大质量比系统(EMRI)

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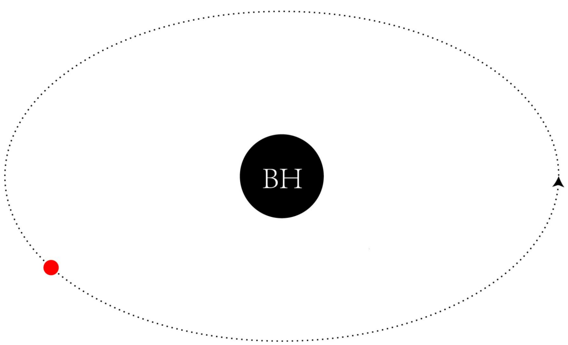
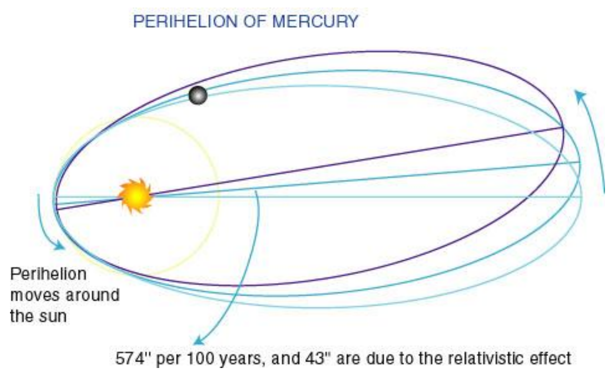
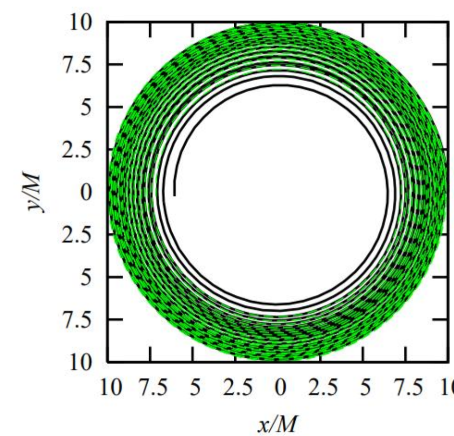
Outline

- 引力波及极端质量比系统介绍
- 中等/极端质量比系统探测暗物质
- 中等/极端质量比系统检验引力理论
- 总结



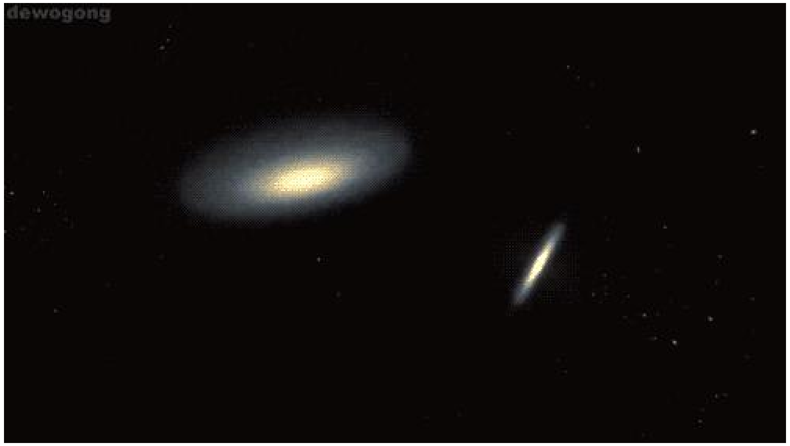


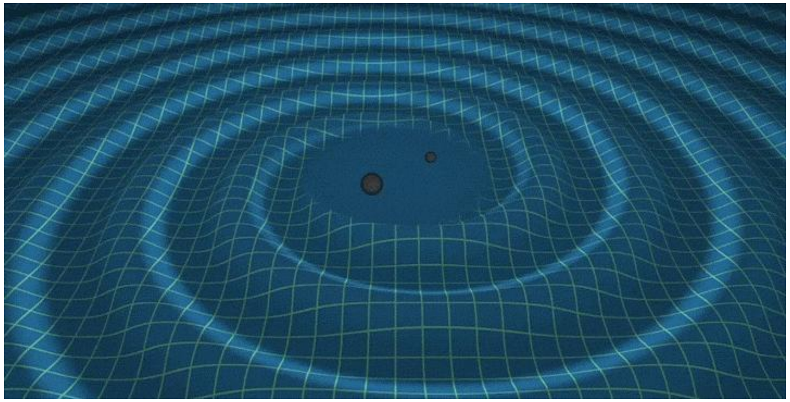


N. Dai, Y. Gong, T. Jiang & D. Liang, PRD 106 (2022) 064033; N. Dai, Y. Gong, Y. Zhao, T. Jiang, 2301.05088; T. Jiang, N. Dai, Y. Gong, D. Liang & C. Zhang, JCAP 2212, 023;

引力波

- 引力波的发现开启了检验引力理论的新时代
  - 聆听宇宙中声音探究宇宙奥秘
  - 强场及非线性区域检验引力理论特性
  - 开启了多信使时代





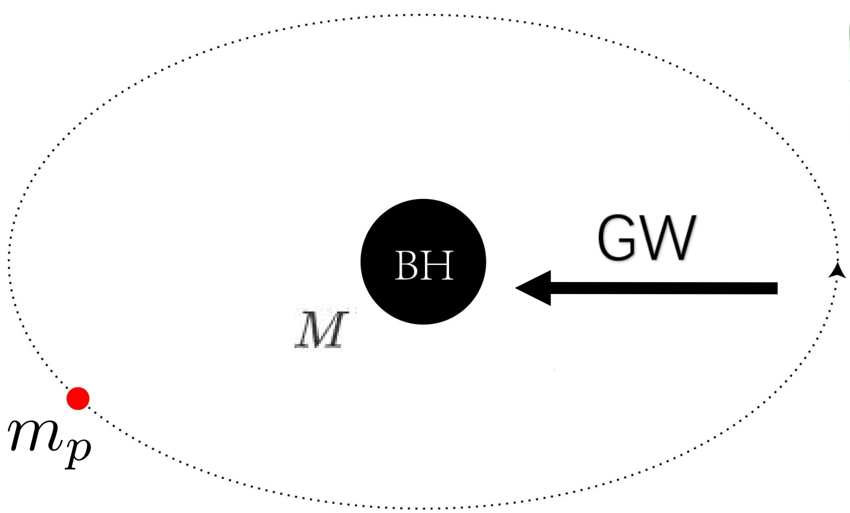
Extreme mass ratio inspirals

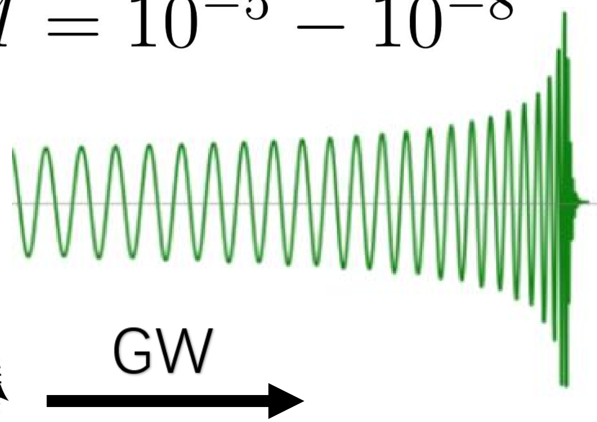
中等质量比系统(Intermediate mass ratio inspirals (IMRI))

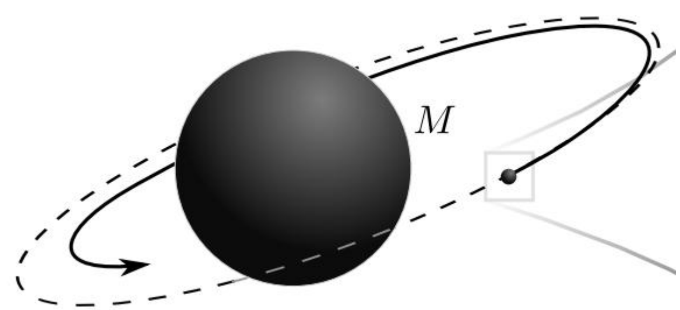
$$m_p/M = 10^{-2} - 10^{-4}$$

极端质量比系统(Extreme mass ratio inspirals (EMRI))

$$m_p/M = 10^{-5} - 10^{-8}$$





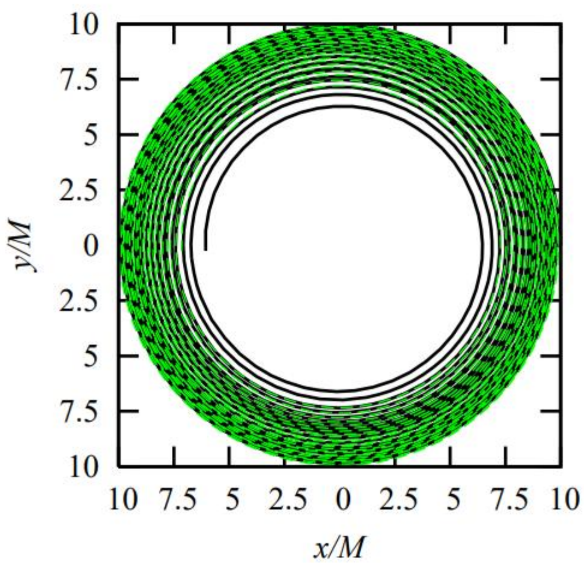


I/EMRI系统

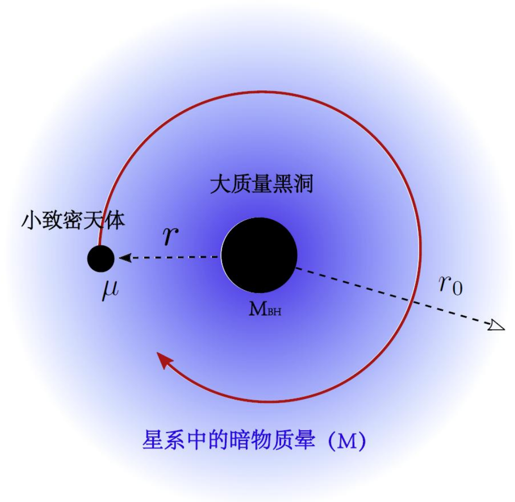
■ 中等/极端质量比系统

- 探测时空几何、黑洞环境、黑洞数目分布
- 检验黑洞无毛定理，额外荷
- 强场及非线性区域检验引力理论
- 波形计算资源消耗巨大

成千上万个周期



精确描绘黑洞周围几何及环境



暗物质环境

- 暗物质环境中等质量：暗物质minispike
- 中等质量比系统检验暗物质minispike

$$\rho_{\rm DM}(r) = \begin{cases} \rho_{\rm sp} \left(\frac{r_{\rm sp}}{r}\right)^{\alpha}, & r_{\rm min} \le r \le r_{\rm sp}, \\ 0, & r \le r_{\rm min}, \end{cases}$$

$$M_{\rm DM} = \begin{cases} \frac{4\pi \rho_{\rm sp} r_{\rm sp}^{\alpha}}{3-\alpha} \left(r^{3-\alpha} - r_{\rm min}^{3-\alpha}\right), & r_{\rm min} \le r \le r_{\rm sp}, \\ 0, & r \le r_{\rm min}. \end{cases}$$

$$\alpha \neq 3$$

K. Eda, Y. Itoh, S. Kuroyanagi and J. Silk, PRL 110 (2013) 221101; PRD 91, 044045 (2015)

暗物质效应

- 牛顿轨道运动

$$\boldsymbol{a}_{\mathrm{G}} = -\frac{GM_{\mathrm{eff}}}{r^2}\boldsymbol{n} - \frac{GF}{r^{\alpha-1}}\boldsymbol{n}$$

$$M_{\mathrm{eff}} = M - 4\pi\rho_{\mathrm{sp}}(r_{\mathrm{sp}})^{\alpha}{r_{\min}}^{3-\alpha}/(3-\alpha)$$

$$F = 4\pi\rho_{\mathrm{sp}}(r_{\mathrm{sp}})^{\alpha}/(3-\alpha)$$

- 动力学摩擦

$$\boldsymbol{f}_{\mathrm{DF}} = -\frac{4\pi G^2\mu^2\rho_{\mathrm{DM}}I_v}{v^3}\boldsymbol{v}, \quad I_v = 3$$

S. Chandrasekhar, ApJ 97 (1943) 255;
V. Cardoso etal, PRD 103 (2021) 023015

- 小黑洞吸积

$$\boldsymbol{f}_{\mathrm{a}} \simeq -\frac{4\pi G^2\mu^2\rho_{\mathrm{DM}}\lambda}{v^3}\boldsymbol{v}, \quad \lambda = 1$$

C. F. B. Macedo etal., ApJ 774 (2013) 48;
P. Mach & A. Odrzywo, PRL 126 (2021) 101104

- 辐射反冲

$$\boldsymbol{a}_{\mathrm{GW}} = \frac{8}{5}\frac{G^2M\mu}{c^5r^3}\left[\left(3v^2 + \frac{17}{3}\frac{Gm}{r}\right)\dot{r}\boldsymbol{n} - \left(v^2 + 3\frac{Gm}{r}\right)\boldsymbol{v}\right]$$

E. Poisson & C. M. Will, Gravity: $\cdots$(Cambridge University Press, 2014).

密切轨道方法

- 轨道方程

$$\boldsymbol{a}_{\mathrm{G}} = -\frac{GM_{\mathrm{eff}}}{r^2}\boldsymbol{n} - \frac{GF}{r^{\alpha-1}}\boldsymbol{n}$$

开普勒轨道　扰动

- 密切(osculating)轨道方法：常数变易法

受迫简谐振子 $\ddot{x} + x = g(t)$

先求解 $\ddot{x} + x = 0$

$$x = x_0 \cos t + v_0 \sin t, \quad \dot{x} = v_0 \cos t - x_0 \sin t$$

常数变易法 $x_0 = x_0(t), \quad v_0 = v_0(t)$

$$x = x_0(t) \cos t + v_0(t) \sin t, \quad \dot{x} = v_0(t) \cos t - x_0(t) \sin t$$

密切条件一 $\dot{x}_0 \cos t + \dot{v}_0 \sin t = 0$

密切轨道方法

- 非齐次方程解 $\ddot{x} + x = g(t)$

$$x = x_0(t)\cos t + v_0(t)\sin t$$

密切条件 $\dot{x}_0 \cos t + \dot{v}_0 \sin t = 0$

密切条件 $-\dot{x}_0 \sin t + \dot{v}_0 \cos t = g(t)$

$$\dot{x}_0 = -g(t)\sin t, \quad \dot{v}_0 = g(t)\cos t$$

- 牛顿引力

$$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n} \qquad \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}_{\mathrm{K}}}{dt} = \boldsymbol{v}_{\mathrm{K}}$$

$$\boldsymbol{r} = \boldsymbol{r}_{\mathrm{K}}(t, \mu^\alpha), \; \boldsymbol{v} = \boldsymbol{v}_{\mathrm{K}}(t, \mu^\alpha)$$ 轨道参数：p, e

$$r(t) = \frac{p}{1 + e\cos\varphi(t)}, \quad \dot{r} = \sqrt{\frac{Gm}{p}}\, e\sin\varphi(t)$$

密切轨道方法

- 牛顿引力 $$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n} + \boldsymbol{f}$$ 扰动

$$\boldsymbol{r} = \boldsymbol{r}_{\mathrm{K}}(t, \mu^\alpha(t)), \ \boldsymbol{v} = \boldsymbol{v}_{\mathrm{K}}(t, \mu^\alpha(t))$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}_{\mathrm{K}}}{\partial t} + \sum_\alpha \frac{\partial \boldsymbol{r}_{\mathrm{K}}}{\partial \mu^\alpha}\frac{d\mu^\alpha}{dt} = \boldsymbol{v}_{\mathrm{K}}$$

密切条件一 $$\sum_\alpha \frac{\partial \boldsymbol{r}_{\mathrm{K}}}{\partial \mu^\alpha}\frac{d\mu^\alpha}{dt} = 0$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}_{\mathrm{K}}}{\partial t} + \sum_\alpha \frac{\partial \boldsymbol{v}_{\mathrm{K}}}{\partial \mu^\alpha}\frac{d\mu^\alpha}{dt} = \boldsymbol{a}$$

密切条件二 $$\sum_\alpha \frac{\partial \boldsymbol{v}_{\mathrm{K}}}{\partial \mu^\alpha}\frac{d\mu^\alpha}{dt} = \boldsymbol{f}$$

密切轨道方法

- 微扰力 $\boldsymbol{f} = \mathcal{R}\boldsymbol{n} + \mathcal{S}\boldsymbol{\lambda} + \mathcal{W}\boldsymbol{e}_z$ $\quad \boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n} + \boldsymbol{f}$

$$r(t) = \frac{p(t)}{1 + e(t)\cos\varphi(t)}, \qquad \dot{r} = \sqrt{\frac{Gm}{p(t)}}\, e(t)\sin\varphi(t)$$

$$\frac{dp}{dt} = 2\sqrt{\frac{p^3}{Gm}}\frac{1}{1+e\cos\varphi}\mathcal{S},$$

$$\frac{de}{dt} = \sqrt{\frac{p}{Gm}}\left[\sin\varphi\mathcal{R} + \frac{2\cos\varphi + e(1+\cos^2\varphi)}{1+e\cos\varphi}\mathcal{S}\right],$$

$$\frac{d\varphi}{dt} = \sqrt{\frac{Gm}{p^3}}(1+e\cos\varphi)^2 + \frac{1}{e}\sqrt{\frac{p}{Gm}}\left[\cos\varphi\mathcal{R} - \frac{2+e\cos\varphi}{1+e\cos\varphi}\sin\varphi\mathcal{S}\right].$$

中等质量比系统探测暗物质

- 暗物质引力效应

$$\boldsymbol{a}_{\mathrm{G}} = -\frac{GM_{\mathrm{eff}}}{r^2}\boldsymbol{n} - \frac{GF}{r^{\alpha-1}}\boldsymbol{n}$$

$$\boldsymbol{f}_{\mathrm{G}} = -\frac{GF}{r^{\alpha-1}}\boldsymbol{n} = \mathcal{R}_{\mathrm{G}}\boldsymbol{n}$$

$$\frac{dp}{d\phi} = 0,$$

$$\frac{de}{d\phi} = -\frac{p^{3-\alpha}F}{M_{\mathrm{eff}}}\frac{\sin\phi}{(1+e\cos\phi)^{3-\alpha}},$$

$$\frac{d\omega}{d\phi} = \frac{p^{3-\alpha}F}{M_{\mathrm{eff}}}\frac{\cos\phi}{e(1+e\cos\phi)^{3-\alpha}},$$

Longitude of pericenter

$$\frac{dt}{d\phi} = \sqrt{\frac{p^3}{GM_{\mathrm{eff}}}}\frac{1}{(1+e\cos\phi)^2} \times \left[1 + \frac{p^{3-\alpha}F\cos\phi}{M_{\mathrm{eff}}\,e(1+e\cos\phi)^{3-\alpha}}\right]$$

暗物质效应

- 一个周期累积效应

$$\boldsymbol{f}_{\rm G} = -\frac{GF}{r^{\alpha-1}}\boldsymbol{n}$$

$$\Delta e = 0 \qquad \Delta\omega_{\rm DM} = \frac{p^{3-\alpha}F}{M_{\rm eff}}W_{\rm DM}(e),$$

反向进动

$$W_{\rm DM}(e) = \int_0^{2\pi} \cos\phi(1+e\cos\phi)^{\alpha-3}e^{-1}d\phi < 0$$

$$0 < e < 1, \ \alpha < 3$$

广义相对论进动效应 $V_{\rm eff} = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}$

$$\Delta\omega_{\rm rp} \simeq \frac{6\pi GM_{\rm eff}}{c^2 p} + \frac{GF}{c^2}p^{2-\alpha}W_{\rm rp}(e) > 0,$$

正向进动

$$W_{\rm rp}(e) = \int_0^{2\pi} \frac{(3-e^2)\cos\phi - 5e\cos 2\phi + 3e}{e(1+e\cos\phi)^{3-\alpha}}d\phi$$

轨道进动

- 暗物质及相对论效应：远轨道暗物质效应明显

  轨道距离大，暗物质参数大

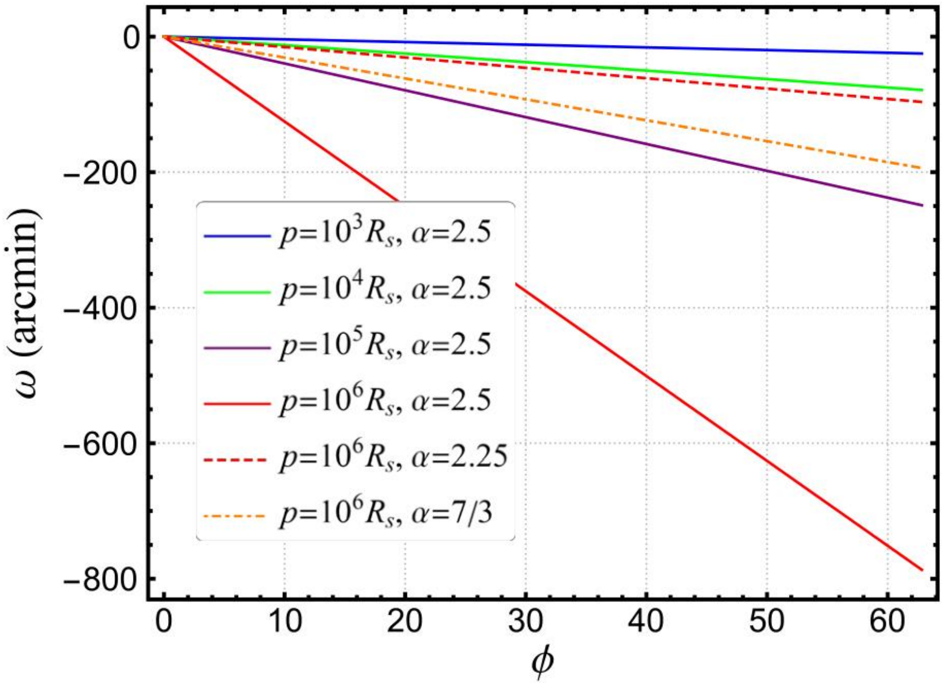


FIG. 1. The accumulated ω versus ϕ for different p and α under the influence of the gravity from the DM minispike. The eccentricity e is 0.6, and the values of α are chosen as 2.25, 7/3 and 2.5. The semilatus rectum p are chosen as 10^3R_s, 10^4R_s, 10^5R_s and 10^6R_s.

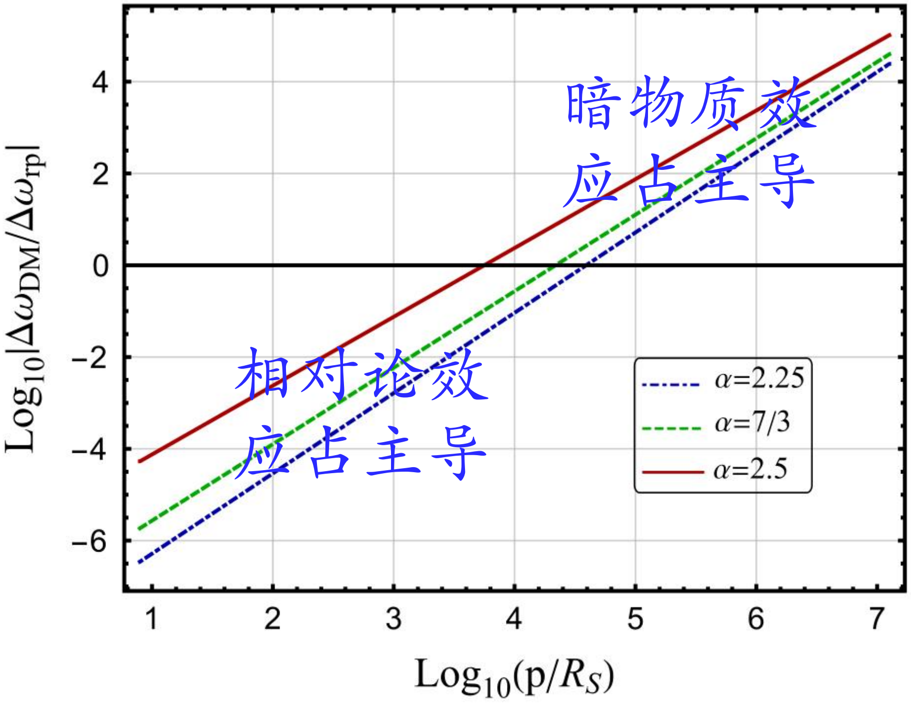


FIG. 2. The ratio $\Delta\omega_{\mathrm{DM}}/\Delta\omega_{\mathrm{rp}}$ versus the semi-latus rectum p in the range $3R_s$ to 10^8R_s for different values of α. We take the orbital eccentricity $e = 0.6$.

$$R_s = 2M_{\mathrm{BH}}$$

动力学摩擦效应

- 动力学摩擦 $$\boldsymbol{f}_{\rm DF} = -\frac{4\pi G^2\mu^2\rho_{\rm DM}I_v}{v^3}\boldsymbol{v}, \quad I_v = 3$$

$$\left\langle \frac{dp}{d\phi} \right\rangle_{\rm DF} = -\frac{4\mu\rho_{\rm sp}r_{\rm sp}^{\alpha}I_v}{M^2}p^{4-\alpha}g(e),$$ 轨道衰减

$$\left\langle \frac{de}{d\phi} \right\rangle_{\rm DF} = -\frac{4\mu\rho_{\rm sp}r_{\rm sp}^{\alpha}I_v}{M^2}p^{3-\alpha}f(e),$$ 偏心率变大

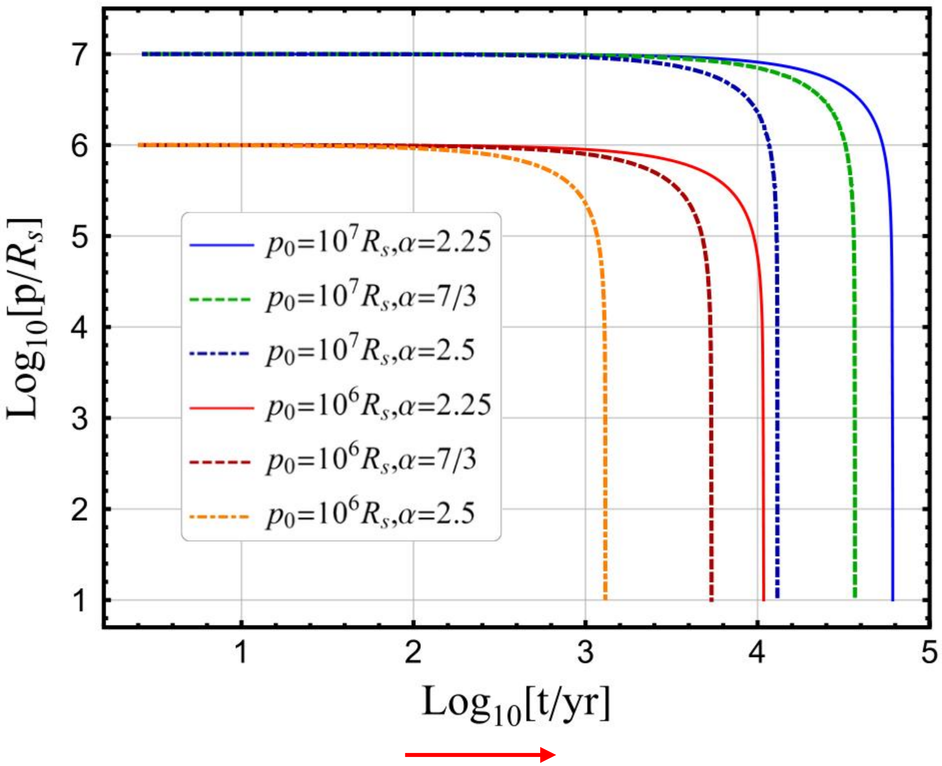
无进动 $$\left\langle \frac{d\omega}{d\phi} \right\rangle_{\rm DF} = 0, \qquad \left\langle \frac{dt}{d\phi} \right\rangle_{\rm DF} = \sqrt{\frac{p^3}{GM}}(1-e^2)^{-3/2},$$

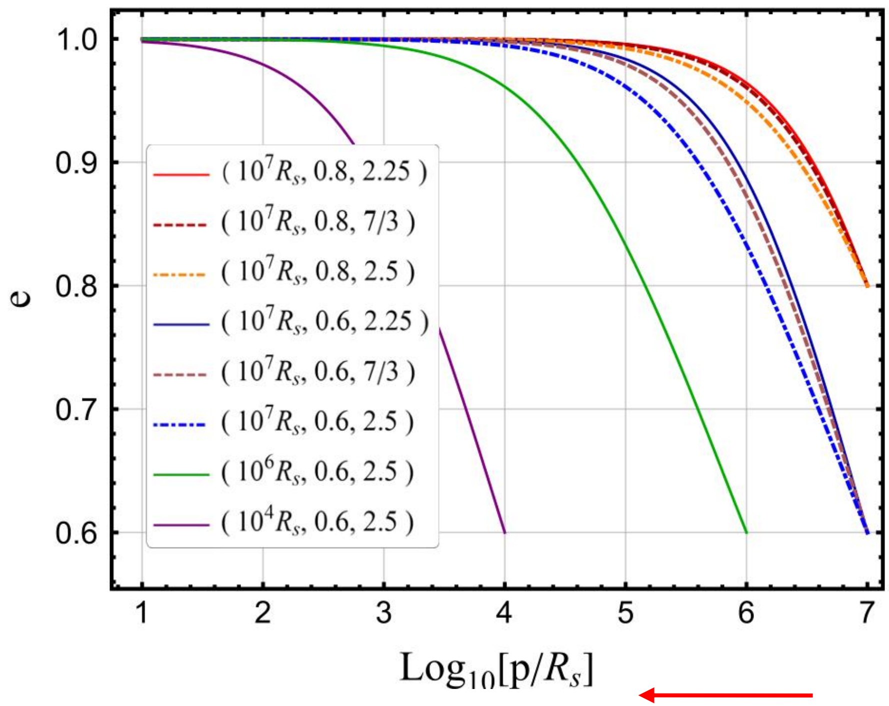
$$g(e) = \int_0^{2\pi} \frac{d\phi}{(1+2e\cos\phi+e^2)^{3/2}(1+e\cos\phi)^{2-\alpha}} > 0,$$

$$f(e) = \int_0^{2\pi} \frac{(\cos\phi+e)d\phi}{(1+2e\cos\phi+e^2)^{3/2}(1+e\cos\phi)^{2-\alpha}} < 0$$

动力学摩擦效应

- 摩擦效应：无进动，轨道减小，偏心率增加





N. Dai, Y. Gong, T. Jiang & D. Liang, PRD 106 (2022) 064033

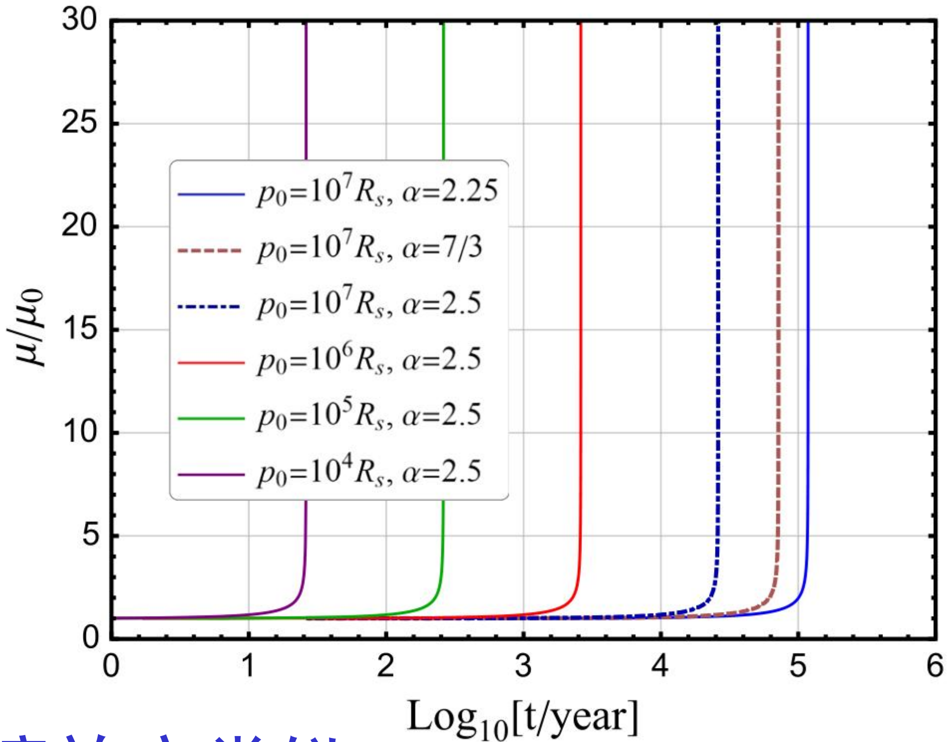
小黑洞吸积效应

- 吸积效应 $\boldsymbol{f}_{\mathrm{a}} \simeq -\frac{4\pi G^2\mu^2\rho_{\mathrm{DM}}\lambda}{v^3}\boldsymbol{v}, \quad \lambda = 1$

$$\left\langle \frac{dp}{d\phi} \right\rangle_{\mathrm{a}} = -\frac{4\mu\rho_{\mathrm{sp}}r_{\mathrm{sp}}^{\alpha}\lambda}{M^2}p^{4-\alpha}g(e),$$

$$\left\langle \frac{de}{d\phi} \right\rangle_{\mathrm{a}} = -\frac{4\mu\rho_{\mathrm{sp}}r_{\mathrm{sp}}^{\alpha}\lambda}{M^2}p^{3-\alpha}f(e),$$

$$\left\langle \frac{d\omega}{d\phi} \right\rangle_{\mathrm{a}} = 0$$



- 吸积效应与动力学摩擦效应类似

无进动，轨道减小，偏心率增加

辐射反冲效应

- 辐射 $$\boldsymbol{a}_{\rm GW}=\frac{8}{5}\frac{G^2M\mu}{c^5r^3}\left[\left(3v^2+\frac{17}{3}\frac{Gm}{r}\right)\dot{r}\boldsymbol{n}-\left(v^2+3\frac{Gm}{r}\right)\boldsymbol{v}\right]$$

四极矩结果

$$\left\langle\frac{dp}{d\phi}\right\rangle_{\rm GW}=-\frac{8}{5}\eta\frac{(Gm)^{5/2}}{c^5p^{3/2}}\left(8+7e^2\right),$$

$$\left\langle\frac{de}{d\phi}\right\rangle_{\rm GW}=-\frac{8}{5}\eta\frac{(Gm)^{5/2}}{c^5p^{5/2}}\left(\frac{304}{24}e+\frac{121}{24}e^3\right),$$

$$\left\langle\frac{d\omega}{d\phi}\right\rangle_{\rm GW}=0,$$

$$\left\langle\frac{dt}{d\phi}\right\rangle_{\rm GW}=\sqrt{\frac{p^3}{Gm}}(1-e^2)^{-3/2}$$

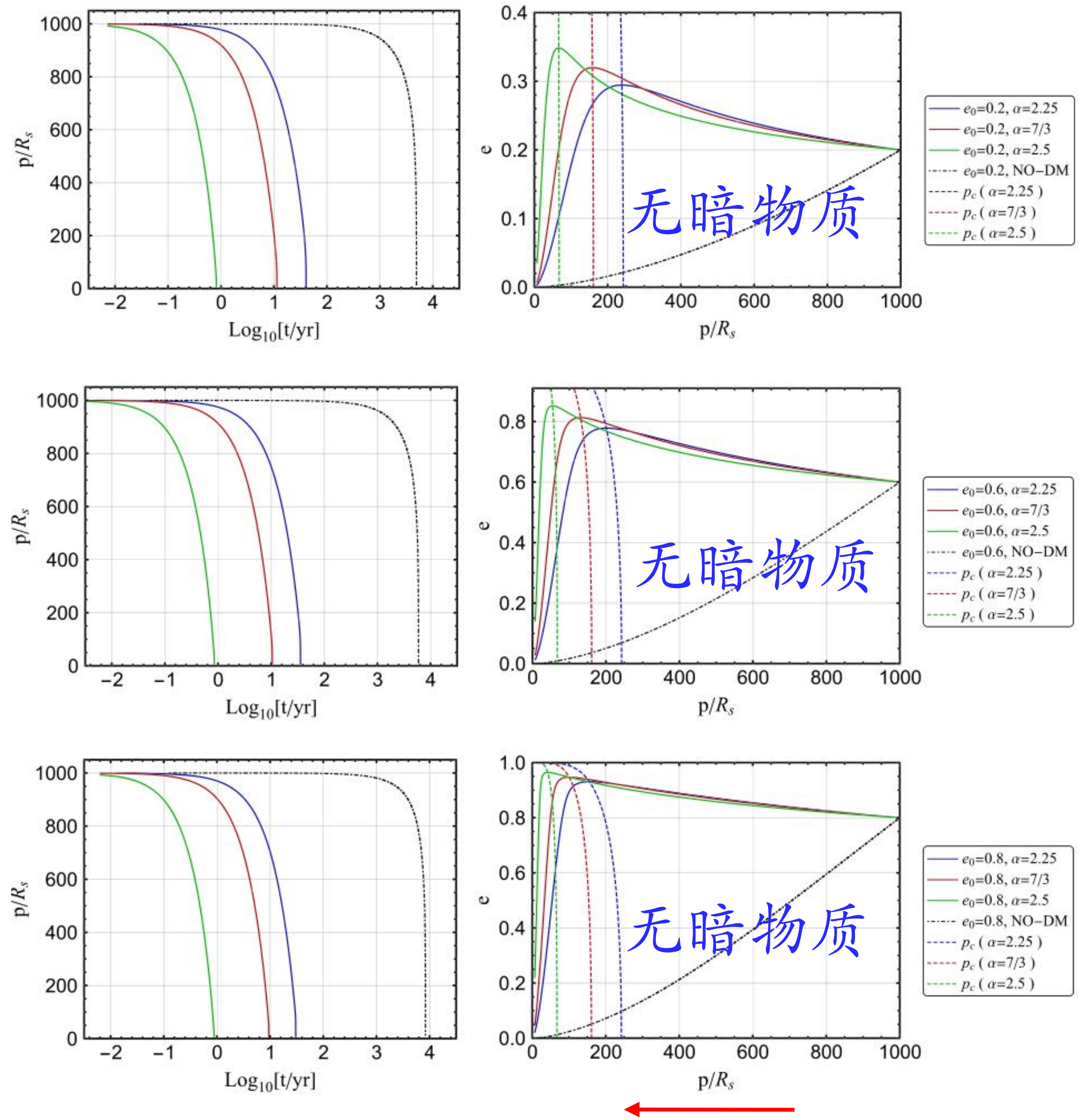
- 无进动、轨道衰减、偏心率减小

E. Poisson & C. M. Will, Gravity: ⋯(Cambridge University Press, 2014).

近距离总体效应

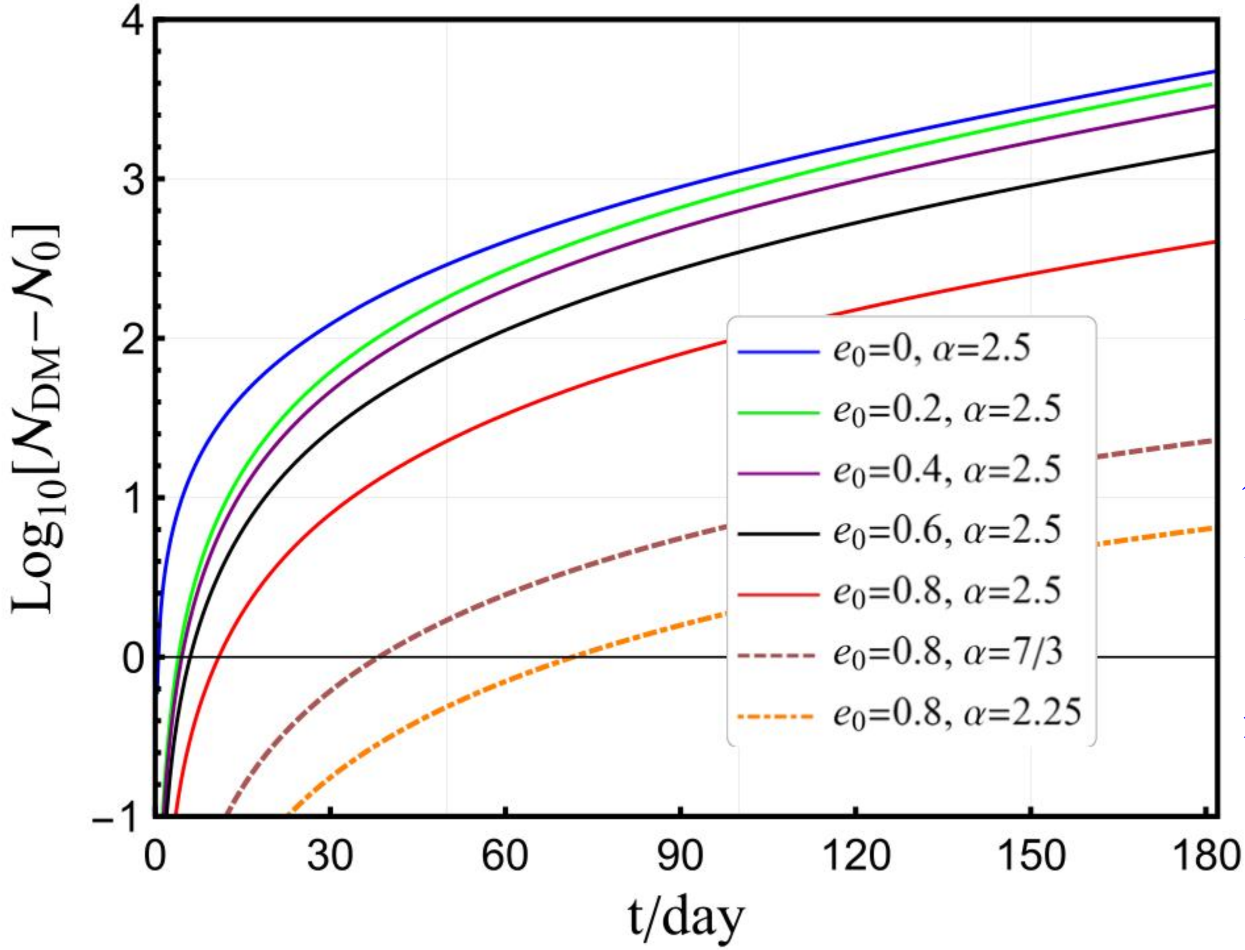
- 轨道演化

轨道半径减小
偏心率先增后减



暗物质对轨道演化影响

- 近距离轨道圈数累积差 $\mathcal{N} = \frac{1}{2\pi}\int_{t_\mathrm{i}}^{t_\mathrm{f}} f(t)\,dt$



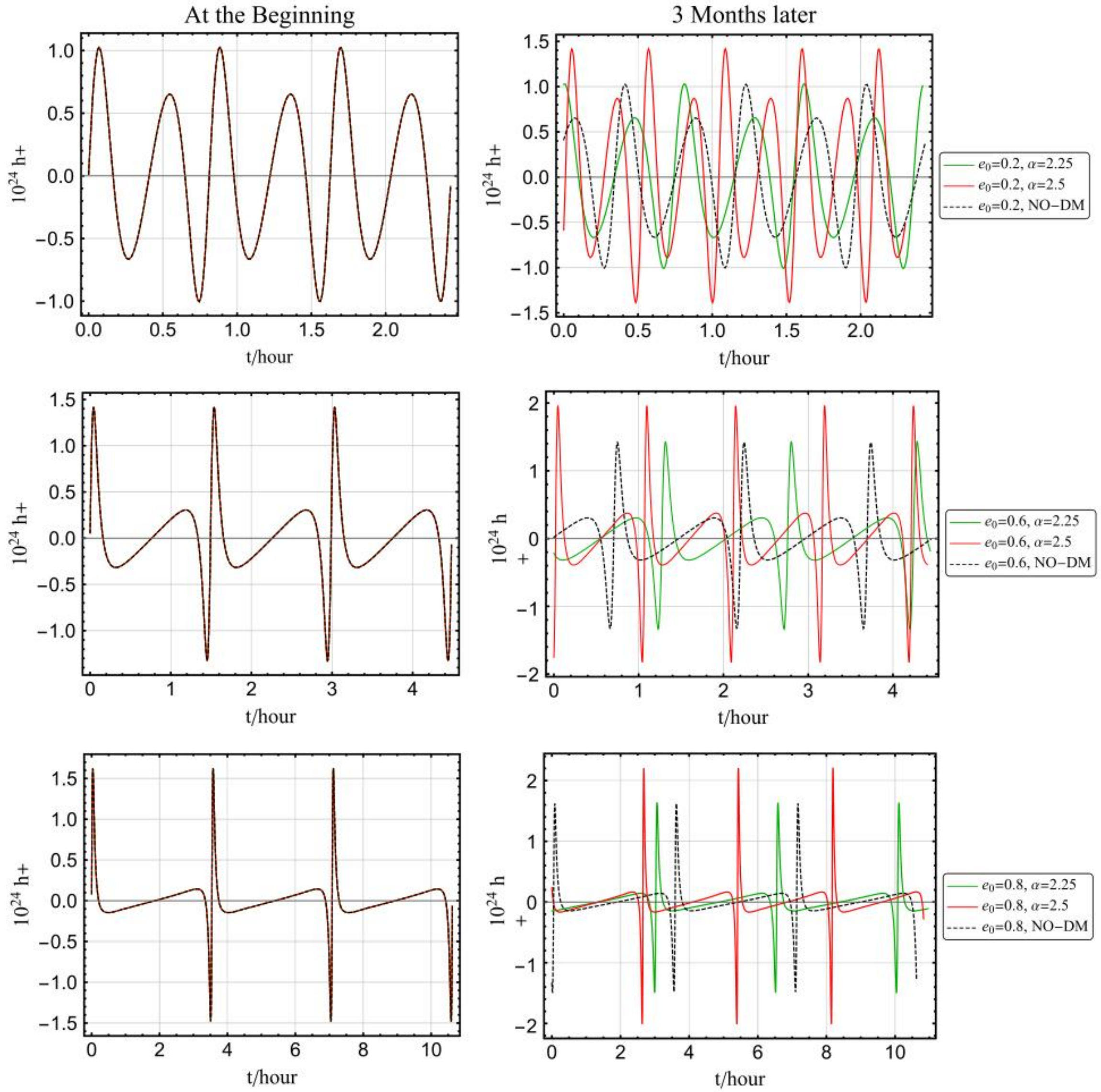
$p_0 = 10^3 R_s$

暗物质多，累积圈数差大

偏心率大，累积圈数差小，参数p选取问题

暗物质环境中等质量比系统引力波

- 近距离
- 波形差



暗物质限制

- Mismatch

$$(h_1|h_2) = 2\int_0^{+\infty} \frac{\tilde{h}_1(f)\tilde{h}_2^*(f) + \tilde{h}_2(f)\tilde{h}_1^*(f)}{S_h(f)} df$$

$$\mathcal{O}(\tilde{h}_1, \tilde{h}_2) = \frac{(\tilde{h}_1|\tilde{h}_2)}{\sqrt{(\tilde{h}_1|\tilde{h}_1)(\tilde{h}_2|\tilde{h}_2)}}$$

$$\text{Mismatch} = 1 - \mathcal{O}_{\max}(\tilde{h}_1, \tilde{h}_2)$$

e_0	SNR_0	SNR_D	Mismatch	$D_{\max}$
0.2	34.13	42.97	0.99992	358.1
0.4	34.19	47.74	0.99944	397.8
0.6	34.44	36.71	0.99965	305.9

$p_0 = 10^3 R_s, e_0$

$\alpha = 7/3$

$p = 10R_s - 1\text{year}$

$d_L = 100\text{Mpc}$

暗物质限制

- FIM参数估计：圆轨道解析波形

$$\Gamma_{ij} = \left\langle \frac{\partial h}{\partial \xi_i} \middle| \frac{\partial h}{\partial \xi_j} \right\rangle_{\xi=\hat{\xi}}$$

$$\mathbf{\Sigma} = \mathbf{\Gamma}^{-1}$$

$$\sigma_i = \Sigma_{ii}^{1/2}$$

K. Eda, Y. Itoh, S. Kuroyanagi, and J. Silk, PRD 91, 044045 (2015)

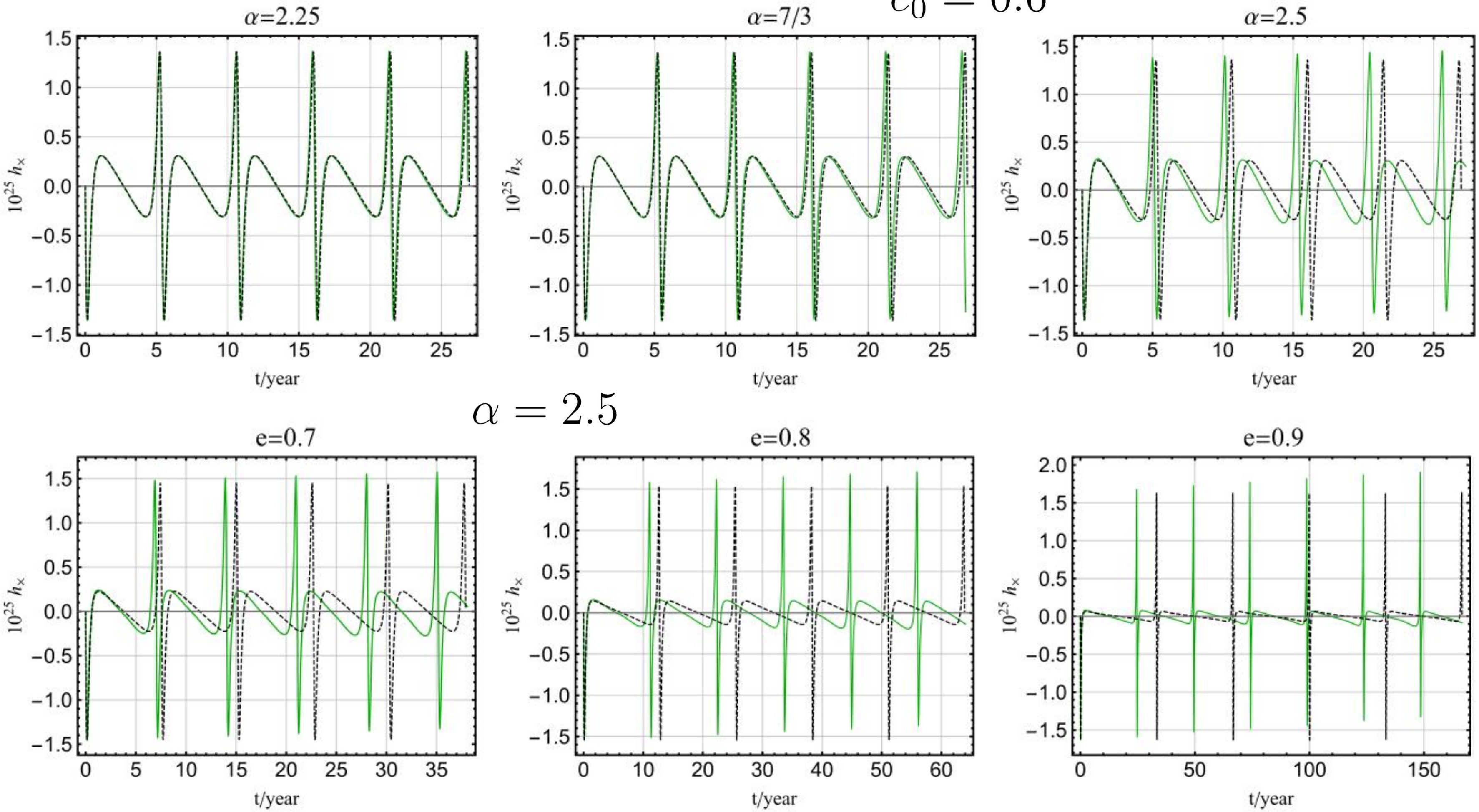
α	$\Delta\Phi_c$	Δt_c	$\Delta \ln \mathcal{M}_c(\%)$	$\Delta\alpha$	$\Delta \ln \kappa(\%)$
No DM	0.496	2.08	4.27×10^{-7}	-	-
2.25	4.40	5.25	1.96×10^{-4}	1.16×10^{-6}	0.000303
7/3	1.88	1.73	9.91×10^{-4}	1.64×10^{-6}	0.000414
2.5	4.88	7.46	1.97×10^{-1}	6.34×10^{-5}	0.0207

大轨道暗物质效应

- 引力波波形 $p_0 = 10^6 R_s$

$e_0 = 0.6$

$\alpha = 2.5$



N. Dai, Y. Gong, T. Jiang & D. Liang, PRD 106 (2022) 064033

极端质量比系统探测暗物质

- 轨道及波形精确计算：暗物质效应通过牛顿引力引进，简单、不精确
- 如何精确求解暗物质情况下爱因斯坦场方程
- 暗物质晕：Hernquist-type，总质量M

$$\rho_{\rm H} = \frac{M r_0}{2\pi r (r+r_0)^3} \sim \left(\frac{M}{r_0}\right)^2 \frac{1}{Mr}$$

- 暗物质晕环境中黑洞解

compactness

$$T^{\mu}_{\nu} = {\rm diag}(-\rho_{\rm DM}, 0, P_t, P_t),$$

$$4\pi\rho_{\rm DM} = \frac{m'}{r^2} = \frac{2M(r_0+2M_{\rm BH})(1-2M_{\rm BH}/r)}{r(r+r_0)^3}$$

$$M_{\rm BH}=0,\ \ \rho_{\rm DM}=\rho_H$$

V. Cardoso etal., PRD 105, L061501 (2022)

暗物质环境黑洞

- 暗物质环境黑洞解

$$4\pi\rho_{\rm DM} = \frac{m'}{r^2} = \frac{2M(r_0+2M_{\rm BH})(1-2M_{\rm BH}/r)}{r(r+r_0)^3}$$

$$m(r) = M_{\rm BH} + \frac{Mr^2}{(r_0+r)^2}\left(1-\frac{2M_{\rm BH}}{r}\right)^2,$$

$$M\to 0,\ m(r)\to M_{\rm BH}$$

$$T^\mu_\nu = {\rm diag}(-\rho_{\rm DM}, 0, P_t, P_t), \qquad 2P_t = \frac{m(r)\rho_{\rm DM}}{r-2m(r)}$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

→ Mr

$$f(r) = \left(1-\frac{2M_{\rm BH}}{r}\right)e^\Upsilon,$$

$$\Upsilon = -\pi\sqrt{\frac{M}{\xi}} + 2\sqrt{\frac{M}{\xi}}\arctan\left(\frac{r+r_0-M}{\sqrt{M\xi}}\right),\ \xi = 2r_0 - M + 4M_{\rm BH}.$$

V. Cardoso etal., PRD 105, L061501 (2022)

暗物质环境极端质量比系统

- 暗物质环境黑洞

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

$$f(r) \approx \left(1 - \frac{2M_{\rm BH}}{r}\right)\left(1 - \frac{2M}{r_0} + \frac{4M^2}{3r_0^2} + \frac{2Mr}{r_0^2} + \mathcal{O}[r_0^{-3}]\right)$$

$$= \left(1 - \frac{2M_{\rm BH}}{r}\right)(1 + \alpha + \beta r), \quad M/r_0 \ll 1$$

$$\alpha = -2M/r_0 + 4M^2/3r_0^2,\ \beta = 2M/r_0^2$$

V. Cardoso etal., PRD 105, L061501 (2022)

- 极端质量比系统：测地线运动

$$\frac{du_\mu}{d\tau} = \frac{1}{2}u^\alpha u^\beta \partial_\mu g_{\alpha\beta}$$

暗物质环境极端质量比系统

- 测地线运动 $$\frac{du_\mu}{d\tau} = \frac{1}{2} u^\alpha u^\beta \partial_\mu g_{\alpha\beta}$$

守恒量 $u_0 = -E/\mu = -\sqrt{1+2\varepsilon},\ u_\phi = L/\mu = h$

$$\frac{dr}{d\tau} + \frac{g_{\theta\theta}}{g_{rr}} \frac{d\theta}{d\tau} + V_{\rm eff} = 0,$$

$$V_{\rm eff} = \frac{1}{g_{rr}} \left(1 + \frac{1}{g_{tt}} E^2 + \frac{1}{g_{\phi\phi}} L^2\right)$$

$$g_{tt}^{-1} \sim -MAr + \frac{B}{r}$$

$$g_{rr}^{-1} \sim -MCr + \frac{D}{r}$$

$$V_{\rm eff} \sim \frac{1}{r},\ \frac{1}{r^2},\ \frac{1}{r^3},\ r,\ r^2$$

K. Destounis etal., 2210.09357

暗物质环境极端质量比系统

- 测地线运动

$$1+\left(\frac{dr}{d\tau}\right)^2\left(1-\frac{2m(r)}{r}\right)^{-1}+\frac{h^2}{r^2}=\frac{1+2\varepsilon}{f}$$

$$r=\frac{p}{1+e\cos\chi}$$

$$\frac{d\phi}{d\chi}=\left[\frac{1}{2}\frac{R_s}{p}(1+\alpha)+\frac{1}{2}p\beta(1-e^2)^{-1}\right]^{\frac{1}{2}}\left\{\frac{1}{2}\frac{R_s}{p}\left[1-\frac{R_s}{p}(3+e\cos\chi)\right]\right.$$
$$\left.+\alpha A+2\alpha^2 A+\beta B\right\}^{-\frac{1}{2}}J_1,$$

$$R_s=2M_{\rm BH}$$

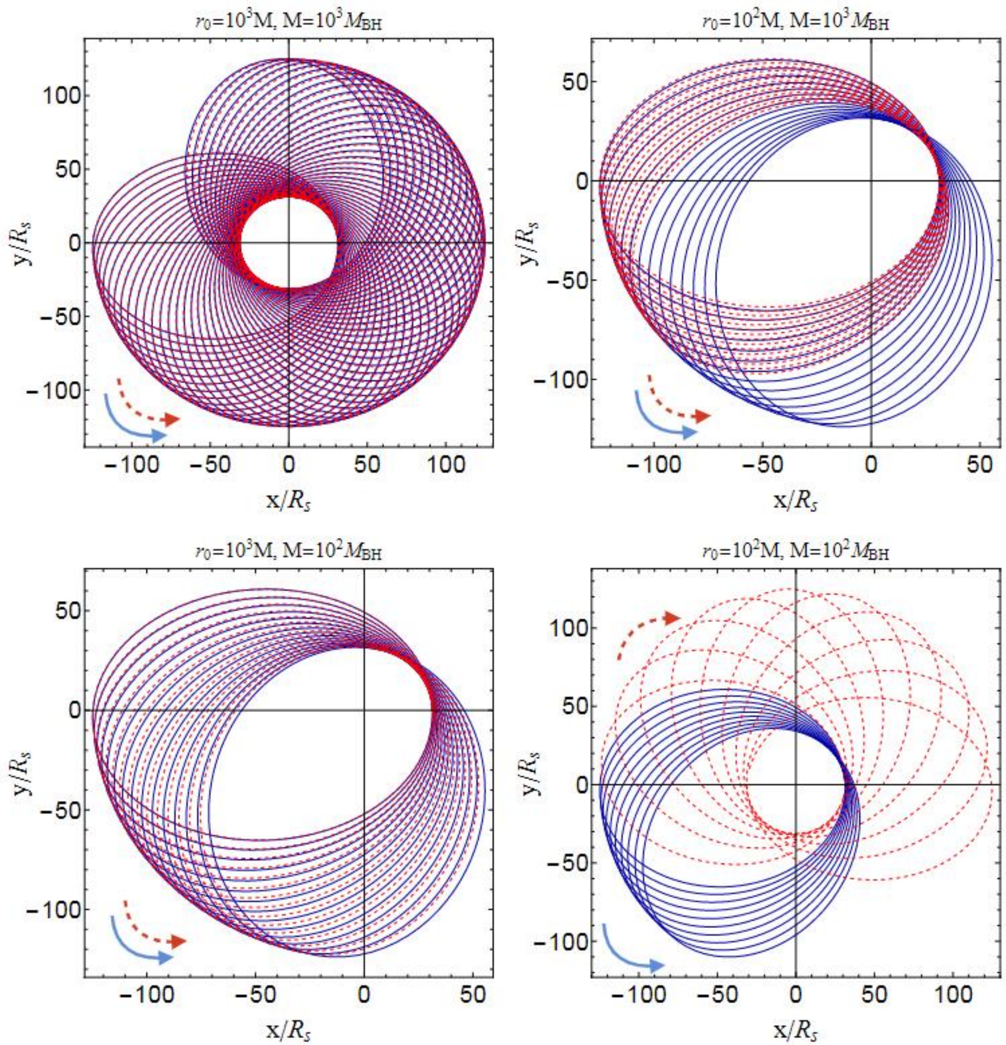
$$\frac{dt}{d\chi}=\frac{p}{(1+e\cos\chi)^2}\left(\left[1-(1+e)\frac{R_s}{p}\right]\left[1-(1-e)\frac{R_s}{p}\right]+C\right)^{\frac{1}{2}}\times$$
$$\left[1-\frac{R_s}{p}(1+e\cos\chi)\right]^{-1}\left(\frac{1}{2}\frac{R_s}{p}\left[1-\frac{R_s}{p}(3+e\cos\chi)+\alpha A+2\alpha^2 A+\beta B\right]\right)^{-\frac{1}{2}}J_2,$$

暗物质环境极端质量比系统

- 轨道运动

红色：暗物质环境
蓝色：无暗物质

$$\rho_{\rm DM} \sim \left(\frac{M}{r_0}\right)^2 \frac{1}{Mr}$$



Retrograde precession

N. Dai, Y. Gong, Y. Zhao, T. Jiang, 2301.05088

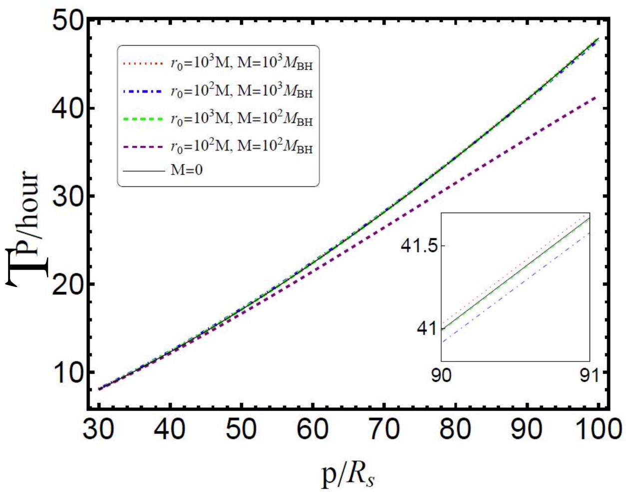
暗物质环境极端质量比系统

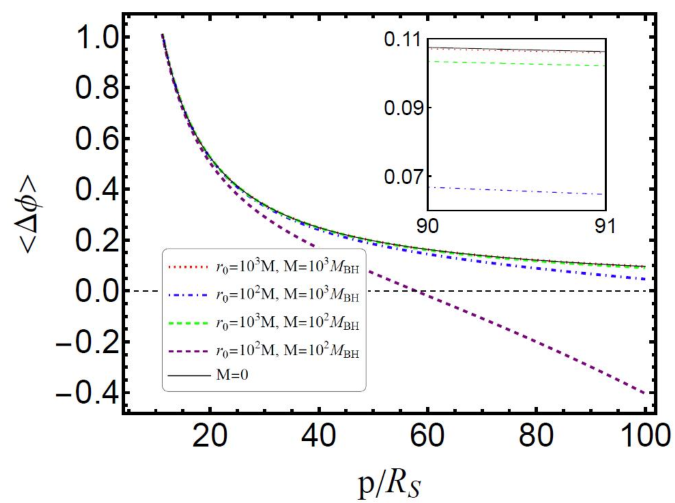
- 轨道周期及进动

周期可能减小

$$T = 2\pi\sqrt{\frac{2p^3}{R_s}\frac{1}{(1-e^2)^{3/2}}}\left\{1+\frac{3}{2}(1-e^2)\frac{R_s}{p}+\frac{3}{2}(1-e^2)\left[1+\frac{5}{4}(1-e^2)^{\frac{1}{2}}\right]\left(\frac{R_s}{p}\right)^2 + \frac{M}{r_0}+\frac{5M^2}{6r_0^2}+\frac{Mp}{r_0^2(1-e^2)}\left(e^2-\frac{11}{2}\right)-\frac{3Mp^2/R_s}{r_0^2(1-e^2)}\right\},$$

$$\Delta\phi = 3\pi\frac{R_s}{p}+\frac{3\pi}{8}(18+e^2)\left(\frac{R_s}{p}\right)^2-\frac{2\pi}{1-e^2}\frac{Mp}{r_0^2}\left[3+\frac{1+e^2+2\frac{R_s}{p}}{(1-e^2)^{1/2}}\right].$$





暗物质减小进动，甚至反向进动

暗物质环境极端质量比系统

- 辐射反冲

四极矩结果

$$\left\langle \frac{dE}{dt} \right\rangle_{\rm GW} \simeq \frac{32}{5}\left(\frac{\mu}{M_{\rm BH}}\right)^2\left(\frac{M_{\rm BH}}{p}\right)^5(1-e^2)^{3/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\left(1-\frac{6M}{r_0}\right),$$

$$\left\langle \frac{dL}{dt} \right\rangle_{\rm GW} \simeq \frac{32}{5}\left(\frac{\mu}{M_{\rm BH}}\right)^2 M_{\rm BH}\left(\frac{M_{\rm BH}}{p}\right)^{7/2}(1-e^2)^{3/2}\left(1+\frac{7}{8}e^2\right)\left(1-\frac{5M}{r_0}\right)$$

$$\frac{dE}{dt}_{\rm orbit} = \mu\frac{R_s}{4}\left(\frac{2e}{p}\frac{de}{dt}+\frac{1-e^2}{p^2}\frac{dp}{dt}\right)\left(1-\frac{2M}{r_0}\right),$$

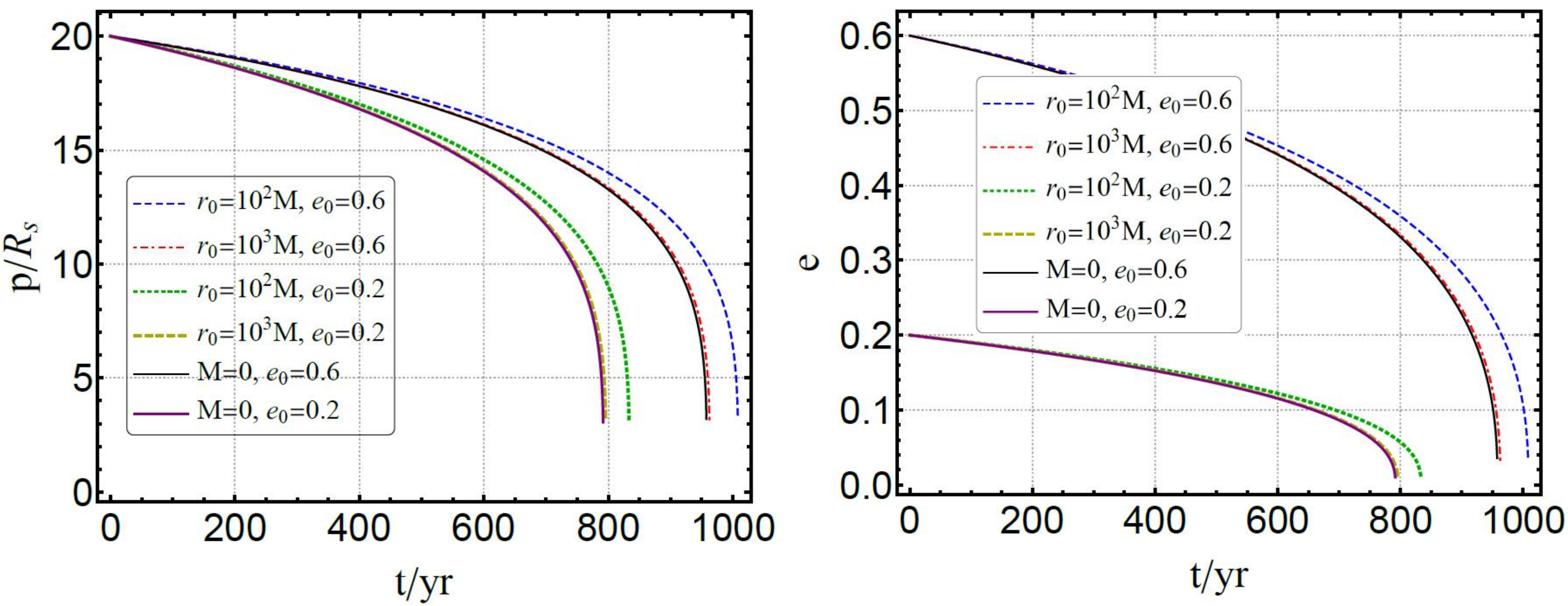
$$\frac{dL}{dt}_{\rm orbit} = \frac{\mu}{4}\left(\frac{2R_s}{p}\right)^{1/2}\frac{dp}{dt}.$$

$$\frac{dp}{dt} = -\frac{64}{5}\frac{\mu}{M_{\rm BH}}\left(\frac{M_{\rm BH}}{p}\right)^3(1-e^2)^{\frac{3}{2}}\left(1+\frac{7}{8}e^2\right)\left(1-\frac{5M}{r_0}\right),$$

$$\frac{de}{dt} = -\frac{304\,e}{15\,p}\frac{\mu}{M_{\rm BH}}\left(\frac{M_{\rm BH}}{p}\right)^3(1-e^2)^{\frac{3}{2}}\left(1+\frac{121}{304}e^2\right)\left(1-\frac{5M}{r_0}\right).$$

暗物质环境极端质量比系统

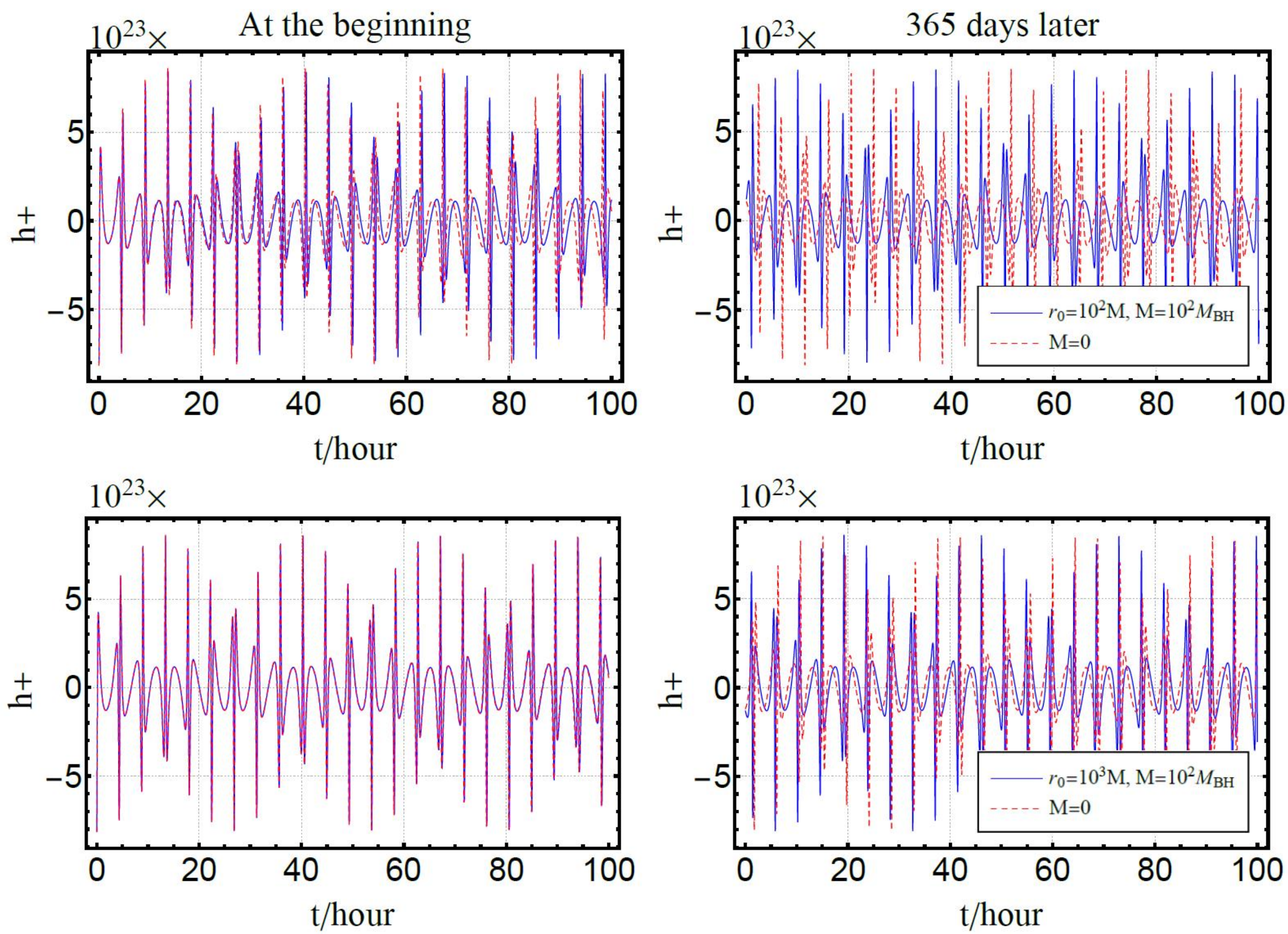
- 辐射反冲：Numerical Kludge方法



轨道衰减，偏心率减小
暗物质引力阻碍衰减

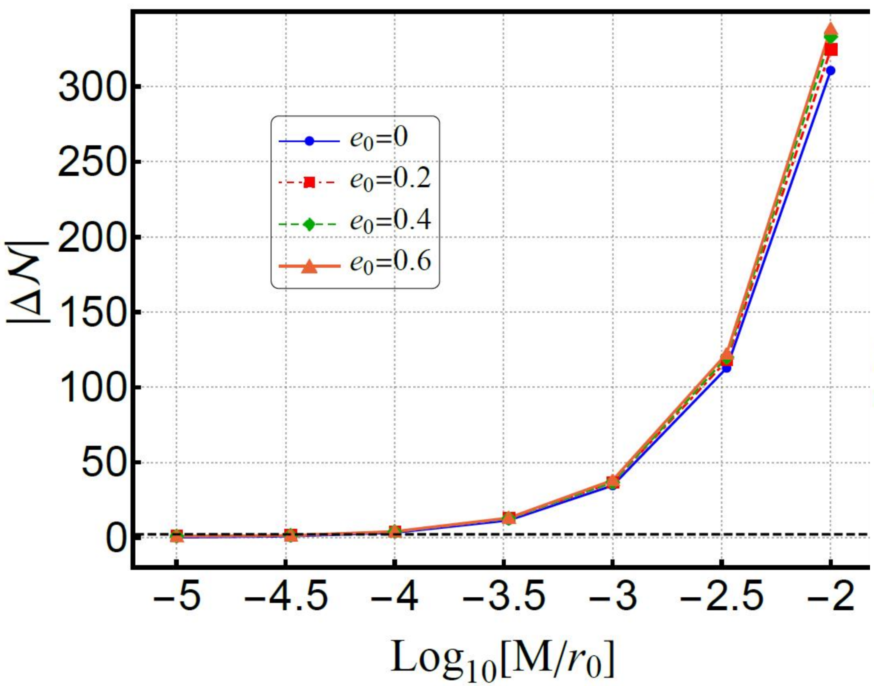
暗物质环境极端质量比系统

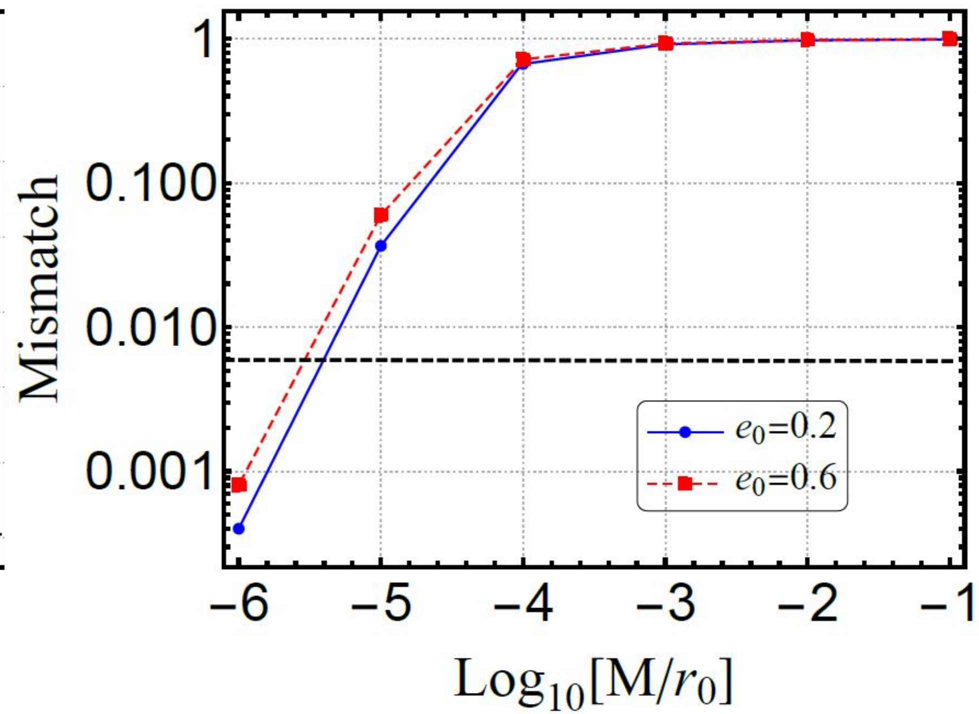
- 引力波波形



暗物质环境极端质量比系统

- 暗物质参数（紧致性）限制





- 偏心率轨道有助于探测暗物质晕，暗物质晕紧致度可以限制到$M/r_0 < 10^{-5}$

引力理论限制

- Brans-Dicke理论: FIM限制（后牛顿近似波形）

$$S = (16\pi)^{-1}\int \left[\varphi R - \varphi^{-1}\omega(\varphi)\varphi^{,\alpha}\varphi_{,\alpha}\right]\sqrt{-g}d^4x + S_m(\Psi, \varphi, g_{\alpha\beta}),$$

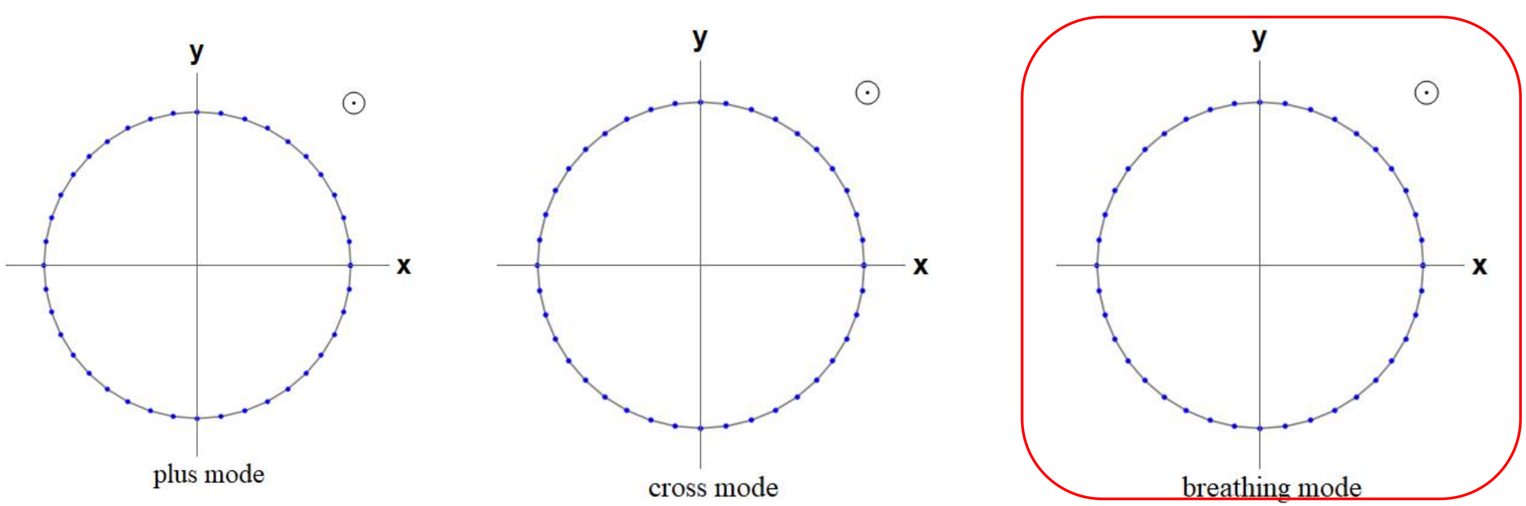
违反等效原理 $s_A = d\ln m_A(\varphi)/d\ln\varphi_0$

$$S_m = -\int m(\varphi)d\tau$$

C. Brans and R. H. Dicke, Phys. Rev. 124 (1961) 925;
D. M. Eardley, ApJ 196 (1975) L59.

$$G = \frac{(4 + 2\omega_{\rm BD})}{\phi_0(3 + 2\omega_{\rm BD})}$$

- 引力波特性：呼吸模，单级及偶极辐射



后牛顿近似

- 标量场能量辐射

$$\left(\frac{dE}{dt}\right)_D = \frac{d^2}{3}\left(\frac{m_p}{M}\right)^2 v^8.$$

圆轨道无单极辐射

单极 $$\frac{dE^{\infty}_{\mathrm{l=0,m=0}}}{dt} = \left(\frac{dE}{dt}\right)_D \left(\frac{25v^2}{6} - \frac{35v^4}{2} + \frac{25}{3}\pi v^5\right) e^2$$

$\sim d^2 e^2 v^{10}$

偶极 $$\frac{dE^{\infty}_{\mathrm{l=1,m=1}}}{dt} = \left(\frac{dE}{dt}\right)_D \left\{1 - \frac{26v^2}{5} + 2\pi v^3 + \frac{1123v^4}{175} - \frac{52}{5}\pi v^5 + e^2\left(-1 + \frac{19v^2}{10} + 3\pi v^3 + \frac{8496v^4}{175} - \frac{321}{5}\pi v^5\right)\right\}.$$

主导项 $\sim d^2 v^8$

弱场 $v \ll 1$ 情况下，偏心率越大，标量场辐射能量越小

四极 $$\frac{dE^{\infty}_{\mathrm{l=2,m=2}}}{dt} = \left(\frac{dE}{dt}\right)_D \left\{\frac{16v^2}{5} - \frac{848v^4}{35} + \frac{64\pi v^5}{5} + e^2\left(\frac{67v^2}{15} - \frac{4481v^4}{42} + \frac{1073\pi v^5}{15}\right)\right\}.$$

$\sim d^2 v^{10}$

$d = 4m'(\varphi_0)/m(\varphi_0)$

Brans-Dicke理论中I/EMRI

- 密切轨道方法

$$\ddot{z}^{\alpha}(\lambda)+\Gamma^{\alpha}_{\beta\gamma}\dot{z}^{\beta}(\lambda)\dot{z}^{\gamma}(\lambda)=f^{\alpha}$$

Self-force（自力），后牛顿

$$f^t=\frac{\dot{t}^2\left[a_p^r\frac{dr}{dt}+a_p^{\phi}r^2F\frac{d\phi}{dt}\right]}{F^2-\left(\frac{dr}{dt}\right)^2-Fr^2\left(\frac{d\phi}{dt}\right)^2},$$

$$f^r=\frac{\dot{t}^2\left[a_\epsilon^r\left(F-r^2\left(\frac{d\phi}{dt}\right)^2\right)+a_\epsilon^{\phi}r^2\frac{dr}{dt}\frac{d\phi}{dt}\right]}{F^{-1}\left(F^2-\left(\frac{dr}{dt}\right)^2-Fr^2\left(\frac{d\phi}{dt}\right)^2\right)},\qquad F=1-2Gm_1/r$$

$$f^{\phi}=\frac{\dot{t}^2\left[a_\epsilon^r\frac{dr}{dt}\frac{d\phi}{dt}+a_\epsilon^{\phi}\left(F^2-\left(\frac{dr}{dt}\right)^2\right)\right]}{F^2-\left(\frac{dr}{dt}\right)^2-Fr^2\left(\frac{d\phi}{dt}\right)^2}.$$

$$r=\frac{pm_1}{1+e\cos(\chi-\xi)}$$

Brans-Dicke理论中EMRI

- 后牛顿2.5PN构造自力：包括高级辐射反冲

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\frac{\alpha m}{r^2}\boldsymbol{n} + \frac{\alpha m}{r^2}\left[\boldsymbol{n}(A_{1\mathrm{PN}} + A_{2\mathrm{PN}}) + \dot{r}\boldsymbol{v}(B_{1\mathrm{PN}} + B_{2\mathrm{PN}})\right] + \frac{8}{5}\eta\frac{(\alpha m)^2}{r^3}\left[\dot{r}\boldsymbol{n}(A_{1.5\mathrm{PN}} + A_{2.5\mathrm{PN}}) - \boldsymbol{v}(B_{1.5\mathrm{PN}} + B_{2.5\mathrm{PN}})\right],$$

$$\alpha = 1 - \zeta + \zeta(1-2s_1)(1-2s_2),\ \zeta = 1/(4+2\omega_0)$$

$$\text{BHs } s = 1/2,\ \text{NS } s = 0.2$$

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\frac{\alpha m}{r^2}(A_{BD}\boldsymbol{n} - B_{BD}\boldsymbol{v})$$

$$A_{BD} = 1 - A_{1\mathrm{PN}} - A_{2\mathrm{PN}} - \frac{8}{5}\eta\frac{\alpha m}{r}\dot{r}(A_{1.5\mathrm{PN}} + A_{2.5\mathrm{PN}}),$$

$$B_{BD} = (B_{1\mathrm{PN}} + B_{2\mathrm{PN}})\dot{r} - \frac{8}{5}\eta\frac{\alpha m}{r}(B_{1.5\mathrm{PN}} + B_{2.5\mathrm{PN}}).$$

Brans-Dicke理论中EMRI

- 密切轨道方法

$$A_{BD} = A_s + \tilde{A},\ B_{BD} = B_s + \tilde{B}$$

$$A_s = 1 - 4\frac{\alpha m_1}{r} + v^2 + 9(\frac{\alpha m_1}{r})^2 - 2\frac{\alpha m_1}{r}(\frac{dr}{dt})^2,$$

微扰构造自力

$$B_s = -\frac{dr}{dt}(4 - 2\frac{\alpha m_1}{r}),\quad a^r_\epsilon = -\frac{m_1}{r^2}(\tilde{A} + \tilde{B}\frac{dr}{dt}),\quad a^\phi_\epsilon = -\frac{m_1}{r^2}\tilde{B}\frac{d\phi}{dt}.$$

$$h^{ij}_{BD} = -\frac{2\zeta\mathcal{A}}{d_L}(\delta^{ij} - \Omega^i\Omega^j) + \frac{4(1-2\zeta)\eta m}{d_L}(v^i v^j - \frac{m}{r}n^i n^j),$$

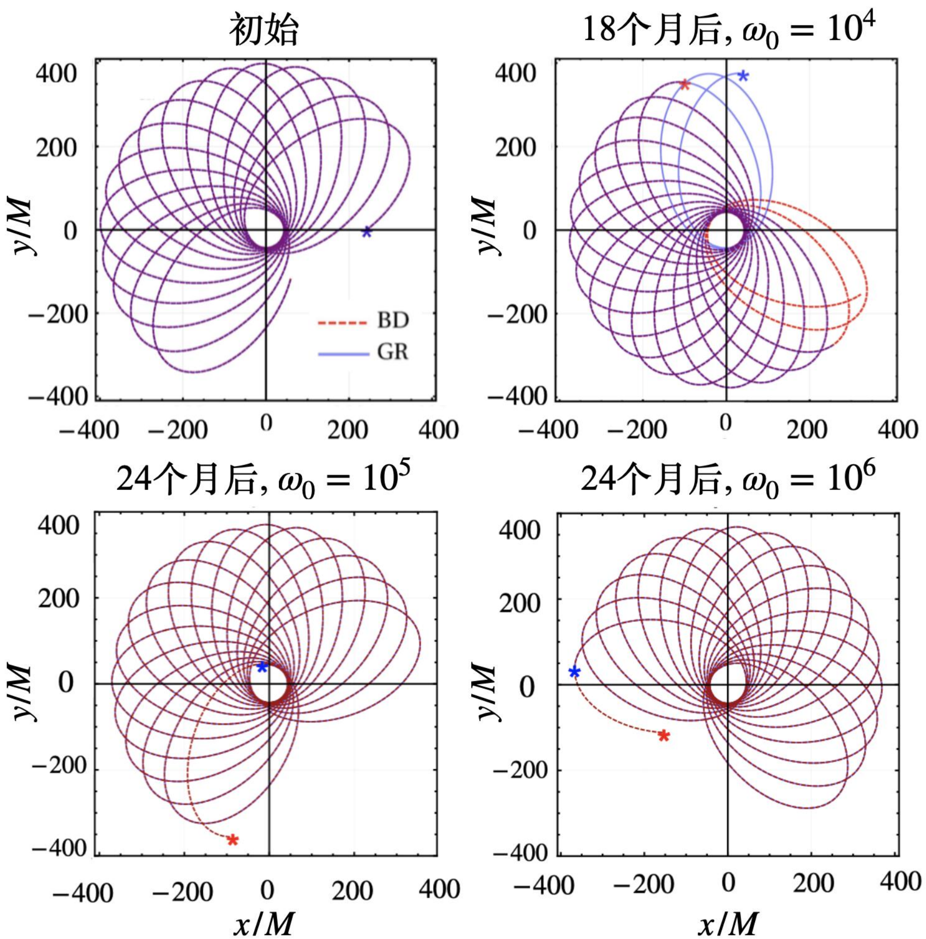
$$\mathcal{A} = \mathcal{E} + \dot{\mathcal{E}}^j\Omega_j - \frac{1}{2}\ddot{I}^{jk}\Omega_j\Omega_k,\ \ddot{I}^{jk} = 2\eta m\left(v^i v^j - \frac{m}{r}n^i n^j\right),$$

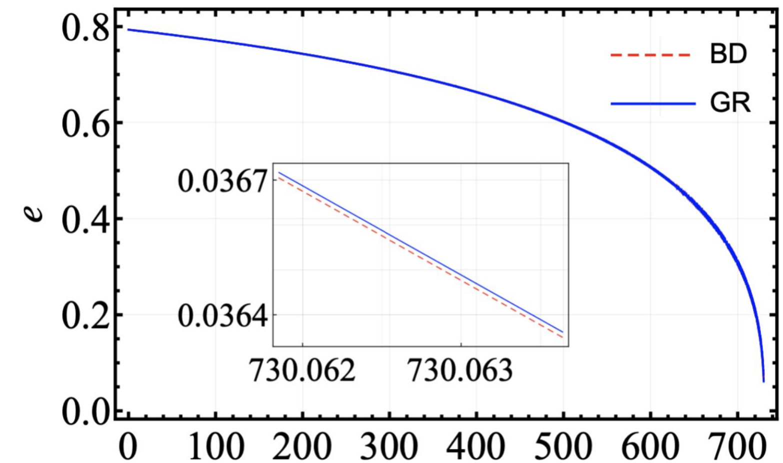
$$\mathcal{E} = 2(1+2\lambda)\frac{M\mu}{r} + \eta m\left[v^2 + (1+4\lambda)\frac{m}{r}\right],$$

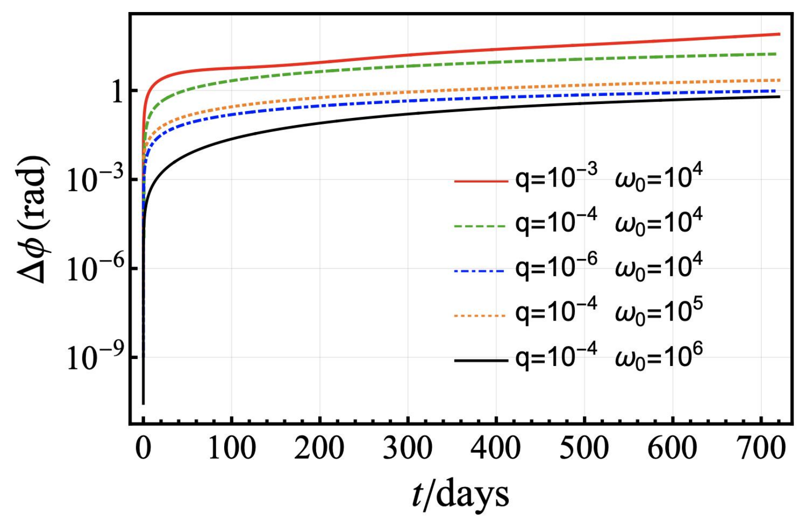
$$\mathcal{E}^j = -2(1+2\lambda)\eta m\mathcal{S}r^j - \eta\Delta m\left[v^2 + (1+4\lambda)\frac{m}{2r}\right]r^j$$

Brans-Dicke理论中EMRI

■ 轨道运动结果

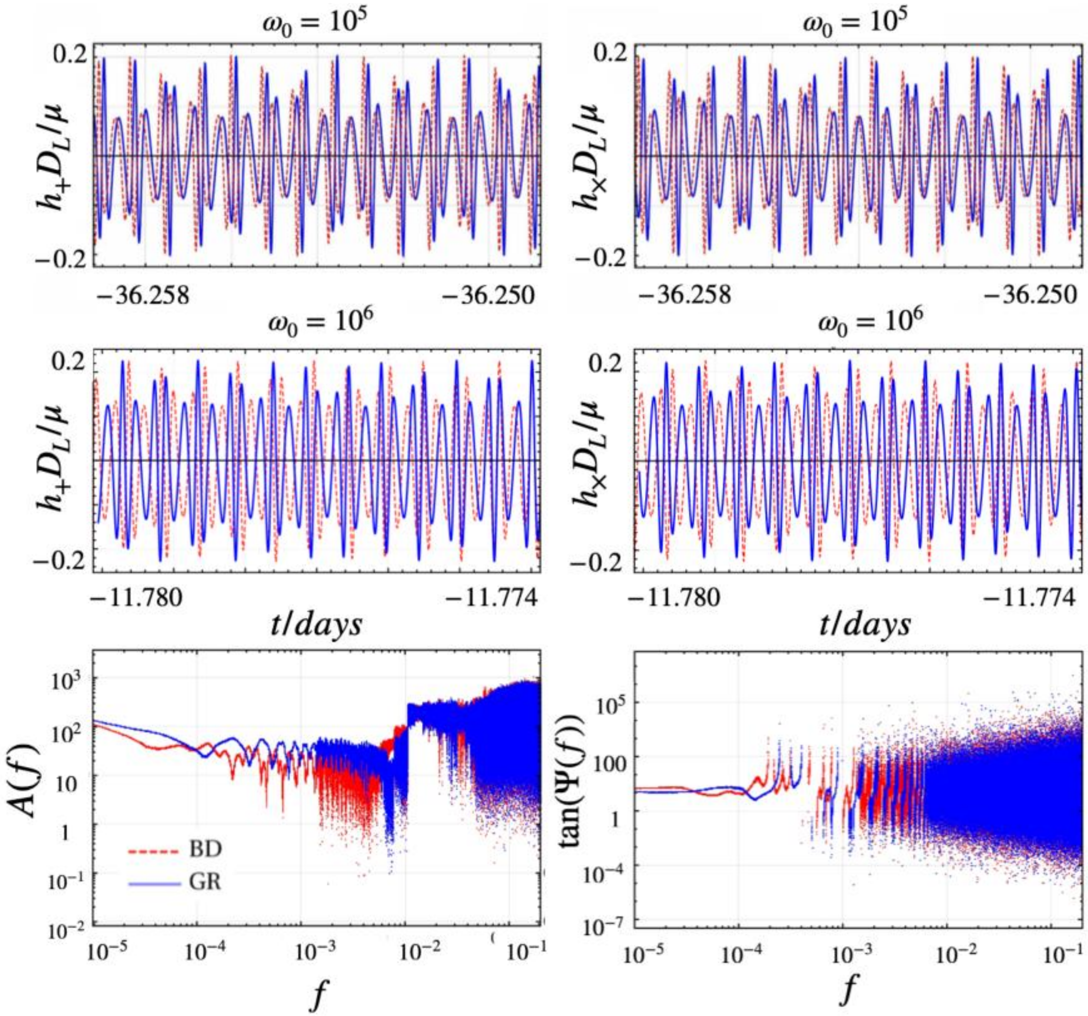






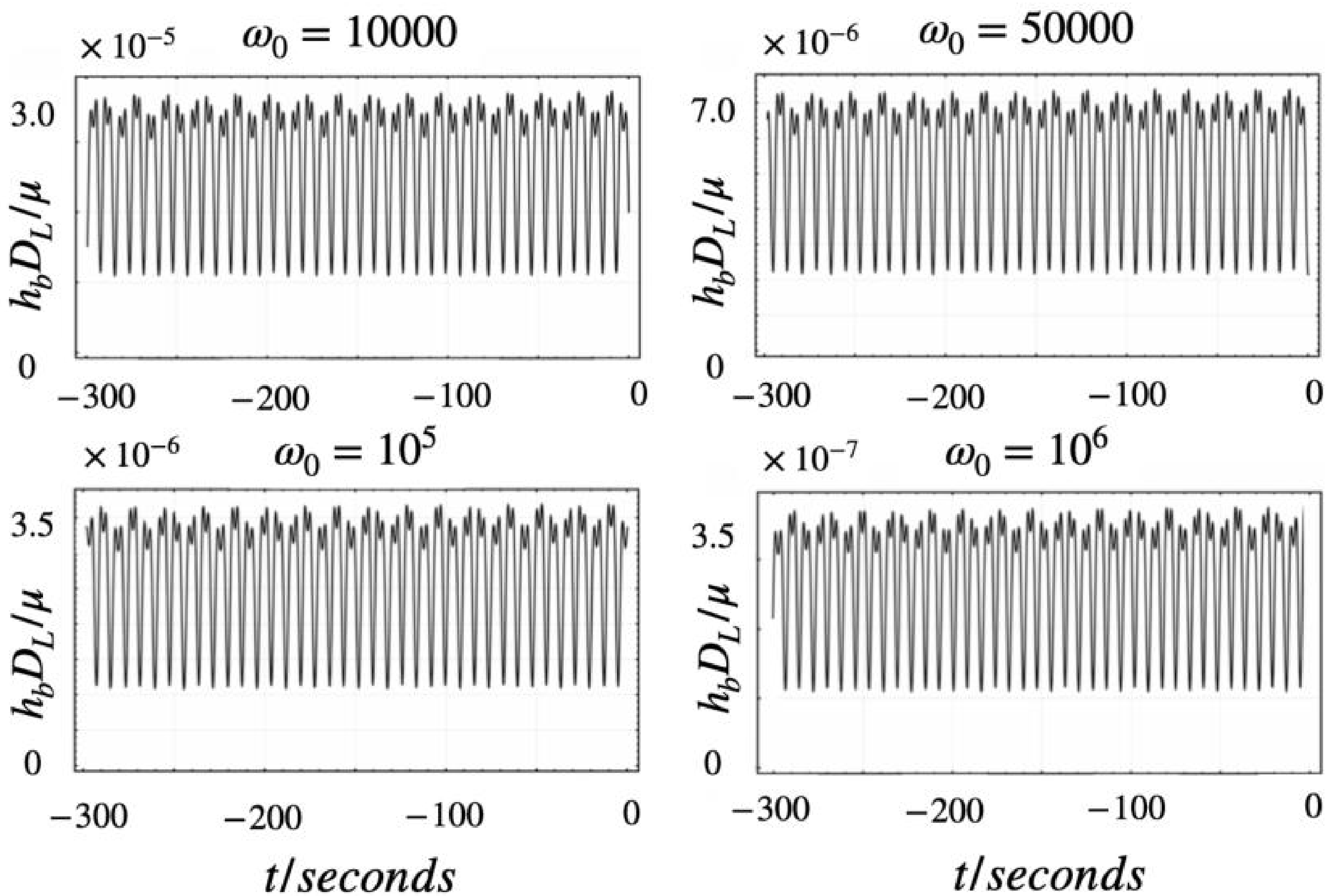
引力波限制Brans-Dicke理论

- 引力波波形



引力波限制Brans-Dicke理论

- 呼吸模



引力波限制Brans-Dicke理论

- 限制结果 $\omega_0 > 10^6$

System	$\rho(\tilde{H}_{\rm GR})$	$\rho(\tilde{H}_{\rm BD})$	$\rho(\Delta\tilde{H})$	Mismatch
$q = 10^{-3}$				
Circular	12.49	12.49 (12.49)	17.88 (17.65)	0.98 (0.71)
Eccentric ($e_0 = 0.8$)	13.22	13.22 (13.22)	18.69 (17.97)	0.99 (0.73)
$q = 10^{-4}$				
Circular	26.86	26.86 (26.87)	38.41 (34.48)	0.97 (0.56)
Eccentric ($e_0 = 0.8$)	23.89	23.88 (23.92)	34.16 (33.86)	0.97 (0.5)
$q = 10^{-6}$				
Circular	197.94	197.95 (197.94)	284.12 (278.51)	0.94 (0.33)
Eccentric ($e_0 = 0.8$)	152.45	152.45 (152.44)	218.14 (211.11)	0.90 (0.26)

$$\omega_0 = 10^5 (10^6)$$

总结

- 计算方法
  - 密切轨道方法+四极矩辐射反冲
  - NK方法：测地线+四极矩辐射反冲
  - 密切轨道方法+后牛顿辐射反冲（自力）
  - TB方法：黑洞微扰、辐射反冲（视界+无穷远，高极矩）对测地线轨道修正
  - 更精确、快速方法：self-force
- IMRI/EMRI可以探究暗物质
- IMRI/EMRI可以限制修改引力理论 $\omega_0 > 10^6$

谢谢