Bulk reconstruction: surface growth approach, tensor networks and bit threads

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with Y.-y. Lin and Y. Sun, C. Yu, J. Zhang, F.-Z. Chen, J.-C. Jin,
arXiv: 2010.01907, 2010.03167, 2012.05737, 2105.09176, 2203.03111

Outline

- Bulk reconstruction in the AdS/CFT correspondence
- Surface growth approach
- Bit threads, EoP and Entanglement contour
- PEE and entanglement island
- Conclusions and Discussions

Bulk reconstruction in the AdS/CFT correspondence

The AdS/CFT correspondence and the more general holographic duality provide a novel connection between different theories, one is a higher dimensional gravitational theory, another is a quantum field theory without gravity on the boundary.

The key equation in the AdS/CFT correspondence is

$$Z_{\mathrm{AdS}}[\phi_0(\vec{x})] = Z_{\mathrm{CFT}}[\phi_0(\vec{x})] = \left\langle \exp \int d^4 x O(\vec{x}) \phi_0(\vec{x}) \right\rangle$$

Important properties:

field/operator duality,

strong/weak duality.

Important applications:

From the bulk to boundary

--studying the strongly coupled systems from their dual classical gravity;

many successful applications:
fluid/gravity duality,
AdS/CMT,
holographic entanglement entropy,
AdS/QCD,
holographic complexity,
etc.

From the boundary to the bulk

bulk matter fields:

using the boundary operators to construct the bulk matter fields.

Banks, Douglas, Horowitz, Martinec, th/9808016;
Hamilton, Kabat, Lifschytz, and Lowe, th/0606141.

$$\phi(z, x) = \int dx' K(x'|z, x)\phi_0(x').$$

bulk local field ~ boundary nonlocal operators

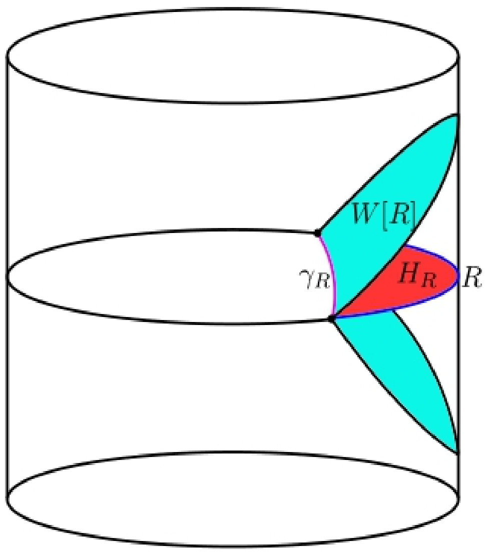
***Question*: what's the largest region that can be constructed from a given boundary region?**

Entanglement wedge reconstruction

Headrick, Hubeny, Lawrence, Rangamani, 2014;
Dong, Harlow, Wall, 2016

$$\mathcal{W}_{\mathcal{E}}[\mathcal{A}] := \tilde{D}[\mathcal{R}_{\mathcal{A}}]\,.$$

subregion-subregion duality.



From the boundary to the bulk

bulk gravity: more difficult to construct the bulk geometry and the gravitational dynamics from the boundary CFT

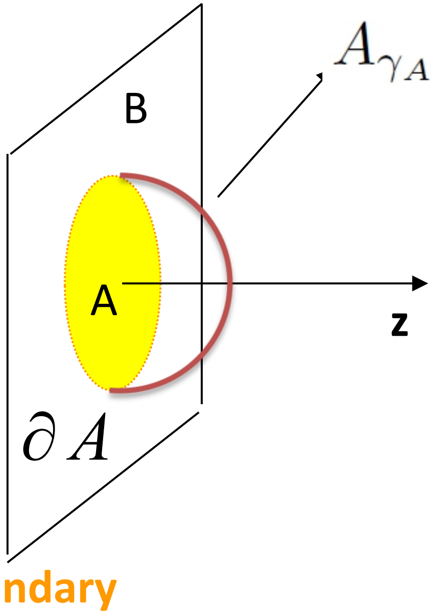
--**an emergent picture of gravity**.

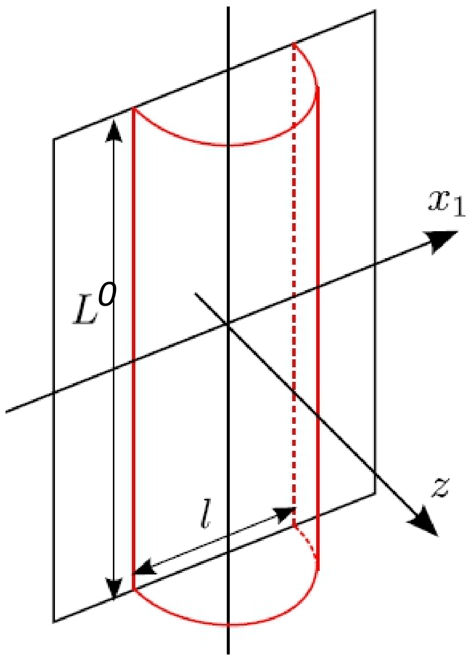
Holographic entanglement entropy

Ryu and Takayanagi 2006

$$S_A = \frac{A_{\gamma_A}}{4G_{d+1}},$$

A_{γ_A} is the *d* dimensional static minimal surface in AdS with boundary ∂A . To calculate entanglement entropy of CFT from the bulk dual gravity.





Lewkowycz and Maldacena 2013

$$S_A = -n\partial_n [\ln Z[n] - n \ln Z[1]]|_{n=1} = \frac{A_{\gamma_A}}{4G}$$

Tensor networks

For quantum manybody systems in condensed matter physics, the ground state wave function is effectively described by a series of tensors which comprise into a network.

Considering a *N* particle quantum manybody system in d-dim flat spacetime, its ground state wave function can be expressed as

$$|\Psi\rangle = \sum_{a_1 \cdots a_N} T_{a_1 \cdots a_N} |a_1\rangle \otimes \cdots \otimes |a_N\rangle,$$

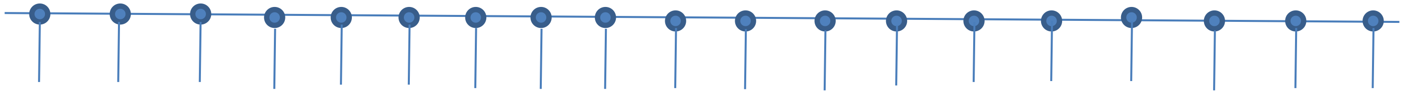
coefficients of a *N*-rank tensor

in the tensor network language, tensor *T* can be reduced into a network of tensors *t* with less rank, e.g.,

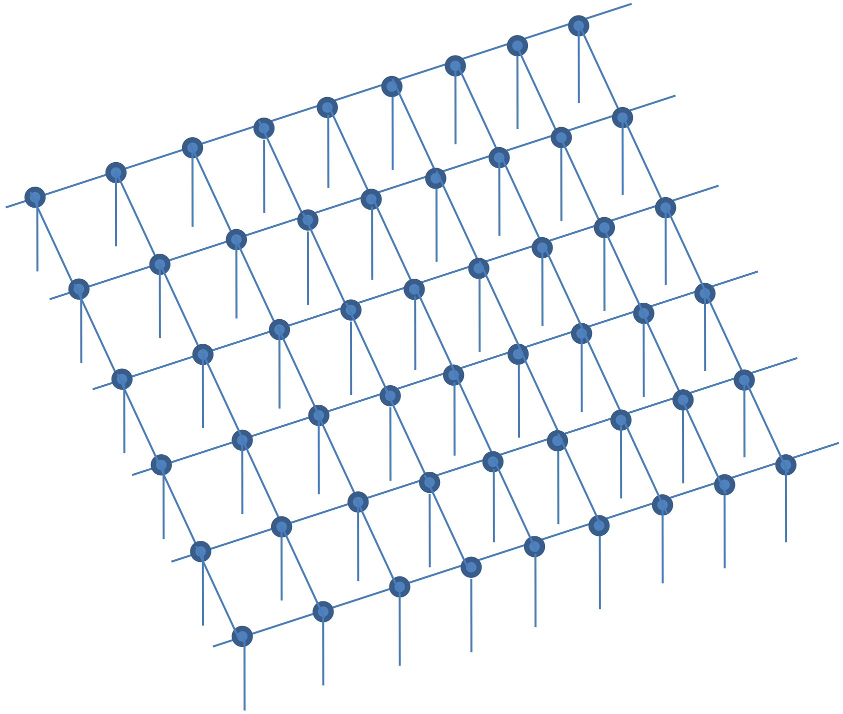
Vidal, 1106.1082;
Orus, 1306.2164

$$T_{a_1 a_2 a_3} = t_{a_1 b} s_{b a_2 c} t_{c a_3}$$

Matrix product state (**MPS**) for ground state of 1d lattice

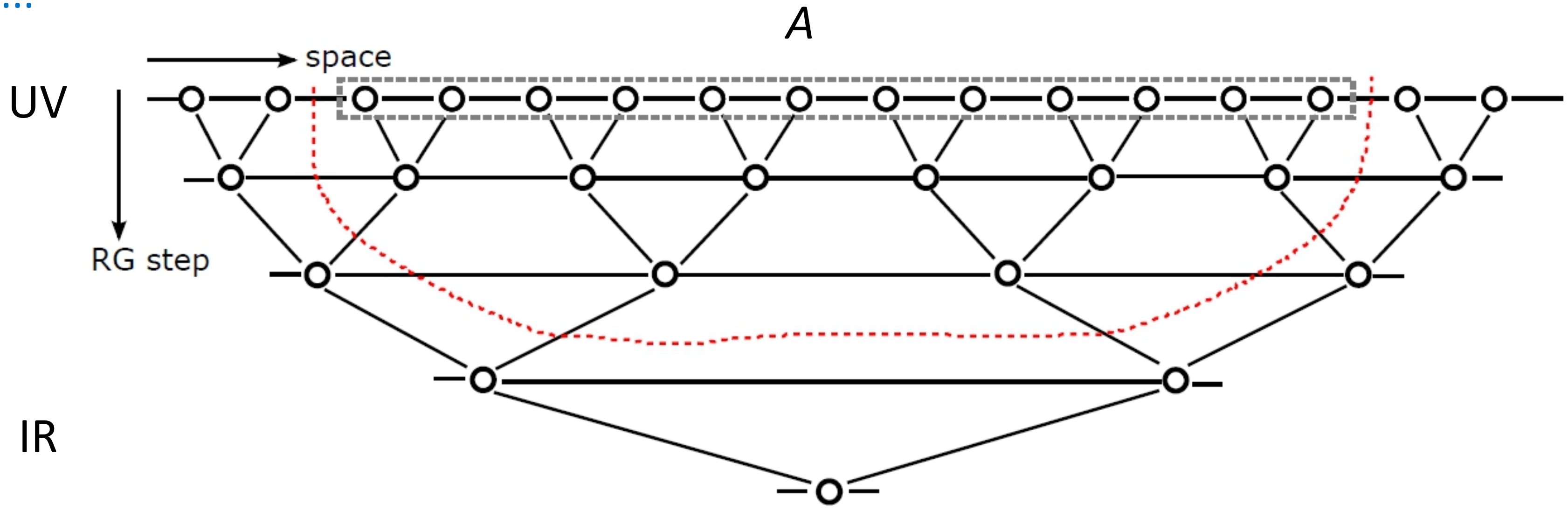


Projected entangled pair state (**PEPS**) for ground state of 2d lattice



Emergence of AdS geometry from MERA tensor networks

Swingle, 0905.1317; 1209.3304;
Qi, 1309.6282;
Almheiri, Dong, Harlow, 1411.7041;
Pastawski, Yoshida, Harlow, Preskill, 1503.06237;
Hayden, Nezami, Qi, Thomas, Walter, Yang, 1601.01694;
Bhattacharyya, Gao, Hung, Liu, 1606.00621;
Gan and Shu, 1705.05750;
Ling, Xiao and Wu, 1907.01215
...



$S_A \propto$ # of external legs of the tensor networks

Assume at each step, *k* # of sites will be coarse grained, then considering two operators O_1 and O_2 separated by *x*, then

$\log_k \frac{x}{a}$ times of renormalization should be taken for them to

connect, and at each step

$$\left\langle O(x_m)\right\rangle = \left|\frac{\partial x_{m-1}}{\partial x_m}\right|^{\Delta} \left\langle O(x_{m-1})\right\rangle = k^{-\Delta}\left\langle O(x_{m-1})\right\rangle$$

then

$$\left\langle O_1 O_2\right\rangle \sim \left(k^{-2\Delta}\right)^{\log_k \frac{x}{a}} = \left(\frac{x}{a}\right)^{-2\Delta} = \exp\left\{-2\Delta \ln\frac{x}{a}\right\}$$

The entanglement RG is consistent with taking a geodesic, i.e. minimal curve in a AdS3 geometry, or the tensor networks can be viewed a **discrete** version of the AdS3.

The entanglement RG is just a scaling transformation, the corresponding geometry (static case) can be expressed as

$$ds^2 = L^2\left(\frac{dr^2 + dx^2}{r^2}\right)$$

for a curve $x(\lambda) = x_0 \cos \pi\lambda$ and $r(\lambda) = x_0 \sin \pi\lambda$

which corresponds to the boundary is a circle, the length of the curve is

$$s \sim L \ln \frac{x_0}{a}$$

Note that the # of external legs (bonds) cut by the geodesic is proportional to the length of the geodesic.

Continueous version-cMERA, using path integral method.

Haegeman, Osborne, Verschelde and Verstraete, 1102.5524;

Miyaji, Numasawa, Shiba, Takayanagi and Watanabe, 1506.01353

MERA-like tensor networks--time slice of AdS3

Milsted, Vidal, 1805.12524; 1812.00529;

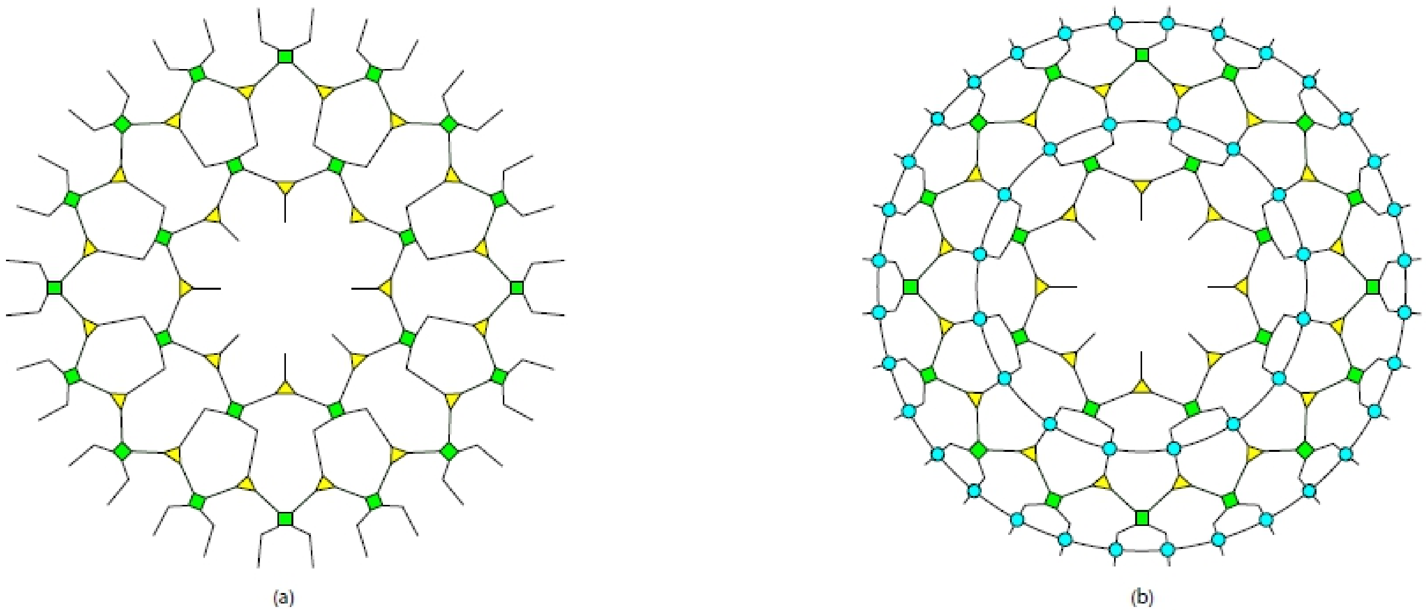
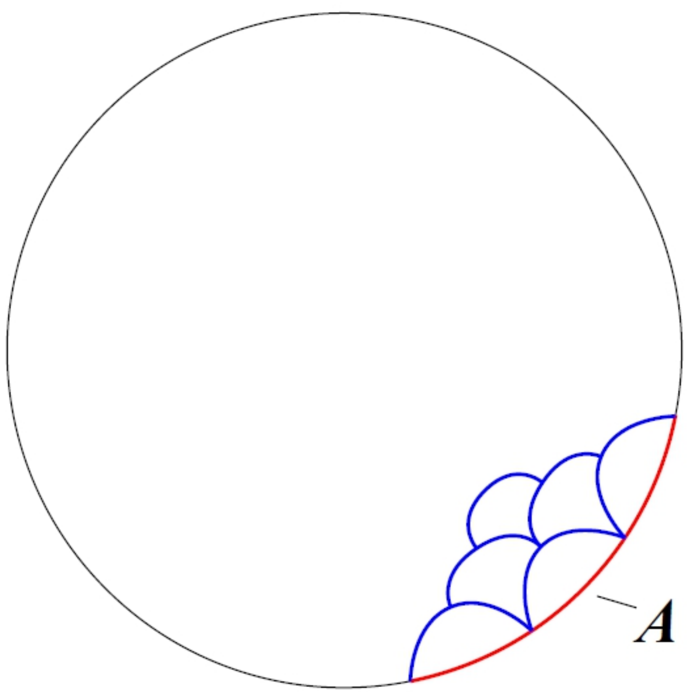


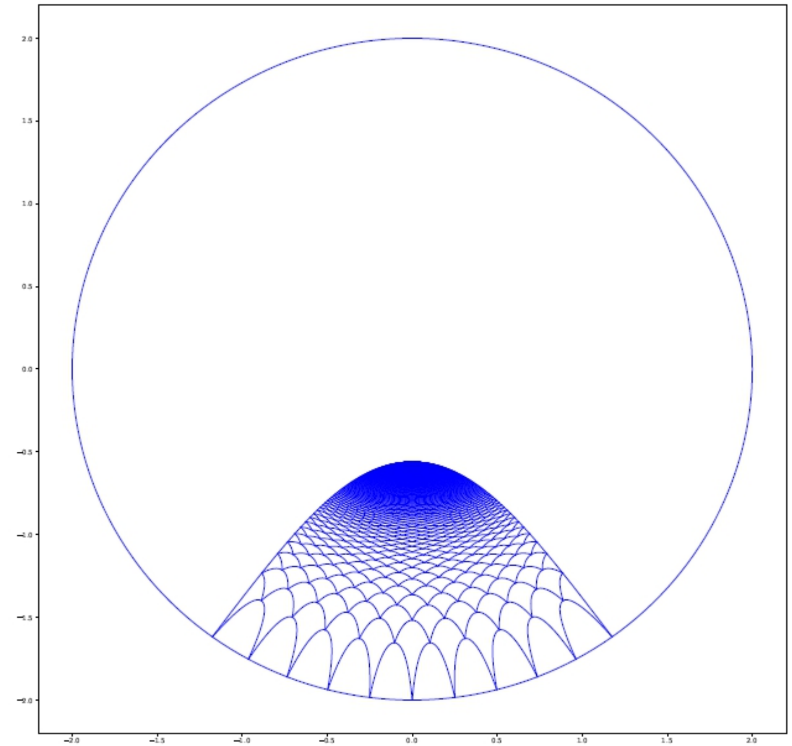
Figure 12. (a) The traditional MERA tensor network on the circle, made of disentanglers u (the green squares) and coarse-grainers w (the yellow triangles). (b) The Euclidean MERA tensor network on the circle, made of disentanglers u (the green squares) and coarse-grainers w (the yellow triangles), and eclideons e (the blue solid circles).

The surface growth approach

Y-y Lin, **JRS**, Y Sun, 2010.01907;
C Yu, F-Z Chen,Y-y Lin, **JRS**, Y Sun, 2010.03167

In previous studies on bulk reconstruction, the methods are indirect, is there a direct and explicit way to construct the bulk geometry and matter fields? Besides, it is interesting to find a more refined structure in the subregion-subregion duality, such as how a given region in the entanglement wedge is dual to a boundary region?

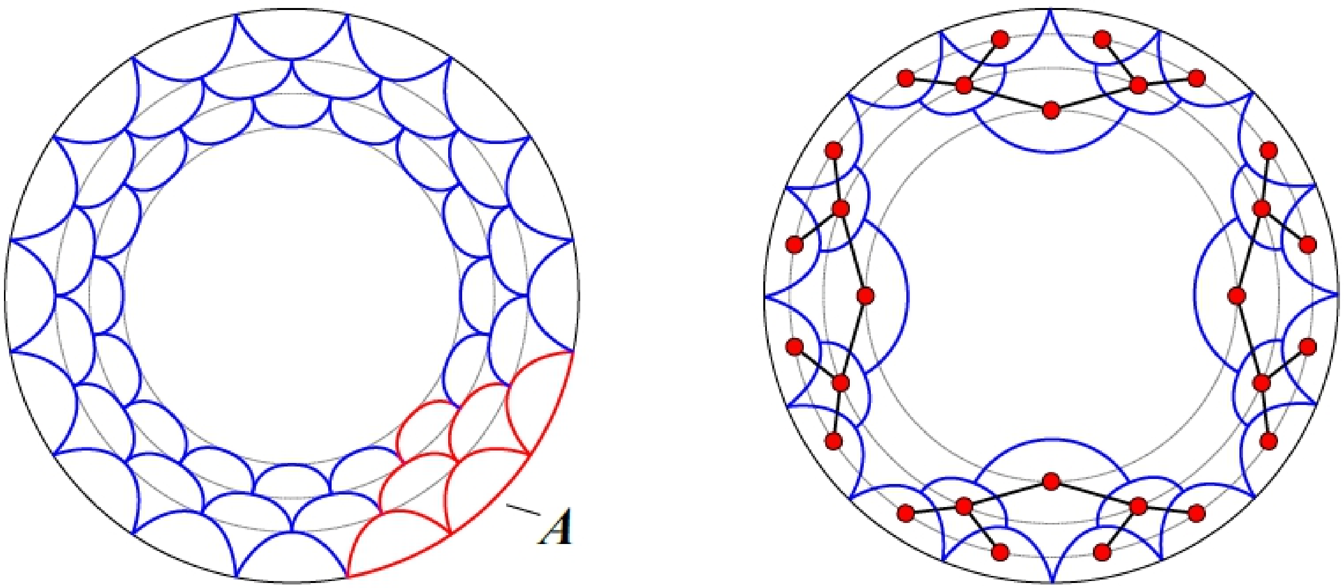




The surface growth approach from tensor networks

Y-y Lin, **JRS**, Y Sun, 2010.01907

Motivated by **Huygens' principle of wave propagation**, we proposed a **novel surface growth scheme** to reconstruct the bulk geometry, which can be explicitly realized with the help of the **surface/state correspondence** and the **one shot entanglement distillation** method.



(a) (b)

One shot entanglement distillation (OSED)

Bao, Penington, Sorce, Wall, 1812.01171

In quantum information theory, $|\varphi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{A^c}$,

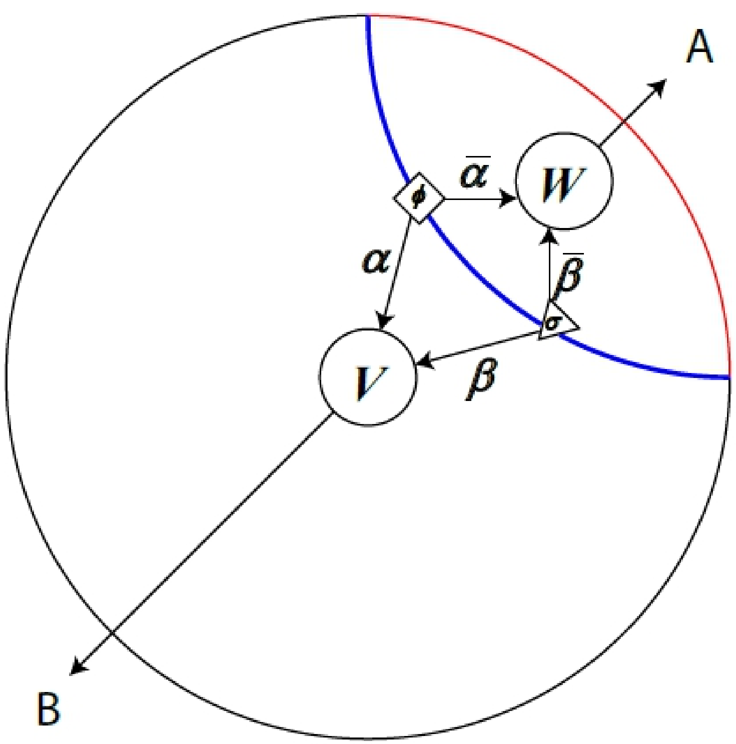
$$|\varphi\rangle^{\otimes m} \approx (W \otimes V)\left[\frac{1}{\sqrt{D}} \sum_{i=0}^{e^{S(A)m-O(\sqrt{m})}} \left|i\bar{i}\right\rangle \otimes \sum_{j=0}^{e^{O(\sqrt{m})}} \sqrt{p_j}\left|j\bar{j}\right\rangle\right],$$

a single holographic state $|\Psi\rangle$ can play the role of $|\varphi\rangle^{\otimes m}$, which has a tensor representation

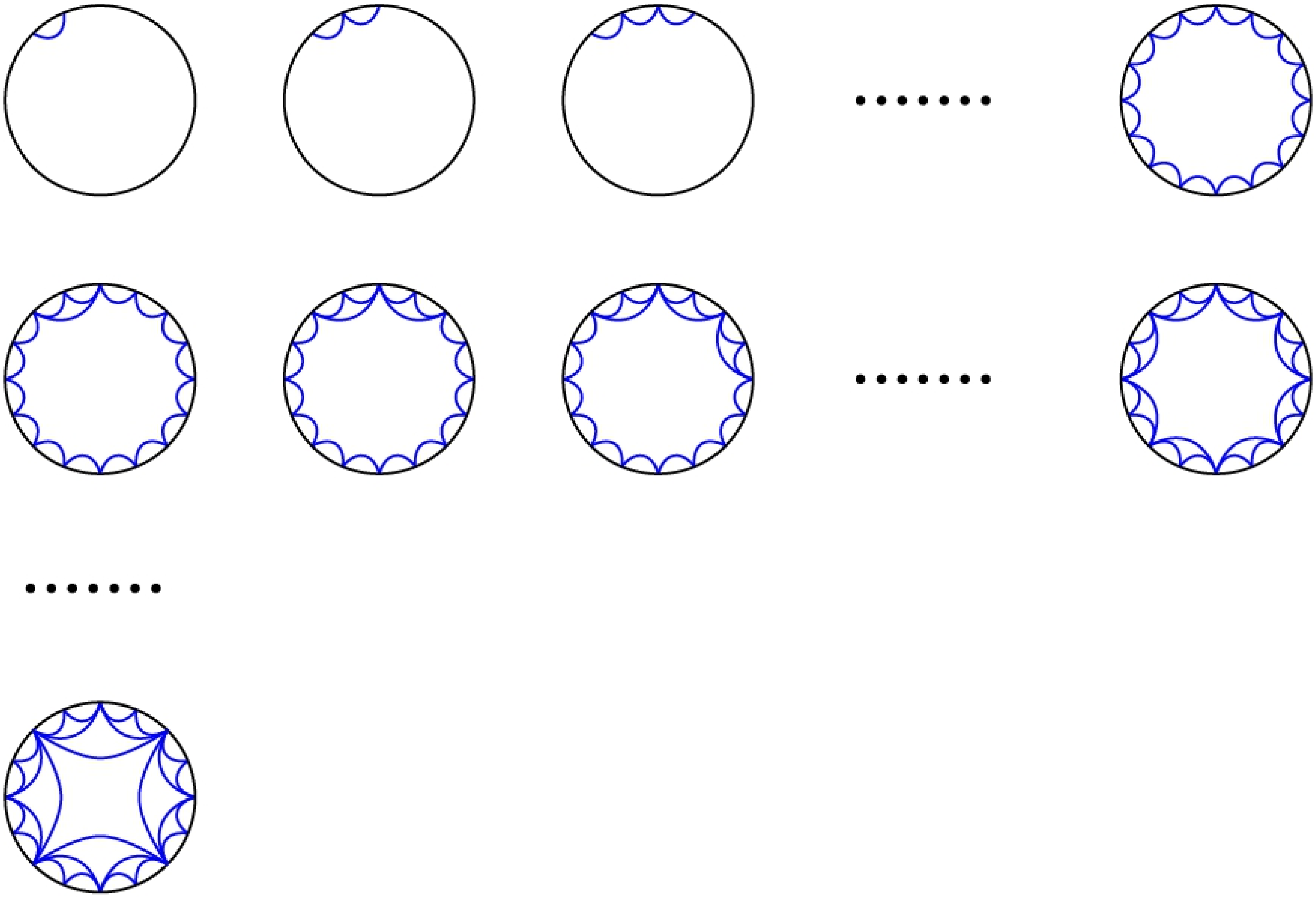
$$\Psi^{AB} = V^B_{\beta\alpha} W^A_{\bar{\beta}\bar{\alpha}} \phi^{\alpha\bar{\alpha}} \sigma^{\beta\bar{\beta}} = (V \otimes W)(|\phi\rangle \otimes |\sigma\rangle)$$

$$|\phi\rangle = \sum_{m=0}^{e^{S-O(\sqrt{S})}} |m\bar{m}\rangle_{\alpha\bar{\alpha}},$$

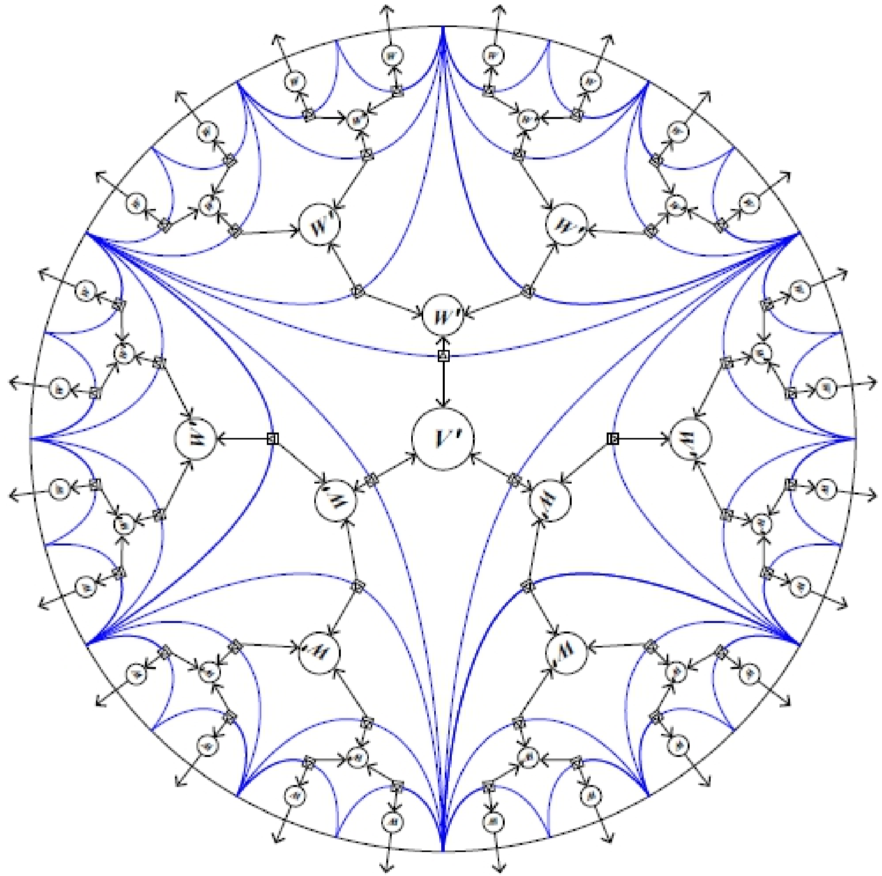
$$|\sigma\rangle = \sum_{n=0}^{e^{O(\sqrt{S})}} \sqrt{\tilde{p}^{avg}_{n\Delta}} |n\bar{n}\rangle_{\beta\bar{\beta}}.$$



Surface growth scheme--a special case

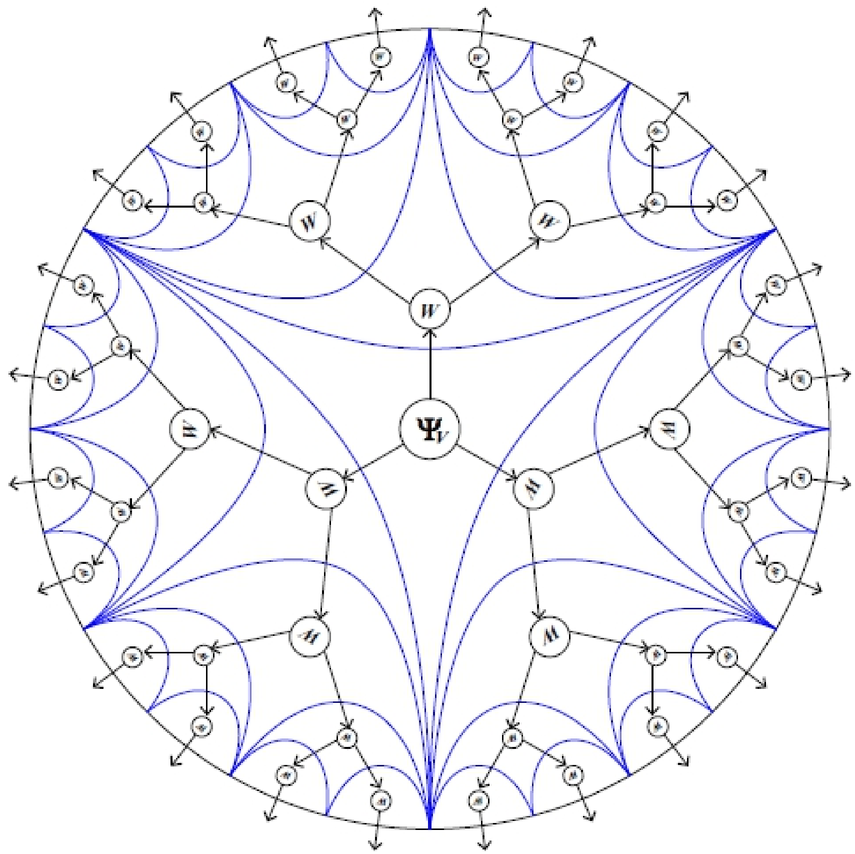


the final surface growth picture corresponding to the OSED tensor network



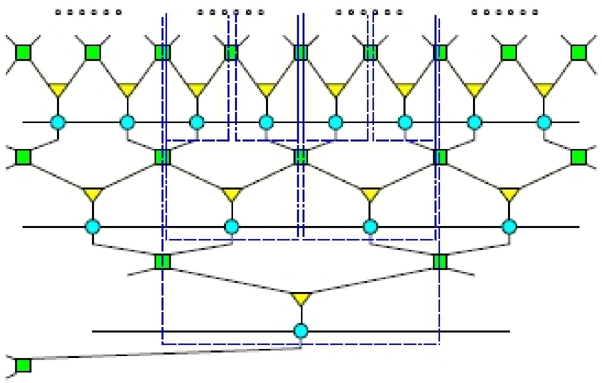
$$|\Psi\rangle^{k\text{th}} = V_M'^{k\text{th}} \prod_{i=1}^{N/2^{k-1}} \prod_{a=1}^{k} W_i'^{a\text{th}} \, |\phi\rangle_i^{a\text{th}} \, |\sigma\rangle_i^{a\text{th}}$$

the final surface growth picture can be identified with the MERA-like tensor network as

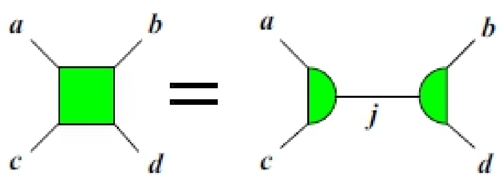


$$|\Psi\rangle^{k\mathrm{th}} = \Psi_V{}^{k\mathrm{th}}_{M} \prod_{i=1}^{N/2^{k-1}} \prod_{a=1}^{k} W_i^{a\mathrm{th}}.$$

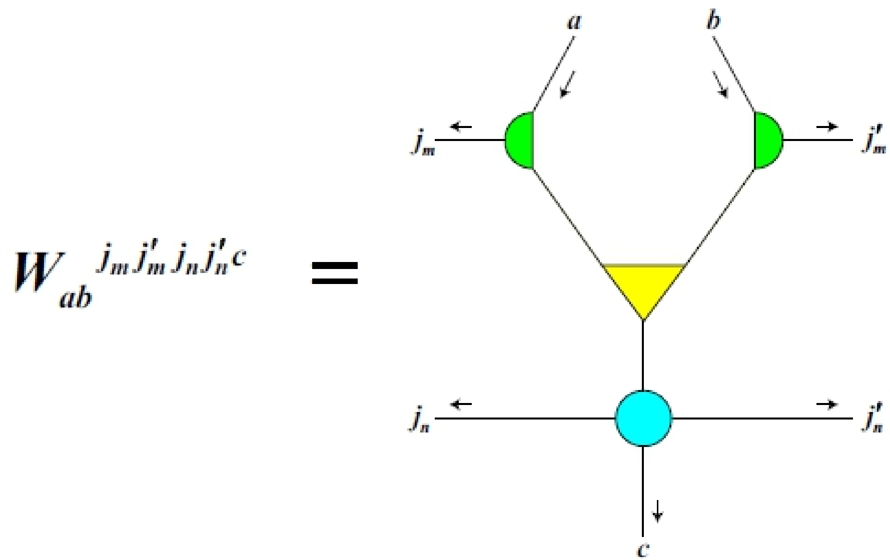
the RT surface in the MERA-like tensor network, and the expression of W tensor is



(a)



(b)



(c)

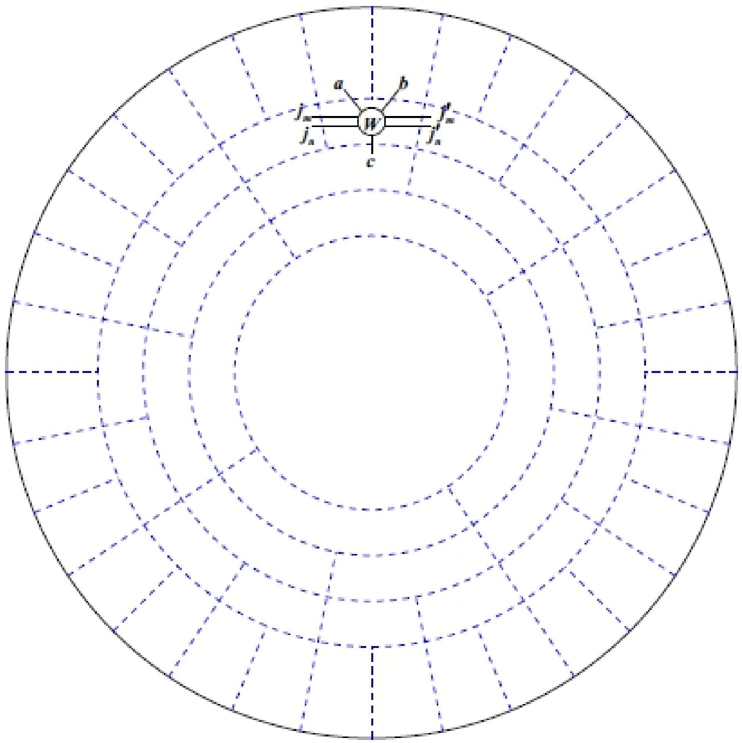
$$W = W_{ab}^{j_m j'_m j_n j'_n c}$$

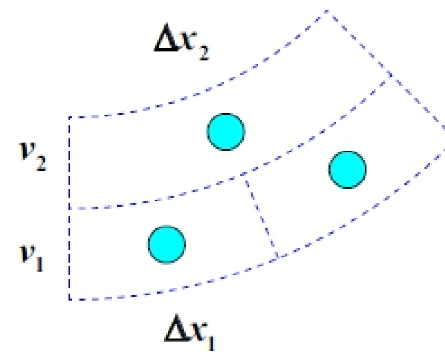
$$u^{abcd} = t^{acj} t^{bd}{}_{j},$$

$$S_A = n \cdot \ln J = (4\log_2 J) \cdot \ln \frac{l}{\varepsilon}.$$

$$\frac{c}{3} = 4\log_2 J$$

the MERA-like tensor network is a kind of discretization of our surface growth picture, in which each euclidean e can be considered as a cell of the bulk spacetime, i.e. the source in the Huygens's picture

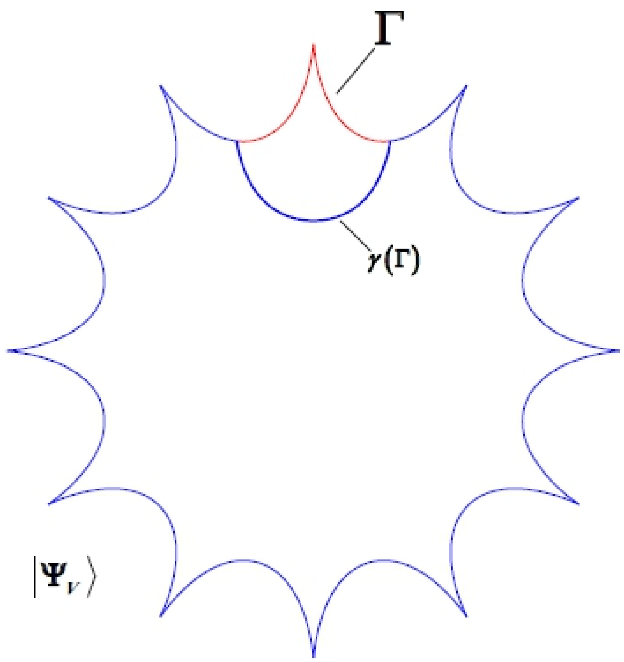




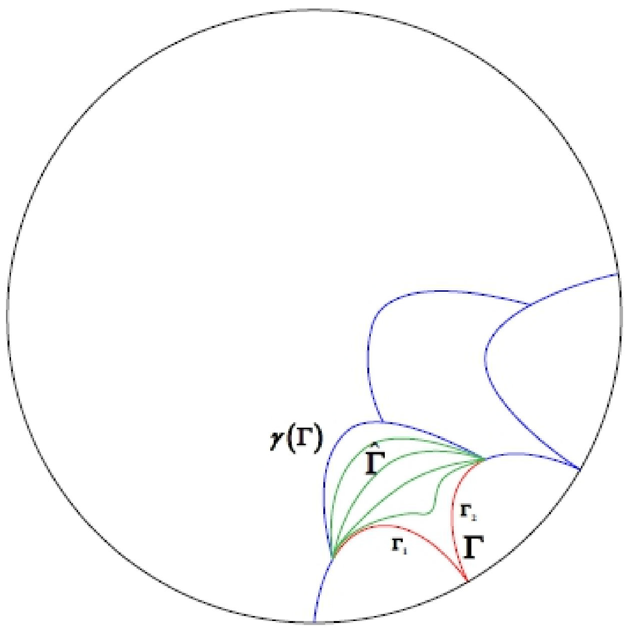
$z = 2^v,$

$$ds^2 = L^2\left[(\ln 2)^2 \cdot dv^2 + \left(2^{-v}\right)^2 \cdot dx^2\right]$$

More general surface growth scheme



(a)



(b)

$$\Psi_V = W^{\Gamma}_{\bar{\beta}\bar{\alpha}} V^{\bar{\Gamma}}_{\beta\alpha} \phi^{\alpha\bar{\alpha}} \sigma^{\beta\bar{\beta}} = W^{\Gamma}_{\bar{\beta}\bar{\alpha}} (V\,|\phi\rangle\,|\sigma\rangle)^{\bar{\Gamma}\bar{\alpha}\bar{\beta}},$$

$$\Psi = \left(W^{1\mathrm{st}}\right)^{N} W^{\Gamma}_{\bar{\beta}\bar{\alpha}} (V\,|\phi\rangle\,|\sigma\rangle)^{\bar{\Gamma}\bar{\alpha}\bar{\beta}}.$$

Direct growth of bulk minimal surfaces

C Yu, F-Z Chen,Y-y Lin, **JRS**, Y Sun, 2010.03167

The surface growth scheme can also be directly checked by the growth of the bulk minimal surfaces layer by layer.

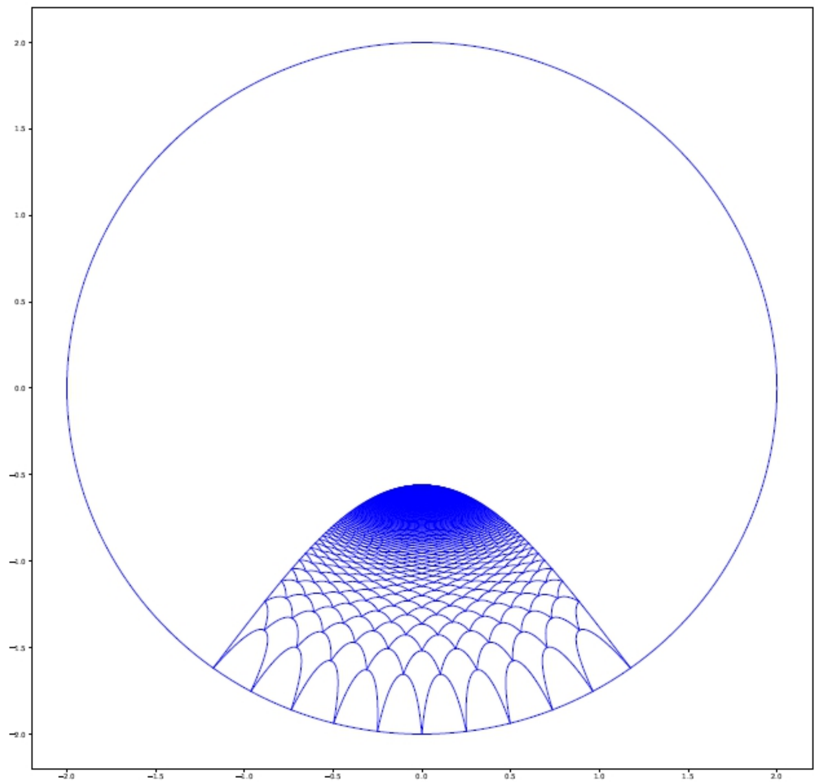
Pure AdS3 case

$$ds^2 = d\rho^2 + L^2 \left(-\cosh^2 \frac{\rho}{L} dt^2 + \sinh^2 \frac{\rho}{L} d\phi^2\right),$$

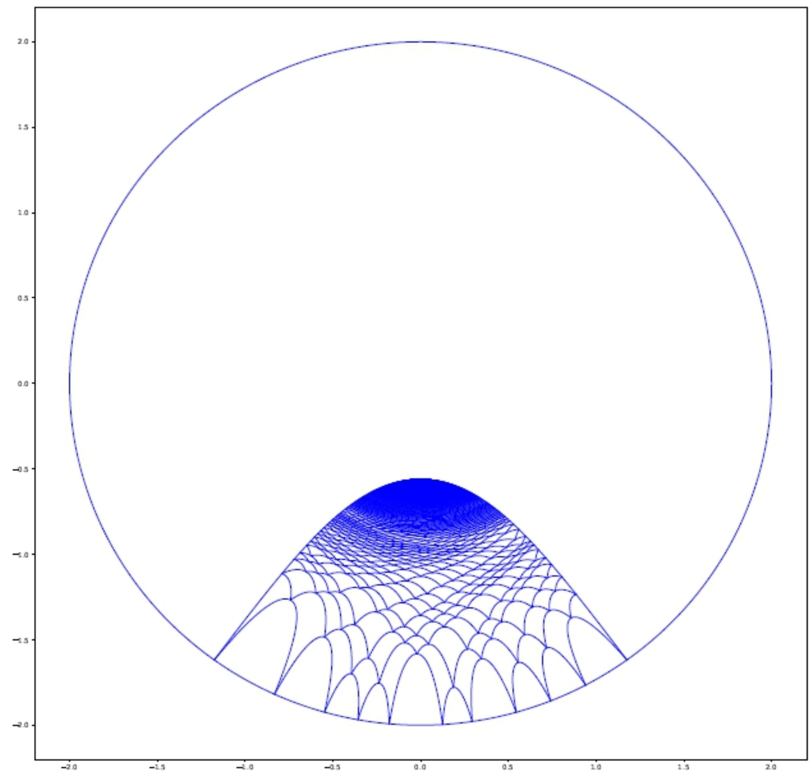
EoM of bulk minimal curve (geodesics) is

$$\begin{aligned}\phi \;&=\; \pm \arctan\left(\frac{\sinh^2 \tilde{\rho}}{\sinh \tilde{\rho}_*} + \cosh \tilde{\rho} \sqrt{\frac{\sinh^2 \tilde{\rho}}{\sinh^2 \tilde{\rho}_*} - 1}\right) \\ &\quad \mp \arctan\left(\sinh \tilde{\rho}_*\right) + \phi_0,\end{aligned}$$

for given angular size of the subsystem, different radial cutoff corresponds to different turning position.



homogenous subregions, with each subregion has angle $\phi = \pi / 25$, the growing steps are 300.



inhomogenous subregions, with growing steps 300.

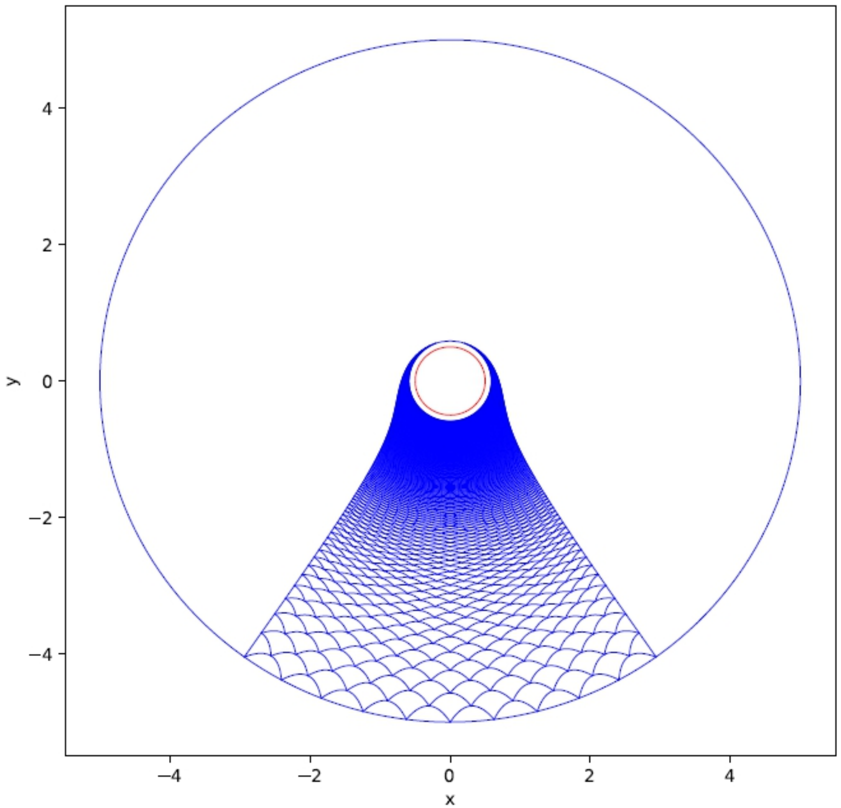
BTZ black hole case

$$ds^2 = -\frac{r^2}{L^2} f(r) dt^2 + \frac{L^2}{r^2 f(r)} dr^2 + r^2 d\phi^2,$$

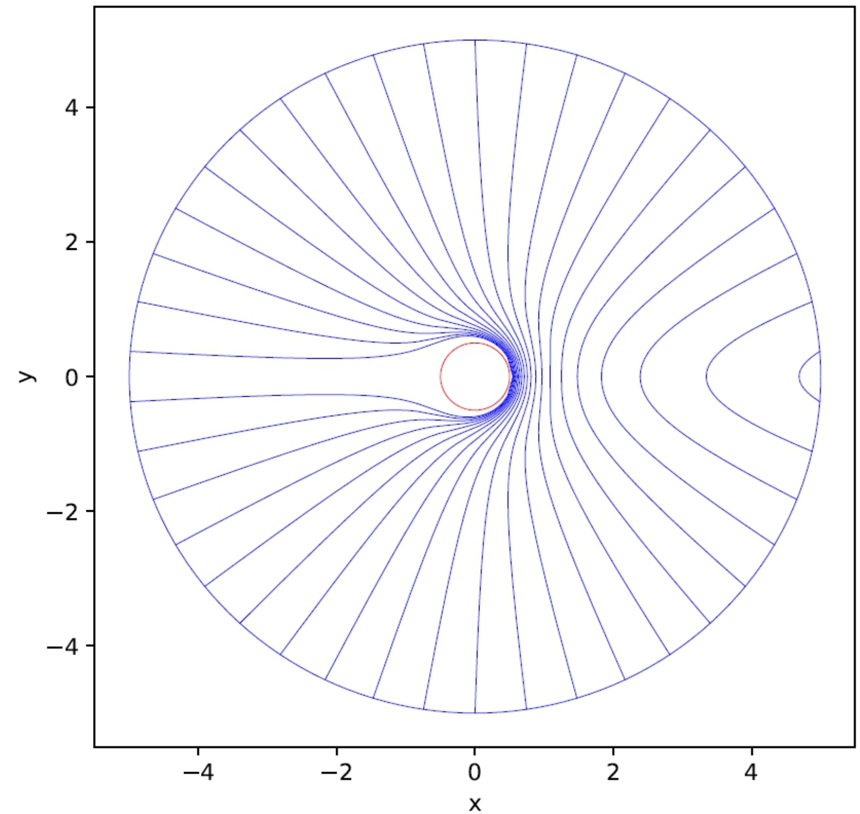
EoM of bulk geodesics is

$$\pm(\phi - \phi_0) = -\frac{L}{r_h} \ln \left(\sqrt{1 - \frac{r_h^2}{r^2}} - \sqrt{\frac{r_h^2}{r_*^2} - \frac{r_h^2}{r^2}} \right) + \frac{L}{r_h} \ln \sqrt{1 - \frac{r_h^2}{r_*^2}},$$

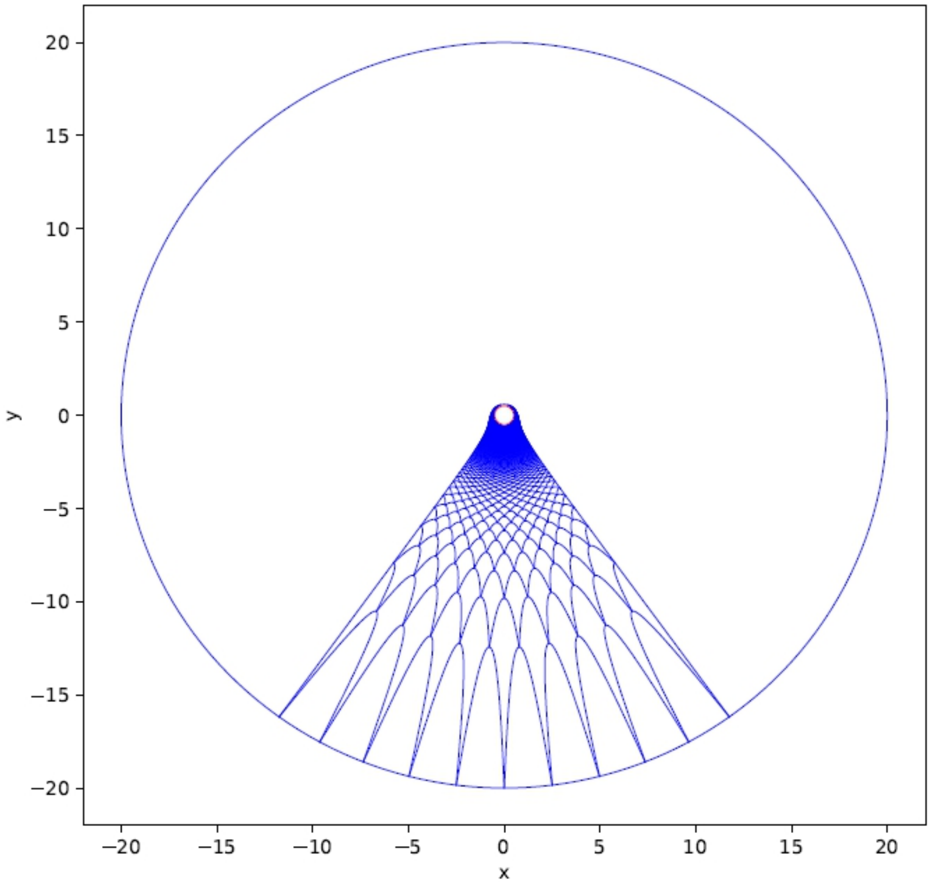
where $r_* < r_1 < r_2$ and $\phi(r_1) < \phi_0 < \phi(r_2)$. Also, for given angular size of the subsystem, different radial cutoff corresponds to different turning position.



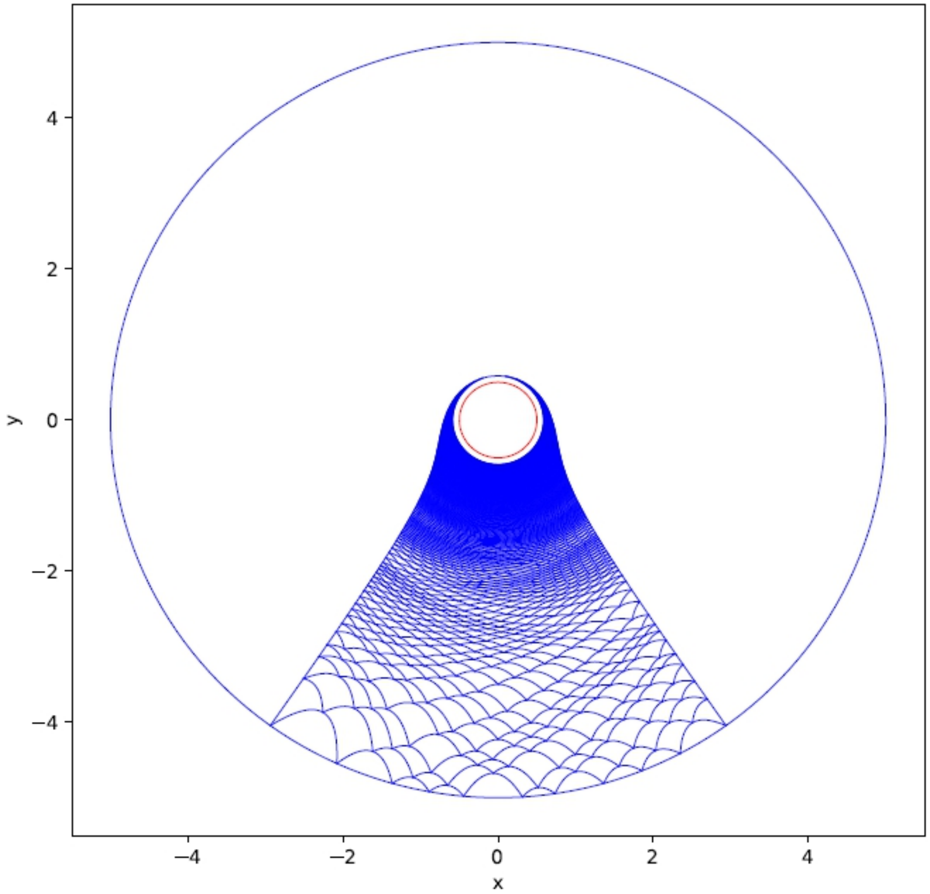
homogenous subregions, with each subregion has angle $\phi = \pi / 25$, cutoff surface is r_c=5, the growing steps are 360.



minimal surfaces wich do not surround the black hole horizon, entanglement plateaux phenomenon.



homogenous subregions, with each subregion has angle $\phi = \pi / 25$, cutoff surface is rc=20, the growing steps are 360.



inhomogenous subregions, whole subsystem with angle $\phi = 2\pi / 5$, with growing steps 358.

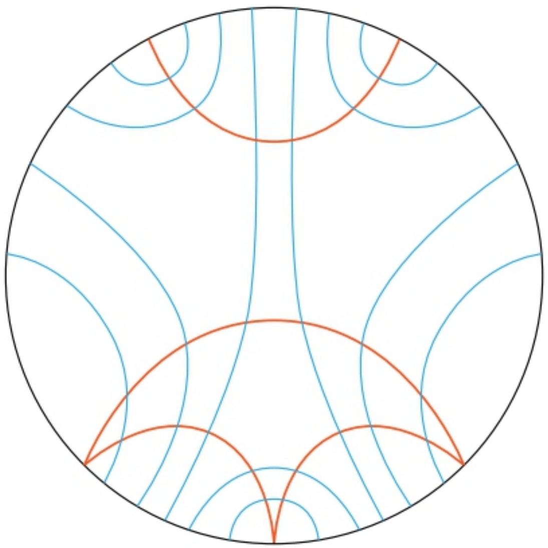
Surface growth, bit threads, Entanglement of purification

Y-y Lin, **JRS**, Y Sun, 2012.05737

Bit thread

Freedman, Headrick, 1604.00354; Cui, Hayden, He, Headrick, Stoica, 1808.05234

Bit thread are unoriented bulk curves that end on the boundary and are subject to the rule that the thread density is less than a constant *C*, say, 1 everywhere. (like the propagator)



$$\nabla \cdot \vec{v} \;=\; 0,$$

$$\rho\left(\vec{v}\right) \;\equiv\; |\vec{v}| \leq 1.$$

$$\int_A \vec{v} = \int_A \sqrt{h}\hat{n} \cdot \vec{v},$$

Riemannian max flow-min cut theorem

$$\max_v \int_A v = C \min_{m \sim A} \text{area}(m).$$

The thread configuration which has the maximal flow is said to lock *A*.

$$S(A) = \text{Flux}_{\text{locking}}(A).$$

Bit thread locking theorem

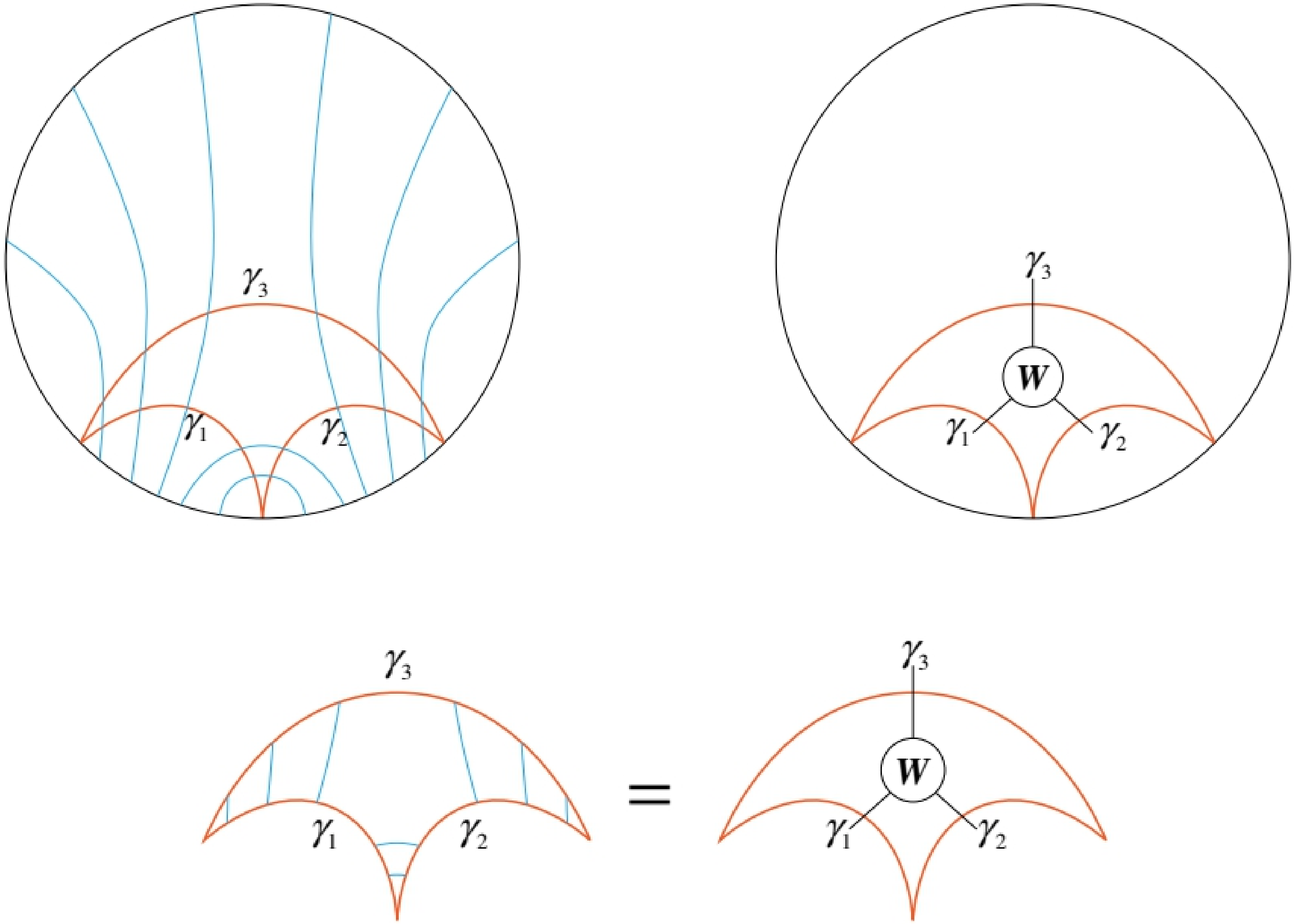
For a *d*-dim Riemannian manifold with boundary *M*, dividing *M* into adjacent nonoverlapping subregions *Ai* (*i*=1,...,*n*)

1. There exists a multiflow that locks all the elementary regions *Ai*.

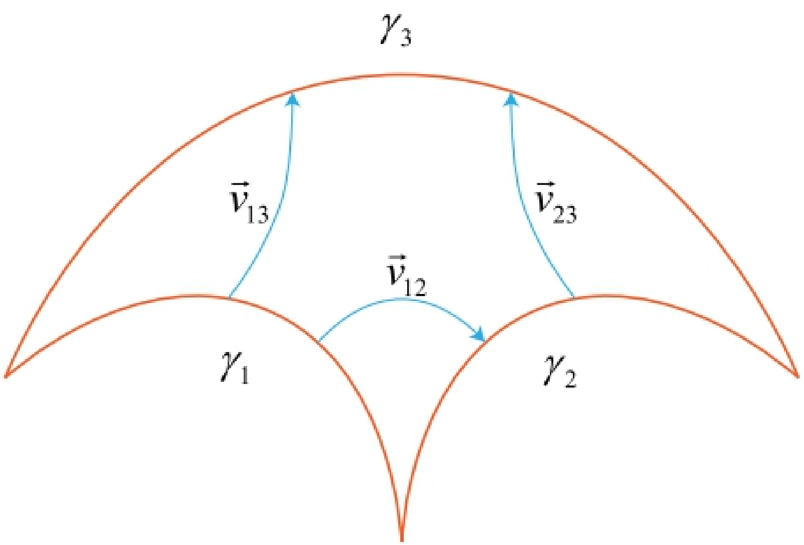
2. There exists a multiflow that can lock all the elementary regions and any single composite region simultaneously.

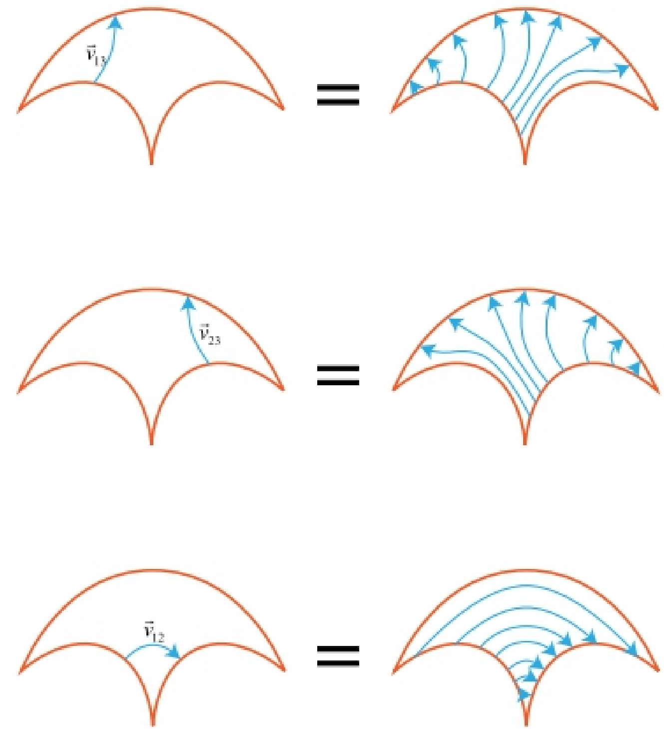
3. There exists a multiflow that can lock all the elementary regions and all noncrossing composite regions simultaneously.

Locking thread configuration corresponds to OSED tensor networks



Locking thread configuration corresponds to OSED tensor networks--*multiflows*





$$\nabla \cdot \vec{v}_{ij} = 0$$

$$\hat{n} \cdot \vec{v}_{ij}\big|_{\gamma_k} = 0 \quad (\text{for } k \neq i, j)$$

$$\rho(V) = \sum_{i<j} |\vec{v}_{ij}| \leq 1,$$

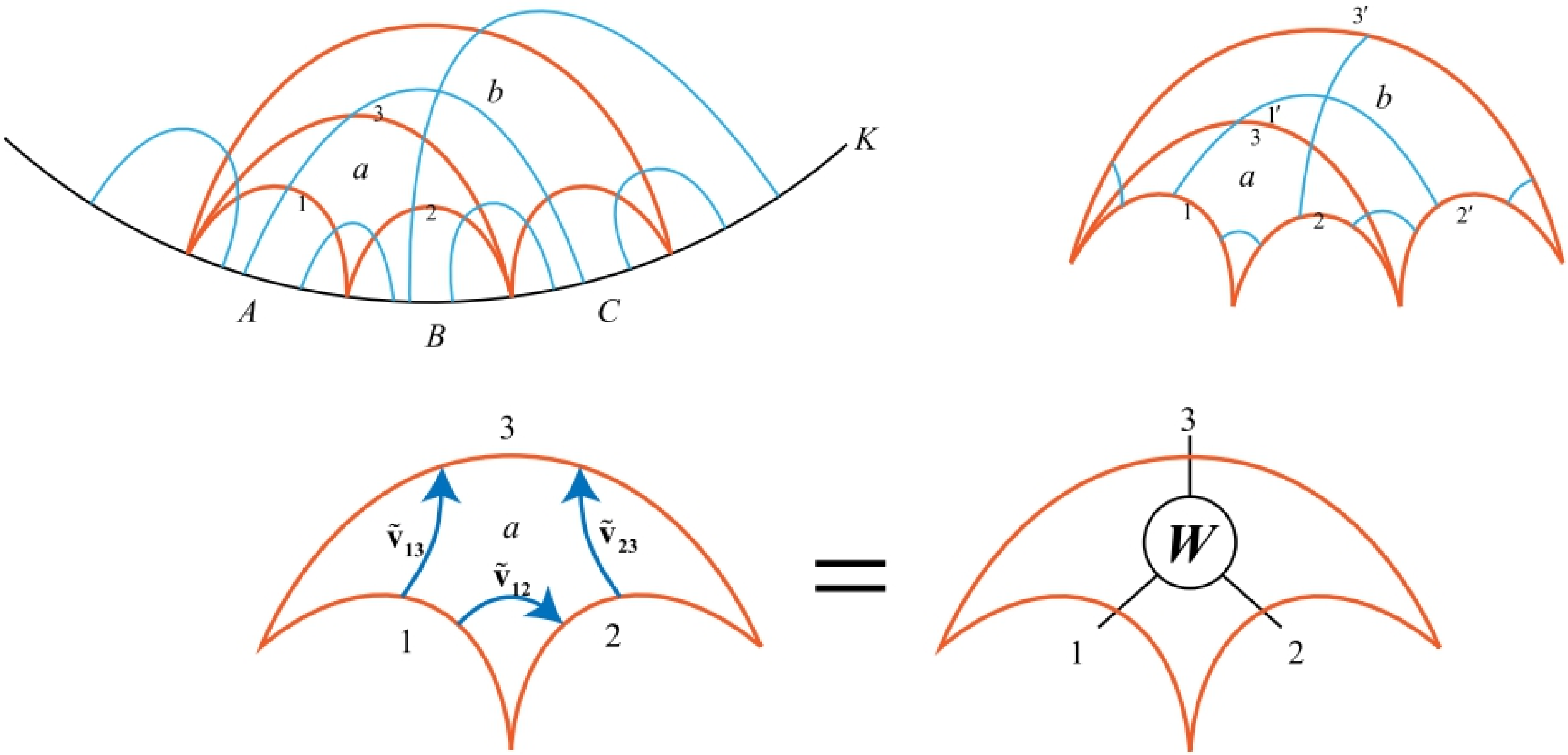
$$\rho(V)\big|_{\gamma} = 1.$$

$$F(i)_{jk} = \left| \int_{\gamma_i} \vec{v}_{jk} \right|,$$

The total flux of bit threads through surface γ_i is

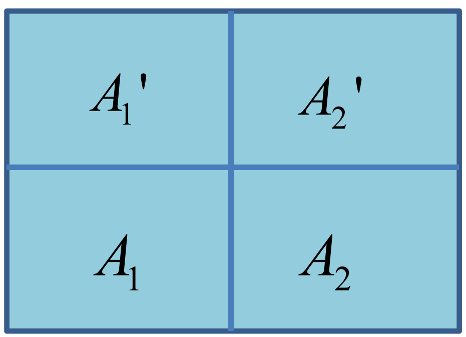
$$N(i) = \sum_{j,k} F(i)_{jk}.$$

a more general surface growth case



Bit thread and EoP

Dividing a quantum system into two parts, a quantity used to describe correlations between A_1 and A_2 is called the entanglement of purification (EoP) $E_P(A_1 : A_2)$



Let $|\psi\rangle \in H_{A_1 A_1'} \otimes H_{A_2 A_2'}$ be a purification of the density matrix

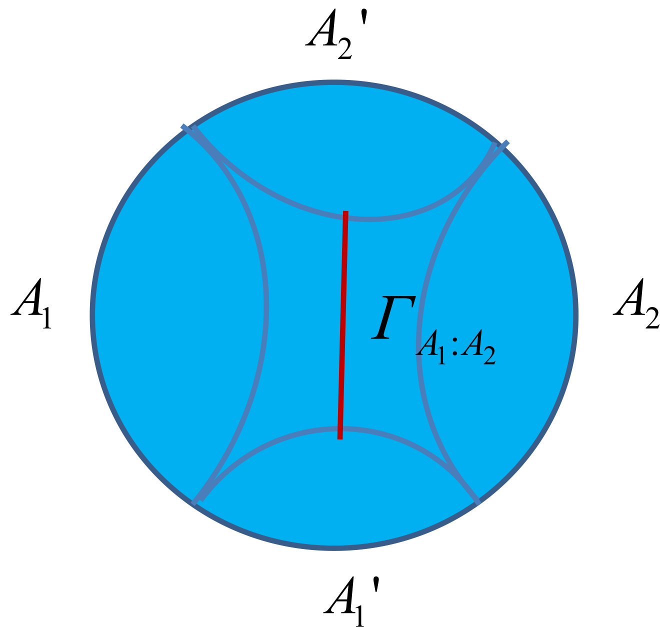
$$\rho_{A_1 A_2} = \mathrm{Tr}_{A_1' A_2'} |\psi\rangle\langle\psi|$$

The EoP is defined as

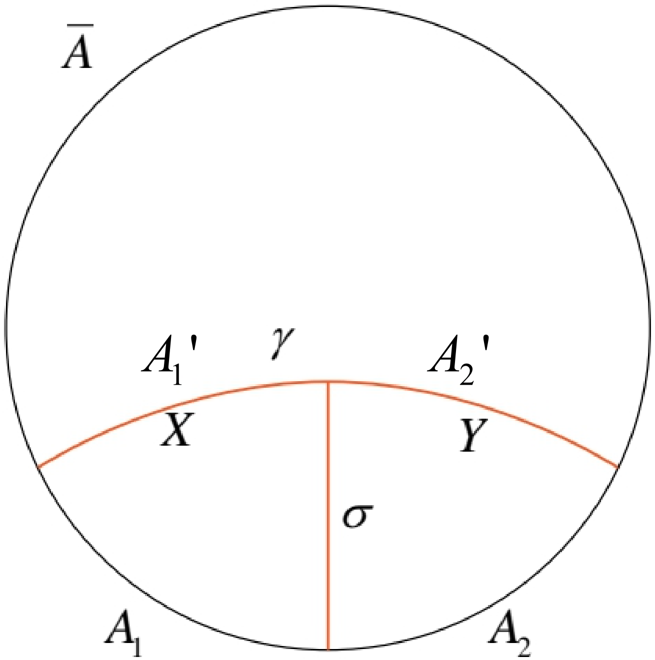
$$E_P(A_1 : A_2) = \min_{|\psi\rangle_{A_1 A_1' A_2 A_2'}} S(A_1 A_1'),$$

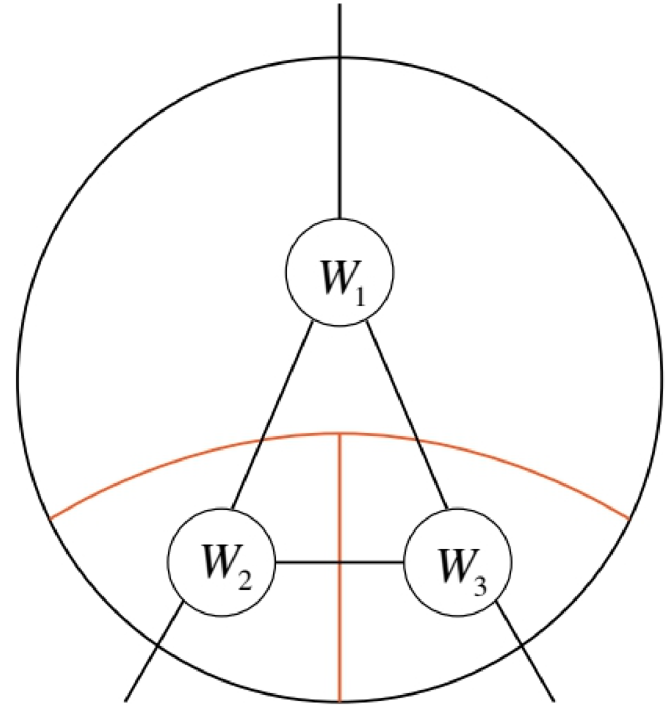
A holographic dual of EoP is Takayanagi, Umemoto, 2018

$$E_P(A_1 : A_2) = \text{Area}(\Gamma_{A_1:A_2}).$$ area of EWCS



The holographic EoP can be natually regarded as a surface growth process.





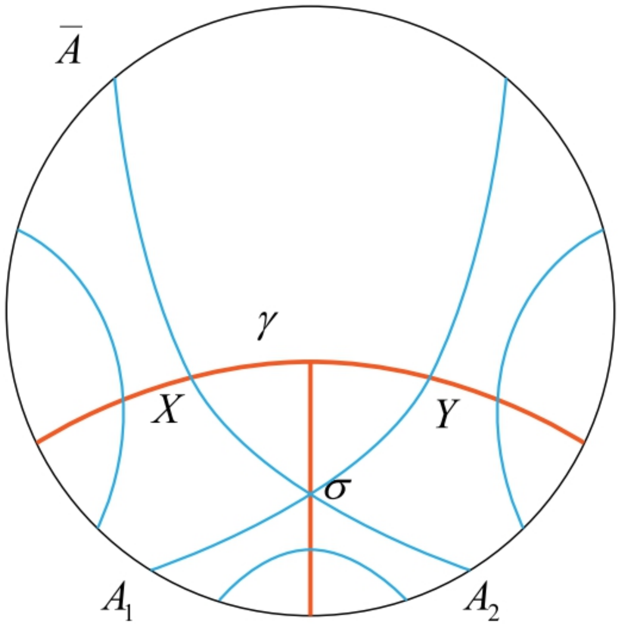
From the surface growth and the generalized RT formula

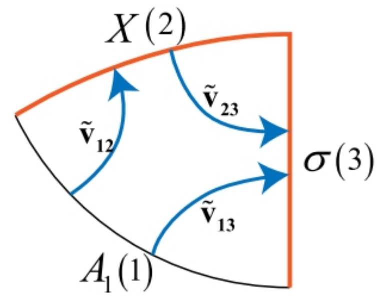
$$\text{Area}(\sigma) = S(XA_1),$$

then $$E_P(A_1 : A_2) = \min_{|\psi\rangle_{A_1A_1'A_2A_2'}} S(A_1A_1') = \min S(XA_1),$$

which gives $$E_P(A_1 : A_2) = \min \text{Area}(\sigma) \equiv \text{Area}(\Gamma_{A_1:A_2}),$$

EoP from surface growth ⟹ OSED tensor network ⟹ new bit thread description of EoP





$$\rho(\tilde{\mathbf{v}}_{12}) = \rho(\vec{v}_{\bar{A}XA_1}),$$

$$\rho(\tilde{\mathbf{v}}_{13}) = \rho(\vec{v}_{\bar{A}Y\sigma A_1}) + \rho(\vec{v}_{A_1\sigma A_2}),$$

$$\rho(\tilde{\mathbf{v}}_{23}) = \rho(\vec{v}_{\bar{A}X\sigma A_2}),$$

which is different from previous descriptions in which $\tilde{\mathbf{v}}_{23} = 0$, since

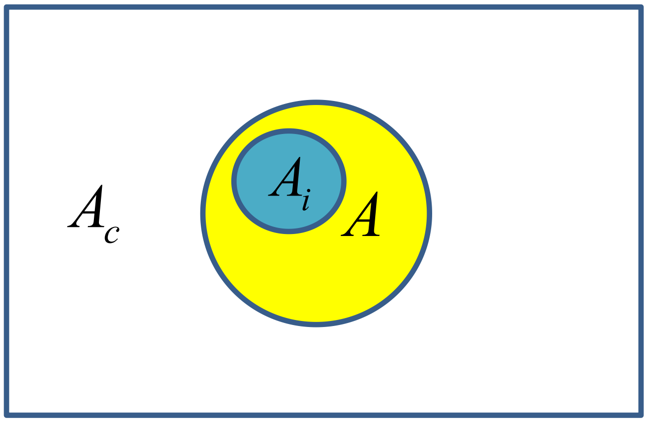
$$S(2) + S(3) - S(1) = 2F(2)_{23}$$

Bit threads and partial entanglement contour

Y-y Lin, **JRS**, J Zhang, 2105.09176

Entanglement contour Chen, Vidal, 1406.1471

a function $f_A(x)$ trying to describe the fine structure the entanglement entropy



$$S(A) = \int_A f_A(x)\, dx$$

Technically, it's more tractable to analyze the **partial entanglement entropy (PEE)** $s_A(A_i)$ of some subsystem of A

$$s_A(A_i) \equiv \int_{A_i} f_A(x)\, dx$$

Conditions required to satisfy

1. Additivity: decomposing A_i as A_i^1 and A_i^2, by definition we should have

$$s_A\left(A_i\right)=s_A\left(A_i^1\right)+s_A\left(A_i^2\right)$$

2. Invariance under local unitary transformations: $s_A\left(A_i\right)$ should be invariant under any local unitary transformations inside A_i or A_c.
3. Symmetry: for any symmetry transformation T under which $TA=A'$ and $TA_i=A_i{}'$, we have $s_A\left(A_i\right)=s_{A'}\left(A_i{}'\right)$.
4. Normalization: $S\left(A\right)=s_A\left(A_i\right)|_{A_i\to A}$.
5. Positivity: $s_A\left(A_i\right)\geq 0$.
6. Upper bound: $s_A\left(A_i\right)\leq S\left(A\right)$.
7. Symmetry under the permutation: the PEE can be expressed as a form with permutation

$$s_A\left(A_i\right)=P\left(A_i,A_c\right)=P\left(A_c,A_i\right)=s_{\left(A_i\right)_c}\left(A_c\right)$$

where $\left(A_i\right)_c$ represents the complement of A_i.

However, the bove requirements are not sufficient to uniquely determine the PEE in general.

PEE proposal Kudler-Flam, MacCormack, Ryu, 1902.04654; Q Wen, 1902.06905, 1803.05552

$$s_A(A_2) = \frac{1}{2}\left(S_{12} + S_{23} - S_1 - S_3\right)$$

In addition, since the HEE can be alternativelly described by the bit threads, it would be interesting to see how bit threads can describe the PEE.

Prelimilary study

Kudler-Flam, MacCormack, Ryu, 1902.04654

$$f_A(x) = |v(x)|$$

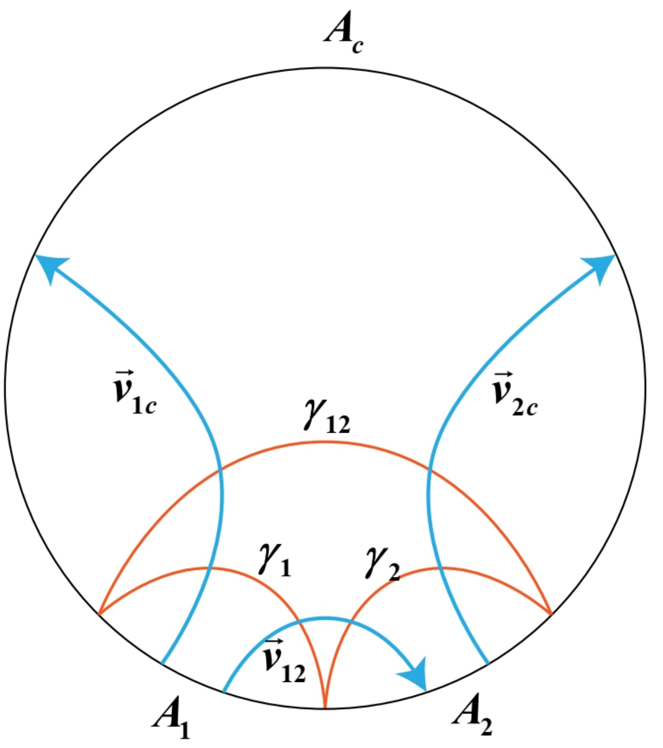
We will use the multiflow and the locking bit thread configurations to further investigate the PEE.

PEE as component flow flux

Dividing subregion A into A_1 and A_2, then PEE gives

$$S(A) = s_A(A_1) + s_A(A_2)$$

which is just like the summation of total bit thread flux.



From properties of multiflow

$$\nabla \cdot \vec{v}_{ij} = 0$$

$$\hat{n}_{A_k} \cdot \vec{v}_{ij} = 0 \quad (\text{for } k \neq i, j)$$

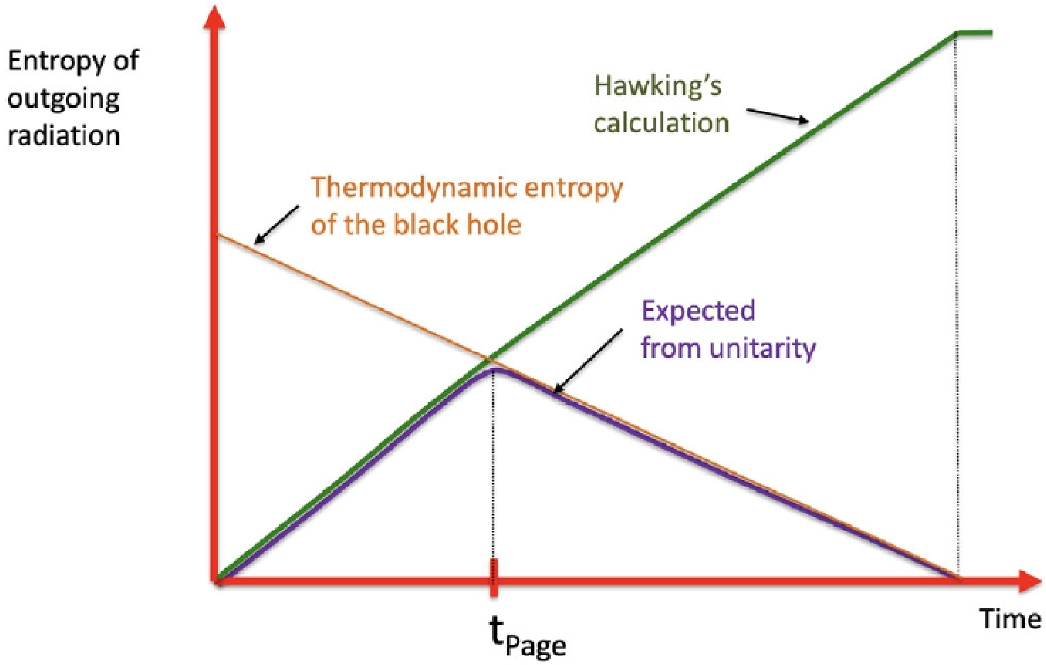
$$\Rightarrow \quad F(A_1)_{1c} = F(\gamma_1)_{1c} = F(\gamma_{12})_{1c} = F(A_c)_{1c}$$

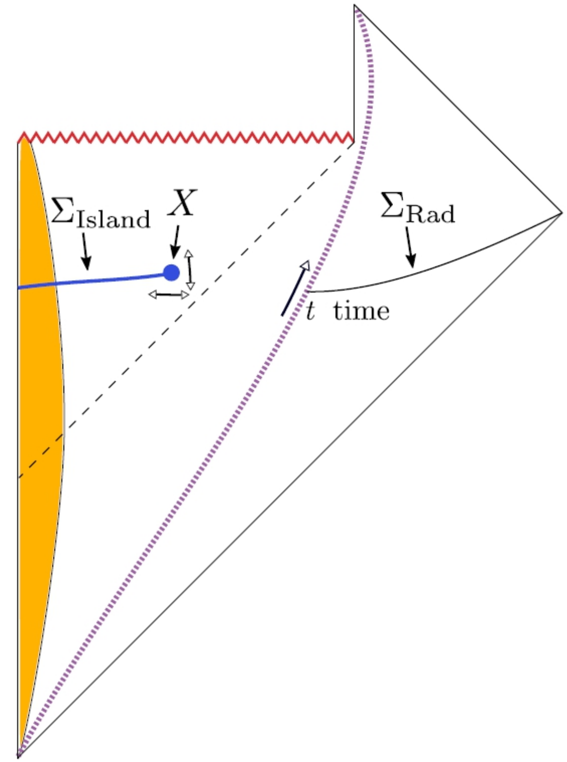
$$F(A_2)_{2c} = F(\gamma_2)_{2c} = F(\gamma_{12})_{2c} = F(A_c)_{2c}$$

$$\Rightarrow \quad F(A_1)_{1c} + F(A_2)_{2c} = S(A) \quad \Rightarrow \quad s_A(A_i) = F(A_i)_{ic} := F_{ic} = P_{ic}$$

Island prescription of Hawking radiation

Penington, 1905.08255; Almheiri, Engelhardt, Marolf, and Maxfield, 1905.08762; Almheiri, Hartman, Maldacena, Shaghoulian and Tajdini, 2006.06872





Bekenstein's generalized entropy

$$S_{\rm gen} = \frac{\text{Area of horizon}}{4\hbar G_N} + S_{\rm outside},$$

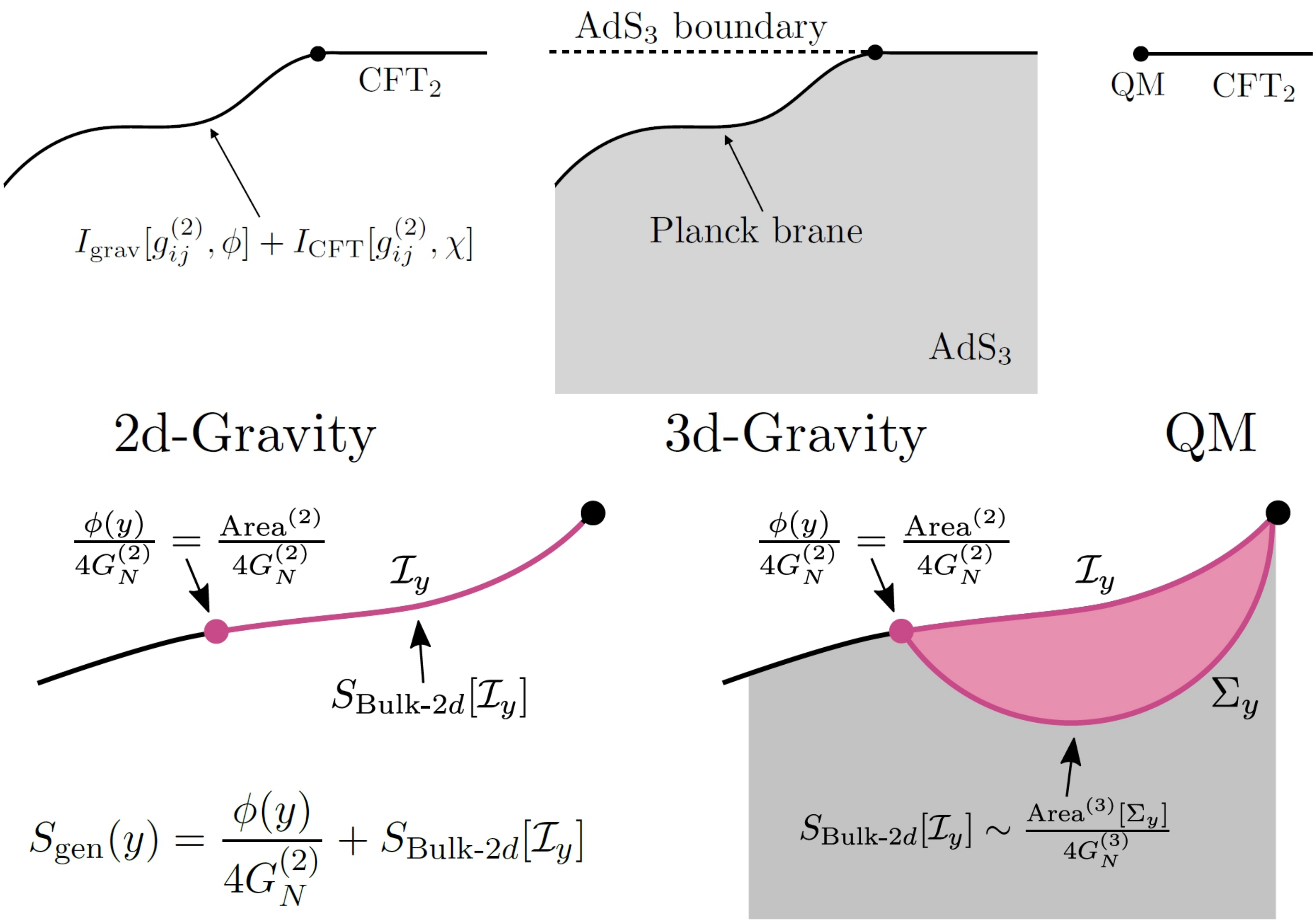
Fine-grained entropy

$$S = \min_X \left\{ \text{ext}_X \left[\frac{\text{Area}(X)}{4G_N} + S_{\text{semi-cl}}(\Sigma_X) \right] \right\}$$

Faulkner, Lewkowycz and Maldacena, 2013; Engelhardt and Wall, 2015

Three viewpoints for gravity+radiation, 2d gravity example

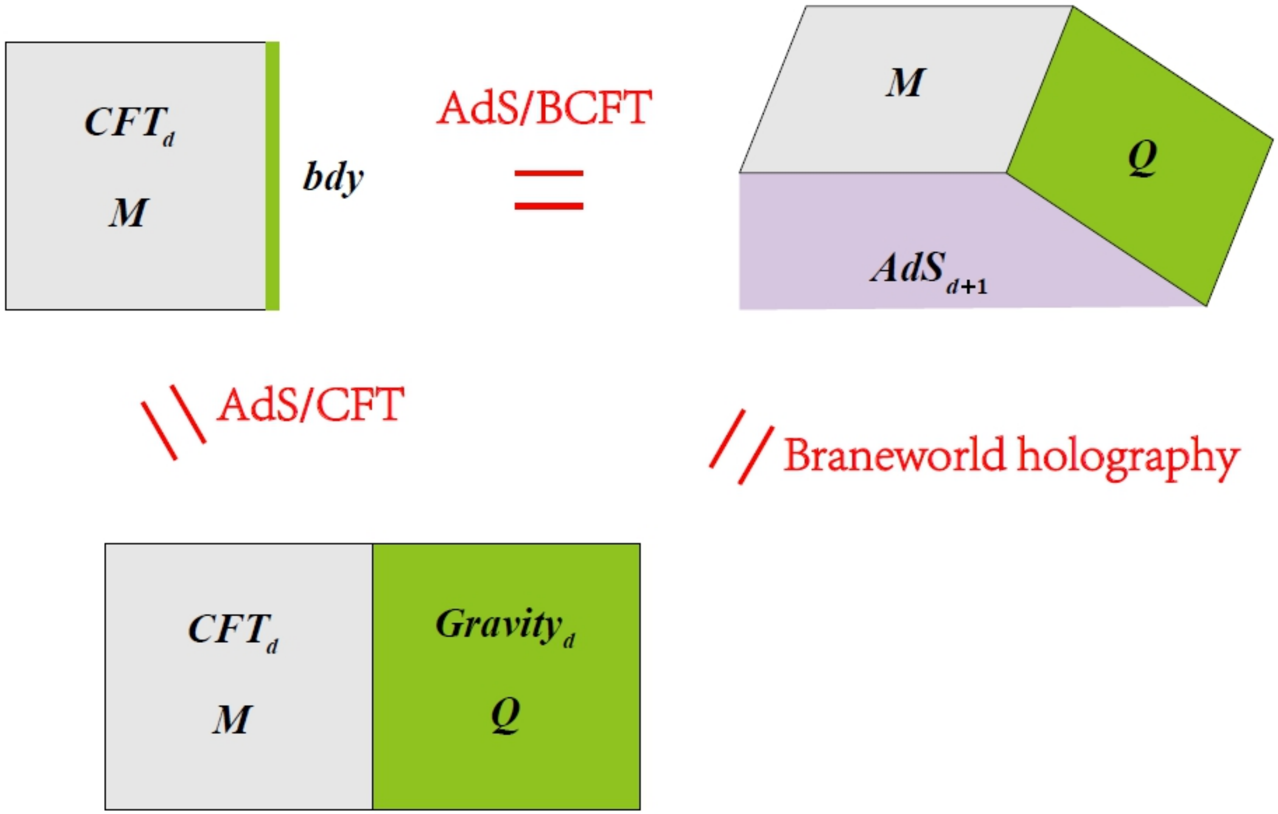
Almheiri, Mahajan, Maldacena and Zhao 2019



$$S_{\rm gen}(y) = \frac{\phi(y)}{4G_N^{(2)}} + S_{\text{Bulk-}2d}[\mathcal{I}_y]$$

AdS/BCFT description for entanglement island

Chen, Myer, Neuenfeld, Reyes, Sandor, 2006.04851, 2010.00018;
Suzuki and Takayanagi, 2202.08462



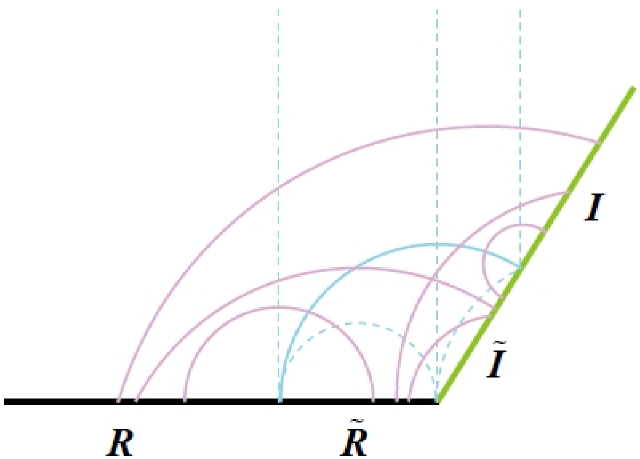
Holographic triality of AdS/BCFT setup.

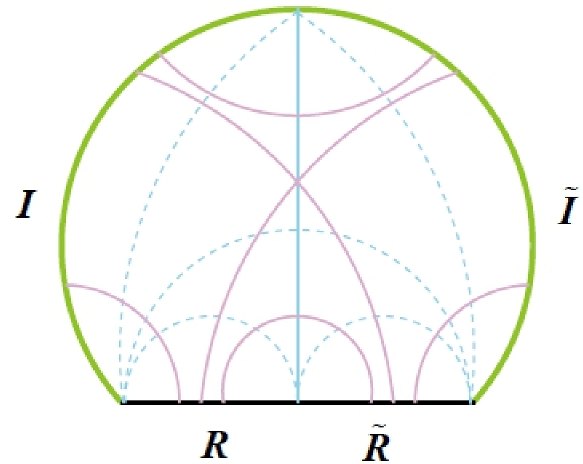
It's interesting to use bit threads to study this model.

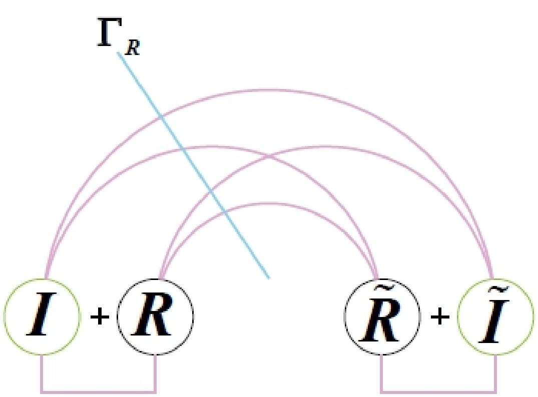
In the AdS/BCFT correspondence, the von Neumann entropy is calculated by

$$\boldsymbol{S}(\boldsymbol{R}) = \min_{\Gamma_R, I} \operatorname{Ext}\left[\frac{\operatorname{Area}(\Gamma_R)}{4G_N^{(d+1)}}\right], \quad \partial\Gamma_R = \partial R \cup \partial I,$$

where region *I* can be used to describe the island.







$$\left\{\gamma(R),\ \gamma\left(\tilde{R}\right),\ \gamma(I),\ \gamma\left(\tilde{I}\right),\ \gamma\left(\tilde{R}\tilde{I}\right),\ \gamma\left(R\tilde{R}\right)\right\}.$$

$$\gamma\left(\tilde{R}\tilde{I}\right) \equiv \boldsymbol{\Gamma_R}.$$

Using the relations between fluxes and entropy

$$F_{R\tilde{R}} + F_{R\tilde{I}} + F_{\tilde{R}I} + F_{I\tilde{I}} = S\left(\tilde{R}\tilde{I}\right) = \boldsymbol{S}\left(\boldsymbol{R}\right)$$

$$F_{R\tilde{R}} + F_{RI} + F_{R\tilde{I}} = S\left(R\right)$$

$$F_{R\tilde{R}} + F_{\tilde{R}I} + F_{\tilde{R}\tilde{I}} = S\left(\tilde{R}\right)$$

$$F_{RI} + F_{\tilde{R}I} + F_{I\tilde{I}} = S\left(I\right)$$

$$F_{R\tilde{I}} + F_{\tilde{R}\tilde{I}} + F_{I\tilde{I}} = S\left(\tilde{I}\right)$$

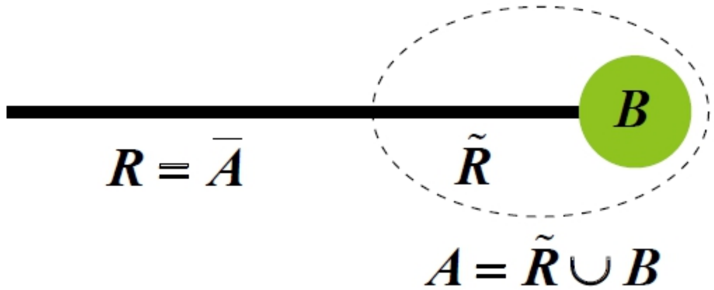
$$F_{RI} + F_{R\tilde{I}} + F_{\tilde{R}I} + F_{\tilde{R}\tilde{I}} = S\left(R\tilde{R}\right)$$

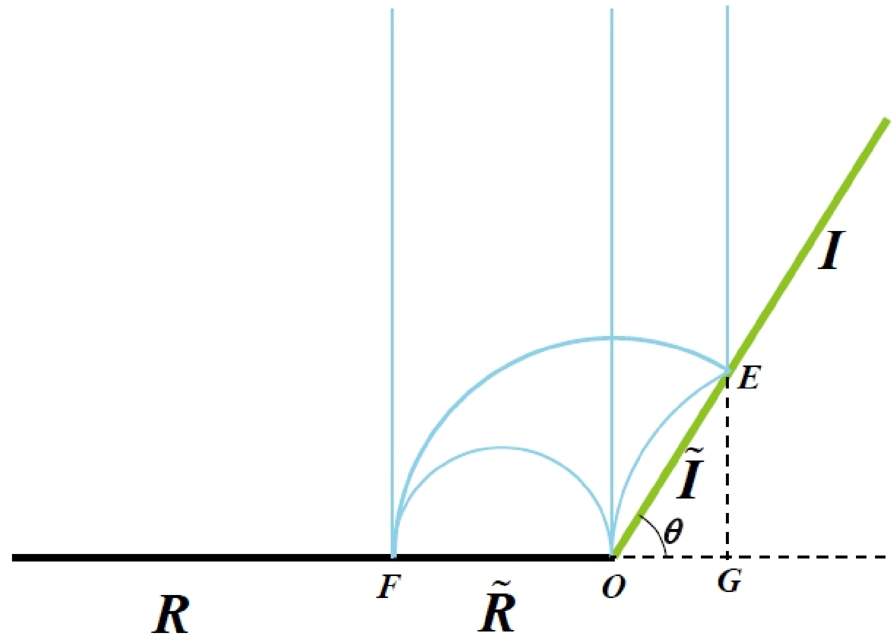
$$F_{R\tilde{R}} = \tfrac{1}{2}\left(S\left(\tilde{R}\right) + S\left(R\right) - S\left(R\tilde{R}\right)\right),$$

$$F_{RI} = \tfrac{1}{2}\left(S\left(R\right) + S\left(I\right) - \boldsymbol{S}\left(\boldsymbol{R}\right)\right),$$

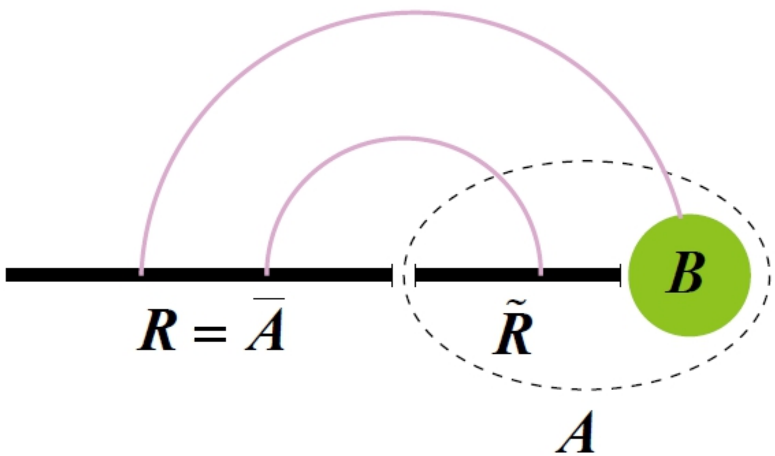
PEE aspects of entanglement island

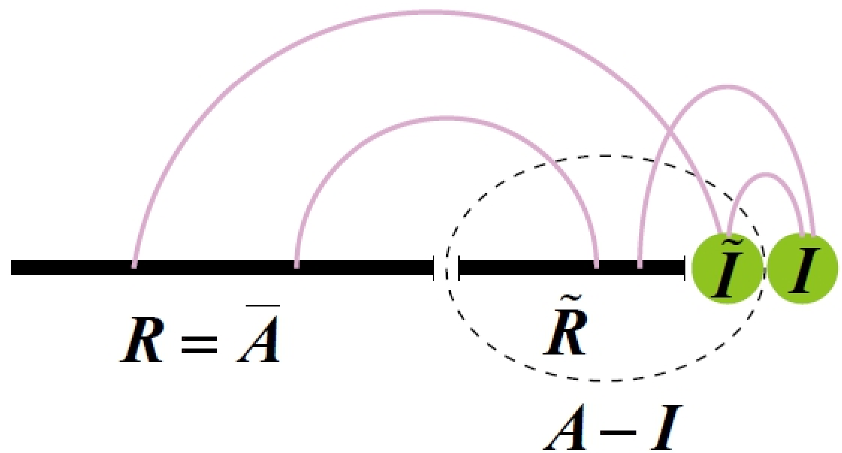
Island rule of PEE for subregion containing the entire boundary





$$s_A\left(\tilde{R}\right) + s_A(B) = S(A)$$





$$s_{\boldsymbol{A}}\left(\tilde{\boldsymbol{R}}\right) = F\left(\tilde{R} \leftrightarrow R\right) + F\left(\tilde{R} \leftrightarrow I\right) \equiv F\left(\tilde{R} \leftrightarrow R \cup I\right)$$

$$s_{\boldsymbol{A}}\left(\boldsymbol{B}\right) = F\left(\tilde{I} \leftrightarrow R\right) + F\left(\tilde{I} \leftrightarrow I\right) \equiv F\left(\tilde{I} \leftrightarrow R \cup I\right)$$

In AdS3 case:

$$ds^2 = L^2 \frac{-dt^2 + dz^2 + dx^2}{z^2},$$

$$\frac{(t_1 - t_2)^2 + (x_1 - x_2)^2 + (z_1 - z_2)^2}{2 z_1 z_2} + 1 = \cosh \frac{d}{L}$$

$$E = (t = 0, x = l\cos\theta, z = l\sin\theta)$$

$$F = (t = 0, x = -l, z = \varepsilon)$$

$$G = (t = 0, x = l\cos\theta, z = \varepsilon)$$

$$O = (t = 0, x = 0, z = \varepsilon)$$

$$\tan\theta = \frac{z}{x} = \frac{1}{\sinh \frac{\rho_*}{L}}$$

$$\boldsymbol{S}(\boldsymbol{R}) = S\left(\tilde{R}\tilde{I}\right) = \frac{d_{EF}}{4G_N^{(d+1)}} = \frac{c}{6}\ln\frac{2l}{\varepsilon} + \frac{c}{6}\ln\frac{(1+\cos\theta)}{\sin\theta}.$$

$$S\left(\tilde{I}\right) = \frac{d_{OA}}{4G_N^{(d+1)}} = \frac{c}{6}\ln\frac{l}{\varepsilon\sin\theta},$$

$$S\left(\tilde{R}\right) = \frac{d_{OF}}{4G_N^{(d+1)}} = \frac{c}{3}\ln\frac{l}{\varepsilon},$$

then from

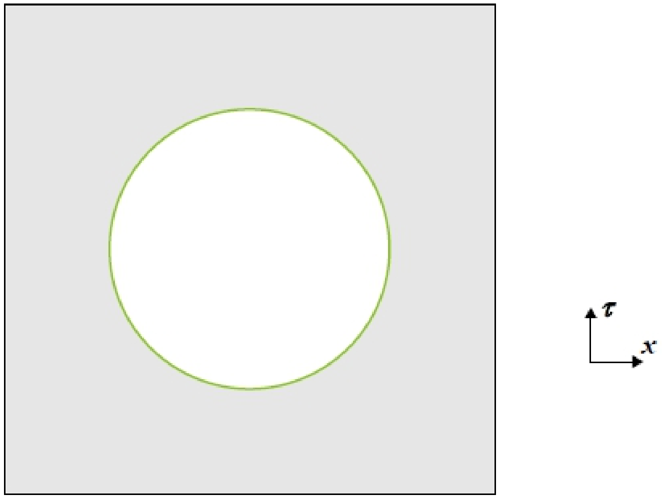
$$\boldsymbol{s_A}\left(\tilde{\boldsymbol{R}}\right) = \frac{1}{2}\left(\boldsymbol{S}(\boldsymbol{R}) + S\left(\tilde{R}\right) - S\left(\tilde{I}\right)\right)$$

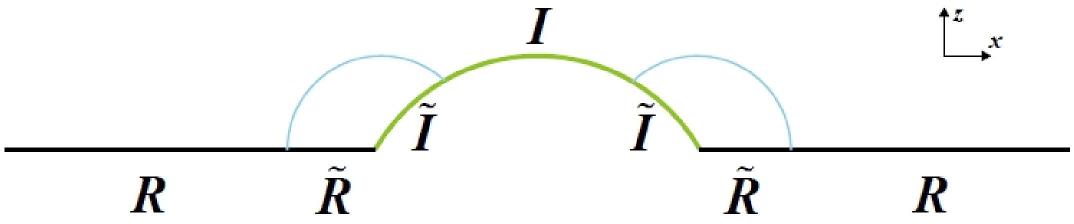
$$\boldsymbol{s_A}(\boldsymbol{B}) = \frac{1}{2}\left(\boldsymbol{S}(\boldsymbol{R}) + S\left(\tilde{I}\right) - S\left(\tilde{R}\right)\right).$$

$$\boldsymbol{s_A}\left(\tilde{\boldsymbol{R}}\right) = \frac{c}{6}\ln\frac{2l}{\varepsilon} + \frac{c}{12}\ln\frac{1+\cos\theta}{2}$$

$$\boldsymbol{s_A}(\boldsymbol{B}) = \frac{c}{12}\ln\frac{2(1+\cos\theta)}{\sin^2\theta}.$$

Insight into black hole inoformation problem





A BCFT setup boundary perspective) that models a two-sided 2d black hole (in green) coupled to a pair of symmetrical auxiliary radiation systems (in grey). The RT surface calculating the true entanglement entropy of R can anchor on the ETW brane (in green, which simulates a black hole) to form an island.

Conclusions and Discussions

*The surface growth approach provides an efficient way to build the bulk geometry in the entanglement wedge far away from the boundary.

*It connects the generalized OSED and the surface/state correspondence and indicates that that the process of growing a new extremal surface is actually a kind of classical encoding operation on the entanglement within the previous extremal surfaces.

*By combining the surface growth approach and the bit threads, we give a new and more reasonable bit thread description for the holographic EoP.

*Many interesting problems to be studied, such as

extending this scheme to cMERA,
relation between surface growth and TTbar deformation,
does bit thread has the physical correspondence, say, a gauge field?
extending the surface growth and bit threads AdS/BCFT setup into black hole cases
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謝謝大家！