黑洞热力学相变的动态化分析

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Z.-M. Xu, B. Wu and W.-L. Yang, SCPMA (2023)
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Introduction

Black Hole Thermodynamics

- Hawking证明，沿时间方向，经历任何过程后，黑洞的视界面积永不减少（黑洞面积定理）

$$\delta A \geq 0$$

- Bekenstein在Hawking的面积定律基础上，提出黑洞具有熵，并且黑洞的熵正比于它的视界面积

$$S \propto A$$

- Hawking用黑洞背景下的半经典量子场论的方法定出了比例系数

$$S = \frac{k_B c^3}{4G\hbar} A$$

- Hawking发现黑洞有辐射

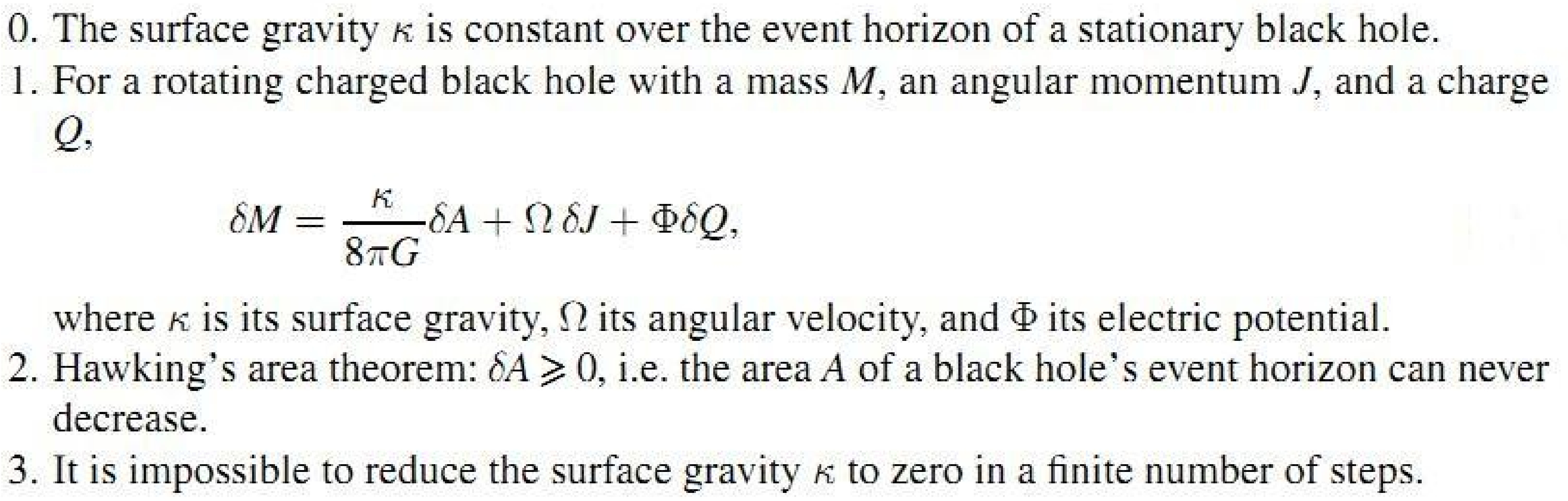
$$T = \frac{\hbar\kappa}{2\pi c k_B}$$

Introduction

Black Hole Thermodynamics

黑洞力学四定律

Bardeen, Carter, Hawking, CMP 31, 161 (1973)



0. The surface gravity κ is constant over the event horizon of a stationary black hole.
1. For a rotating charged black hole with a mass M, an angular momentum J, and a charge Q,

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\,\delta J + \Phi\delta Q,$$

   where κ is its surface gravity, Ω its angular velocity, and Φ its electric potential.
2. Hawking's area theorem: $\delta A \geqslant 0$, i.e. the area A of a black hole's event horizon can never decrease.
3. It is impossible to reduce the surface gravity κ to zero in a finite number of steps.

Introduction

Black Hole Thermodynamics

Thermodynamics Gravity

Energy $E \leftrightarrow M$ Mass

Temperature $T \leftrightarrow \frac{\hbar\kappa}{2\pi}$ Surface gravity

Entropy $S \leftrightarrow \frac{A}{4\hbar}$ Horizon Area

$$dE = TdS + VdP + \text{work terms} \leftrightarrow dM = \frac{\kappa}{8\pi}dA + \Omega dJ + \Phi dQ$$

First Law First Law

?

Introduction Black Hole Thermodynamics

Schwarzschild-AdS Black hole

Kastor, Ray, Traschen, CQG 26, 195011 (2009)

Dolan, CQG 28, 125020 (2011)

$$E = M = \frac{l^2 + r_+^2}{2l^2} r_+ \quad T = \frac{l^2 + 3r_+^2}{4\pi r_+ l^2} \quad S = \pi r_+^2 \qquad (D=4)$$

$$M = 2(TS - VP)$$

Smarr ✓

$$dE = TdS + VdP$$

First Law ✓

Provided

Extended phase space

$$P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi}\frac{1}{l^2}$$

Thermodynamic Pressure

$$V = -8\pi\frac{\partial M}{\partial \Lambda} = \frac{4\pi}{3} r_+^3$$

Thermodynamic Volume

Introduction

Thermodynamics Gravity

Mass as Enthalpy

Enthalpy $H \leftrightarrow M$ Mass

Temperature $T \leftrightarrow \frac{\hbar\kappa}{2\pi}$ Surface gravity

Entropy $S \leftrightarrow \frac{A}{4\hbar}$ Horizon Area

$$dH = TdS + VdP + \cdots \leftrightarrow dM = \frac{\kappa}{8\pi} dA + VdP + \cdots$$

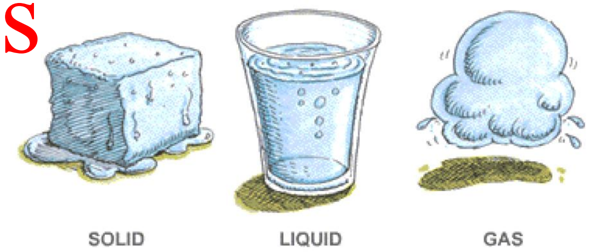
First Law First Law

$$H = E + PV + \cdots \leftrightarrow M = E - \rho V$$

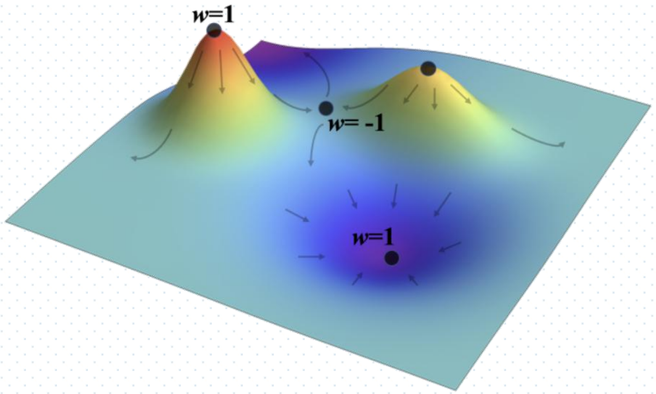
Mass
= Total Energy
– Vacuum
Contribution
(infinite)

Kastor, Ray, Traschen, CQG 26, 195011 (2009)

Everyday AdS Black Hole Thermodynamics



- Hawking Page Transition **Hawking, Page, CMP 87, 577 (1983)**
- Van der Waals Fluid and Charged AdS Black Holes **Kubiznak, Mann, JHEP 07, 033 (2012)**
- Black Hole Triple Points ←→ Solid/Liquid/Gas **Altamirano, Kubiznak, Mann, Sherkatghanad, CQG 31, 042001 (2014)**
- Holographic Heat Engines **Johnson, CQG 31, 205002 (2014)**
- Thermodynamics geometry **Ruppeiner, RMP 67, 605 (1995)**
- Holographic Thermodynamics **Visser, PRD 105, 106014 (2022)**
- Free Energy Landscape **R. Li and J. Wang, PRD 102, 024085 (2020)**
- Black hole topological thermodynamic **S.-W. Wei, Y.-X. Liu and R. B. Mann, PRL 129, 191101 (2022)**



Introduction Black Hole Thermodynamics

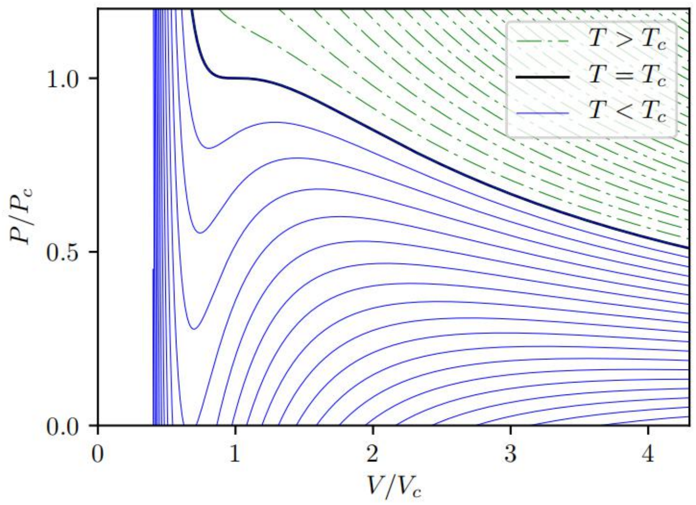
Equation of State of RN-AdS BH

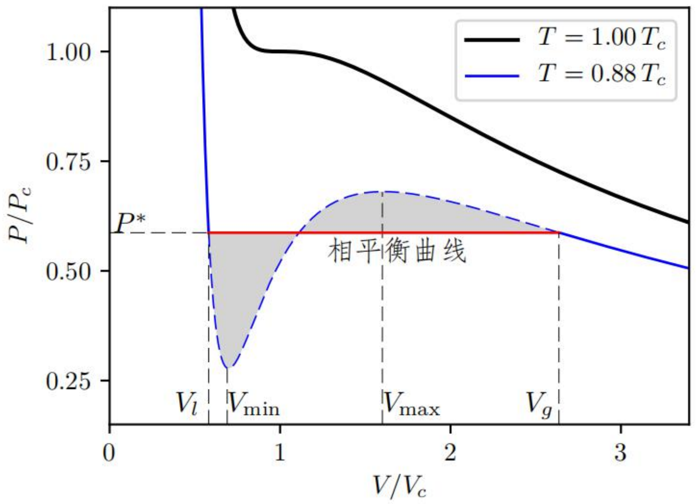
$$T = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{l^2} - \frac{Q^2}{r_+^2}\right)$$

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi}\frac{1}{l^2}$$

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} \qquad r_+ = \left(\frac{3V}{4\pi}\right)^{1/3}$$

P-*V* Diagram





Maxwell's
Equal Area Law

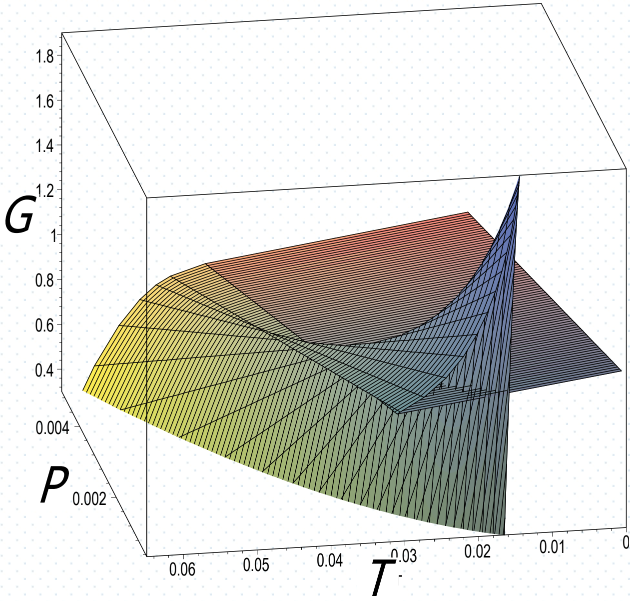
Introduction

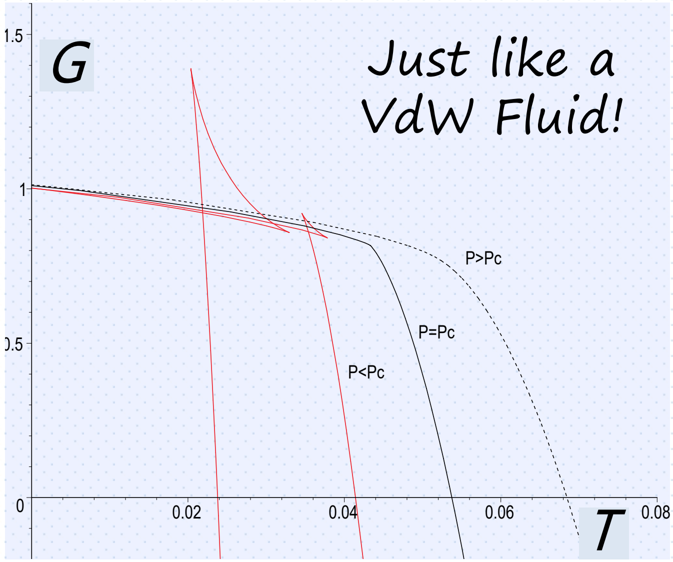
Black Hole Thermodynamics

Gibbs Free Energy of RN-AdS BH

$$I = -\frac{1}{16\pi}\int_M \sqrt{-g}\left(R - F^2 + \frac{6}{l^2}\right) - \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}K - \frac{1}{4\pi}\underbrace{\int_{\partial M} d^3x\sqrt{h}n_aF^{ab}A_b}_{\text{Fixed Charge}} + I_c$$

$$G = G(T,P) = \frac{1}{4}\left(r_+ - \frac{8\pi}{3}Pr_+^3 + \frac{3Q^2}{r_+}\right)$$





Just like a VdW Fluid!

Kubiznak, Mann, JHEP 07, 033 (2012)

Landau functional

Landau approximate the free energy of a system

it exhibits the non-analyticity of a phase transition and turns out to capture much of the physics

Landau believed that the order parameter m near the critical point T_c is a small amount; thus the free energy function $F(T,m)$ can be expanded to the power of m near T_c *(second-order phase transition)*

$$F = a(T) + \frac{1}{2}b(T)\mathfrak{m}^2 + \frac{1}{4}c(T)\mathfrak{m}^4 - \mathscr{B}\mathfrak{m} + \cdots .$$

（伊辛模型）

$$a(T) = a_0 + a_1(T - T_c) + \cdots ,$$

$$b(T) = b_0(T - T_c) + \cdots ,$$

$$c(T) = c_0 + c_1(T - T_c) + \cdots .$$

For black holes: **X.-Y. Guo, H.-F. Li, L.-C. Zhang and R. Zhao, PRD 100, 064036 (2019).**
X.-P. Li, Y.-B. Ma, Y. Zhang, L.-C. Zhang, and H.-F. Li, CJP 83, 123 (2023) .

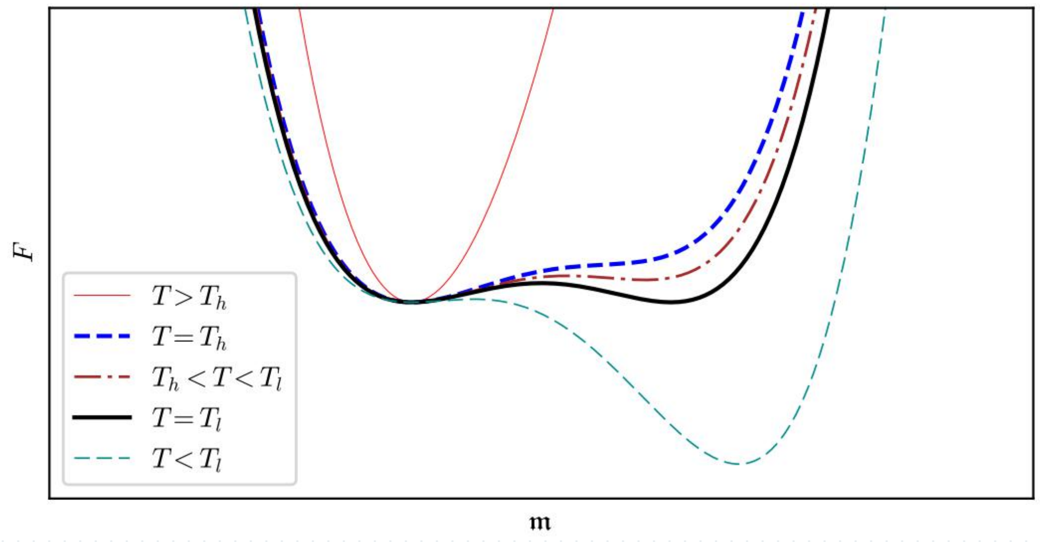
Landau functional

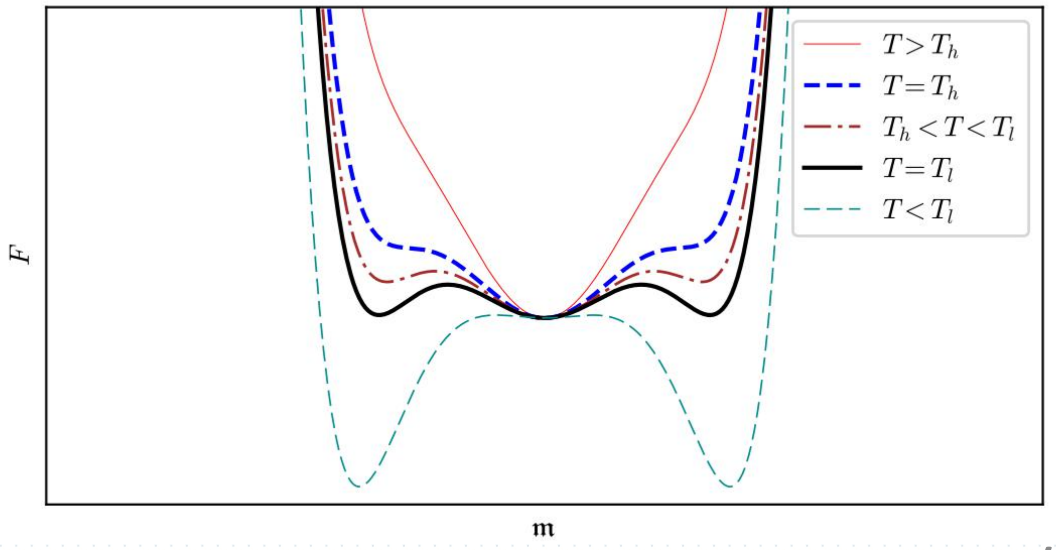
朗道理论是否只能描述二级以上的连续相变而不能描述一级相变？这个问题的回答是否定的。至少有两种方法对朗道理论进行微调后都可适用于描述一级相变。

$$F = a(T) + \frac{1}{2}b(T)\mathfrak{m}^2 + \boxed{\frac{1}{3}\xi(T)\mathfrak{m}^3} + \frac{1}{4}c(T)\mathfrak{m}^4$$

$$F = a(T) + \frac{1}{2}b(T)\mathfrak{m}^2 + \frac{1}{4}c(T)\mathfrak{m}^4 \boxed{+ \frac{1}{6}\xi(T)\mathfrak{m}^6,}$$

$$\boxed{c(T) < 0, \qquad \xi(T) > 0.}$$





Landau functional free energy landscape

4D Schwarzschild-AdS BH

$$M = \frac{r_+}{2}\left(1+\frac{r_+^2}{L^2}\right) \qquad S = \pi r_+^2. \qquad T_H = \frac{1}{4\pi r_+}\left(1+\frac{3r_+^2}{L^2}\right)$$

Gibbs Free Energy $G = M - \underset{\blacktriangle}{T_H} S = \frac{r_+}{2}\left(1+\frac{r_+^2}{L^2}\right) - \frac{r_+}{4}\left(1+\frac{3r_+^2}{L^2}\right)$

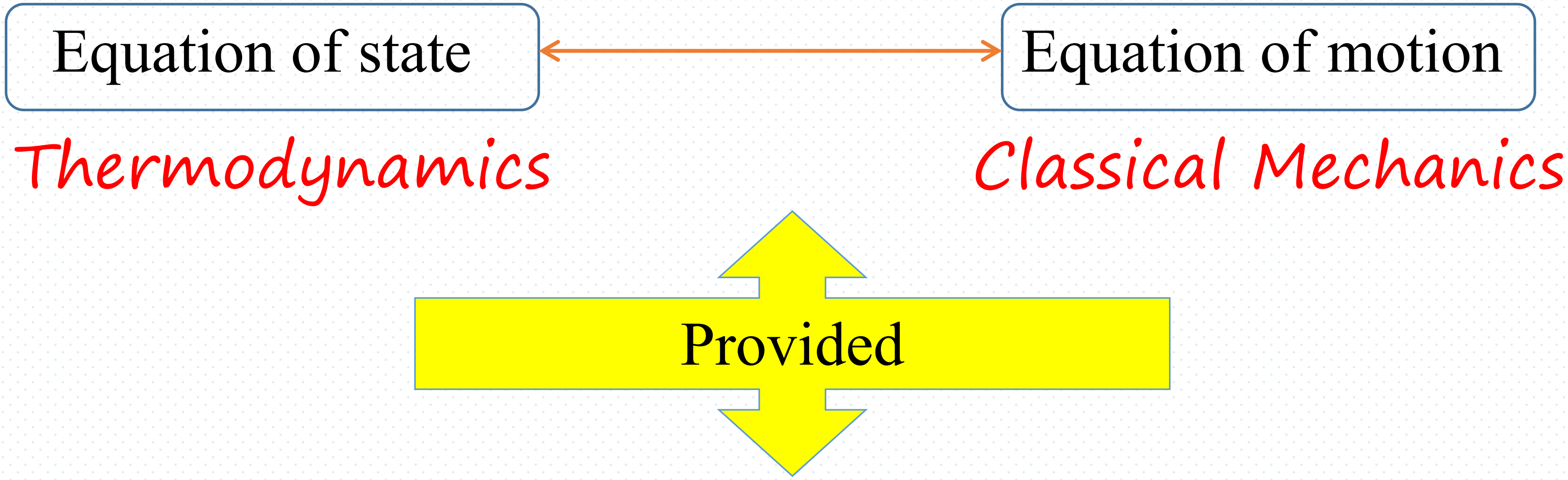
◆ On-shell Gibbs free energy: $G = M - T_H S$ or calculated directly from the Euclidean action

◆ Off-shell Gibbs free energy: **replacing the Hawking temperature T_H with the ensemble temperature T**

Free energy landscape $G = M - \underset{\blacktriangle}{T} S = \frac{r_+}{2}\left(1+\frac{r_+^2}{L^2}\right) - \pi T r_+^2$

R. Li and J. Wang, PRD 102, 024085 (2020)

Thermal potential Landau functional



Equation of state

Thermodynamics

Equation of motion

Classical Mechanics

Provided

The process of a system from an unknown state to an equilibrium state:

selecting a relation (equation of state in equilibrium) from all possible relations

$$\boxed{L = \int F(X,T,P)\mathrm{d}X}$$

$$\boxed{F(X,T,P) \equiv P - f(X,T)}$$

Z.-M. Xu, B. Wu and W.-L. Yang, CQG (2021)

Thermal potential

- Canonical ensemble at temperature T composed of a large number of states

◆ **The real black hole state (on shell) is the solution of the Einstein field equation**

◆ **while others (off shell) are not**

- $T=T_h$: the ensemble is made up of real black hole states

Thermal potential

$$\boxed{f(x) = \int (T_h - T)\mathrm{d}S.}$$

The integrand： the deviation of all possible states from the real black hole state

Z.-M. Xu, PRD (2021)

- **Degree of the thermal motion is measured by the product of temperature and the entropy.**
- **$T = T_h$ is just one of the ways to get the value of the ensemble temperature T.**
- **For a simple thermodynamic system, according to the first law of thermodynamics $dE = T_h dS - PdV$**

➡ $f(x) = L$

Thermal potential

Thermodynamic Properties ═ **Geometric Characteristics**

- **Equilibrium state is the one that makes the potential take the minimum value**

$$\frac{\mathrm{d}f(x)}{\mathrm{d}S}=0\Rightarrow T=T_h$$

- **Thermal stabilityis related to the convexity and convexity of the extreme point**

◆$\partial t(S,\ Y)/\partial S > 0$, potential well, stable

◆$\partial t(S,\ Y)/\partial S < 0$, potential barrier, unstable

van der Waals fluid

van der Waals fluid

$$\left(P+\frac{a}{v^2}\right)(v-b)=\tau \qquad \tau = k_B T$$

Chemical potentials μ, or equivalently the Gibbs free energy $G = \mu N$

$$\mu = -\tau\ln(v-b)+\frac{b\tau}{v-b}-\frac{2a}{v}-\tau\ln n_Q$$

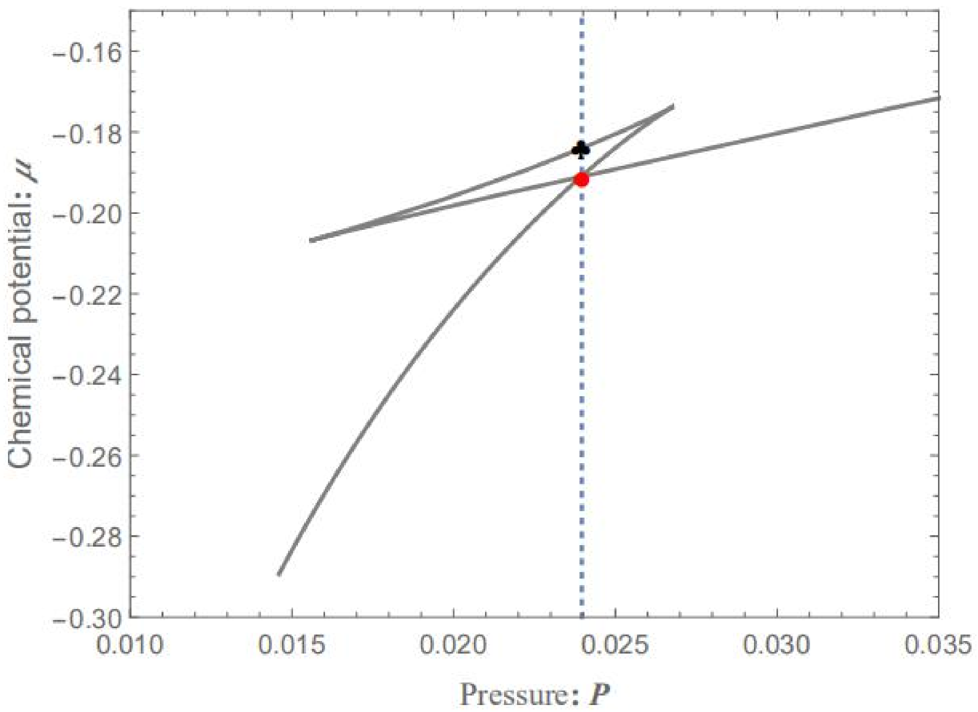
Thermal potential

$$L = Px - \frac{a}{x} - \tau\ln(x-b)$$

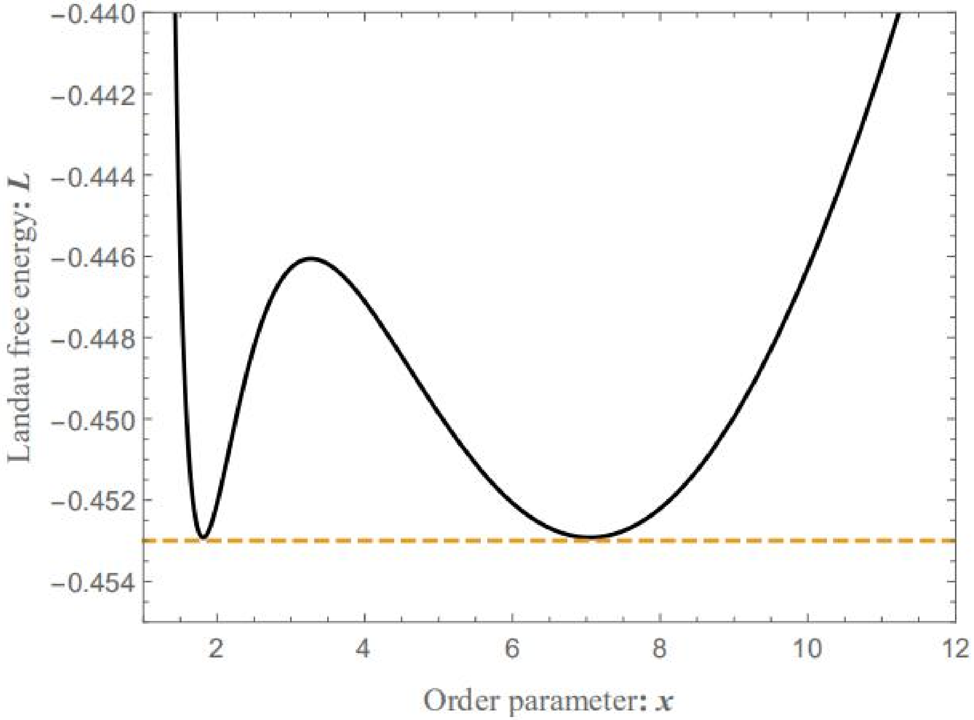
$$\frac{\mathrm{d}L}{\mathrm{d}x}=0 \qquad \Rightarrow \qquad P=\frac{\tau}{v-b}-\frac{a}{v^2}.$$

van der Waals fluid

van der Waals fluid



(a)$\tau = 0.900\tau_c$



(b)$\tau = 0.900\tau_c$ and $P = 0.647P_c$

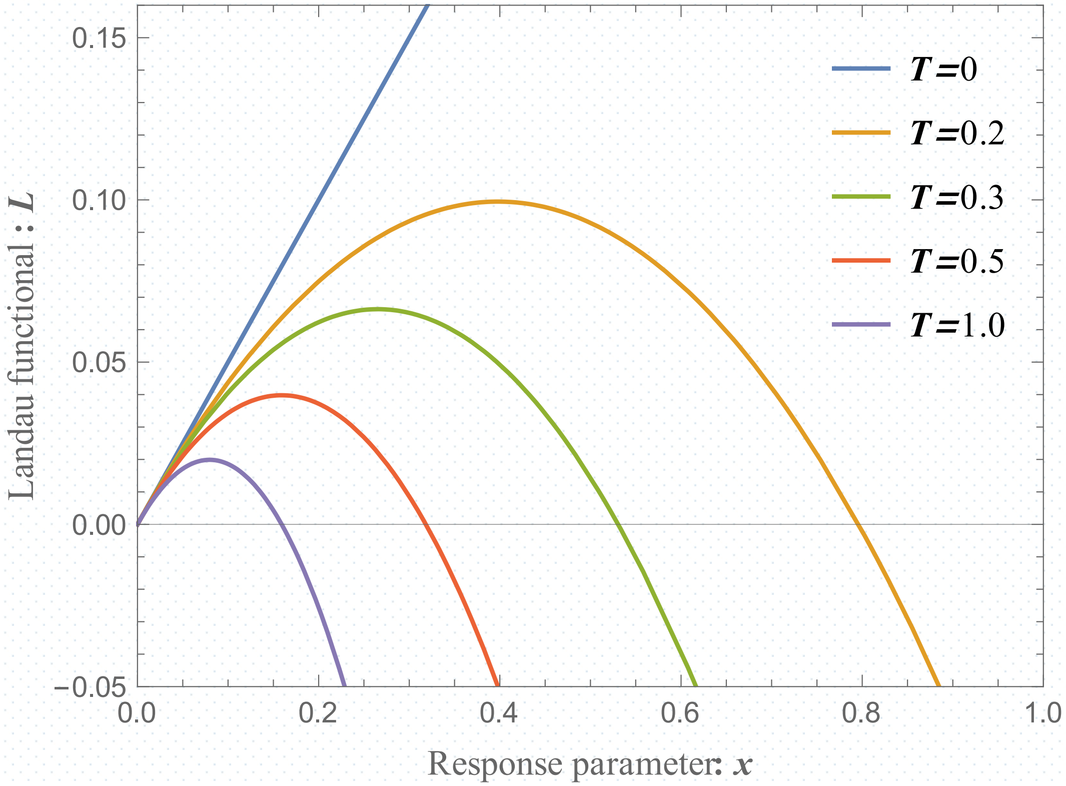
Schwarzschild black hole

4D Schwarzschild BH

Instability

Thermal potential

$$f(x) = L = \frac{x}{2} - \pi T x^2$$



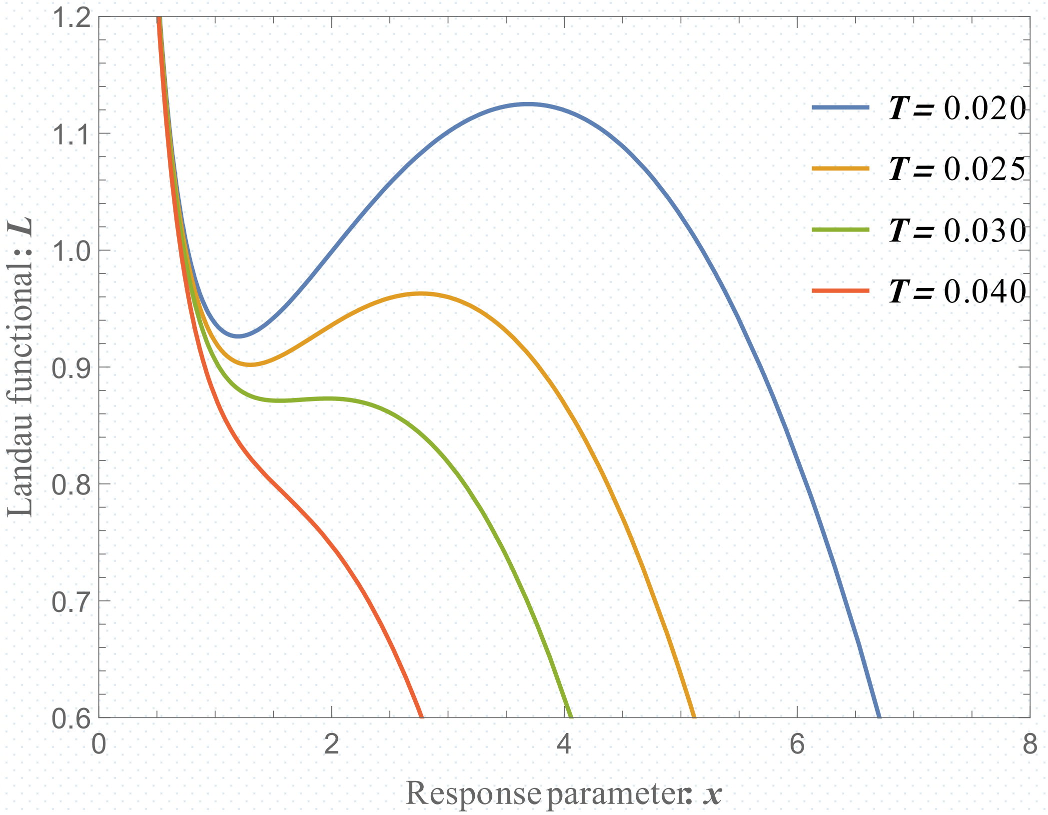
Charged black hole

4D Reissner-Nordström BH

Thermal potential

$$f(x) = L = \frac{x}{2} - \pi T x^2 + \frac{Q^2}{2x}$$

Metastability



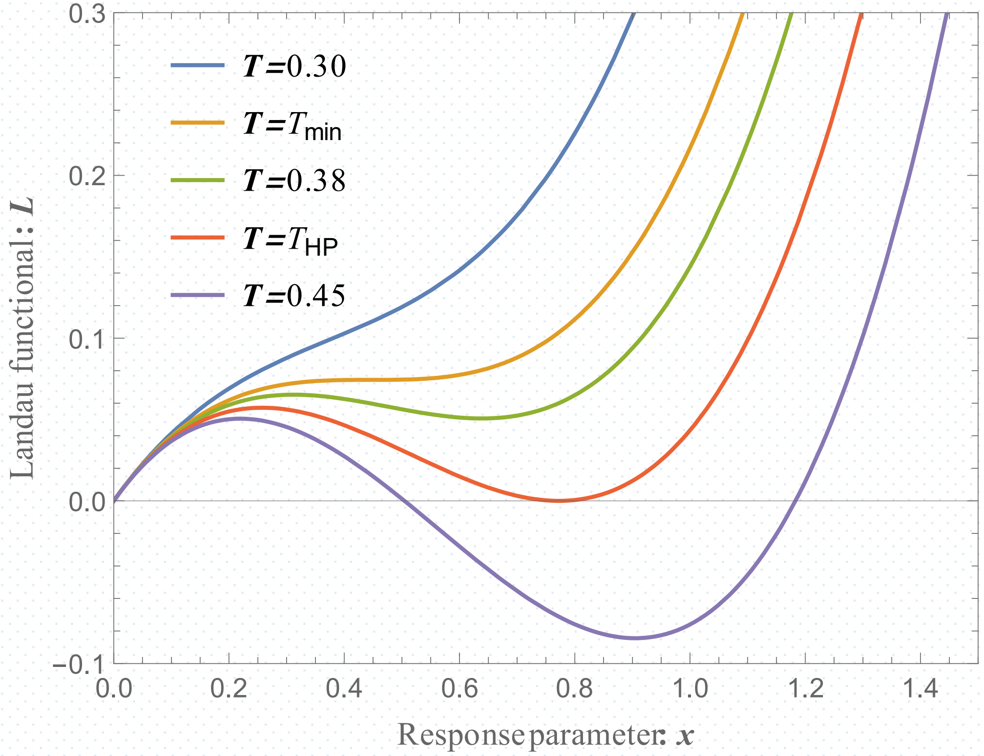
Schwarzschild-AdS black hole

4D Schwarzschild-AdS BH

Stability

Thermal potential

$$f(x) = L = \frac{x}{2} - \pi T x^2 + \frac{4\pi P}{3} x^3$$



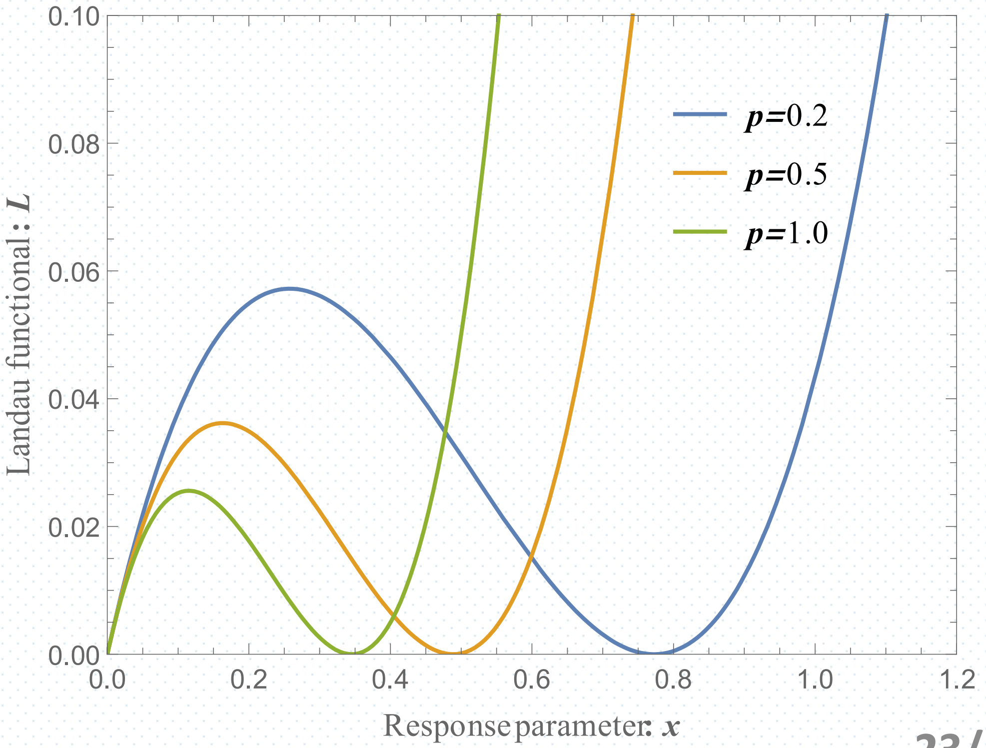
Schwarzschild-AdS black hole

4D Schwarzschild-AdS BH

Thermal potential

$$f(x) = L = \frac{x}{2} - \pi T x^2 + \frac{4\pi P}{3} x^3$$

Hawking-Page phase transition



Charged AdS black hole

4D RN-AdS BH

Thermal potential

$$f(x) = L = \frac{r}{2} - \pi T r^{2} + \frac{4\pi P}{3} r^{3} + \frac{Q^{2}}{2r}$$

dimensionless

$$\psi(x,t,p) := \frac{L}{G_c} = \frac{1}{4}\left(\frac{1}{x} + 6x + px^{3} - 4tx^{2}\right)$$

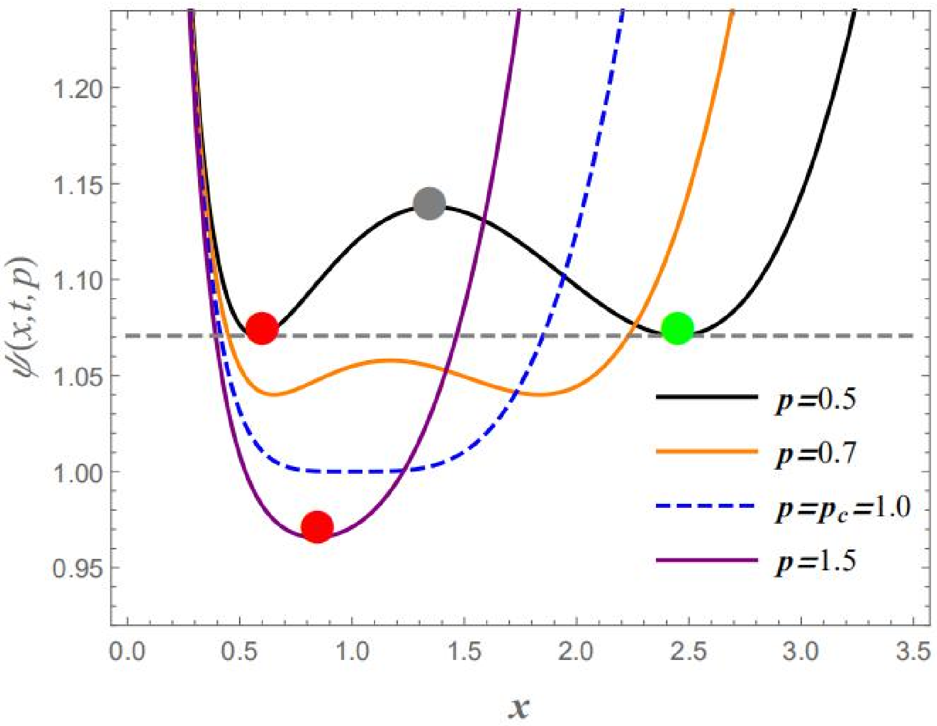
Charged AdS black hole

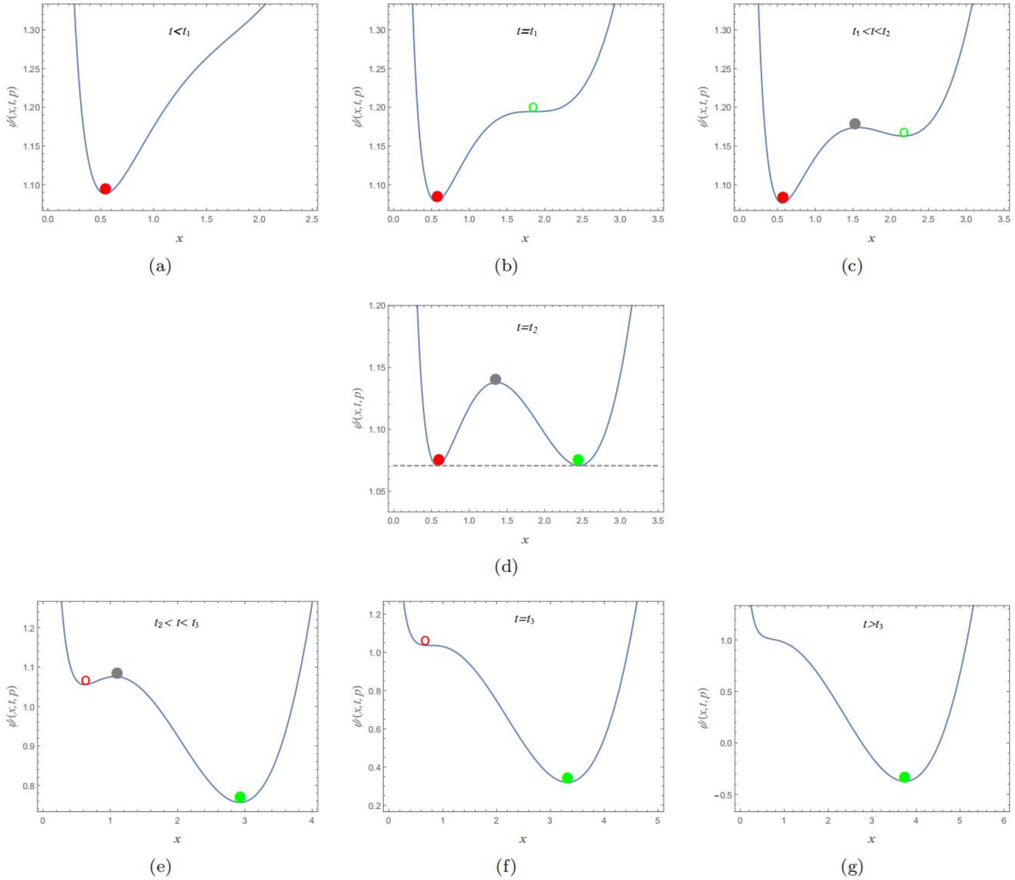
4D RN-AdS BH

$$f(x)=\frac{\sqrt{6}Q}{3}\psi(x)=\frac{\sqrt{6}Q}{3}\left(\frac{1}{4x}+\frac{3x}{2}+\frac{px^3}{4}-tx^2\right)$$

second-order phase transition

first-order phase transition



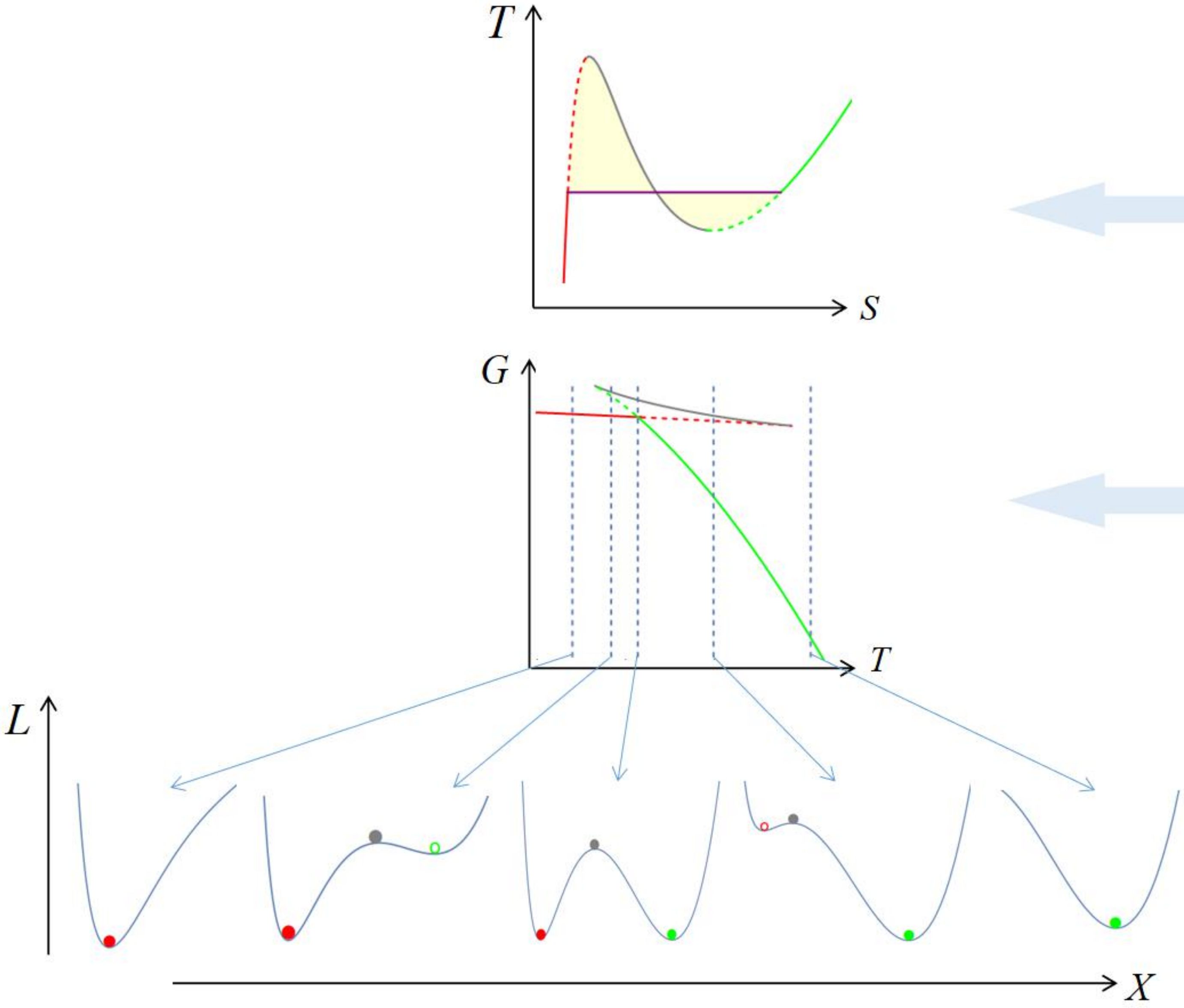


(a) (b) (c)

(d)

(e) (f) (g)

Thermal potential



Maxwell equal area law

$$T\Big|_{S_1} = T\Big|_{S_2} = T^*$$

$$T^* = \frac{1}{S_2 - S_1}\int_{S_1}^{S_2} T(S,P)\,dS$$

Swallow tail intersection

$$G\Big|_{S_1} = G\Big|_{S_2}$$

$$T\Big|_{S_1} = T\Big|_{S_2} = T^*$$

Global minimum

$$\left.\frac{\partial L}{\partial X}\right|_{T,P} = 0$$

1. Fokker-Planck equation

Due to thermal fluctuations, the black hole moves stochastically in the thermal potential, which leads to different phase transition characteristics.

The probability distribution *W(x,t)* of these black hole states (on-shell states and off-shell states) evolving in time under the thermal fluctuation should be described by the probabilistic Fokker-Planck equation

$$\boxed{\frac{\partial W(x,t)}{\partial t} = \left[\frac{\partial}{\partial x} f'(x) + D\frac{\partial^2}{\partial x^2}\right] W(x,t)}$$

one-variable

time independent drift coefficient

constant diffusion coefficient

$$L\psi(x) = \varepsilon\psi(x),$$

$$L = -D\frac{\partial^2}{\partial x^2} + V_s(x), \quad V_s(x) = \frac{1}{4D}[f'(x)]^2 - \frac{1}{2}f''(x).$$

1. Fokker-Planck equation

4D Schwarzschild BH

Thermal potential

$$f(x) = \frac{x}{2} - \pi T x^2$$

$$V_s(x) = \frac{\pi^2 T^2}{D} z^2 + \pi T, \qquad z = x - \frac{1}{4\pi T}.$$

$$\frac{\partial^2}{\partial \xi^2}\psi(x) + \left(\frac{\varepsilon}{\pi T} - 1 - \xi^2\right)\psi(x) = 0.$$

$$\varepsilon_n = 2\pi T(n+1), \quad n = 0, 1, 2, \cdots$$

$$\psi_n(x) = \left(\frac{T}{D}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2},$$

If the ensemble temperature is exactly in accordance with the temperature expression of a Schwarzschild black hole, that is $T = T_h = 1/(8\pi M)$

$$\varepsilon_n = \frac{n+1}{4M}, \quad n = 0, 1, 2, \ldots,$$

1. Fokker-Planck equation

3D BTZ BH

Thermal potential

$$f(x) = \frac{x^2}{8l^2} - \frac{\pi T x}{2}$$

$$\frac{\partial^2}{\partial \xi^2}\psi(x) + (8l^2\varepsilon + 1 - \xi^2)\psi(x) = 0.$$

$$\varepsilon_n = \frac{n}{4l^2}, \quad n = 0, 1, 2, \cdots$$

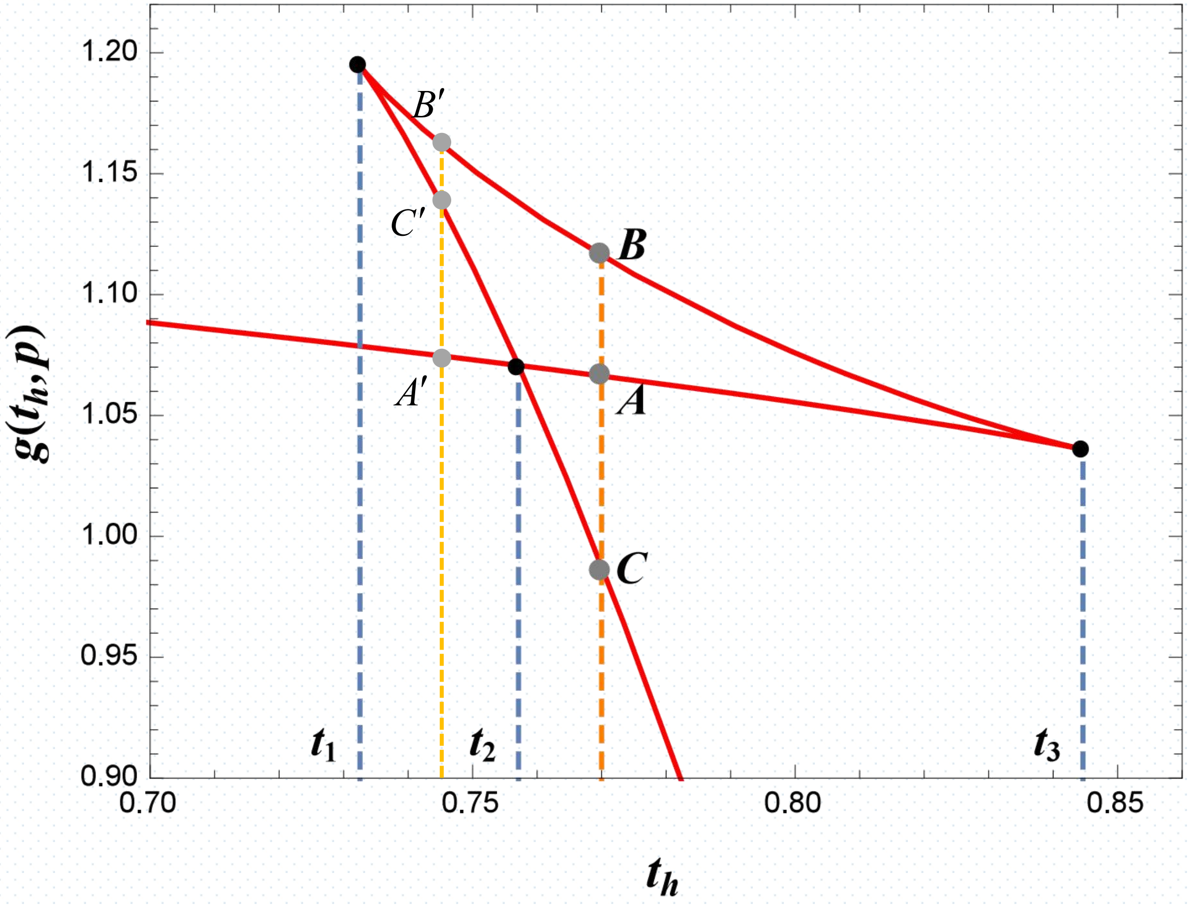
$$\psi_n(x) = \left(\frac{1}{8\pi D l^2}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}.$$

The most prominent difference is that the ground state is zero

The energy spectrum only depends on the parameter of the black hole itself: the AdS radius *l*

2. Phase transition rate

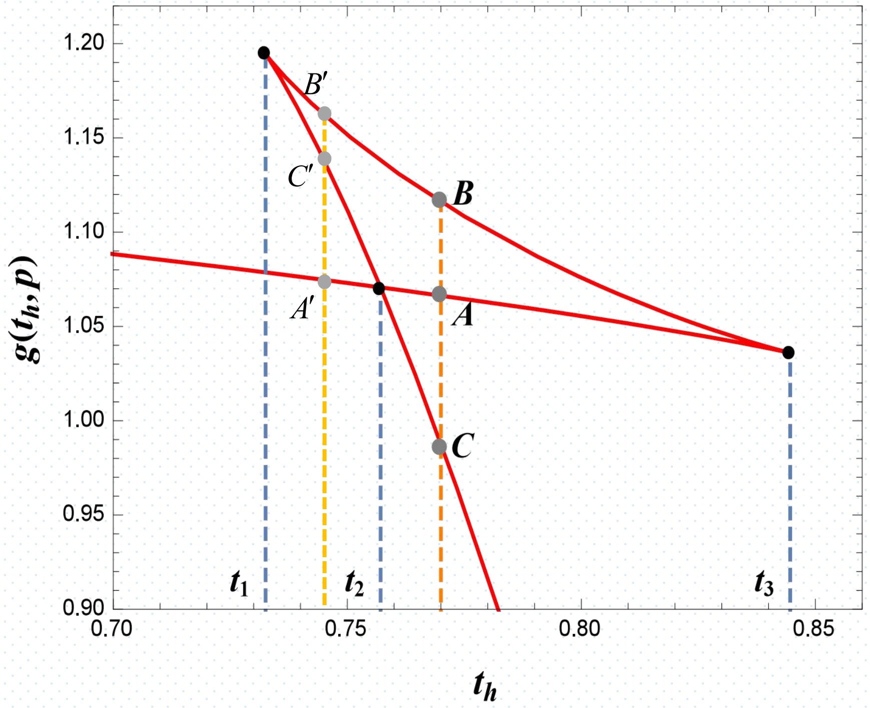
First-order phase transition



◆Which of the two processes dominates in the phase transition of the AdS black hole?

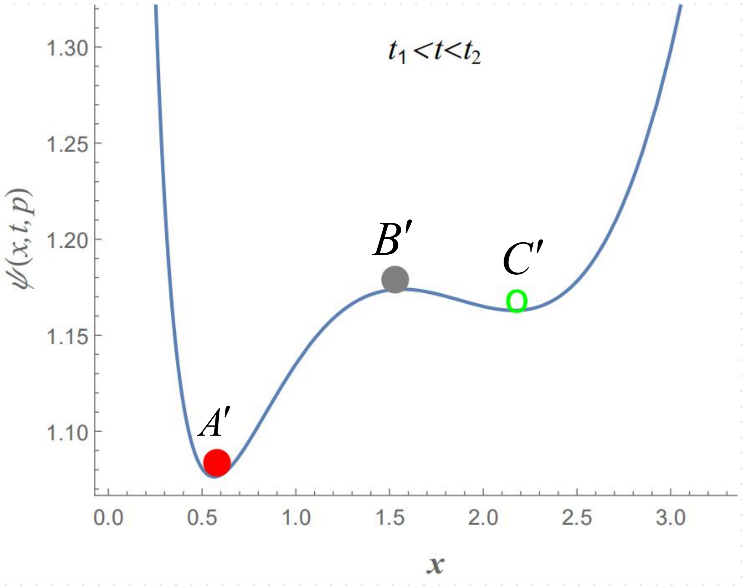
◆Under what circumstances will the two processes achieve dynamic balance?

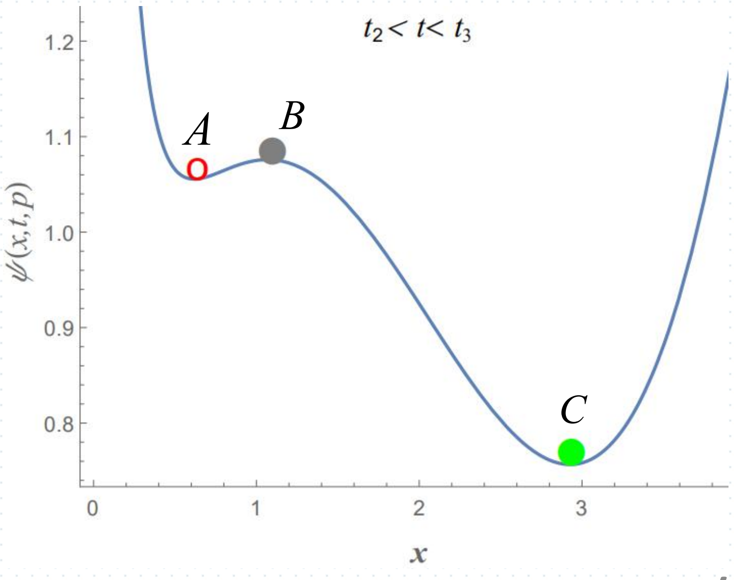
2. Phase transition rate



First-order phase transition

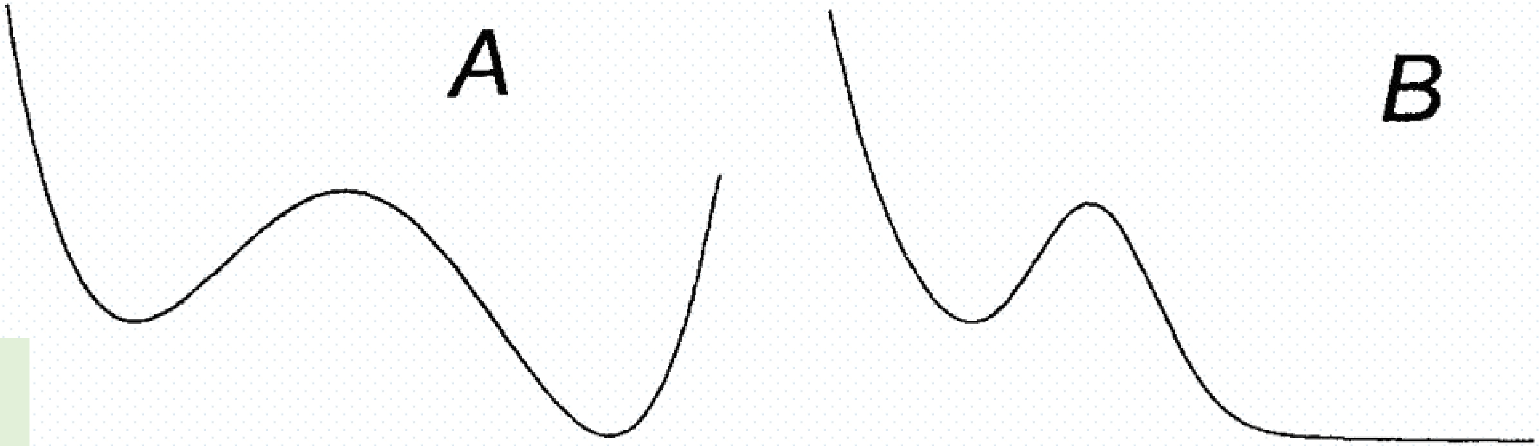
Black hole state in potential





2. Phase transition rate

Kramer's rate



To determine the rate at which a Brownian particle escapes from a potential well

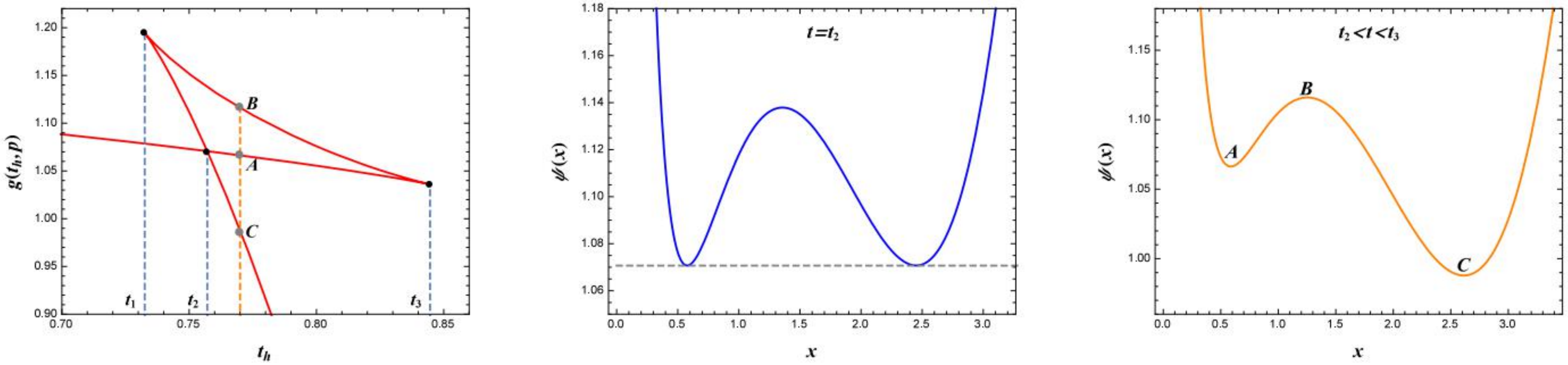
A: it might describe a molecular rearrangement

B: it might describe a molecular dissociation

$$r_k = \frac{\sqrt{|f''(x_{\min}) f''(x_{\max})|}}{2\pi} e^{-\dfrac{f(x_{\max}) - f(x_{\min})}{D}}$$

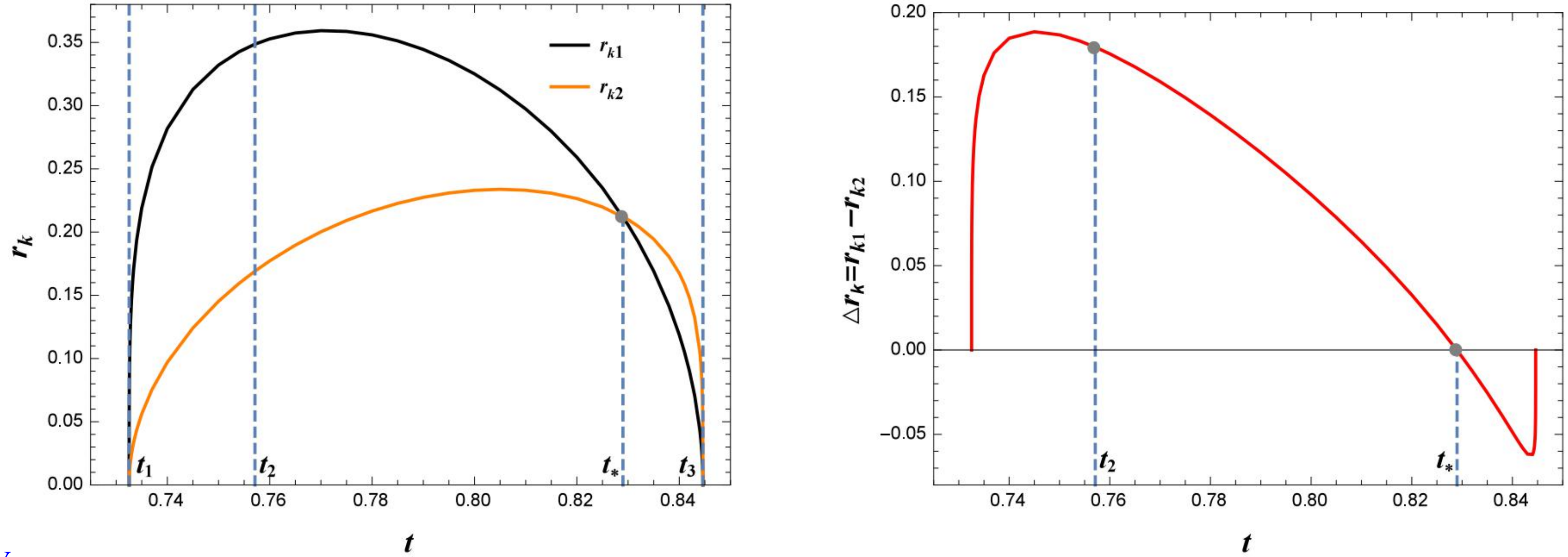
2. Phase transition rate

4D RN-AdS



$A \rightarrow C$: r_{k1}

$C \rightarrow A$: r_{k2}



(a) transition rate

(b) net transition rate

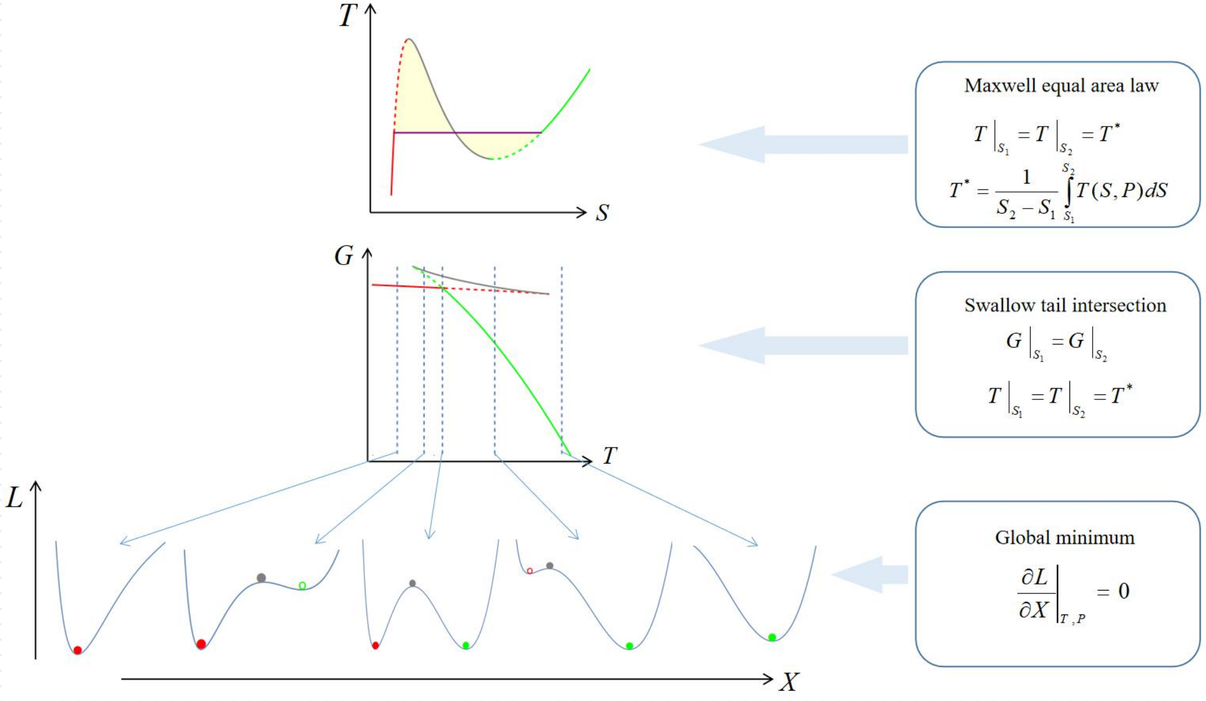
Z.-M. Xu, B. Wu and W.-L. Yang,
SCPMA (2023)

Summary

Thermal potential can directly reflect the physical process of black hole phase transition

- Geometric representation of the first and second order phase transition of the black hole
- Observing the thermal-motion behavior of states of the canonical ensemble in thermal potential due to the thermal fluctuation
- Phase transition between small and large black holes for a charged AdS black hole presents a serious asymmetric feature
- Overall process is dominated by the transition from a small black hole to a large black hole

Summary



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Swallow tail intersection

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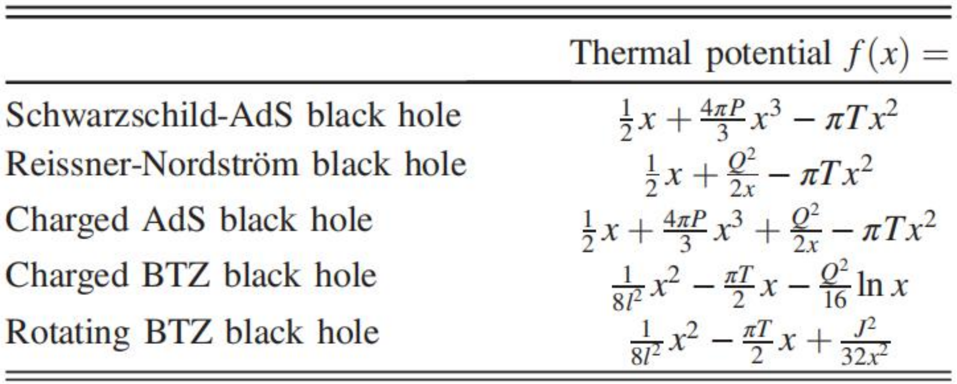
$$T\big|_{S_1} = T\big|_{S_2} = T^*$$

Global minimum

$$\left.\frac{\partial L}{\partial X}\right|_{T,P} = 0$$



	Thermal potential $f(x) =$
Schwarzschild-AdS black hole	$\frac{1}{2}x + \frac{4\pi P}{3}x^3 - \pi T x^2$
Reissner-Nordström black hole	$\frac{1}{2}x + \frac{Q^2}{2x} - \pi T x^2$
Charged AdS black hole	$\frac{1}{2}x + \frac{4\pi P}{3}x^3 + \frac{Q^2}{2x} - \pi T x^2$
Charged BTZ black hole	$\frac{1}{8l^2}x^2 - \frac{\pi T}{2}x - \frac{Q^2}{16}\ln x$
Rotating BTZ black hole	$\frac{1}{8l^2}x^2 - \frac{\pi T}{2}x + \frac{J^2}{32x^2}$

Thank you !