$

- **The metric:**

$$f(r) = -8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}$$

- **The thermodynamic variables**

$\kappa = 8\pi G, c = 1$

$$M = \frac{r_0^{\ 2}}{8Gl^2} + \frac{2GJ^2}{r_0^{\ 2}} \qquad \Omega = \left(\frac{\partial M}{\partial J}\right)_{S,l} = \frac{4GJ}{r_0^{\ 2}}$$

$$S = \frac{A_1}{4G} = \frac{\pi r_0}{2G} \qquad T = \frac{f'(r_0)}{4\pi} = \frac{r_0}{2\pi l^2} - \frac{8G^2J^2}{\pi r_0^{\ 3}}$$

- **The heat capacity :**

$$C_J = T\left(\frac{\partial S}{\partial T}\right)_J \geq 0$$

Stable

2. Holographic thermodynamics of the BTZ black hole

Additive thermodynamics of the BTZ BH $J \neq 0, \; Q = 0$

➢ Order homogeneity

The thermodynamical variables with central charge

$$M = \frac{S^2}{16\pi^2 lC} + \frac{4\pi^2 J^2 C}{lS^2} \qquad T = \left(\frac{\partial M}{\partial S}\right)_{J,C} = \frac{S}{8\pi^2 lC} - \frac{8\pi^2 J^2 C}{lS^3}$$

$$\Omega = \left(\frac{\partial M}{\partial J}\right)_{S,C} = \frac{8\pi^2 JC}{lS^2} \qquad \mu = \left(\frac{\partial M}{\partial C}\right)_{S,J} = -\frac{S^2}{16\pi^2 lC^2} + \frac{4\pi^2 J^2}{lS^2}$$

➢ Thermodynamic behavior

$$dM = TdS + \Omega dJ + \mu dC, \quad M = TS + \Omega J + \mu C$$

Neither an extremal point nor an inflection point, there are ***no*** phase transition.

2. Holographic thermodynamics of the dRGT black hole

Extensive thermodynamics of the dRGT BH

- The bulk action for dRGT massive gravity on the *d=(n+2)*-dimensional background manifold *M*

$$I_b = \frac{1}{16\pi G_d}\int_M d^d x\sqrt{-g}[R-2\Lambda - F_{\mu\nu}F^{\mu\nu} + m_g^2\sum_{i=1}^{4} c_i U_i(g,f)]$$

- The total Euclidian action

$$I_E = I_b' + I_s + I_{ct} = I_{BH} - I_{AdS}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{U}_i = \sum_{y=1}^{i}(-1)^{y+1}\frac{(i-1)!}{(i-y)!}\mathcal{U}_{i-y}[\mathcal{K}^y]$$

(n=2) →

$$\mathcal{U}_1 = [\mathcal{K}],$$

$$\mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2],$$

$$\mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3],$$

$$\mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 8[\mathcal{K}][\mathcal{K}^3] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]$$

$$\mathcal{K}^\mu{}_\nu = \sqrt{g^{\mu\lambda} f_{\lambda\nu}}$$

2. Holographic thermodynamics of the dRGT black hole

Extensive thermodynamics of the dRGT BH

- **The equation of field**

$$\delta I_b = \frac{1}{16\pi G_d}\int_M d^d x\sqrt{-g}[G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 \mathcal{X}_{\mu\nu} - T_{\mu\nu}]\delta g^{\mu\nu}$$
$$-\frac{1}{8\pi G_d}\int_{\partial M} d^{d-1}x\sqrt{-h}\,n^\alpha \delta g_{\mu\nu,\alpha}$$
$$+\frac{1}{4\pi G_d}\int_M d^d x\sqrt{-g}[\nabla_\mu F^{\mu\nu}]\delta A_\upsilon$$
$$-\frac{1}{4\pi G_d}\int_{\partial M} d^{d-1}x\sqrt{-h}\,n_\mu F^{\mu\nu} A_\upsilon$$

$$\mathcal{X}_{\mu\nu} = -\sum_{i=1}^{d-2}\frac{c_i}{2}[\mathcal{U}_i g_{\mu\nu} + \sum_{y=1}^{i}(-1)^y \frac{i!}{(i-y)!}\mathcal{U}_{i-y}\mathcal{K}_{\mu\nu}^y] = \begin{aligned} &-\frac{c_1}{2}(\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) \\ &-\frac{c_2}{2}(\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1\mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2) \\ &-\frac{c_3}{2}(\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2\mathcal{K}_{\mu\nu} + 6\mathcal{U}_1\mathcal{K}_{\mu\nu}^2 - 6\mathcal{K}_{\mu\nu}^3) \\ &-\frac{c_4}{2}(\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3\mathcal{K}_{\mu\nu} + 12\mathcal{U}_2\mathcal{K}_{\mu\nu}^2 - 24\mathcal{U}_1\mathcal{K}_{\mu\nu}^3 + 24\mathcal{K}_{\mu\nu}^4) \end{aligned}$$

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}F_{\mu\nu}F^{\mu\nu} + 2F_{\mu\lambda}F_\nu^{\ \lambda}$$

2. Holographic thermodynamics of the dRGT black hole

Extensive thermodynamics of the dRGT BH

➢ The equation of field

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 \mathcal{X}_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} F_{\mu\nu} F^{\mu\nu} + 2 F_{\mu\lambda} F_{\nu}^{\lambda} \qquad \nabla_{\mu} F^{\mu\nu} = 0$$

➢ The metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 h_{ij} dx_i dx_j (i, j = 1.2 \ldots n)$$

$$h_{ij}dx_i dx_j = \begin{cases} dx_1^2 + \sum_{i=2}^{d-2} \prod_{j=1}^{i-1} \sin^2 x_j dx_i^2 & (k=+1) \\ \sum_{i=1}^{d-2} dx_i^2 & (k=0) \\ dx_1^2 + \sinh^2 x_1 \sum_{i=2}^{d-2} dx_i^2 \prod_{j=2}^{i-1} \sin^2 x_j & (k=-1) \end{cases}$$

$$f_{\mu\nu} = \mathrm{diag}(0, 0, c_0^2 h_{ij})$$

➢ The metric function

$$d_i = d - i$$

$$f(r) = k - \frac{2\Lambda r^2}{d_1 d_2} - \frac{m}{r^{d_3}} + m_g^2 \sum_{i=1}^{4} \left(\frac{c_0^i c_i}{d_2 r^{i-2}} \prod_{j=2}^{i} d_j \right) + \frac{2q^2}{d_2 d_3 r^{2d_3}}$$

2 Holographic thermodynamics of the dRGT black hole

Extensive thermodynamics of the dRGT BH

- **Four-dimensional case**

$$f(r) = k + \frac{r^2}{l^2} - \frac{m}{r} + \frac{q^2}{r^2} + \frac{c_0 c_1 m_g^2}{2} r + c_0^2 c_2 m_g^2$$

- **The thermodynamic variables**

V_2 is the solid Angle

$$M = \frac{V_2 r_+}{8\pi G}\left(k + \frac{r_+^2}{l^2} + \frac{q^2}{r_+^2} + \frac{c_0 c_1 m_g^2}{2} r_+ + c_0^2 c_2 m_g^2\right) \qquad S = \frac{V_2}{4G} r_+^2 \qquad \hat{Q} = Ql = \frac{V_2 q l}{4\pi G}$$

$$T = \frac{1}{4\pi r_+}\left(k + \frac{3r_+^2}{l^2} - \frac{q^2}{r_+^2} + c_0 c_1 m_g^2 r_+ + c_0^2 c_2 m_g^2\right) \qquad \hat{\Phi} = \frac{\Phi}{l} = \frac{q}{r_+ l} \qquad C = \frac{V_2 l^2}{16\pi G} \propto \frac{l^2}{G}$$

2. Holographic thermodynamics of the dRGT black hole

- **The on-shell action of the black hole spacetime**

$$I_{BH} = \frac{\beta V_n}{16\pi G_d}\left(\left[\frac{2}{l^2} r^{d-1} - m_g^2 \sum_{i=1}^{d-2} \frac{(i-2)c_0^i c_i}{d-i-1} r^{d-i-1} \prod_{j=3}^{i+1} d_j\right]_{r_+}^{R=\infty} - 4\frac{d_3}{d_2}\Phi^2 r_+^{d_3}\right)$$

- **The on-shell action of the AdS background without black hole**

$$I_{AdS} = \frac{\beta V_n}{16\pi G_d}\left[\frac{2}{l^2} R^{d_1} - m_g^2 \sum_{i=1}^{d-2} \frac{(i-2)c_0^i c_i}{d-i-1} R^{d-i-1} \prod_{j=3}^{i+1} d_j\right]$$

- **On-shell Euclidean action**

$$TI_E = \frac{1}{\beta}(I_{BH} - I_{AdS}) = \mu C$$

$$I_{on-shell} = \lim_{R\to\infty}(I_{BH} - I_{AdS}) = \frac{\beta V_n r_+^{d_3}}{16\pi G_d}\left[k - \frac{r_+^2}{l^2} - 2\frac{d_3}{d_2}\Phi^2 + m_g^2 \sum_{i=1}^{d-2} \frac{(i-1)c_0^i c_i}{r_+^{i-2}} \prod_{j=3}^{i+1} d_j\right]$$

$$\xrightarrow{(d=4)} \quad = \frac{\beta V_2 r_+}{16\pi G_d}\left(1 - \frac{r_+^2}{l^2} - \frac{q^2}{r_+^2} + c_0 c_1 m_g^2\right)$$

2. Holographic thermodynamics of the dRGT black hole

- **The on-shell action of the black hole spacetime**

$$I_{BH} = \frac{\beta V_n}{16\pi G_d}\left(\left[\frac{2}{l^2} r^{d-1} - m_g^2 \sum_{i=1}^{d-2} \frac{(i-2)c_0^i c_i}{d-i-1} r^{d-i-1} \prod_{j=3}^{i+1} d_j\right]_{r_+}^{R=\infty} - 4\frac{d_3}{d_2}\Phi^2 r_+^{d_3}\right)$$

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$(d=4)$ →

$$= \frac{\beta V_2 r_+}{16\pi G_d}\left(1 - \frac{r_+^2}{l^2} - \frac{q^2}{r_+^2} + c_0 c_1 m_g^2\right)$$

$$dM = TdS + \hat{\Phi} d\hat{Q} + \mu dC, \quad M = TS + \hat{\Phi}\hat{Q} + \mu C$$

2. Holographic thermodynamics of the dRGT black hole

➢ **Whether coupling parameters can be treated as thermodynamic variables ?**

$$I_b = \frac{1}{16\pi G_d}\int_M d^d x\sqrt{-g}[R - 2\Lambda - F_{\mu\upsilon}F^{\mu\upsilon} + m_g^2\sum_{i=1}^{4} c_i U_i(g,f)]$$

$$\begin{aligned}\mathrm{d}M = T\mathrm{d}S + V\mathrm{d}P &+ \frac{c_0 m^2 V_{d-2} r_+^{d-2}}{16\pi}\mathrm{d}c_1 \\ &+ \frac{(d-2)c_0^2 m^2 V_{d-2} r_+^{d-3}}{16\pi}\mathrm{d}c_2 \\ &+ \frac{(d-2)(d-3)c_0^3 m^2 V_{d-2} r_+^{d-4}}{16\pi}\mathrm{d}c_3 \\ &+ \frac{(d-2)(d-3)(d-4)c_0^4 m^2 V_{d-2} r_+^{d-5}}{16\pi}\mathrm{d}c_4\end{aligned}$$

$$\begin{aligned}(d-3)M = (d-2)TS - 2VP &- \frac{c_0 c_1 m^2 V_{d-2}}{16\pi} r_+^{d-2} \\ &+ \frac{(d-2)(d-3)c_0^3 c_3 m^2 V_{d-2}}{16\pi} r_+^{d-4} \\ &+ \frac{(d-2)(d-3)(d-4)c_0^4 c_4 m^2 V_{d-2} r_+^{d-5}}{8\pi}\end{aligned}$$

Eur. phys. J.C, 77,256(2017).

$$M = 2TS + Q\Phi - \mu m_g + 2\mu_{\alpha'}\alpha' + \mu_{\beta'}\beta'$$

Eur. phys. J.C, 79,342(2019).

➢ **First law and Smarr relation:**

$$dM \neq TdS + \hat{\Phi}d\hat{Q} + \mu dC + C_i dc_i, \quad M \neq TS + \hat{\Phi}\hat{Q} + \mu C + C_i c_i$$

2. Holographic thermodynamics of the dRGT black hole

Extensive thermodynamics of the dRGT BH

➢ **The thermodynamical variables**

$$M=\frac{S\tilde{c}_1 l}{4\pi l}+\frac{SCV_2(\tilde{c}_2+\mathrm{k})+4\pi^2\hat{Q}^2+4C^2}{4\pi l\sqrt{SCV_2}} \qquad \hat{\Phi}=\left(\frac{\partial M}{\partial \hat{Q}}\right)_{S,C}=\frac{2\pi\hat{Q}}{l\sqrt{SCV_2}}$$

$$T=\left(\frac{\partial M}{\partial S}\right)_{\hat{Q},C}=\frac{\tilde{c}_1 l}{4\pi l}+\frac{SCV_2(\tilde{c}_2+\mathrm{k})-4\pi^2\hat{Q}^2+12S^2}{8\pi lS\sqrt{SCV_2}} \qquad \mu=\left(\frac{\partial M}{\partial C}\right)_{S,\hat{Q}}=\frac{SCV_2(\tilde{c}_2+\mathrm{k})-4\pi^2\hat{Q}^2-4S^2}{8\pi lC\sqrt{SCV_2}}$$

$$S=\frac{V_2C}{72}[(\tilde{c}_1 l-4\pi Tl)\sqrt{\tilde{c}_1{}^2l^2-8\pi T\tilde{c}_1 l^2+12\hat{\Phi}^2l^2-12(\tilde{c}_2+\mathrm{k})}+\tilde{c}_1{}^2l^2-8\pi T\tilde{c}_1 l^2$$
$$+\tilde{c}_1{}^2l^2-8\pi T\tilde{c}_1 l^2+16\pi^2\mathrm{T}^2l^2+6\hat{\Phi}^2l^2-6(\tilde{c}_2+\mathrm{k})]$$

$$\hat{Q}=\frac{\sqrt{2}\hat{\Phi}lV_2C}{24\pi}[(\tilde{c}_1 l-4\pi Tl)\sqrt{\tilde{c}_1{}^2l^2-8\pi T\tilde{c}_1 l^2+16\pi^2\mathrm{T}^2l^2+12\hat{\Phi}^2l^2-12(\tilde{c}_2+\mathrm{k})}$$
$$+\tilde{c}_1{}^2l^2-8\pi T\tilde{c}_1 l^2+16\pi^2\mathrm{T}^2l^2+6\hat{\Phi}^2l^2-6(\tilde{c}_2+\mathrm{k})]^{\frac{1}{2}}$$

2. Holographic thermodynamics of the dRGT black hole

Extensive thermodynamics of the dRGT BH

➢ **Order homogeneity**

$$M = \frac{S\tilde{c}_1 l}{4\pi l} + \frac{SCV_2(\tilde{c}_2 + \mathrm{k}) + 4\pi^2\hat{Q}^2 + 4C^2}{4\pi l\sqrt{SCV_2}}$$

$$\hat{\Phi} = \left(\frac{\partial M}{\partial \hat{Q}}\right)_{S,C} = \frac{2\pi\hat{Q}}{l\sqrt{SCV_2}}$$

$$T = \left(\frac{\partial M}{\partial S}\right)_{\hat{Q},C} = \frac{\tilde{c}_1 l}{4\pi l} + \frac{SCV_2(\tilde{c}_2 + \mathrm{k}) - 4\pi^2\hat{Q}^2 + 12S^2}{8\pi l S\sqrt{SCV_2}}$$

$$\mu = \left(\frac{\partial M}{\partial C}\right)_{S,\hat{Q}} = \frac{SCV_2(\tilde{c}_2 + \mathrm{k}) - 4\pi^2\hat{Q}^2 - 4S^2}{8\pi l C\sqrt{SCV_2}}$$

$$\tilde{c}_1 = c_0 c_1 m_g^2 \qquad \hat{Q} = Ql = \frac{V_2 q l}{4\pi G} \qquad \mathcal{S} = \frac{S}{C}$$

$$\tilde{c}_2 = c_0^2 c_2 m_g^2 \qquad \hat{\Phi} = \frac{\Phi}{l} = \frac{q}{r_+ l} \qquad \hat{\mathcal{Q}} = \frac{\hat{Q}}{C}$$

- When $S \to \lambda S, \hat{Q} \to \lambda \hat{Q},$
  $C \to \lambda C, \; M \to \lambda M.$
- $[T] = [\hat{\Phi}] = [\mu] = \frac{1}{[l]}$

2. Holographic thermodynamics of the dRGT black hole

Extensive thermodynamics of the dRGT BH

- **Thermodynamic behavior**

< *T-S* curves >

$$\tilde{c}_1 = c_0 c_1 m_g^2 \quad \tilde{c}_2 = c_0^2 c_2 m_g^2 \quad \hat{\Phi} = \frac{\Phi}{l} = \frac{q}{r_+ l} \quad \hat{Q} = Ql = \frac{V_2 q l}{4\pi G}$$

1. Find the inflection point .

$$\left(\frac{\partial T}{\partial S}\right)_{\hat{Q},C} = 0 \quad \left(\frac{\partial^2 T}{\partial^2 S}\right)_{\hat{Q},C} = 0$$

$$Q_c = \frac{V_2 C}{24\pi}(\tilde{c}_2 + k) = \mathcal{Q}_c C$$

$$S_c = \frac{V_2 C}{24}(\tilde{c}_2 + k) = \mathcal{S}_c C$$

$$T_c = \frac{\tilde{c}_1}{4\pi} + \frac{\sqrt{6}\sqrt{(\tilde{c}_2 + k)}}{3 l \pi}$$

2. Introducing the reduced parameters .

$$t = \frac{T}{T_c} \quad s = \frac{\mathcal{S}}{\mathcal{S}_c} \quad q = \frac{\hat{\mathcal{Q}}}{\mathcal{Q}_c} \quad \Rightarrow \quad t = \frac{\sqrt{6}\tilde{c}_1 l s^{3/2} - \sqrt{\tilde{c}_2 + k}\,(q^2 - 3s^2 - 6s)}{s^{3/2}(8\sqrt{\tilde{c}_2 + k} + \sqrt{6}\tilde{c}_1 l)}$$

2. Holographic thermodynamics of the dRGT black hole

Extensive thermodynamics of the dRGT BH

- **Thermodynamic behavior**

<*F-T* curves>

$$\tilde{c}_1 = c_0 c_1 m_g^2 \quad \tilde{c}_2 = c_0^2 c_2 m_g^2 \quad \hat{\Phi} = \frac{\Phi}{l} = \frac{q}{r_+ l} \quad \hat{Q} = Ql = \frac{V_2 ql}{4\pi G}$$

1. Introduce the Helmholtz free energy by a Legendre transform

$$F(T, \hat{Q}, C) = M(S, \hat{Q}, C) - TS$$

2. Introducing the relative parameters .

$$t = \frac{T}{T_c} \quad s = \frac{S}{S_c} \quad q = \frac{\hat{Q}}{Q_c} \quad f = \frac{F}{F_c}$$

$\Rightarrow$

$$f = \frac{-8ts\sqrt{\tilde{c}_2 + \mathrm{k}} - \sqrt{6}\tilde{c}_1 ls(t-1)}{8\sqrt{\tilde{c}_2 + \mathrm{k}}} + \frac{q^2 + s^2 + 6s}{4\sqrt{s}}$$

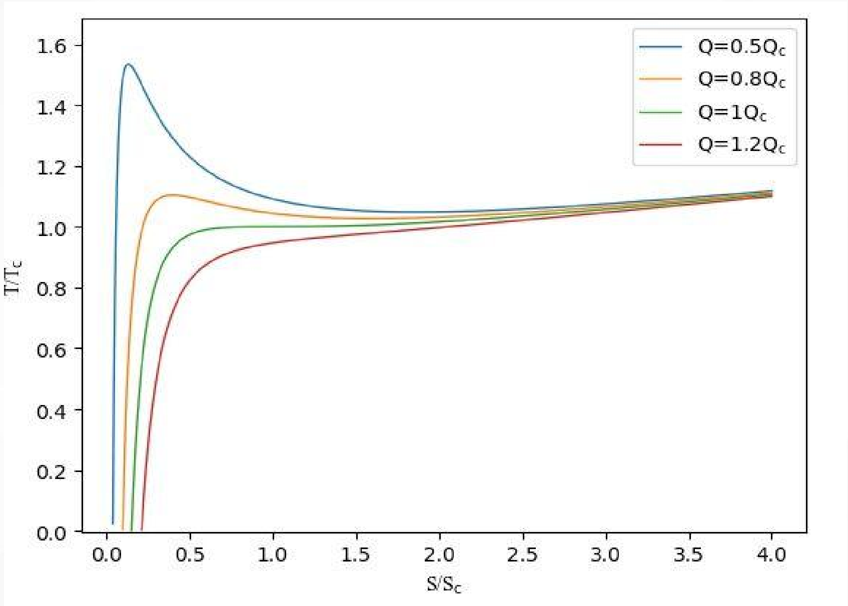
2. Holographic thermodynamics of the dRGT black hole

➢ Thermodynamic behavior

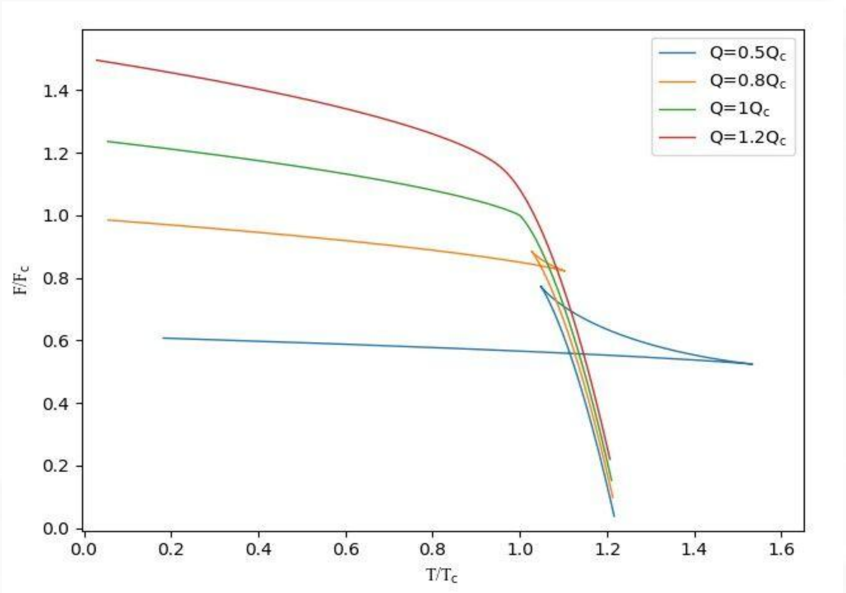
$$t = \frac{\sqrt{6}\tilde{c}_1 l s^{3/2} - \sqrt{\tilde{c}_2 + \mathrm{k}}\,(q^2 - 3s^2 - 6s)}{s^{3/2}(8\sqrt{\tilde{c}_2 + \mathrm{k}} + \sqrt{6}\tilde{c}_1 l)}$$

$$f = \frac{-8ts\sqrt{\tilde{c}_2 + \mathrm{k}} - \sqrt{6}\tilde{c}_1 l s(t-1)}{8\sqrt{\tilde{c}_2 + \mathrm{k}}} + \frac{q^2 + s^2 + 6s}{4\sqrt{s}}$$

Phase Transition



< *T-S* >



< *F-T* >

<*Φ-Q* >curves There is neither inflection point nor extremum on the curve.

3 Summary and Outlook

➢ Extensive Black Hole Thermodynamics

- BTZ black hole thermodynamics becomes additive.
- Euler relation of BTZ black hole and dRGT black hole is established.
- Homogeneous Function are verified.
- Extensive thermodynamics of BTZ black hole and dRGT black hole.
- Thermodynamic role of the coupling parameters are determined.
- Thermodynamic phase transition is analysed.

➢ Outlook

- The universality of extensive thermodynamics.

Thank you!