中山大學
SUN YAT-SEN UNIVERSITY

Towards gravitational theories with two tensorial degrees of freedom

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中国科学技术大学

[Based on: 1910.13995, 2011.00805, 2104.07615, 2302.02090]

Introduction and motivation

Why all these?

Phenomenological side:

- (2011 Nobel prize)
  To explain the early and late accelerated expansion of our universe.
- (2017 Nobel prize)
  The gravitational waves have been detected, which are new tools to test gravity theories.

Theoretical side:

To understand if and why GR is unique.

“The best way to understand something is to modify it.”

Playing with DoF's

The key question (task):

Keeping the correct degrees of freedom (DoF's).

To introduce the wanted DoF's.

To eliminate the unwanted DoF's.

Two faces of modified gravity

Extra mode(s) without ghost(s).

(2011 Nobel prize)
Dark energy / Inflation

↓

"Gravity" is partly described by GR.

(Horndeski, DHOST, Horava, EFT of inflation, dRGT, SCG ...)

Non-GR theory for the TTDOFs.

(2017 Nobel prize)
Gravitational waves (GWs)

↓

"Gravity" is carried by the same DoF's of GR, but behave differently from that of GR.

(Cuscuton, MMG, 4dEGB, TTDOF, ...)

Uniqueness of GR

GR is the unique theory (kinetic term) for the TTDOFs, if we require Lorentz invariance and locality.

Degrees of freedom in GR:

10 - 4 - 4 = 2

4 gauge modes

4 constraints

2 polarizations of a massless graviton

T**T**DOFs = **T**ransverse and **T**raceless DOFs
= **T**wo **T**ensorial DOFs

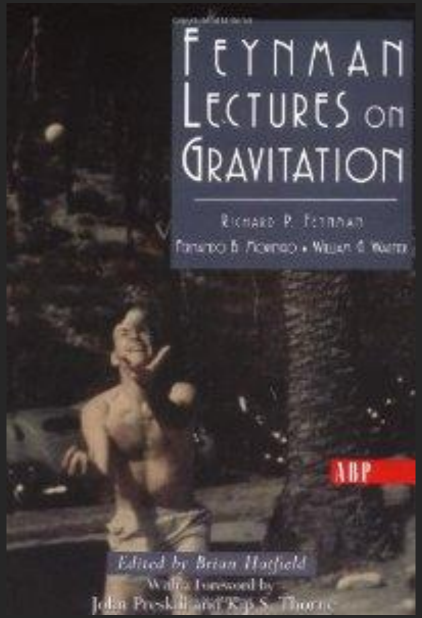
Uniqueness of GR

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$$

$$\mathcal{L}_2 = c_1\,\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} + c_2\,\partial_\mu h^{\mu\nu}\partial_\nu h + c_3\,\partial_\nu h^{\mu\nu}\partial^\lambda h_{\mu\lambda} + c_4\,\partial_\mu h \partial^\mu h$$

$$\delta_{\boldsymbol{\xi}} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$c_1 = -\frac{1}{4}, \qquad c_2 = -\frac{1}{2}, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{1}{4}.$$



Uniqueness of GR

$$
\begin{aligned}
\mathcal{L}_2 \simeq\ & -\underline{(c_1+c_2+c_3+c_4)}\,\dot{h}_{00}^2 + \underline{(c_1+c_4)}\,\partial_i h_{00}\partial_i h_{00} \\
& + \underline{(2c_1+c_3)}\,\dot{h}_{0i}\dot{h}_{0i} - c_3\partial_i h_{0i}\partial_j h_{0j} - 2c_1\partial_j h_{0i}\partial_j h_{0i} \\
& + 2\,\underline{(c_2+c_3)}\,\dot{h}_{00}\partial_i h_{0i} \\
& + 2h_{0i}\left(c_2\partial_i\dot{h}_{jj} + c_3\partial_j\dot{h}_{ij}\right) \\
& + h_{00}\left[2c_4\partial_i^2 h_{kk} - \underline{(c_2+2c_4)}\,\ddot{h}_{ii} + c_2\partial_i\partial_j h_{ij}\right] \\
& - c_1\,\dot{h}_{ij}^2 - c_4\dot{h}_{ii}^2 + c_4\partial_j h_{kk}\partial_j h_{ii} + c_3\partial_j h_{ij}\partial_k h_{ik} + c_2\partial_j h_{ii}\partial_k h_{kj} + c_1\partial_k h_{ij}\partial_k h_{ij}.
\end{aligned}
$$

“resummation” ⟶ GR

$$
\mathcal{L} = \sum_n \mathcal{L}_n = \sqrt{-\det\left(\eta_{\mu\nu}+h_{\mu\nu}\right)}R\left[\eta_{\mu\nu}+h_{\mu\nu}\right] \equiv \sqrt{g}R[g]
$$

Uniqueness of GR

How about for the massive gravitons (5 DOFs)?

Mass (potential) terms:

$$\mathcal{L}_2^{(\mathrm{p})} = -\frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) \equiv \mathcal{L}_{\mathrm{FP}}$$

(Fierz-Pauli mass term)

$$\sum_n \mathcal{L}_n^{(\mathrm{p})} \simeq \sqrt{-g}\mathcal{L}_{\mathrm{dRGT}}$$

[C. de Rham, G. Gabadadze, A. Tolley, 1011.1232]

Uniqueness of GR

How about for the massive gravitons (5 DOFs)?

Kinetic terms:

Very likely GR is also the unique kinetic term for the massive gravitons, with Lorentz invariance and locality.

[C. de Rham, A. Matas, A. Tolley, 1311.6485]
*[**XG**, 1403.6781]*

New kinetic interactions for massive gravity?

Claudia de Rham, Andrew Matas and Andrew J Tolley

Abstract

We show that there can be no new Lorentz invariant kinetic interactions free from the Boulware–Deser ghost in four dimensions in the metric formulation of gravity, beyond the standard Einstein–Hilbert, up to total derivatives. We

Uniqueness of GR

Einstein equation is quite unique.

$$\alpha G_{\mu\nu} + \lambda g_{\mu\nu} = 0$$

[Lovelock, 1971]

Any metric theory of gravity alternative to GR must satisfy (at least):

- extra degrees of freedom,
- extra dimensions (e.g., brane world),
- higher derivative terms (e.g., $f(R)$),
- non-Riemannian geometry (e.g., $f(T)$),
- giving up locality.

Non-GR theories for the TTDOFs

How about to abandon Lorentz invariance?

Only spatial covariance,
but without introducing the unwanted scalar mode?

Spatially covariant gravity

$$S=\int \mathrm{d}t\mathrm{d}^3x\,N\sqrt{h}\mathcal{L}(t,N,h_{ij},R_{ij},K_{ij},\nabla_i)$$

[XG, et al: 1406.0822 , 1409.6708, 1806.02811, 1902.07702, 1910.13995, 2004.07752, 2006.15633, 2111.08652 …]

2 tensor + 1 scalar DoFs with higher derivative EoMs.

Non-GR theories for the TTDOFs

Reduction of symmetries	→	Extra mode(s)
Breaking U(1)	→	Massive vector (2 vector + 1 scalar)
Breaking whole spacetime diff	→	Massive gravity (2 tensor + 2 vector + 1 scalar)
Breaking time diff, respecting only spatial symmetries	→	2 tensor + 1 scalar

Non-GR theories for the TTDOFs

2007 • Cuscuton

[N. Afshordi, D. J.H. Chung, G. Geshnizjani, hep-th/0609150]

$$S=\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}R+\mu^2\sqrt{|-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi|}-V\left(\phi\right)\right)$$

Causal field theory with an infinite speed of sound

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(Received 5 December 2006; revised manuscript received 19 February 2007; published 13 April 2007)

We introduce a model of scalar field dark energy, *Cuscuton*, which can be realized as the incompressible (or infinite speed of sound) limit of a scalar field theory with a noncanonical kinetic term (or *k*-essence). Even though perturbations of Cuscuton propagate superluminally, we show that they have a locally

(cuscuta, 菟丝子)

Non-GR theories for the TTDOFs

2007 • Cuscuton

[N. Afshordi, D. J.H. Chung, G. Geshnizjani, hep-th/0609150]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}R + \mu^2\sqrt{\left|-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right|} - V(\phi)\right)$$

Only 2 DOFs when the scalar field is timelike (e.g., in a cosmological background).

[H. Gomes, D. Guariento, 1703.08226]

The scalar mode becomes an instantaneous mode (with an infinite speed of sound) and effectively non-dynamical.

[A. De Felice, D. Langlois, S. Mukohyama, K. Noui, A. Wang, 1803.06241]

$$L = \frac{1}{2}\left(\dot{\psi}^2 - c_s^2\left(\partial_i\psi\right)^2\right)$$

Non-GR theories for the TTDOFs

2007 — Cuscuton
[N. Afshordi, D. J.H. Chung, G. Geshnizjani, hep-th/0609150]

2017 — Minimally modified gravity
[C. Lin, S. Mukohyama, 1708.03757]

$$S = \int \mathrm{d}t\mathrm{d}^3x\sqrt{h}N\, F\left(K_{ij}, R_{ij}, h_{ij}, \nabla_i; t\right)$$

with some conditions

Non-GR theories for the TTDOFs

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$$S = \int \mathrm{d}t\mathrm{d}^3x\sqrt{h}N\,F\left(K_{ij}, R_{ij}, h_{ij}, \nabla_i; t\right)$$

linear

with some conditions

Non-GR theories for the TTDOFs

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2018 — Extended Cuscuton

[A. Iyonaga, K. Takahashi, T. Kobayashi, 1809.10935]

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2019 **Spatially covariant gravity with TTDOFs**
*[**XG**, Z.-B. Yao, 1910.13995]*

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Perturbative approach *[Yu-Min Hu, **XG**, 2104.07615]*

With the dynamical lapse *[J. Lin, Y. Gong, Y. Lu, F. Zhang, 2011.05739]*

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2020 TTDOFs with an auxiliary constraint
*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

Non-GR theories for the TTDOFs

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2020 **TTDOFs with an auxiliary constraint**
*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

2023 **TTDOFs with multiple auxiliary constraints**
*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

Spatially covariant gravity with TTDOFs

General case of spatially covariant gravity

[**XG**, Z.-B. Yao, 1910.13995]

$$\tilde{S} = \int \mathrm{d}t\mathrm{d}^3x\, N\sqrt{h}\mathcal{L}\left(N, h_{ij}, K_{ij}, R_{ij}, \nabla_i; t\right)$$

$$S = \tilde{S}_B + \int \mathrm{d}t\mathrm{d}^3x \frac{\delta \tilde{S}_B}{\delta B_{ij}}\left(K_{ij} - B_{ij}\right) \qquad \tilde{S}_B \equiv \left.\tilde{S}\right|_{K_{ij}\to B_{ij}}$$

$$\{\Phi_I\} \equiv \left\{N, N^i, h_{ij}, B_{ij}\right\}$$

$$\left\{\Pi^I\right\} \equiv \left\{\pi, \pi_i, \pi^{ij}, p^{ij}\right\}$$

16 primary constraints:

$$\pi := \frac{\delta S}{\delta \dot{N}} \approx 0 \qquad \pi^i := \frac{\delta S}{\delta \dot{N}^i} \approx 0$$

$$\tilde{\pi}^{ij} := \pi^{ij} - \frac{1}{2N}\frac{\delta \tilde{S}_B}{\delta B_{ij}} \approx 0 \qquad p^{ij} := \frac{\delta S}{\delta \dot{B}_{ij}} \approx 0$$

Playing with the Poisson brackets

[**XG**, Z.-B. Yao, 1910.13995]

16 primary

	π	π^i	$\tilde{\pi}^{ij}$	p^{ij}
π				
π^i				
$\tilde{\pi}^{ij}$				
p^{ij}				

Playing with the Poisson brackets

[**XG**, Z.-B. Yao, 1910.13995]

16 primary: π, π^i, $\tilde{\pi}^{ij}$, p^{ij}

4 secondary: C, C_i

	π	π^i	$\tilde{\pi}^{ij}$	p^{ij}	C	C_i
π						
π^i						
$\tilde{\pi}^{ij}$						
p^{ij}						
C						
C_i						

Playing with the Poisson brackets

*[**XG**, Z.-B. Yao, 1910.13995]*

	π	π^i	$\tilde{\pi}^{ij}$	p^{ij}	C	C_i
π		0				0
π^i	0	0	0	0	0	0
$\tilde{\pi}^{ij}$		0				0
p^{ij}		0				0
C		0				0
C_i	0	0	0	0	0	0

16 primary (π, π^i, $\tilde{\pi}^{ij}$, p^{ij})

4 secondary (C, C_i)

Spatial covariance

$$
\begin{aligned}
\#_{\mathrm{DOF}} &= \frac{1}{2}\left(2 \times \#_{\mathrm{var}} - 2 \times \#_1 - \#_2\right) \\
&= \frac{1}{2}\left(2 \times 16 - 2 \times 6 - 14\right) \\
&= 3
\end{aligned}
$$

[XG, 1409.6708, …]

Playing with the Poisson brackets

[**XG**, Z.-B. Yao, 1910.13995]

16 primary: π, π^i, $\tilde{\pi}^{ij}$, p^{ij}

4 secondary: C, C_i

	π	π^i	$\tilde{\pi}^{ij}$	p^{ij}	C	C_i
π		0				0
π^i	0	0	0	0	0	0
$\tilde{\pi}^{ij}$		0				0
p^{ij}		0				0
C		0				0
C_i	0	0	0	0	0	0

must be degenerate!

Spatial covariance

$$\begin{aligned}\#_{\mathrm{DOF}} &= \frac{1}{2}\left(2\times\#_{\mathrm{var}} - 2\times\#_1 - \#_2\right)\\ &= \frac{1}{2}\left(2\times 16 - 2\times 6 - 14\right)\\ &= 3\end{aligned}$$

[XG, 1409.6708, ...]

Playing with the Poisson brackets

[**XG**, Z.-B. Yao, 1910.13995]

	π	π^i	$\tilde{\pi}^{ij}$	p^{ij}	C	C_i
π		0				0
π^i	0	0	0	0	0	0
$\tilde{\pi}^{ij}$		0				0
p^{ij}		0				0
C		0				0
C_i	0	0	0	0	0	0

16 primary (π, π^i, $\tilde{\pi}^{ij}$, p^{ij})

4 secondary (C, C_i)

Case 1:
Nullity = 2

$$\begin{aligned}\#_{\rm DOF} &= \frac{1}{2}\left(2\times\#_{\rm var} - 2\times\#_1 - \#_2\right)\\ &= \frac{1}{2}\left(2\times 16 - 2\times 8 - 12\right)\\ &= 2\end{aligned}$$

Playing with the Poisson brackets

[**XG**, Z.-B. Yao, 1910.13995]

		π	π^i	$\tilde{\pi}^{ij}$	p^{ij}	C	C_i
16 primary	π	0	0	0	0	0	0
	π^i	0	0	0	0	0	0
	$\tilde{\pi}^{ij}$	0	0			0	0
	p^{ij}	0	0			0	0
4 secondary	C	0	0	0	0	0	0
	C_i	0	0	0	0	0	0

Case 1:
Nullity = 2

Case of GR.
(Enhancement of the time symm.)

$$
\begin{aligned}
\#_{\mathrm{DOF}} &= \frac{1}{2}(2\times\#_{\mathrm{var}} - 2\times\#_1 - \#_2) \\
&= \frac{1}{2}(2\times 16 - 2\times 8 - 12) \\
&= 2
\end{aligned}
$$

Playing with the Poisson brackets

[**XG**, Z.-B. Yao, 1910.13995]

	π	π^i	$\tilde{\pi}^{ij}$	p^{ij}	C	C_i	Φ
π		0				0	
π^i	0	0	0	0	0	0	0
$\tilde{\pi}^{ij}$		0				0	
p^{ij}		0				0	
C		0				0	
C_i	0	0	0	0	0	0	0
Φ		0				0	

16 primary

5 secondary

Case 2:

Nullity = 1

$$
\begin{aligned}
\#_{\mathrm{DOF}} &= \frac{1}{2}\left(2 \times \#_{\mathrm{var}} - 2 \times \#_1 - \#_2\right) \\
&= \frac{1}{2}\left(2 \times 16 - 2 \times 7 - 14\right) \\
&= 2
\end{aligned}
$$

TTDOF conditions

We got 2 TTDOF conditions.

*[**XG**, Z.-B. Yao, 1910.13995]*

(1) degeneracy condition:

The $\{N, K_{ij}\}$-sector must be degenerate.

$$0 \approx \mathcal{S}(\vec{x},\vec{y}) \equiv \frac{\delta^2 S_B}{\delta N(\vec{x})\,\delta N(\vec{y})} - \int \mathrm{d}^3x' \int \mathrm{d}^3y' N(\vec{x}')\,\frac{\delta}{\delta N(\vec{x})}\left(\frac{1}{N(\vec{x}')}\frac{\delta S_B}{\delta B_{i'j'}(\vec{x}')}\right) \times \mathcal{G}_{i'j',k'l'}(\vec{x}',\vec{y}')\,N(\vec{y}')\,\frac{\delta}{\delta N(\vec{y})}\left(\frac{1}{N(\vec{y}')}\frac{\delta S_B}{\delta B_{k'l'}(\vec{y}')}\right)$$

(2) consistency condition:

To completely eliminate the single (extra scalar) DOF, otherwise the dimension of the phase space would be odd.

$$0 \approx \mathcal{J}(\vec{x},\vec{y}) \equiv \int \mathrm{d}^3x' \int \mathrm{d}^3y' \int \mathrm{d}^3x'' \int \mathrm{d}^3y'' \frac{\delta C(\vec{x})}{\delta B_{ij}(\vec{x}')}\mathcal{G}_{ij,i'j'}(\vec{x}',\vec{x}'') \times N(\vec{x}'')\,\frac{\delta^2 S_B}{\delta h_{i'j'}(\vec{x}'')\,\delta B_{k'l'}(\vec{y}'')}\mathcal{G}_{k'l',kl}(\vec{y}'',\vec{y}')\,\frac{\delta C(\vec{y})}{\delta B_{kl}(\vec{y}')} - \int \mathrm{d}^3x' \int \mathrm{d}^3y' \frac{\delta C(\vec{x})}{\delta B_{ij}(\vec{x}')}\mathcal{G}_{ij,kl}(\vec{x}',\vec{y}')\,N(\vec{y}')\,\frac{\delta C(\vec{y})}{\delta h_{kl}(\vec{y}')} - (\vec{x}\leftrightarrow\vec{y})$$

Special cases

(1) "minimally modified gravity"

[C. Lin, S. Mukohyama, 1708.03757]

$$S = \int \mathrm{d}t\mathrm{d}^3x\sqrt{h}N\, F\left(K_{ij}, R_{ij}, h_{ij}, \nabla_i; t\right)$$

The "degeneracy" condition is automatically satisfied, while the "consistency" condition must be satisfied.

(2) "extended cuscuton" (SCG without spatial der.)

[A. Iyonaga, K. Takahashi, T. Kobayashi, 1809.10935]

$$S = \int \mathrm{d}t\mathrm{d}^3x\, N\sqrt{h}\,\mathcal{L}\left(N, h_{ij}, K_{ij}, R_{ij}; t\right)$$

The "degeneracy" condition must be satisfied, then the "consistency" condition is also needed.

A concrete example

*[**XG**, Z.-B. Yao, 1910.13995]*

Quadratic in K_{ij}:

$$S^{(\mathrm{quad})} = \int \mathrm{d}t\mathrm{d}^3x\, N\sqrt{h}\Big[\frac{N}{\beta_2+N}K^{ij}K_{ij} - \frac{1}{3}\left(\frac{2N}{\beta_1+N}+\frac{N}{\beta_2+N}\right)K^2 + \rho_1 + \rho_2 R + \frac{1}{N}\left(\rho_3+\rho_4 R\right)\Big],$$

GR: $\beta_1=\beta_2=\rho_3=\rho_4=0,\qquad \rho_1=\mathrm{const.}\qquad \rho_2=1$

Cuscuton: $\beta_1=\beta_2=\rho_4=0,\qquad \rho_2=1$

A concrete example

[A. Iyonaga, T. Kobayashi, 2109.10615]

II. SPATIALLY COVARIANT GRAVITY WITH TWO TENSORIAL DEGREES OF FREEDOM

In this section, we introduce a class of spatially covariant modified gravity in which there are just two tensorial d.o.f. [19] and in particular there is no scalar d.o.f.

A. "Scalarless" scalar-tensor theories

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x N\sqrt{\gamma}\left[\frac{\beta_0 N}{\beta_2 + N}K_{ij}K^{ij} - \frac{\beta_0}{3}\left(\frac{2N}{\beta_1 + N} + \frac{N}{\beta_2 + N}\right)K^2 + \alpha_1 + \alpha_2 R + \frac{1}{N}(\alpha_3 + \alpha_4 R)\right], \quad (3)$$

- evade solar system tests as far as the ppN parameter γ is concerned;
- $c_{GW} = 1$ (as $\alpha_2 = 1, \alpha_4 = 0$)
- no modification to asymptotically flat black holes (at rest with respect to the preferred frame);
- cosmological dynamics can be identical to ΛCDM.

A concrete example

[N. Bartolo, A. Ganz, S. Matarrese, 2111.06794]

3 Generalized Cuscuton

In [4] the authors constructed a spatial covariant gravity theory up to the quadratic level in derivatives of the metric with just two tensor degrees of freedom which can be understood as a generalization of the cuscuton model in the unitary gauge. The action is given by

$$S = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{h}N\left[\frac{N}{b_2+N}K_{ij}K^{ij} - \frac{1}{3}\left(\frac{2N}{b_1+N}+\frac{N}{b_2+N}\right)K^2 + \rho_1 + (1+\rho_2)\bar{R} + \frac{1}{N}(\rho_3+\rho_4\bar{R})\right], \tag{3.1}$$

$$\delta_2 S = \int \mathrm{d}^3k\mathrm{d}\tau\,\frac{1}{2}z^2(k,\tau)\left[\tilde{\xi}'^2 - c_s^2(k,\tau)\frac{k^2}{a^2}\tilde{\xi}^2\right], \tag{3.6}$$

where

$$z^2(k,\tau) = 2a^2\alpha\frac{b(1+\rho_2)^2k^4 + d_1\mathcal{H}^2k^2 + d_2\mathcal{H}^4}{b(1+\rho_2)^2k^4 + d_3\mathcal{H}^2k^2 + d_4\mathcal{H}^4}, \tag{3.7}$$

$$c_s^2(k,\tau) = \frac{b^2(1+\rho_2)^4k^8 + bd_5\mathcal{H}^2k^6 + d_6\mathcal{H}^4k^4 + d_7\mathcal{H}^6k^2 + d_8\mathcal{H}^8}{b^2(1+\rho_2)^2k^8 + bd_9\mathcal{H}^2k^6 + d_{10}\mathcal{H}^4k^4 + d_{11}\mathcal{H}^6k^2 + d_{12}\mathcal{H}^8}, \tag{3.8}$$

Perturbative analysis

Construct the Lagrangian perturbatively

*[Y. Hu, **XG**, 2104.07615]*

TTDOF conditions are too complicated.

Question: how to find concrete Lagrangians?

A simple example

[Y. Hu, **XG**, 2104.07615]

The most general local and spatially covariant Lagrangian with d=2:

$$\mathcal{L} = c_1 K_{ij} K^{ij} + c_2 a_i a^i + c_3 K^2 + c_4 R$$

Expanding around an FRW background:

$$N = \bar{N} e^{A}, \qquad N_i = \bar{N} a\, \partial_i B, \qquad h_{ij} = a^2 e^{2\zeta} \delta_{ij}.$$

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_2(A, B, \zeta, \dot{\zeta}) + \mathcal{L}_3(A, B, \zeta, \dot{\zeta}) + \mathcal{L}_4(A, B, \zeta, \dot{\zeta}) + \cdots \\ & \Downarrow\, A=A(\zeta,\dot{\zeta}), \quad B=B(\zeta,\dot{\zeta}) \\ &= \mathcal{L}_2(\zeta, \dot{\zeta}) + \mathcal{L}_3(\zeta, \dot{\zeta}) + \mathcal{L}_4(\zeta, \dot{\zeta}) + \end{aligned}$$

By killing the kinetic term $\dot{\zeta}$ order by order, we can determine the coefficients.

A simple example

[Y. Hu, **XG**, 2104.07615]

- Killing $\dot{\zeta}$ at the 2nd order:

$$\mathcal{L} = c_1 \hat{K}_{ij}\hat{K}^{ij} + \frac{1}{3}\frac{C_1 N}{1+C_2 N}K^2 + c_4 R, \qquad \hat{K}_{ij} = K_{ij} - \frac{1}{3}K h_{ij}$$

GR is a special case:

$$c_1 = c_4 = 1, \qquad |C_1|, |C_2| \to \infty, \qquad \text{keeping} \qquad \frac{C_1}{C_2} = -2.$$

- Killing $\dot{\zeta}$ at the 3rd order:

$$\mathcal{L} = \frac{C_5 C_1 N}{1 - 2C_5 C_2 N}\left(K_{ij}K^{ij} - \frac{1}{3}K^2\right) + \frac{1}{3}\frac{C_1 N}{1+C_2 N}K^2 + \left(C_3 + \frac{C_4}{N}\right)R$$

A simple example

*[Y. Hu, **XG**, 2104.07615]*

Lagrangian got by the perturbative analysis:

$$\mathcal{L} = \frac{C_5 C_1 N}{1 - 2C_5 C_2 N}\left(K_{ij}K^{ij} - \frac{1}{3}K^2\right) + \frac{1}{3}\frac{C_1 N}{1 + C_2 N}K^2 + \left(C_3 + \frac{C_4}{N}\right)R$$

↕ Reproduce the full TTDOF Lagrangian.

Lagrangian got by constraint analysis:

*[**XG**, Z.-B. Yao, 1910.13995]*

$$S^{(\mathrm{quad})} = \int \mathrm{d}t\mathrm{d}^3x\, N\sqrt{h}\Bigg[\frac{N}{\beta_2 + N}K^{ij}K_{ij} - \frac{1}{3}\left(\frac{2N}{\beta_1 + N} + \frac{N}{\beta_2 + N}\right)K^2 + \rho_1 + \rho_2 R + \frac{1}{N}\left(\rho_3 + \rho_4 R\right)\Bigg],$$

TTDOFs with an auxiliary constraint

Working directly with the Hamiltonian

(local) Lagrangian

↓

Hamiltonian

↓

TTDOF conditions
(complicated)

(local) Hamiltonian

↓

TTDOF conditions
(simplified)

↓

Lagrangian

Working directly with the Hamiltonian

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]

A 20-dim phase space: $\Phi_I = \{N, N^i, h_{ij}\}$ $\Pi^I = \{\pi, \pi_i, \pi^{ij}\}$

$$H_\mathrm{T} = \int \mathrm{d}^3x\sqrt{h}\left[\mathcal{H}\left(N, h_{ij}, \pi^{ij}; \nabla_i\right) + \nu\varphi\left(N, h_{ij}, \pi^{ij}; \nabla_i\right) + \lambda\pi + N^i\mathcal{H}_i + \lambda^i\pi_i\right]$$

free function — auxiliary constraint — N is non-dyn. — spatial diff.

$$\mathcal{H}_i = \sum_I \Pi^I \pounds_{\vec{N}}\Phi_I$$

8 constraints: $\pi \approx 0,$ $\pi_i \approx 0,$ $\mathcal{H}_i \approx 0,$ $\varphi \approx 0$

	π	π_i	$\mathcal{H}_i$	φ
π	X	0	0	X
π_i	0	0	0	0
$\mathcal{H}_i$	0	0	0	0
φ	X	0	0	X

2 second class

6 first class (spatial diff.)

Working directly with the Hamiltonian

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]

A 20-dim phase space: $\Phi_I = \{N, N^i, h_{ij}\}$ $\Pi^I = \{\pi, \pi_i, \pi^{ij}\}$

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free function — auxiliary constraint — N is non-dyn. — spatial diff.

$$\mathcal{H}_i = \sum_I \Pi^I \pounds_{\vec{N}}\Phi_I$$

8 constraints: $\pi \approx 0,$ $\pi_i \approx 0,$ $\mathcal{H}_i \approx 0,$ $\varphi \approx 0$

$$\begin{aligned}\#_{\mathrm{DOF}} &= \frac{1}{2}\left(2\times\#_{\mathrm{var}} - 2\times\#_1 - \#_2\right)\\ &= \frac{1}{2}\left(2\times 10 - 2\times 6 - 2\right)\\ &= 3\end{aligned}$$

Further requirement on φ and/or $\mathcal{H}$.

Case 1

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]

If $\pi \approx 0$ is pushed to be a first-class constraint.

We need to require (constraint on both φ and $\mathcal{H}$)

$$\frac{\delta\varphi\left(\vec{y}\right)}{\delta N\left(\vec{x}\right)}\approx 0,\qquad \frac{\delta^{2}\int \mathrm{d}^{3}z\,\mathcal{H}\left(\vec{z}\right)}{\delta N\left(\vec{x}\right)\delta N\left(\vec{y}\right)}\approx 0$$

The general solution:

$$H_{\mathrm{T}} = \int \mathrm{d}^{3}x\Big[\mathcal{V}\left(h_{ij},\pi^{ij};\nabla\right)+N\mathcal{H}_{0}\left(h_{ij},\pi^{ij};\nabla\right)+\nu\varphi_{0}\left(h_{ij},\pi^{ij};\nabla\right)$$
$$+\lambda\pi+N^{i}\mathcal{H}_{i}+\lambda^{i}\pi_{i}\Big]$$

	π	π_i	$\mathcal{H}_i$	φ
π	X	0	0	X
π_i	0	0	0	0
$\mathcal{H}_i$	0	0	0	0
φ	X	0	0	X

Case 1

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

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$$\frac{\delta\varphi(\vec{y})}{\delta N(\vec{x})} \approx 0, \qquad \frac{\delta^2 \int \mathrm{d}^3 z\, \mathcal{H}(\vec{z})}{\delta N(\vec{x})\,\delta N(\vec{y})} \approx 0$$

The general solution:

$$H_{\mathrm{T}} = \int \mathrm{d}^3 x \Big[\mathcal{V}\left(h_{ij},\pi^{ij};\nabla\right) + N\mathcal{H}_0\left(h_{ij},\pi^{ij};\nabla\right) + \nu\varphi_0\left(h_{ij},\pi^{ij};\nabla\right) + \lambda\pi + N^i\mathcal{H}_i + \lambda^i\pi_i\Big]$$

	π	π_i	$\mathcal{H}_i$	φ	$\dot{\pi}$
π	0	0	0	0	0
π_i	0	0	0	0	0
$\mathcal{H}_i$	0	0	0	0	0
φ	0	0	0	X	X
$\dot{\pi}$	0	0	0	X	X

7 first class (π, π_i, $\mathcal{H}_i$)

2 second class (φ, $\dot{\pi}$)

Case 1

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

If $\pi \approx 0$ is pushed to be a first-class constraint.

We need to require (constraint on both φ and $\mathcal{H}$)

$$\frac{\delta\varphi\left(\vec{y}\right)}{\delta N\left(\vec{x}\right)}\approx0,\qquad\frac{\delta^{2}\int\mathrm{d}^{3}z\,\mathcal{H}\left(\vec{z}\right)}{\delta N\left(\vec{x}\right)\delta N\left(\vec{y}\right)}\approx0$$

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$$\begin{aligned}H_{\mathrm{T}} &= \int\mathrm{d}^{3}x\Big[\mathcal{V}\left(h_{ij},\pi^{ij};\nabla\right)+N\mathcal{H}_{0}\left(h_{ij},\pi^{ij};\nabla\right)+\nu\varphi_{0}\left(h_{ij},\pi^{ij};\nabla\right)\\ &\quad+\lambda\pi+N^{i}\mathcal{H}_{i}+\lambda^{i}\pi_{i}\Big]\end{aligned}$$

$$\begin{aligned}\#_{\mathrm{DOF}} &= \frac{1}{2}\left(2\times\#_{\mathrm{var}}-2\times\#_{1}-\#_{2}\right)\\ &= \frac{1}{2}\left(2\times10-2\times(6+1)-(2-1+1)\right)\\ &= 2\end{aligned}$$

Case 2

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

If $\pi \approx 0$ remains to be a second-class constraint.

We need to require (constraint on φ only)

$$\frac{\delta\varphi(\vec{y})}{\delta N(\vec{x})} \approx 0, \qquad \int \mathrm{d}^3 z \left(\frac{\delta\varphi(\vec{x})}{\delta h_{ij}(\vec{z})} \frac{\delta\varphi(\vec{y})}{\delta \pi^{ij}(\vec{z})} - (\vec{x} \leftrightarrow \vec{y}) \right) \approx 0$$

A special solution:

$$H_{\mathrm{T}} = \int \mathrm{d}^3 x \left[\mathcal{H}(N, h_{ij}, \pi^{ij}; \nabla) + \nu\tilde{\varphi}(h_{ij}, \pi^{ij}) + \lambda\pi + N^i \mathcal{H}_i + \lambda^i \pi_i \right]$$

	π	π_i	$\mathcal{H}_i$	$\mathcal{H}_i$
π	X	0	0	X
π_i	0	0	0	0
$\mathcal{H}_i$	0	0	0	0
φ	X	0	0	X

Case 2

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]

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$$\frac{\delta\varphi(\vec{y})}{\delta N(\vec{x})} \approx 0, \qquad \int \mathrm{d}^3 z \left(\frac{\delta\varphi(\vec{x})}{\delta h_{ij}(\vec{z})} \frac{\delta\varphi(\vec{y})}{\delta\pi^{ij}(\vec{z})} - (\vec{x} \leftrightarrow \vec{y}) \right) \approx 0$$

A special solution:

$$H_{\mathrm{T}} = \int \mathrm{d}^3 x \left[\mathcal{H}\left(N, h_{ij}, \pi^{ij}; \nabla\right) + \nu\tilde{\varphi}\left(h_{ij}, \pi^{ij}\right) + \lambda\pi + N^i \mathcal{H}_i + \lambda^i \pi_i \right]$$

	π	π_i	$\mathcal{H}_i$	φ	$\dot{\pi}$	$\dot{\varphi}$
π	X	0	0	X	X	X
π_i	0	0	0	0	0	0
$\mathcal{H}_i$	0	0	0	0	0	0
φ	X	0	0	X	X	X
$\dot{\pi}$	X	0	0	X	X	X
$\dot{\varphi}$	X	0	0	X	X	X

6 first class

4 second class

Case 2

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2011.00805]*

If $\pi \approx 0$ remains to be a second-class constraint.

We need to require (constraint on φ only)

$$\frac{\delta\varphi(\vec{y})}{\delta N(\vec{x})} \approx 0, \qquad \int \mathrm{d}^3 z \left(\frac{\delta\varphi(\vec{x})}{\delta h_{ij}(\vec{z})} \frac{\delta\varphi(\vec{y})}{\delta \pi^{ij}(\vec{z})} - (\vec{x} \leftrightarrow \vec{y}) \right) \approx 0$$

A special solution:

$$H_{\mathrm{T}} = \int \mathrm{d}^3 x \left[\mathcal{H}\left(N, h_{ij}, \pi^{ij}; \nabla\right) + \nu \tilde{\varphi}\left(h_{ij}, \pi^{ij}\right) + \lambda\pi + N^i \mathcal{H}_i + \lambda^i \pi_i \right]$$

$$\begin{aligned} \#_{\mathrm{DOF}} &= \frac{1}{2}\left(2 \times \#_{\mathrm{var}} - 2 \times \#_1 - \#_2\right) \\ &= \frac{1}{2}\left(2 \times 10 - 2 \times 6 - (2+2)\right) \\ &= 2 \end{aligned}$$

With multiple auxiliary constraints

Multiple auxiliary constraints

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

A general Hamiltonian with multiple auxiliary constraints:

$$
\begin{aligned}
H_{\mathrm{T}} \quad = \quad & \int \mathrm{d}^3x \Big[\mathscr{H}\left(N, \pi, h_{ij}, \pi^{ij}; \nabla_i\right) + N^i \mathcal{H}_i + \lambda^i \pi_i \\
& + \mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + \nu^i_{\mathrm{m}} \mathcal{V}^{\mathrm{m}}_i + \rho^{ij}_{\mathrm{r}} \mathcal{T}^{\mathrm{r}}_{ij} \Big]
\end{aligned}
$$

Multiple auxiliary constraints

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]

A general Hamiltonian with multiple auxiliary constraints:

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \Big[\mathscr{H}\left(N, \pi, h_{ij}, \pi^{ij}; \nabla_i\right) + \boxed{N^i \mathcal{H}_i + \lambda^i \pi_i} + \mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + \nu^i_{\mathrm{m}} \mathcal{V}^{\mathrm{m}}_i + \rho^{ij}_{\mathrm{r}} \mathcal{T}^{\mathrm{r}}_{ij} \Big]$$

$$N^i \mathcal{H}_i \simeq \sum_I \Pi^I \pounds_{\vec{N}} \Phi_I = \pi \pounds_{\vec{N}} N + \pi_i \pounds_{\vec{N}} N^i + \pi^{ij} \pounds_{\vec{N}} h_{ij}$$

$$[\mathcal{H}_i, Q] \approx 0, \quad \forall Q \approx 0 \qquad\qquad [\pi_i, Q] = -\frac{\delta Q}{\delta N^i} \equiv 0$$

$$\mathcal{H}_i \approx 0_i, \quad \pi_i \approx 0_i$$

first class ➡ spatial diffeomorphism

Multiple auxiliary constraints

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

A general Hamiltonian with multiple auxiliary constraints:

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \Big[\mathscr{H}\left(N, \pi, h_{ij}, \pi^{ij}; \nabla_i\right) + N^i \mathcal{H}_i + \lambda^i \pi_i$$
$$+\mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + \nu^i_{\mathrm{m}} \mathcal{V}^{\mathrm{m}}_i + \rho^{ij}_{\mathrm{r}} \mathcal{T}^{\mathrm{r}}_{ij} \Big]$$

Auxiliary constraints:

(scalar)	$\mathcal{S}^{\mathrm{n}} \approx 0^{\mathrm{n}},$	$\mathrm{n} = 1, \cdots, \mathcal{N}$
(vector)	$\mathcal{V}^{\mathrm{m}}_i \approx 0^{\mathrm{m}}_i,$	$\mathrm{m} = 1, \cdots, \mathcal{M}$
(tensor)	$\mathcal{T}^{\mathrm{r}}_{ij} \approx 0^{\mathrm{r}}_{ij},$	$\mathrm{r} = 1, \cdots, \mathcal{R}$

We assume them to be second class (the general case).

Multiple auxiliary constraints

[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]

A general Hamiltonian with multiple auxiliary constraints:

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \Big[\mathscr{H}\left(N, \pi, h_{ij}, \pi^{ij}; \nabla_i\right) + N^i \mathcal{H}_i + \lambda^i \pi_i + \mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + \nu^i_{\mathrm{m}} \mathcal{V}^{\mathrm{m}}_i + \rho^{ij}_{\mathrm{r}} \mathcal{T}^{\mathrm{r}}_{ij} \Big]$$

$$\begin{aligned}
\#_{\mathrm{dof}} &= \frac{1}{2}\left(\#_{\mathrm{var}} \times 2 - \#_{\mathrm{1st}} \times 2 - \#_{\mathrm{2nd}}\right) \\
&= \frac{1}{2}\Big[\left(4_{\mathrm{s}} + 4_{\mathrm{v}} + 2_{\mathrm{t}}\right) \times 2 - \left(1_{\mathrm{s}} + 2_{\mathrm{v}}\right) \times 2 \times 2 \\
&\quad - 1_{\mathrm{s}} \times \mathcal{N} - \left(1_{\mathrm{s}} + 2_{\mathrm{v}}\right) \times \mathcal{M} - \left(2_{\mathrm{s}} + 2_{\mathrm{v}} + 2_{\mathrm{t}}\right) \times \mathcal{R} \Big] \\
&= \left(2_{\mathrm{t}} - \mathcal{R}_{\mathrm{t}}\right) - \left(\mathcal{M}_{\mathrm{v}} + \mathcal{R}_{\mathrm{v}}\right) + \frac{1}{2}\left(4_{\mathrm{s}} - \mathcal{N}_{\mathrm{s}} - \mathcal{M}_{\mathrm{s}} - 2 \times \mathcal{R}_{\mathrm{s}}\right)
\end{aligned}$$

If no auxiliary constraints → 4 dof = 2t+2s

Multiple auxiliary constraints

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

A general Hamiltonian with multiple auxiliary constraints:

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \Big[\mathscr{H}\left(N, \pi, h_{ij}, \pi^{ij}; \nabla_i\right) + N^i \mathcal{H}_i + \lambda^i \pi_i + \mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + \nu^i_{\mathrm{m}} \mathcal{V}^{\mathrm{m}}_i + \rho^{ij}_{\mathrm{r}} \mathcal{T}^{\mathrm{r}}_{ij} \Big]$$

$$2 - \mathcal{R} \geq 0, \quad \mathcal{M} + \mathcal{R} \leq 0 \qquad\qquad 4 - \mathcal{N} - \mathcal{M} - 2\mathcal{R} \geq 0$$

$$\mathcal{R} = 0, \quad \mathcal{M} = 0, \quad \mathcal{N} \leq 4$$

No vector nor tensor constraints;
No more than 4 scalar constraints.

Multiple auxiliary constraints

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

A general Hamiltonian with scalar auxiliary constraints:

$$H_{\mathrm{T}} = \int \mathrm{d}^3x \left(\mathscr{H} + \mu_{\mathrm{n}} \mathcal{S}^{\mathrm{n}} + N^i \mathcal{H}_i + \lambda^i \pi_i \right)$$

$$\begin{aligned}
\#_{\mathrm{dof}} &= \frac{1}{2}\left(\#_{\mathrm{var}} \times 2 - \#_{\mathrm{1st}} \times 2 - \#_{\mathrm{2nd}}\right) \\
&= \frac{1}{2}\Big[(4_{\mathrm{s}} + 4_{\mathrm{v}} + 2_{\mathrm{t}}) \times 2 - (1_{\mathrm{s}} + 2_{\mathrm{v}}) \times 2 \times 2 \\
&\qquad - \#^{\mathrm{s}}_{\mathrm{1st}} \times 2 - \#^{\mathrm{s}}_{\mathrm{2nd}} \Big] \\
&= 2_{\mathrm{t}} + \frac{1}{2}\left(4_{\mathrm{s}} - \#^{\mathrm{s}}_{\mathrm{1st}} \times 2 - \#^{\mathrm{s}}_{\mathrm{2nd}}\right)
\end{aligned}$$

$$4 - \#^{\mathrm{s}}_{\mathrm{1st}} \times 2 - \#^{\mathrm{s}}_{\mathrm{2nd}} = 0$$

$$\mathcal{N} \leq \#^{\mathrm{s}}_{\mathrm{1st}} + \#^{\mathrm{s}}_{\mathrm{2nd}} \leq 4$$

Classification of TTDOF theories

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

The "minimalizing" and "symmetrizing" conditions.

# ACs	Minimalizing cond.	Symmetrizing cond.	Classifications
$\mathcal{N}=4$	none	none	$\#^{\mathrm{s}}_{\mathrm{1st}}=0,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=4$
$\mathcal{N}=3$	$[\mathcal{S}^1,\mathcal{S}^{\mathrm{n}}]$	$[\mathcal{S}^1,\mathscr{H}]$	$\#^{\mathrm{s}}_{\mathrm{1st}}=1,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=2$
		none	$\#^{\mathrm{s}}_{\mathrm{1st}}=0,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=4$
$\mathcal{N}=2$	$[\mathcal{S}^1,\mathcal{S}^{\mathrm{n}}]$ & $[\mathcal{S}^2,\mathcal{S}^2]$	$[\mathcal{S}^1,\mathscr{H}]$ & $[\mathcal{S}^2,\mathscr{H}]$	$\#^{\mathrm{s}}_{\mathrm{1st}}=2,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=0$
		$[\mathcal{S}^2,\dot{\mathcal{S}}^1]$ & $[\mathcal{S}^2,\mathscr{H}]$	$\#^{\mathrm{s}}_{\mathrm{1st}}=1,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=2$
		none	$\#^{\mathrm{s}}_{\mathrm{1st}}=0,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=4$
	$[\mathcal{S}^1,\mathcal{S}^{\mathrm{n}}]$ & $[\mathcal{S}^1,\dot{\mathcal{S}}^1]$	$[\dot{\mathcal{S}}^1,H_{\mathrm{P}}]$	$\#^{\mathrm{s}}_{\mathrm{1st}}=1,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=2$
		none	$\#^{\mathrm{s}}_{\mathrm{1st}}=0,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=4$
$\mathcal{N}=1$	$[\mathcal{S}^1,\mathcal{S}^1]$, $[\mathcal{S}^1,\dot{\mathcal{S}}^1]$ & $[\dot{\mathcal{S}}^1,\dot{\mathcal{S}}^1]$	$[\dot{\mathcal{S}}^1,\mathscr{H}]$	$\#^{\mathrm{s}}_{\mathrm{1st}}=2,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=0$
		$[\dot{\mathcal{S}}^1,\ddot{\mathcal{S}}^1]$	$\#^{\mathrm{s}}_{\mathrm{1st}}=1,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=2$
	$[\mathcal{S}^1,\mathcal{S}^1]$, $[\mathcal{S}^1,\dot{\mathcal{S}}^1]$ & $[\mathcal{S}^1,\ddot{\mathcal{S}}^1]$	$[\ddot{\mathcal{S}}^1,\mathscr{H}]$	$\#^{\mathrm{s}}_{\mathrm{1st}}=1,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=2$
		none	$\#^{\mathrm{s}}_{\mathrm{1st}}=0,\ \#^{\mathrm{s}}_{\mathrm{2nd}}=4$

Cayley-Hamilton construction

*[Z.-B. Yao, M. Oliosi, **XG**, S. Mukohyama, 2302.02090]*

$$\begin{aligned}\mathscr{R}^{\mathrm{I}} &\equiv \left\{R^i_i, R^i_j R^j_i, R^i_j R^j_k R^k_i\right\}\\ \mathit{\Pi}^{\mathrm{I}} &\equiv \left\{\pi^i_i, \pi^i_j \pi^j_i, \pi^i_j \pi^j_k \pi^k_i\right\}\\ \mathscr{Q}^{\mathrm{I}} &\equiv \left\{R^i_j \pi^j_i, R^i_j \pi^j_k \pi^k_i, R^i_j R^j_k \pi^k_i\right\}\end{aligned}$$

$$\mathcal{S}^{\mathrm{I}} = \mathscr{Q}^{\mathrm{I}} - \mathscr{P}^{\mathrm{I}}(N) \approx 0, \quad \mathrm{I} = 1,2,3$$

$$H_{\mathrm{T}}^{(\mathrm{C.H.})} = \int \mathrm{d}^3 x \left[\mathscr{H}^{(\mathrm{C.H.})} + N^i \mathcal{H}_i + \lambda^i \pi_i + \lambda \pi + \mu_{\mathrm{I}}\left(\mathscr{Q}^{\mathrm{I}} - \mathscr{P}^{\mathrm{I}}\right)\right],$$

$$\mathscr{H}^{(\mathrm{C.H.})} = \mathscr{H}^{(\mathrm{C.H.})}\left(N, R_{ij}, \pi^{ij}\right)$$

Conclusion

Main message from this talk

There are non-GR theories propagating TTDOF's respecting only spatial covariance.

We find the TTDOF conditions in 2 approaches:

1. (Lagrangian side) spatially covariant gravity with TTDOFs;
2. (Hamiltonian side) TTDOFs with auxiliary constraint(s).

We also show a simpler perturbative analysis to construct concrete Lagrangians.

Open questions:

Concrete Lagrangian (so that can be applied in practice);
Different from/equivalent to GR;
To be tested against the observations;
…

Thank you for your attention!