Thermodynamics of Taub-NUT Spacetime

刘海山

2023@中国科技大学

Talk is based on JHEP10(2022)174

in collaboration with Hong Lu and Liang Ma

Outline

- Background
- Mass and NUT charges of Taub-NUT
- Thermodynamics of Kerr-Taub-NUT

Taub-NUT spacetime

- Taub-NUT spacetime is a solution to Einstein's general relativity, first discovered by Taub in 1951, subsequently rediscovered by Newman, Tamburino and Unti in 1963.

- The solution is a simple generalization of the Schwarzschild spacetime. In addition to a Schwarzschild-like mass parameter m, it contains one more parameter which is called NUT parameter n.

- The nature of this NUT parameter n was viewed as a "gravitational magnetic charge" by some people, thus the solution is a sort of gravitational dyon and its Euclidean continuation is the solution known as Kaluza-Klein monopole.

Taub-NUT spacetime

- The theory is just Einstein's general relativity

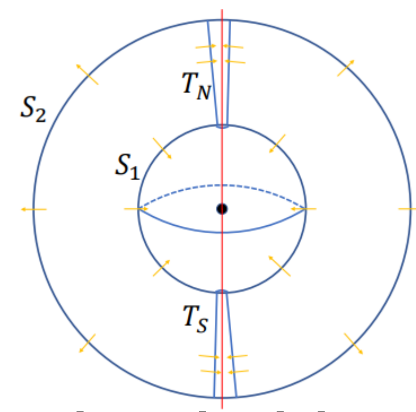
$$\mathcal{L} = \sqrt{-g}\mathrm{R}$$

- Taub-NUT metric is

$$\mathrm{ds^2 = -f(dt + 2n\cos\theta d\varphi)^2 + \frac{dr^2}{f} + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2)}$$

$$\mathrm{f = \frac{r^2 - 2mr - n^2}{r^2 + n^2}}$$

with two parameters m and n.



- This metric does not have curvature singularities, but has the so-called Minser string singularities at north and south pole ($\theta = 0, \pi$), where the metric fails to be invertible.

Taub-NUT spacetime

- The Misner string singularity can be avoided by imposing periodic condition on the time coordinate. Then the NUT parameter is not independent and will not contribute directly to first law.

- However, it is showed that the periodic time condition is not necessary. The spacetime is geodesically complete and Misner string remains transparent and physical. Recently, a lot of work has been done in this direction, and we shall focus on this direction, too.

NUT charge and Thermodynamics

Due to NUT parameter n, the first law is expected to be

$$\delta \mathrm{M} = \mathrm{T}\delta \mathrm{S} + \Phi_{\mathrm{n}}\delta \mathrm{Q}_{\mathrm{n}}$$

- The temperature and entropy can be derived by standard method. There is not a well accepted way of defining the corresponding NUT charge and the mass of Taub-NUT spacetime.

- Thus, the first law is not fixed, though 70 years has passed since the construction of Taub-NUT solution!

- Here are two initial treatment of the first law of Taub-NUT spacetime.

Example 1

- The mass, temperature and entropy are

$$\mathrm{M = m} \qquad \mathrm{T} = \frac{1}{4\pi \mathrm{r_+}} \qquad \mathrm{S} = \pi(\mathrm{r_+^2 - n^2})$$

- The NUT charge and NUT potential

$$\mathrm{Q_N} = \frac{\mathrm{n}}{\mathrm{r_+}} \qquad \Phi_\mathrm{N} = -\frac{\mathrm{n}}{2}$$

- The first law and Smarr relation are

$$\mathrm{M = 2TS} \qquad \delta\mathrm{M} = \mathrm{T}\delta\mathrm{S} + \Phi_\mathrm{n}\delta\mathrm{Q_n}$$

- The entropy is not proportional to area of the horizon.

Hennigar, Kubiznak and Mann, Phys.Rev.D 100 (2019) 064055

Example II

- The mass, temperature and entropy are

$$\mathbf{M} = \mathbf{m} \qquad \mathbf{T} = \frac{1}{4\pi \mathbf{r}_+} \qquad \mathbf{S} = \pi(\mathbf{r}_+^2 + \mathbf{n}^2)$$

- The NUT charge and NUT potential are

$$\mathbf{Q}_\mathbf{N} = -\frac{4\pi \mathbf{n}^3}{\mathbf{r}_+} \qquad \Phi_\mathbf{N} = \frac{1}{8\pi \mathbf{n}}$$

- The first law and Smarr relation are

$$\mathbf{M} = 2(\mathbf{TS} - \Phi_\mathbf{n}\mathbf{Q}_\mathbf{n}) \qquad \delta\mathbf{M} = \mathbf{T}\delta\mathbf{S} + \Phi_\mathbf{n}\delta\mathbf{Q}_\mathbf{n}$$

- When n approaches 0, infinity emerges, thus there is no smooth limit to Schwartzschild limit.

- There exist other choice of mass and NUT charge definition in the literature.

NUT charge and Thermodynamics

- There is no systematic way of defining NUT charge and thus calculating the NUT charge directly.

- In fact, the definition of mass is difficult, since the Taub-NUT metric is not asymptotic to Minkowski spacetime. $M = m$ is mostly used, but other choices are also proposed.

- Due to the uncertainty of definition of both mass and NUT charge, the first law is also unclear.

- We aim at this puzzle, and propose a systematic way of calculating the NUT charge and mass, and then find that the first law is automatically satisfied.

Taub-NUT spacetime

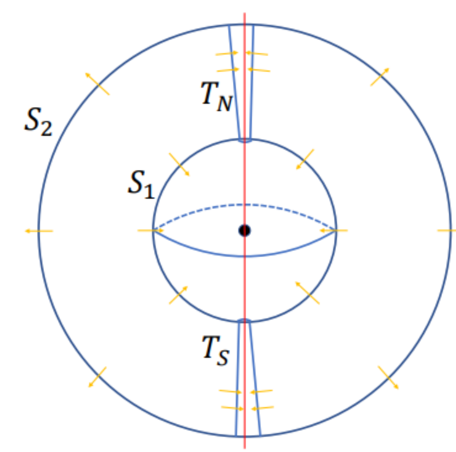
- Let us first take a close look at the Taub-NUT spacetime

$$ds^2 = -f(dt + 2n\cos\theta d\varphi)^2 + \frac{dr^2}{f} + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$f = \frac{r^2 - 2mr - n^2}{r^2 + n^2}$$

- The spacetimes contains three Killing horizons: one event horizon and other two Killing horizons on the poles ($\theta = 0, \pi$) with Killing vectors

$$\xi = \partial_t \qquad l_\pm = \partial t \pm \frac{1}{2n}\partial_\phi$$



- The famous Komar integration gives

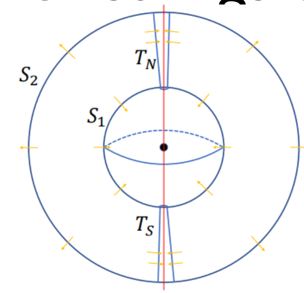
$$-\frac{1}{8\pi}\int_\infty *d\xi = m$$

This may be the reason why many people choose $M = m$.

Taub-NUT spacetime

- We can also count the contribution of the Misner strings on the poles

$$-\frac{1}{8\pi}\left(\int_{T_+} *d\xi - \int_{T_-} *d\xi\right) = \frac{n^2}{r_+}$$



- Combinating them together, we choose the mass of the Taub-NUT spacetime as

$$M = m + \frac{n^2}{r_+} = \sqrt{m^2 + n^2}$$

In this expression, we can see the parameter n contribute to the total mass, too. And the mass parameter m and NUT parameter n are on an equal footing.

Taub-NUT spacetime

- Now we are in the stage of introducing a systematic way of defining and thus calculating the mass of Taub-NUT spacetime by generalizing the Komar integration.

- Through Wald formalism, we can get a 2-form potential with respect to the Killing vector ξ for constant t

$$\mathbf{Q}[\xi] = \mathbf{V}(\mathbf{r})\mathbf{\Omega}_{(2)} + \mathbf{2n}\cos\theta\mathbf{U}(\mathbf{r})\mathbf{dr}\wedge\mathbf{d}\phi$$

$$\mathbf{\Omega}_{(2)} = \sin\theta\mathbf{d}\theta\wedge\mathbf{d}\phi \qquad \mathbf{V} = (\mathbf{r}^2+\mathbf{n}^2)\mathbf{f}' \qquad \mathbf{U} = \frac{\mathbf{2nf}}{\mathbf{r}^2+\mathbf{n}^2}$$

- The Komar 2-forms are closed due to the integrability condition

$$\mathbf{V} + \mathbf{2nU}' = \mathbf{0}$$

Taub-NUT spacetime

- Then we can define mass through the integration

$$\mathbf{M(r)} = \frac{\mathbf{1}}{\mathbf{8\pi}}\int \mathbf{d\phi}\left(\int_0^\pi \mathbf{V d\theta} + \int_{\mathbf{r_+}}^{\mathbf{r}} \mathbf{2n}\cos\theta \mathbf{U(r')}\Big|_{\theta=0}^{\theta=\pi} \mathbf{dr'}\right) = \sqrt{m^2+n^2}$$

- On the other hand, we can turn to coordinate ϕ, for constant ϕ, the Komar 2-form takes

$$\mathbf{Q[\xi] = U(r) dr \wedge dt}$$

- The integrability condition is obvious

$$\partial_\theta \mathbf{U = 0}$$

- Integrating out the r coordinate, gives a u-invariant quantity

$$\int_{\mathbf{r_+}}^{\infty} \mathbf{U(r')dr'} = \frac{\mathbf{n}}{\mathbf{r_+}}$$

- Which we define as the NUT charge.

Taub-NUT spacetime

- Together with the temperature and entropy

$$\mathbf{T} = \frac{\mathbf{1}}{\mathbf{4\pi r_+}} \qquad \mathbf{S} = \pi(\mathbf{r}_+^2 + \mathbf{n}^2)$$

- The first law is given by

$$\delta \mathbf{M} = \mathbf{T}\delta\mathbf{S} + \mathbf{\Phi_N}\delta\mathbf{Q_N}$$

with NUT potential as

$$\mathbf{\Phi_N} = \frac{\mathbf{n}}{\mathbf{2}}$$

- And the Smarr relation is

$$\mathbf{M} = \mathbf{2TS}$$

- Till now, all the thermodynamical quantities in the first law are independently calculated, except the NUT potential.

A useful formula

- A metric is cohomogeneity 2, depending on two coordinates (r, u). For simplicity, we focus on the relevant 1-form

$$\Xi = X(r,u)\,du - Y(r,u)\,dr$$

We assume that it is closed, $d\Xi = 0$, which implies that there exist a scalar quantity such that $\Xi = d\Upsilon$, therefore we integrability conditions

$$X = \partial_u \Upsilon \qquad Y = -\partial_r \Upsilon$$

- Then there exist two u-independent quantities

$$-\left(\int_{r_+}^{\infty} dr'\, Y(r',u) + \int_{\pm 1}^{u} du'\, X(r,u') \Big|_{r_+}^{\infty} \right) = \Upsilon(u=\pm 1)\Big|_{\infty}^{r_+}$$

- And one r-independent quantity

$$\int_{-1}^{+1} du X(r,u) + \int_{r_+}^{r} dr'\, Y(r',u) \Big|_{u=-1}^{u=+1} = \Upsilon(r_+)\Big|_{u=-1}^{u=1}$$

Kerr-Taub-NUT spacetime

- The Kerr-Taub-NUT metric is

$$\mathbf{ds^2 = (r^2+v^2)(\frac{dr^2}{\Delta_r}+\frac{du^2}{1-u^2})+\frac{1}{r^2+v^2}((1-u^2)e_1^2+\Delta_r e_2^2)}$$

$$\mathbf{e_1 = adt-(r^2+a^2+n^2)d\phi \qquad e_2 = dt+(2nu-a(1-u^2))d\phi}$$

$$\mathbf{\Delta_r = r^2-2mr+a^2-n^2 \qquad v = n+au\,,\, u = \cos\theta \in [-1,1]}$$

- There are three integration constants (m, n, a)

- The metric has three Killing horizons, too, with three Killing vectors

$$\xi = \partial_t + \Omega_+\partial_\phi \qquad \Omega_+ = \frac{a}{r_+^2+n^2+a^2}$$

$$l_\pm = \partial t \pm \frac{1}{2n}\partial_\phi$$

Kerr-Taub-NUT spacetime

- We can write the Killing vectors on the poles as

$$\mathbf{l}_\pm = \partial_\phi \mp \mathbf{2n}\partial\mathbf{t} \qquad \leftarrow \qquad \mathbf{l}_\pm = \partial\mathbf{t} \pm \frac{\mathbf{1}}{\mathbf{2n}}\partial_\phi$$

- In this expression n has smooth limit to zero.

Comparing with the Killing vector on the event horizon

$$\mathbf{l}_\pm = \partial_\phi \mp \mathbf{2n}\partial\mathbf{t} \qquad \xi = \partial_\mathrm{t} + \mathbf{\Omega}_+\partial_\phi$$

- We find a correspondence

$$\mathbf{t} \leftrightarrow \phi \qquad \mathbf{\Omega}_+ \leftrightarrow \mathbf{n}$$

together with $r \leftrightarrow u$ suggest a duality.

From this view, we take the NUT parameter n as a potential, like angular momentum Ω_+, not as an conserved charge. It agrees with the result in the previous Taub-NUT case.

Kerr-Taub-NUT spacetime

- Following the similar procedure, we can calculate the mass and angular momentum through the Komar 2-forms of constant t with ∂_t and ∂_ϕ

$$\mathbf{Q[\partial_t] = V(r,u)\Omega_2 + U(r,u)dr \wedge d\phi}$$

$$\mathbf{Q[\partial_\phi] = X(r,u)\Omega_2 + Y(r,u)dr \wedge d\phi}$$

$$V = \frac{2(r^2+a^2+n^2)(2nrv+m(r^2-v^2))}{(r^2+v^2)^2} \quad U = \frac{2(a^2+n^2-v^2)(2mrv-n(r^2-v^2))}{a(r^2+v^2)^2}$$

$$\begin{aligned} X &= \frac{2}{a\left(r^2+v^2\right)^2}\Big(m\left(a^2+n^2-v^2\right)\left(a^2\left(v^2-r^2\right)+n^2\left(v^2-r^2\right)-r^2\left(3r^2+v^2\right)\right) \\ &\quad -2nr\left(2n^2v\left(a^2+r^2\right)+v\left(a^2+r^2\right)^2+n^4v-n\left(r^2+v^2\right)^2\right)\Big), \\ Y &= \frac{2}{a^2\left(r^2+v^2\right)^2}\Big(n(n-v)^2\left(n^2\left(r^2-v^2\right)+2nv\left(r^2-v^2\right)+r^4+3r^2v^2\right) \\ &\quad +n\left(a^4\left(r^2-v^2\right)+a^2\left(2n^2\left(r^2-v^2\right)+r^4+3v^4\right)\right)-2mrv\left(a^2+n^2-v^2\right)^2\Big) \end{aligned}$$

Kerr-Taub-NUT spacetime

- The Komar 2-forms are closed due to

$$\partial_\mathbf{r}\mathbf{V} + \partial_\mathbf{u}\mathbf{U} = \mathbf{0} \qquad \partial_\mathbf{r}\mathbf{X} + \partial_\mathbf{u}\mathbf{Y} = \mathbf{0}$$

- The mass and angular momentum are

$$\mathcal{M}(r) \equiv \frac{\int d\phi}{8\pi}\Big(\int_{-1}^{+1} V(r,u)du + \int_{r_+}^{r} U(r',u)\Big|_{u=-1}^{u=+1} dr'\Big) = M = m + \frac{n^2}{r_+}$$

$$\mathcal{J}(r) \equiv -\frac{\int d\phi}{16\pi}\Big(\int_{-1}^{+1} X(r,u)du + \int_{r_+}^{r} Y(r',u)\Big|_{u=-1}^{u=+1} dr'\Big) = Ma$$

- For constant ϕ, the Komar 2-form is

$$\mathbf{Q}[\partial_\mathbf{t}] = (\zeta(\mathbf{r},\mathbf{u})\mathbf{dr} - \eta(\mathbf{r},\mathbf{u})\mathbf{du}) \wedge \mathbf{dt}$$

$$\zeta = \frac{2\left(n\left(r^2 - v^2\right) - 2mrv\right)}{\left(r^2 + v^2\right)^2} \qquad \eta = -\frac{2a\left(m\left(r^2 - v^2\right) + 2nrv\right)}{\left(r^2 + v^2\right)^2}$$

Kerr-Taub-NUT spacetime

- The integrability condition is

$$\partial_{\mathbf{u}}\zeta + \partial_{\mathbf{r}}\eta = \mathbf{0}$$

- Integrating out the r coordinate associated with Misner singularity at the two poles gives

$$\mathbf{Q}_{\mathbf{N}}^{\pm} = \frac{\mathbf{1}}{\mathbf{2}}\left(\int_{\mathbf{r}_+}^{\infty}\zeta(\mathbf{r}',\mathbf{u})\mathbf{d}\mathbf{r}' + \int_{\pm\mathbf{1}}^{\mathbf{u}}\eta(\mathbf{r},\mathbf{u}')|_{\mathbf{r}_+}^{\infty}\right) = \frac{\mathbf{n}\mp\mathbf{a}}{\mathbf{2}\mathbf{r}_+}$$

- Since the potential is the same for the two poles, the NUT charge is

$$\mathbf{Q}_{\mathbf{N}} = \mathbf{Q}_{\mathbf{N}}^{+} + \mathbf{Q}_{\mathbf{N}}^{-} = \frac{\mathbf{n}}{\mathbf{r}_+}$$

Kerr-Taub-NUT spacetime

- Together with the temperature and entropy

$$\mathbf{T} = \frac{\mathbf{r}_+^2 + \mathbf{n}^2 - \mathbf{a}^2}{4\pi\mathbf{r}_+(\mathbf{r}_+^2 + \mathbf{n}^2 + \mathbf{a}^2)} \qquad \mathbf{S} = \pi(\mathbf{r}_+^2 + \mathbf{a}^2 + \mathbf{n}^2)$$

- The first law is automatically satisfied

$$\delta\mathbf{M} = \mathbf{T}\delta\mathbf{S} + \boldsymbol{\Omega}_+\mathbf{J} + \boldsymbol{\Phi}_\mathbf{N}\delta\mathbf{Q}_\mathbf{N}$$

- The Smarr relation is

$$\mathbf{M} = \mathbf{2}(\mathbf{TS} + \boldsymbol{\Omega}_+\mathbf{J})$$

- All the thermodynamical quantities are calculated independently.

Asymmetric Misner Strings

- We can make a linear coordinate transformation

$$\mathbf{t \to t - 2n\alpha\phi} \qquad \phi \to \phi$$

- α is a constant.
- The Kerr-Taub-NUT metric turns out to be

$$\mathbf{ds^2 = (r^2+v^2)(\frac{dr^2}{\Delta_r} + \frac{du^2}{1-u^2}) + \frac{1}{r^2+v^2}((1-u^2)e_1^2 + \Delta_r e_2^2)}$$

$$\mathbf{e_1 = adt - (r^2+a^2+n^2+2na\alpha)d\phi \, e_2 = dt + (2nu - 2n\alpha - a(1-u^2))d\phi}$$

$$\mathbf{\Delta_r = r^2 - 2mr + a^2 - n^2} \qquad \mathbf{v = n + au\,,\, u = \cos\theta \in [-1,1]}$$

- The Killing vectors at the poles change to

$$\mathbf{l_\pm = \partial_\phi \mp 4\Phi_N^\pm \partial_t} \qquad \mathbf{\Phi_N^\pm = \frac{1}{2}n(1 \pm \alpha)}$$

- Thus the NUT potentials is changed.

Asymmetric Misner Strings

- The rest thermodynamical quantities can be calculated through the similar procedure

$$M = m + 2\Phi_N^+ Q_N^+ + 2\Phi_N^- Q_N^- \qquad J = M\Big(a + (\Phi_N^- - \Phi_N^+)\Big),$$

$$T = \frac{r_+^2 + n^2 - a^2}{4\pi r_+(r_+^2 + n^2 + a^2 + 2\alpha n a)} \qquad S = \pi(r_+^2 + n^2 + a^2 + 2\alpha n a)$$

$$\Omega_+ = \frac{a}{r_+^2 + n^2 + a^2 + 2\alpha n a} \qquad \Phi_N^\pm = \tfrac{1}{2} n(1 \pm \alpha) \qquad Q_N^\pm = \frac{n \mp a}{2r_+}$$

- The first law is

$$\delta \mathbf{M} = \mathbf{T}\delta\mathbf{S} + \mathbf{\Omega}_+ \delta\mathbf{J} + \mathbf{\Phi}_N^+ \delta\mathbf{Q}_N^+ + \mathbf{\Phi}_N^- \delta\mathbf{Q}_N^-$$

- α doesn't contribute to the first law directly.
- The Smarr relation is unchanged $\mathbf{M} = 2(\mathbf{TS} + \mathbf{\Omega}_+\mathbf{J})$
- Comment: when $a = 0, \Omega_+ = 0$, but the angular momentum doesn't vanish

$$\mathbf{J} = \mathbf{M}(\mathbf{\Phi}_N^- - \mathbf{\Phi}_N^+)$$

The Plebanski Solution

- Theory

$$\mathcal{L} = \sqrt{-g}(\mathrm{R} - \mathbf{F}^2) \qquad \mathbf{F}_{(2)} = \mathbf{dA}_{(1)}$$

- Solution in original form

$$\begin{aligned}
ds^2 &= (p^2+q^2)\left(\frac{dp^2}{P(p)} + \frac{dq^2}{Q(q)}\right) + \frac{1}{p^2+q^2}\left(P(p)\sigma_q^2 - Q(q)\sigma_p^2\right) \\
A_{(1)} &= \frac{1}{p^2+q^2}(eq\sigma_p + gp\sigma_q) \qquad B_{(1)} = \frac{1}{p^2+q^2}(gq\sigma_p - ep\sigma_q) \\
P(p) &= b - g^2 + 2np - \epsilon p^2 \qquad Q(q) = b + e^2 - 2mq + \epsilon q^2 \\
\sigma_p &= d\tau - p^2 d\sigma \qquad \sigma_q = d\tau + q^2 d\sigma
\end{aligned}$$

- The solution has six integration constants $(m, n, e, g, b, \epsilon)$.

The Plebanski Solution

- Solution in our familiar form

$$\mathbf{ds^2 = (r^2 + v^2)(\frac{dr^2}{\Delta_r} + \frac{du^2}{1 - u^2}) + \frac{1}{r^2 + v^2}((1 - u^2)e_1^2 + \Delta_r e_2^2)}$$

$$A_{(1)} = \frac{g(v-n)}{a\left(r^2+v^2\right)} e_1 + \frac{(er+gn)}{r^2+v^2} e_2 \qquad B_{(1)} = \frac{e(n-v)}{a\left(r^2+v^2\right)} e_1 + \frac{(gr-en)}{r^2+v^2} e_2$$

$$\mathbf{e_1 = adt - (r^2 + a^2 + n^2)d\phi} \qquad \mathbf{e_2 = dt + (2nu - a(1 - u^2))d\phi}$$

$$\mathbf{\Delta_r = r^2 - 2mr + a^2 - n^2 + e^2 + g^2}$$

The Plebanski Solution

- Thermodynamics

$$T=\frac{r_+^2+n^2-a^2-e^2-g^2}{4\pi r_+(r_+^2+a^2+n^2)} \qquad S=\pi(r_+^2+a^2+n^2)$$

$$M=m+nQ_{\mathrm{N}} \qquad \Omega_+=\frac{a}{r_+^2+a^2+n^2} \qquad J=Ma$$

$$Q_{\mathrm{N}}=\frac{n}{r_+}\left(1-\frac{(e^2+g^2)(r_+^2+n^2-a^2)}{(r_+^2+n^2+a^2)^2-4n^2a^2}\right) \qquad \Phi_N=\frac{n}{2}$$

$$Q_e=e+4\Phi_{\mathrm{N}}^+Q_{e\mathrm{N}}^+ +4\Phi_{\mathrm{N}}^-Q_{e\mathrm{N}}^- \qquad \Phi_e=\frac{er_++ng}{r_+^2+a^2+n^2}$$

$$Q_g=g-4\Phi_{\mathrm{N}}^+Q_{g\mathrm{N}}^+ -4\Phi_{\mathrm{N}}^-Q_{g\mathrm{N}}^- \qquad \Phi_g=\frac{gr_+-ne}{r_+^2+a^2+n^2}$$

$$Q_{e\mathrm{N}}^{\pm}=\frac{gr_+-e(n\pm a)}{2(r_+^2+(a\pm n)^2)} \qquad \Phi_{e\mathrm{N}}^{\pm}=-\frac{n(er_++g(n\mp a))}{r_+^2+(n\mp a)^2}$$

$$Q_{g\mathrm{N}}^{\pm}=\frac{er_++g(n\pm a)}{2(r_+^2+(a\pm n)^2)} \qquad \Phi_{g\mathrm{N}}^{\pm}=\frac{n(gr_+-e(n\mp a))}{r_+^2+(n\mp a)^2}$$

- First law

$$\begin{aligned}\delta M \;=\;& T\delta S+\Omega_+\delta J+\Phi_e\delta Q_e+\Phi_g\delta Q_g\\ &+\Phi_{\mathrm{N}}\delta Q_{\mathrm{N}}+\Phi_{e\mathrm{N}}^+\delta Q_{e\mathrm{N}}^+ +\Phi_{e\mathrm{N}}^-\delta Q_{e\mathrm{N}}^- +\Phi_{g\mathrm{N}}^+\delta Q_{g\mathrm{N}}^+ +\Phi_{g\mathrm{N}}^-\delta Q_{g\mathrm{N}}^-\end{aligned}$$

Conclusion

- We introduced a systematic way of defining Mass and NUT charges for NUTty spacetimes. Together with other thermodynamical quantities calculated through standard method, the first law and the Smarr relation are automatically satisfied. Thus our approach is not ad hoc.

- We believe we give the right definition of NUT charge and then obtain the consist first law. However, it is not an end, but a beginning. The physical meaning of NUT charge and the NUT induced electromagnetic charges are still under exploring.

Thank you!