Localization of Fermions in Higher-Dimensional Spacetime

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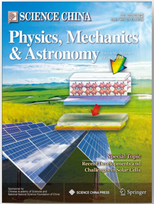

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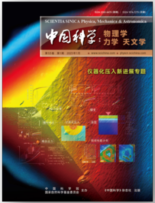

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- 2. Clifford algebra $\mathbb{C}\ell(2n+1,1)$
- 3. Decomposition of fermion in (2n+2)-dimensions
- 4. Localization of fermion in 6D spacetime
- 5. Summary

Exta dimensions and braneworld models

- Compact extra dimensions—Kaluza-Klein (KK) theory
- Infinite flat extra dimensions—Domain wall
- Infinite warped extra dimensions—Thin brane
- Infinite warped extra dimensions—Thick brane

Compact extra dimensions—KK theory

Kaluza-Klein (KK) theory (1920) [Kaluza et al. 1921, Klein, 1926]

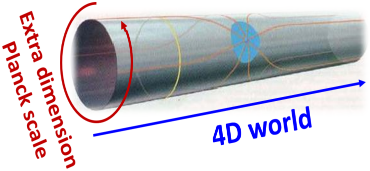


Figure: Picture of KK theory with topology $M_4 \times S^1$

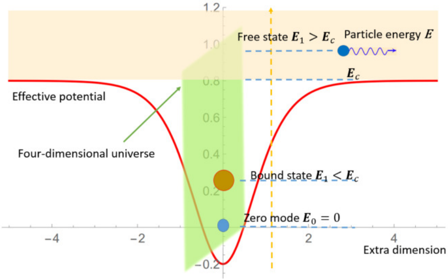
- The motivation is to unity Maxwell's electromagnetism theory and Einstein's general relativity.
- It contains a compact extra dimension.
- Reduction of 5D fermion does not result in a 4D chiral theory.
- KK theory only allows 4D neutral particles in the Standard Model.

Infinite flat extra dimension—Domain wall

Domain wall (DW) scenario [Akama, Rubakov, Shaposhnikov, 1983]

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dy^2$$

- Our 4D world is a DW embedded in 5D flat space-time.
- Generated by a scalar field: $\phi(y) = v_0 \tanh(ky)$.
- Fermions can be localized on DW by Yukawa coupling $\eta\phi\bar{\Psi}\Psi$.



[Y.-X. Liu, Y. Zhong, and K. Yang, Progress in Physics 37 (2017) 41.]

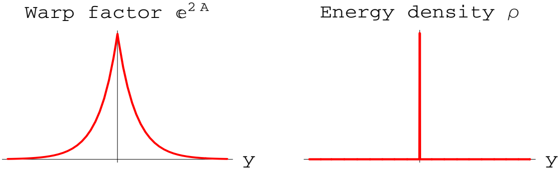
- Newton's law can not be recovered on DW.

Infinite warped extra dimensions—Thin brane

Thin brane scenario [Randall and Sundrum (RS), 1999]

$$ds^2 = e^{-2k|y|}\,\eta_{\mu\nu}\,dx^\mu dx^\nu + dy^2$$

The energy density: $\rho(y) \propto \sigma\delta(y)$



- Our 4D world is a brane embedded in a 5D spacetime.
- Newton's law can be recovered on brane

$$U(r) = G_N \frac{m_1 m_2}{r}\left(1 + \frac{1}{k^2 r^2}\right).$$

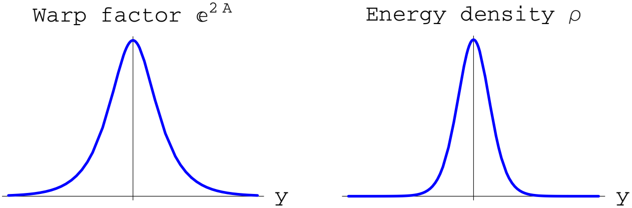
- The brane has no inner structure.

Infinite warped extra dimensions—Thick brane

Thick brane scenario

Combined the Randall-Sundrum-2 model and the domain wall model, a thick brane has thickness and inner structure.

$$ds^2 = e^{2A(y)}\,\hat{g}_{\mu\nu}(x)\,dx^\mu dx^\nu + dy^2$$



- It includes infinite but warped extra dimensions.
- The brane is generated by one or more scalar fields.
- Newton's law can be recovered on the brane.

Motivation for higher dimensions

Localization of graviton and other fields

- The standard model particles (zero modes of various bulk matter and gauge fields) should be Localized on the brane.
- For Randall-Sundrum-like brane models, graviton and a free massless scalar field can be localized on the brane.
- Localization condition for a free vector field is [1]
  # of extra dimensoins $>$ # of infinite extra dimensions.
  Thus, a free vector field can not be localized on the brane in AdS_5.
- It is not possible to simultaneously localize graviton, free scalar and vector fields in a 5D Randall-Sundrum-like brane model.
  We should consider six or higher-dimensional spacetime.

[1] L. F. Freitas, G.Alencar, and R. Landim, JHEP 02 , 035 (2019), arXiv:1809.07197 [hep-th]

Difficulty for fermion localization in higher dimensions

- In 5D brane models, the Yukawa coupling allows for localization of a fermion field on the brane. There are a lot of related works [2].
- In higher dimensional brane models, a free fermion can not be localized on the brane. While the Yukawa coupling results in that the left-handed and right-handed massive KK modes are not decoupled, and hence the fermion can not be localized.
- This leads to the difficulty in obtaining a 4D free fermion theory.

[2] arXiv: 1707.08541 [hep-th]

The usual ways for fermion localization

- Suppose that the 4D spinor comes from a 4D Weyl spinor [3]

$$\Psi^{(8)} = \begin{pmatrix} \Psi^{(4)} \\ 0 \end{pmatrix},$$

which assumes that the underlying theory is chiral. It cannot explain the origin of chirality.

- Localization mechanism is not introduced, and the extra dimension parts of each component of the spinor is the same[4]:

$$\Psi^{(8)}\left(x^M\right) = \psi^{(8)}\left(x^\mu\right)\alpha(r)\sum e^{il\theta}.$$

For conformal flat spacetime, fermions cannot be localized in this way. And under this assumption, the difference between left and right chiralities is ignored.

[3] [Nucl. Phys. B 767 (2007) 54, hep-th/0608074; JHEP 04 (2007) 097, hep-th/0701010]

[4] [Phys. Lett. B 496 (2000), hep-th/0006203]

What we would like to do?

- A new localization mechanism is introduced to ensure the Lorentz symmetry and the decoupling of higher-dimensional left-handed and right-handed fermions.
- The fermion zero mode can be Localized on the brane, and hence a 4D effective action of free spinor fields can be obtained.
- In order to explain the chirality of fermions, it is better to get a 4D effective chirality theory by reducing from a fundamental theory.

Contents

2. Clifford algebra $\mathbb{C}\ell(2n+1,1)$

Given a $(2n+2)$-dimensional complex space $\mathbb{C}^N$, the $(2n+2)$-dimensional Clifford algebra $\mathbb{C}\ell(2n+1,1)$ generated by Dirac matrices Γ_A $(A = 0, 1, 2, \cdots, 2n+1)$ is defined by

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} I_{2n+2}. \tag{1}$$

The above definition can be generalized to the scenario of a curved spacetime

$$\left\{E_M{}^A\Gamma_A, E_N{}^B\Gamma_B\right\} = 2g_{MN} I_{2n+2} \tag{2}$$

with

$$g_{MN} = E_M{}^A E_N{}^B \eta_{AB}. \tag{3}$$

2. Clifford algebra $\mathbb{C}\ell(2n+1,1)$

Construct $\mathbb{C}\ell(2n+1,1)$ generators Γ_A from $\mathbb{C}\ell(2n-1,1)$ [5]:

Weyl representation $\Gamma_A^{(1)}$ or $\Gamma_A^{(2)}$:

$$\Gamma_\mu^{(1)} = \sigma_1 \otimes \gamma_\mu, \Gamma_{2n+1}^{(1)} = -i\sigma_1 \otimes \gamma_{2n+1}, \Gamma_{2n+2}^{(1)} = -i\sigma_2 \otimes 1_{2^n}, \Gamma_{2n+3}^{(1)} = \sigma_3 \otimes 1_{2^n}$$

$$\Gamma_\mu^{(2)} = \sigma_2 \otimes \gamma_\mu, \Gamma_{2n+1}^{(2)} = -i\sigma_2 \otimes \gamma_{2n+1}, \Gamma_{2n+2}^{(2)} = -i\sigma_1 \otimes 1_{2^n}, \Gamma_{2n+3}^{(2)} = \sigma_3 \otimes 1_{2^n}$$

Pauli representation $\Gamma_A^{(3)}$:

$$\Gamma_\mu^{(3)} = \sigma_3 \otimes \gamma_\mu, \quad \Gamma_{2n+1}^{(3)} = -i\sigma_1 \otimes 1_{2^n}, \quad \Gamma_{2n+2}^{(3)} = -i\sigma_2 \otimes 1_{2^n}, \quad \Gamma_{2n+3}^{(3)} = \sigma_3 \otimes \gamma_{2n+1}$$

Dirac representation $\Gamma_A^{(0)}$:

$$\Gamma_\mu^{(0)} = 1_2 \otimes \gamma_\mu, \; \Gamma_{2n+1}^{(0)} = -i\sigma_1 \otimes \gamma_{2n+1}, \; \Gamma_{2n+2}^{(0)} = -i\sigma_2 \otimes \gamma_{2n+1}, \; \Gamma_{2n+3}^{(0)} = \sigma_3 \otimes \gamma_{2n+1}$$

[5] P. Budinich, Found. Phys. 32 (2002) 1347

These representations correspond the following spinor embeddings

$$\psi_D \simeq \psi_P \hookrightarrow \psi_P \oplus \psi_P \simeq \psi_W \oplus \psi_W = \Psi_D \simeq \psi_D \oplus \psi_D$$

which means that

- a 2^n component Dirac spinor is isomorphic to a 2^n component Pauli spinor,
- the direct sum of two such Pauli spinors is equivalent to that of two Weyl spinors,
- a Dirac spinor with 2^{n+1} components may be then considered as a doublet of 2^n component Dirac, Weyl or Pauli spinors.

Relationship between these representations:

The Gamma matrices satisfy the similar transformation

$$U_j\Gamma_A^{\{0\}}U_j^{-1}=\Gamma_A^{\{j\}},\quad (A=1,2,\cdots,2n+2,\quad j=1,2,3)$$

The transformation between the corresponding spinors

$$U_j\Psi^{\{0\}}=\Psi^{\{j\}}.$$

Same form for Dirac equation

$$\left(\Gamma_M^{\{j\}}D^M-m\right)\Psi^{\{j\}}=0$$

This means that different 4D spinors may comes from the same fundamental theory in higher dimensional spacetime.

Contents

3. Decomposition of fermion in (2n+2)-dimensions

Introduce tensor coupling mechanism [6]:

$$\bar{\Psi}\Gamma^M\Gamma^N\Gamma^P\cdots T_{MNP\cdots}\Psi. \tag{4}$$

If we choose weyl representations, then

left and right chiralities coupling for even order tensors;

left and right chiralities decoupling for odd order tensors.

As a special case of first-order tensors, the action is

$$S=\int d^{2n+2}x\sqrt{-g}\left[\bar{\Psi}\Gamma^M D_M\Psi+\varepsilon\bar{\Psi}\Gamma^M\xi_M\Psi\right]. \tag{5}$$

The Dirac equation is

$$\left[\Gamma^M(\partial_M+\Omega_M)+\varepsilon\Gamma^M\xi_M(z)\right]\Psi(x^N)=0. \tag{6}$$

[6] J.-J. Wan and Y.-X. Liu, arXiv: 2303.06278 [hep-th]

Define two operators on the left and right chiralities:

$$\hat{D}_L^{(2^{n+1})} = P_R \Gamma^A E_A^M (\partial_M + \varepsilon \xi_M + G_M), \tag{7a}$$

$$\hat{D}_R^{(2^{n+1})} = P_L \Gamma^A E_A^M (\partial_M + \varepsilon \xi_M + G_M). \tag{7b}$$

For weyl representations

$$P_L = \frac{I_{2^{n+1}} + \Gamma_{2n+3}}{2} = \begin{pmatrix} I_{2^n} & 0 \\ 0 & 0_{2^n} \end{pmatrix}, \tag{8a}$$

$$P_R = \frac{I_{2^{n+1}} - \Gamma_{2n+3}}{2} = \begin{pmatrix} 0_{2^n} & 0 \\ 0 & I_{2^n} \end{pmatrix}. \tag{8b}$$

The bulk action can be independently decomposed into the left and right chiral parts

$$S = S_L + S_R, \tag{9}$$

where

$$S_L = \int d^{2n}x\sqrt{-g}\left[\bar{\Psi}_1^{(2^n)}\hat{D}_L^{(2^n)}\Psi_1^{(2^n)}\right], \tag{10a}$$

$$S_R = \int d^{2n}x\sqrt{-g}\left[\bar{\Psi}_2^{(2^n)}\hat{D}_R^{(2^n)}\Psi_2^{(2^n)}\right]. \tag{10b}$$

The equations of motion are also independent

$$\hat{D}_L^{(2^{n+1})}\Psi_L^{(2^{n+1})} = \hat{D}_L^{(2^{n+1})}\begin{pmatrix}\Psi_1^{(2^n)}\\ 0\end{pmatrix} = 0 \Rightarrow \hat{D}_L^{(2^n)}\Psi_1^{(2^n)} = 0, \tag{11a}$$

$$\hat{D}_R^{(2^{n+1})}\Psi_R^{(2^{n+1})} = \hat{D}_R^{(2^{n+1})}\begin{pmatrix}0\\ \Psi_2^{(2^n)}\end{pmatrix} = 0 \Rightarrow \hat{D}_R^{(2^n)}\Psi_2^{(2^n)} = 0. \tag{11b}$$

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The localization mechanism

The vector ξ_M in the localization mechanism can be constructed from partial derivatives of a scalar function:

$$\varepsilon\bar{\Psi}\Gamma^M\xi_M\Psi = \varepsilon\bar{\Psi}\Gamma^M\partial_M F(\phi, R, R^{\mu\nu}R_{\mu\nu}, \cdots)\Psi. \tag{12}$$

The effects of geometry and the coupling term on fermion localization can be equivalent described by four effective potentials, and can be analyzed independently.

Separate variables

Consider the following 6D line element

$$ds^2 = a_4^2(x^5, x^6)\eta_{\mu\nu}dx^\mu dx^\nu + a_5^2(x^5, x^6)dx_5^2 + a_6^2(x^5, x^6)dx_6^2. \tag{13}$$

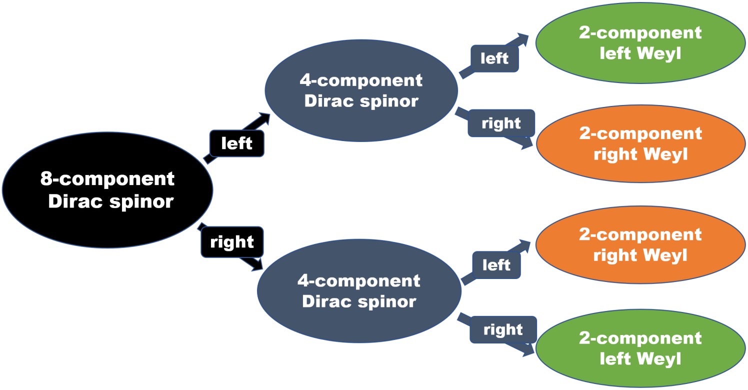
We separate the 4-dimensional and the extra-dimensional parts of the spinor

$$\Psi_1^{(4)} = \psi_1^{(4)}(x^\mu)\,\phi_1(x^5, x^6), \tag{14a}$$

$$\Psi_2^{(4)} = \psi_2^{(4)}(x^\mu)\,\phi_2(x^5, x^6). \tag{14b}$$

4. Localization of fermion in 6D spacetime

Decomposition of the 6D Dirac spinor



The localization condition

The action can be decomposed as

$$S_{L,R} = \int dx^5 dx^6 \tilde{\alpha}_4 \int d^4x \; \bar{\psi}^{(\iota)}_{1,2} \gamma^a \partial_a \psi^{(\iota)}_{1,2}$$
$$+ \int dx^5 dx^6 \tilde{\alpha}_5 \left(\frac{\partial_5 \phi_1}{\phi_1} + \varepsilon \xi_5 + G_5 \right) \int d^4x \, \bar{\psi}^{(\iota)}_{1,2} \gamma^5 \psi^{(\iota)}_{1,2}$$
$$\mp \int dx^5 dx^6 \tilde{\alpha}_6 \left(\frac{\partial_6 \varphi_1}{\varphi_1} + \varepsilon \xi_5 + G_6 \right) \int d^4x \, \bar{\psi}^{(\iota)}_{1,2} \psi^{(\iota)}_{1,2}. \qquad (15)$$

where

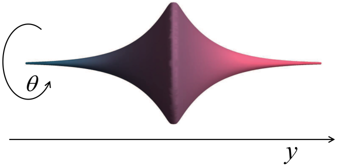
$$\tilde{\alpha}_{4,5,6} = \sqrt{-g} \left[\phi_1^*(x^5, x^6) \phi_1(x^5, x^6) \right] a^{-1}_{4,5,6}(x^5, x^6). \qquad (16)$$

Comparing with the kinetic energy term in 4D effective theory, we obtain the localization condition

$$\int dx^5 dx^6 \tilde{\alpha}_4 = 1. \qquad (17)$$

Background spacetime

A model containing a compact dimension and a non-compact dimension with topology $\mathcal{M}_4 \times \mathcal{R}_1 \times \mathcal{S}_1$, where $\mathcal{M}_4$ is a 4D Minkowski manifold and $\mathcal{R}_1 \times \mathcal{S}_1$ is a transverse manifold.



The metric can be written as

$$ds^2 = a^2(z)(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 + d\Theta^2). \tag{18}$$

Asymptotically AdS spacetime

If this spacetime is generated by a background dynamic field, the action can be written as

$$S = \frac{M^4}{2} \int d^6x \sqrt{-g} \left(R - \Lambda + \mathcal{L}_m \right) \tag{19}$$

with $\mathcal{L}_m$ the Lagrangian of the background scalar field ϕ

$$\mathcal{L}_m = -\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - V(\phi). \tag{20}$$

For asymptotically AdS spacetime, the asymptotic behavior of the background scalar field can be described by three cases:

$$\text{case I:} \quad |\phi(z)| \rightarrow \infty \quad \text{and} \quad \frac{|\phi(z)|}{\log|z|} \rightarrow 0 \tag{21a}$$

$$\text{case II:} \quad \phi(z) \rightarrow v_{\pm} \tag{21b}$$

$$\text{case III:} \quad \phi(z) \rightarrow 0 \tag{21c}$$

Example 1: $\mathcal{M}_4 \times \mathcal{R}_1 \times \mathcal{S}_1$

The extra dimensional parts of these four 2-component spinor fields satisfy the following equations

$$m_1{}^2\phi_{11} = -\partial_5\partial_5\phi_{11} + V_{11}\phi_{11}, \tag{22a}$$
$$m_1{}^2\phi_{12} = -\partial_5\partial_5\phi_{12} + V_{12}\phi_{12}, \tag{22b}$$
$$m_2{}^2\phi_{21} = -\partial_5\partial_5\phi_{21} + V_{21}\phi_{21}, \tag{22c}$$
$$m_2{}^2\phi_{22} = -\partial_5\partial_5\phi_{22} + V_{22}\phi_{22}. \tag{22d}$$

If $F = 0$, then

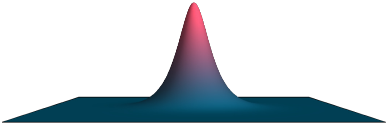
$$V_{11} = V_{22} = V_{12} = V_{21} = 0,$$

which means gravity does not distinguish the right and left chiralities, and spinor fields cannot be localized by minimal coupling with gravity in this geometry.

If $F = \phi^n$, the localization condition is
$n > 1$ for $\phi(z \to \pm\infty) \to \infty$,
$n < 0$ for $\phi(z \to \pm\infty) \to 0$.

Background spacetime

A model containing two non-compact dimension with topology $\mathcal{M}_4 \times \mathcal{R}_2$, where $\mathcal{R}_2$ is a transverse manifold.



We assume that the spacetime is conformally flat, the metric can be written as

$$ds^2 = a_4^2(r)(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + r^2 d\theta^2). \tag{23}$$

Example 2: $\mathcal{M}_4 \times \mathcal{R}_2$

The extra dimensional parts of these four 2-component spinor fields satisfy the following equation

$$m_1{}^2\phi_{11} = -\partial_5\partial_5\phi_{11} + V_{11}\phi_{11}, \tag{24a}$$
$$m_1{}^2\phi_{12} = -\partial_5\partial_5\phi_{12} + V_{12}\phi_{12}, \tag{24b}$$
$$m_2{}^2\phi_{21} = -\partial_5\partial_5\phi_{21} + V_{21}\phi_{21}, \tag{24c}$$
$$m_2{}^2\phi_{22} = -\partial_5\partial_5\phi_{22} + V_{22}\phi_{22}, \tag{24d}$$

where the effective potentials in the $F = 0$ case are

$$V_{11}(r) = V_{22}(r) = \frac{l_6(l_6 - 1)}{r^2}, \tag{25a}$$
$$V_{12}(r) = V_{21}(r) = \frac{l_6(l_6 + 1)}{r^2}. \tag{25b}$$

This means gravity distinguish the right and left chiralities.

Comparison of two topologies of extra dimensions

Similarities

1. If spacetime is conformally flat, a minimal coupling to gravity without other interactions can not localize fermions.
2. If the bulk is conformally AdS, a derivative coupling mechanism (such as $\varepsilon\bar{\Psi}\Gamma^M\partial_M\phi^n\Psi$) is effective for localization of fermions.

Differences

1. The topology $\mathcal{R}_1 \times \mathcal{S}_1$ does not distinguish left- and right-handed fermions.
2. The topology $\mathcal{R}_2$ distinguishes left- and right-handed fermions.
   The 4D chiral theory may be restored with suitable coupling function $F(\phi)$.

5. Summary

1. In a conformally flat extra-dimensional spacetime, fermions cannot be localized through minimal coupling with gravity.
2. Therefore, we propose using a coupling mechanism with
$$\bar{\Psi}\Gamma^M\Gamma^N\Gamma^P\cdots T_{MNP\cdots}\Psi$$
to preserve Lorentz symmetry and decouple the components of higher-dimensional spinors to obtain a 4D effective free field theory.
3. For the manifold with a topology of $\mathcal{M}_4\times\mathcal{R}_2$, the minimal coupling between fermions and gravity will distinguish left and right chiralities, and the fermion may be localized on the brane with the tensor coupling.

Thank you!