Recent adventures with amplitudes: gravitons, gluons & Chern-Simons

何 颂

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Partly based on works with A. Edison, H. Johansson, O. Schlotterer, 滕飞, 张勇, to appear
& with Chia-Kai Kuo, 李振杰， 张耀奇, 2204.08297 + work in progress

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北京
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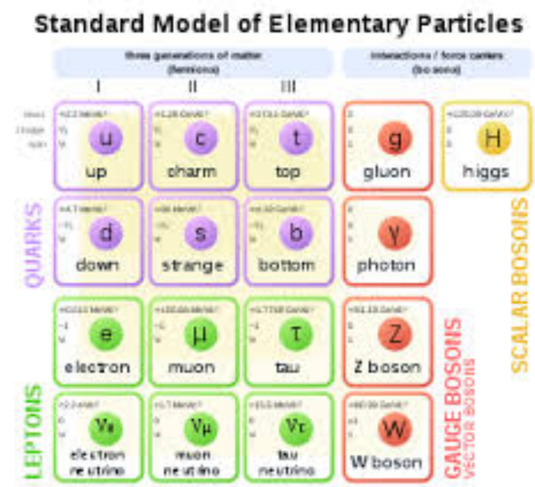
QFT & scattering amplitudes

Theoretical framework to describe Nature: particle physics, condensed matter, cosmology, strings

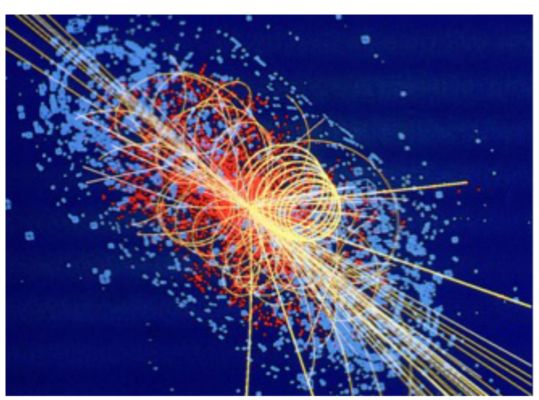


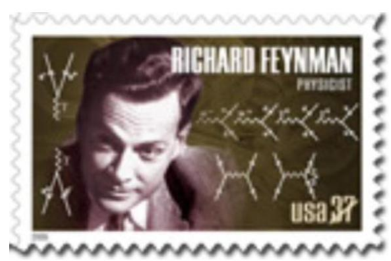
$$g_e^{\text{theory}} = 2.00231930435801\ldots \quad [2012]$$
$$g_e^{\text{expt}} = 2.002319304361\ldots \quad [2011]$$

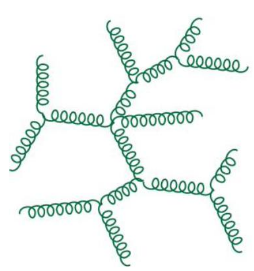
incredible accuracy!

inevitable: consequence of QM & relativity!
fundamental interactions unified @ high energy

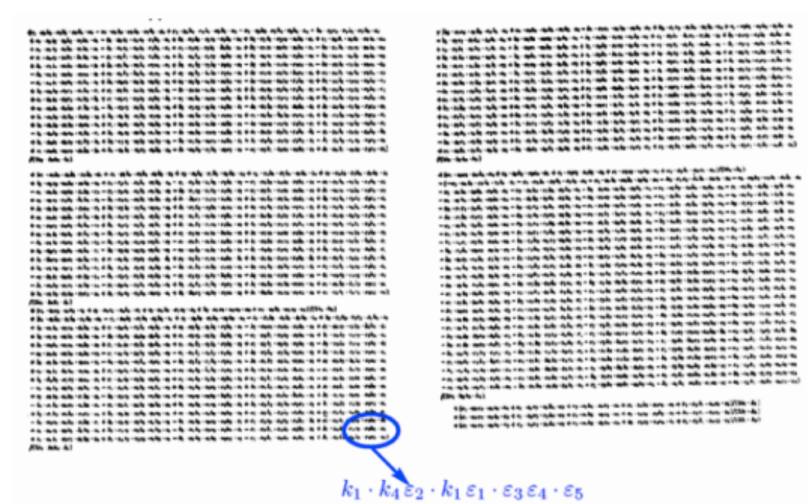
- S-matrix: most basic calculation, connects theory to experiment e.g. colliders at high energies need amplitudes of many gluons/quarks







+ ...



Process	N_{FG}
$gg \to 2g$	4
$gg \to 3g$	25
$gg \to 4g$	220
$gg \to 5g$	2485
$gg \to 6g$	34300
$gg \to 7g$	559405
$gg \to 8g$	10525900
$gg \to 9g$	224449225
$gg \to 10g$	5348843500

- Fundamental level our understanding of QFT & gravity incomplete: strong coupling, dualities, hidden symmetries, quantum gravity & cosmology…
- simplicity, new structures & relations seen in perturbative scattering amplitudes!

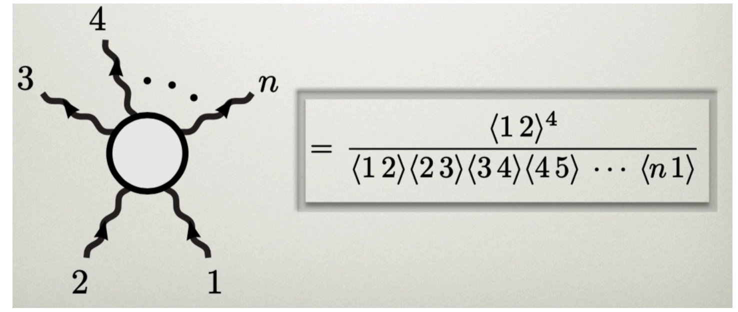
A brief history of amplitudes

“Chinese magic” & Parke-Taylor formula for MHV tree (1986)

Spinor-helicity variables

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \tilde{\lambda}_{\dot{a}}$$
$$\langle 12 \rangle = \epsilon_{ab} \lambda_a^{(1)} \lambda_b^{(2)}$$
$$[12] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_{\dot{a}}^{(1)} \tilde{\lambda}_{\dot{b}}^{(2)}$$

(Mangano, Parke, Xu 1987)



90’s: developing generalized unitarity (w. methods for trees)
e.g. tree & one-loop gluon amps in QCD, N=4 SYM & N=8 SG

- Twistor strings (2003) … **BCFW recursion**: all trees in QCD

new unitarity methods → all one-loop QCD & a lot more

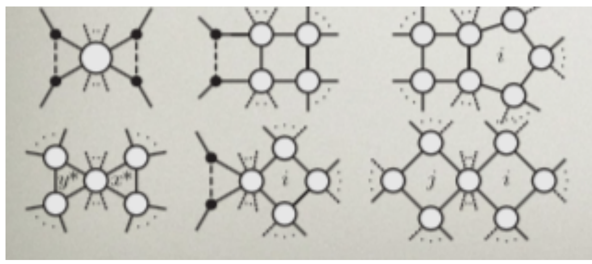
→NLO revolution -> NNLO, loop integrands, integrals & polylogs, …

New math structures: Grassmannian for all-loop integrands in N=4 SYM (hydrogen atom of QFT) + bootstrap, integrability, AdS/CFT…

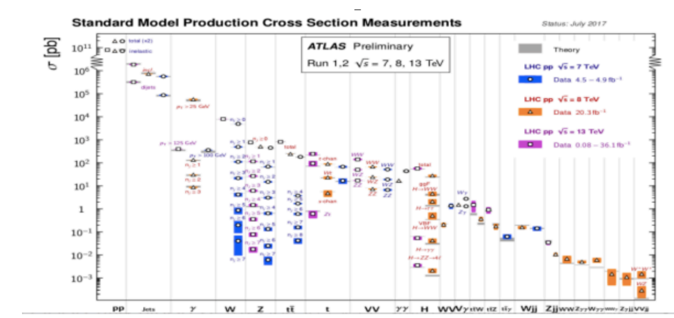
Double copy, gravity & perturbative strings —> e.g. CHY formulation
Geometric picture for scattering—> e.g. amplituhedron, associahedron, etc.

(numerous topics & names omitted here…)

Amplitudes today



Multi-loop integrands, integrals,…



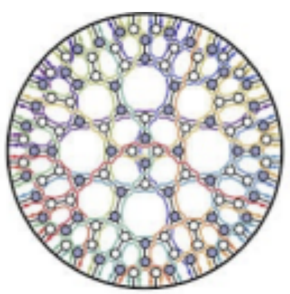
Formal QFT

Mathematics

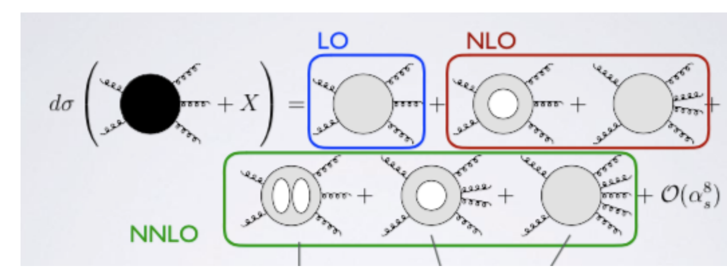
Amplitudes

Collider Phenomenology

precision frontier

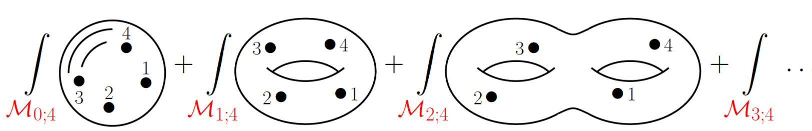


Geometries, combinatorics,..



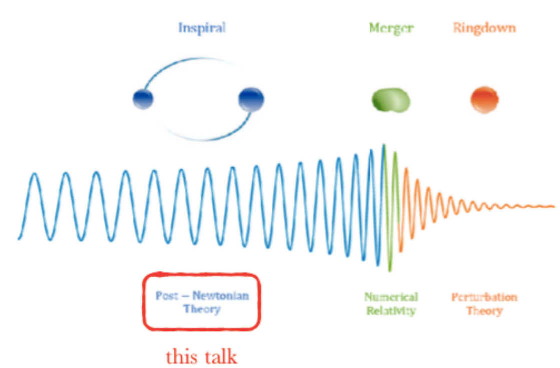
String Theory

Gravity



$$\mathcal{A}^{g\text{-loop}}_{\text{string}}(1,2,\ldots,n) \sim \int_{\mathcal{M}_{g,n}} \Big\langle \big(\prod_j \mathrm{PCO}(w_j)\big)\, V_1(z_1)\, V_2(z_2) \ldots V_n(z_n) \Big\rangle_g$$

CFT, string perturbation,…

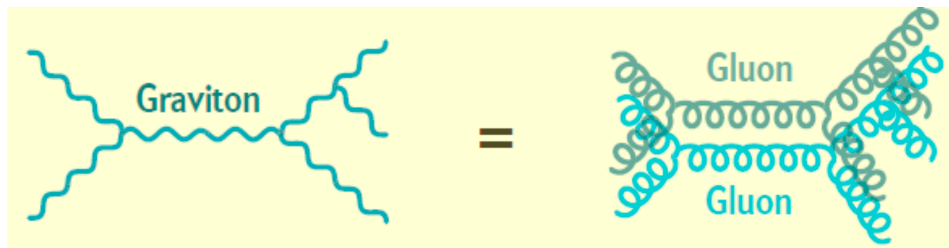


gravity amps
-> post-Newtonian

Gravity=(Gauge Theory)^2

1985: Kawai, Lewellen, Tye (KLT): **"closed string amp=open-string amp^2"**

Field-theory limit:



$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)\,A_4^{\text{tree}}(1,2,4,3)\,,$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)A_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\,A_5^{\text{tree}}(3,1,4,2,5)\,,$$

2008: Bern, Carrasco, Johansson (BCJ): **double-copy construction**

$$\mathscr{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$$n_s + n_t = n_u$$

$$\mathscr{A}_4^{\text{tree}}\Big|_{c_i \to n_i} \equiv \mathscr{M}_4^{\text{tree}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

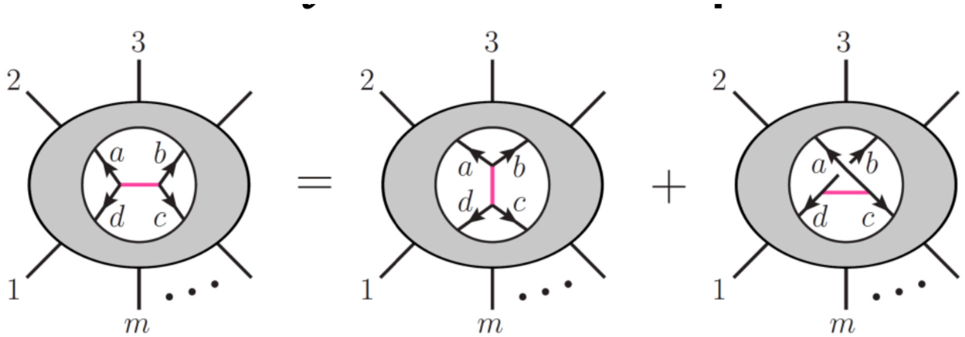
kinematic factor

color factor

$$\sum_{i\in\Gamma}\int\prod_{l=1}^{L}\frac{d^D p_l}{(2\pi)^D}\frac{1}{S_i}\frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

symmetry factor

scalar propagator for each edge in Γ



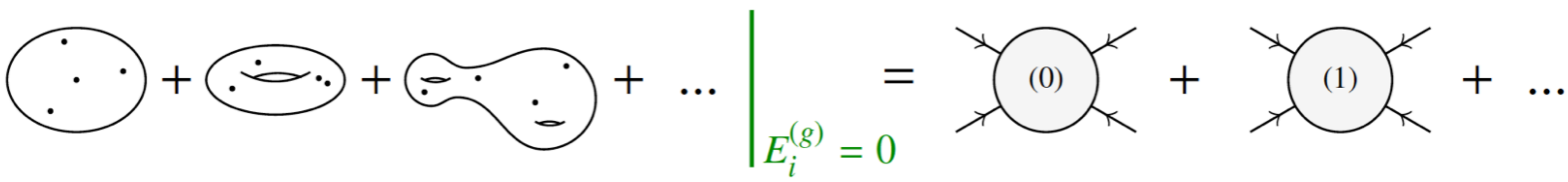
$$C_i = C_j + C_k$$

$$n_i = n_j + n_k$$

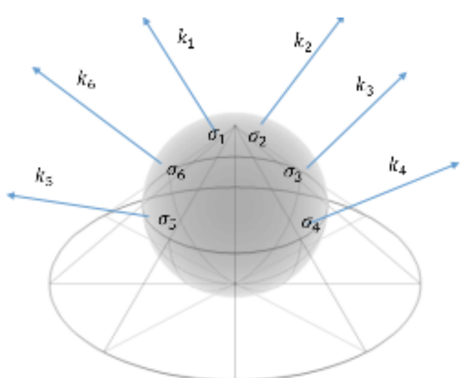
color-kinematics duality
=> double copy

New formulation of QFT

- CHY formulation: scattering of massless particles in any dim[Cachazo, SH, Yuan 2013]
  - *compact formulas* for amps of gluons, gravitons, scalars, (fermions?!) etc.
  - *manifest* gauge (diff) invariance, soft theorems, double-copy & new relations, etc.
  - *worldsheet picture:* ambitwsitor strings etc. [Mason, Skinner; Adamo et al; Berkovits; Siegel…]



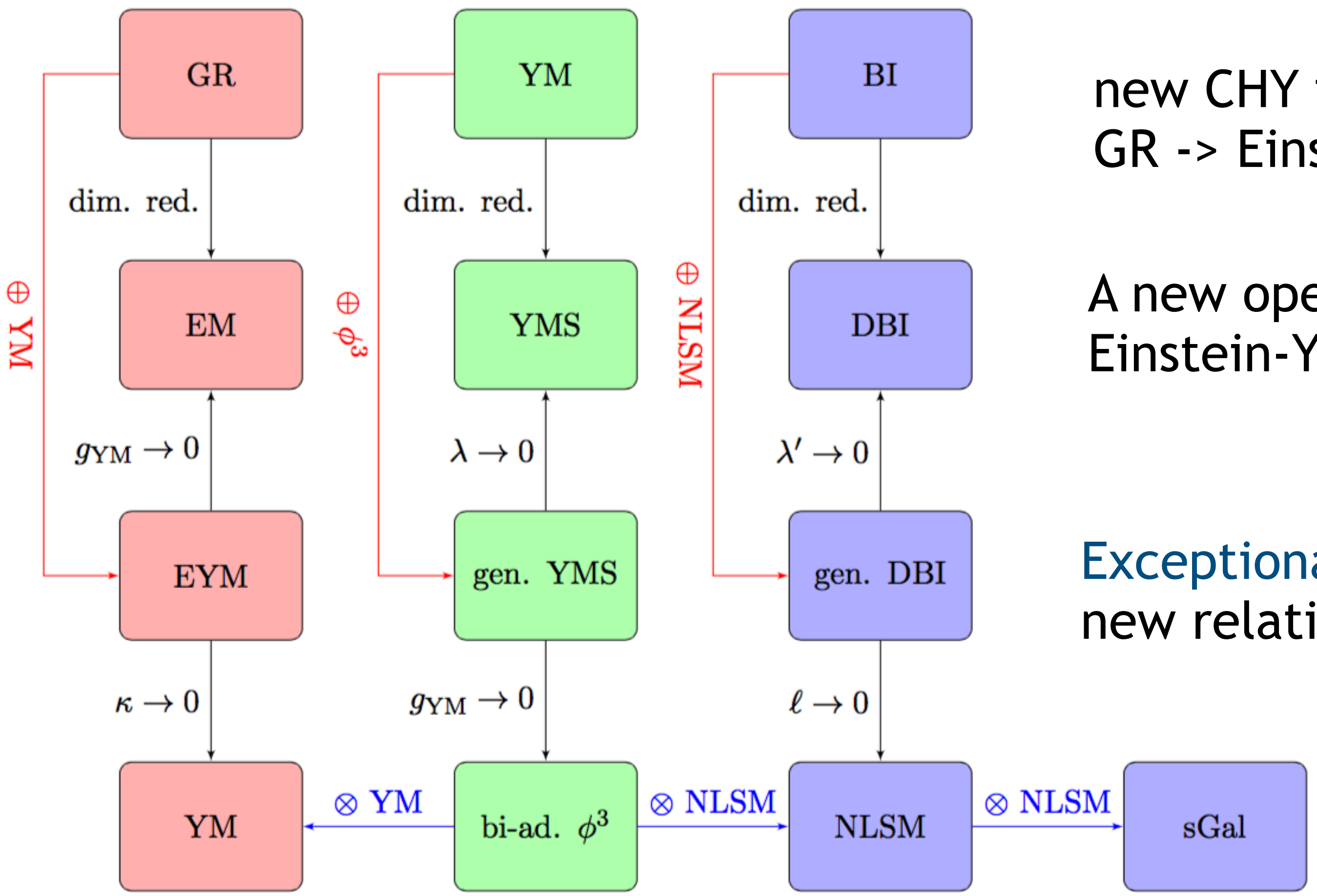
Scattering equations $E_a := \sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \qquad a = 1, 2, \ldots, n$ $\qquad SL(2, \mathbb{C})$ symmetry:



$$M_n = \underbrace{\int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k, \epsilon, \sigma\}) = \sum_{\{\sigma\}\in \text{solns.}} \frac{\mathcal{I}(\{k, \epsilon, \sigma\})}{J(\{\sigma\})}$$

- New picture: scattering of massless particles via worldsheet correlators
- Feynman diagrams, Lagrangians, even spacetime itself become emergent

A landscape of massless theories



new CHY from old ones by e.g. dim reduction
GR -> Einstein-Maxwell, YM -> YM-scalar

A new operation as direct sum of two particles ->
Einstein-Yang-Mills, Yang-Mills + bi-adjoint scalars

Exceptional EFTs of e.g. pions, DBI, Galileon [CHY 14] [Cheung et al 14]
new relations e.g. pions from special dim. red. of gluons!

These amplitudes are strongly constrained (even uniquely determined) by symmetries:
gauge invariance & Adler zero; deeply connected to each other!

	Gauge Theories		
I	GR (s=2)	YM (s=1)	BI (s=1)
II	YM (s=1)	ϕ^3 (s=0)	NLSM (s=0)

	Effective Field Theories			
I	sGal (τ^3)	NLSM (τ^1)	BI (τ^1)	DBI (τ^2)
II	NLSM (τ^1)	ϕ^3 (τ^{-1})	YM (τ^{-1})	YMs (τ^0)

Double-copy as direct product

- (Tree-level) double copy explained by CHY: $GR = YM^2/\phi^3$ (inverse of bi-adjoint amps)
- Direct product of amplitudes in two theories: discover new double-copies

Double copies from CHY

$$A \equiv L \otimes R = \int d\mu_n \, I_L \, I_R$$

$$A_L(\alpha) = \int d\mu_n \, I_L \, PT(\alpha)$$

$$A_R(\beta) = \int d\mu_n \, I_R \, PT(\beta)$$

$$m(\alpha|\beta) = \int d\mu_n \, PT(\alpha) \, PT(\beta)$$

$$A = \sum_{\alpha,\beta \in S_{n-3}} A_L(\alpha) m^{-1}(\alpha|\beta) A_R(\beta)$$

L⊗R	L	R
GR	YM	YM
BI	YM	NLSM
DBI	YMS	NLSM
sGal	NLSM	NLSM

- Special cases in D=3,4,6, e.g. D=6 amps for M5/D5 brane & D=3 super-gravity=(ABJM)^2
- a web of relations totally invisible in Feynman diagrams/Lagrangian [Cheung et al] e.g. "Interpolating" between ⊗ and ⊕: expand GR/YM … amps into EYM/YMS … [w. Dong, Hou]

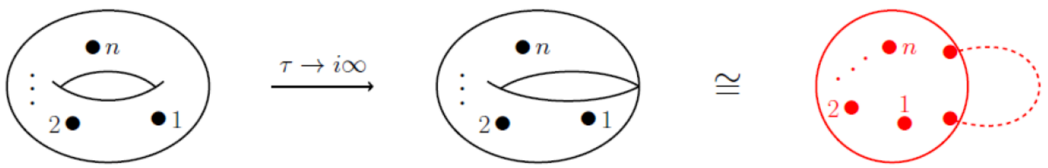
Ambitwistor strings & 1-loop CHY+KLT

[Mason,Skinner; Adamo,Casali,Skinner; Geyer, Mason, Monteiro,Tourkine,…]

- (tree) CHY <= 2d chiral CFTs with “ambitwistor” target-space

- higher-genus -> loop CHY but impossible to use; res. theorem: torus -> nodal Riemann spheres

$$S = \frac{1}{2\pi}\int P\cdot\bar{\partial}X + \frac{1}{2}\sum_r \Psi_r\cdot\bar{\partial}\Psi_r - \frac{e}{2}P^2 - \chi_r P\cdot\Psi_r$$



- Equivalently, 1-loop from forward limit of trees: linear propagators [SH, Yuan; Cachazo, SH, Yuan, 15]

$$M^{(1)} = \int \frac{d^d\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\int \underbrace{\prod_{i=2}^{n}\delta\left(\frac{\ell\cdot k_i}{\sigma_i}+\sum_{j=1,j\neq i}^{n}\frac{k_i\cdot k_j}{\sigma_{ij}}\right)}_{d\mu_{n+2}^{\text{tree}}}\hat{I}(\ell)$$

- Derive one-loop KLT-like formula for gravity integrand as double copy of gauge-theory integrands [SH Schlotterer, 17;+ Y. Zhang 18; Edison, SH Schlotterer, Teng 20] higher loops?

- Recombine diagrams with linear propagators -> quadratic ones [B. Feng, SH, Y. Zhang^2, 22]

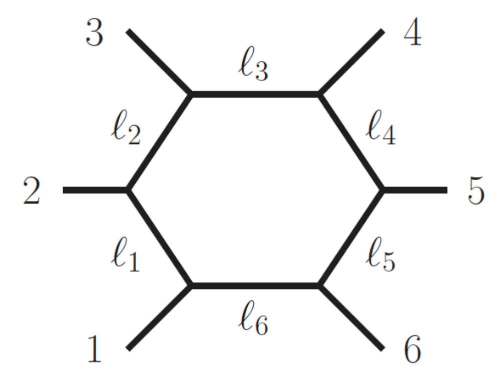
Cuts & numerators from CHY

- Tree-level numerators extracted from CHY(closed form) [CHY 13; Du, Feng, Teng, … Edison, Teng,…]; Forward-limit gives maximal cut at 1-loop: only need to fix contact terms!

- For max-SYM (in D=10): famous results for n=4, 5 numerators = box/pentagon cut

- For n>5, n-gon cut from forward limit of tree amps (CHY), e.g. n=6

$$
\begin{aligned}
N_{123456} \supset \Big\{ & \left(\varepsilon_1 \cdot \ell_1\right)\left(\varepsilon_2 \cdot \ell_2\right) t_8\left(f_3, f_4, f_5, f_6\right) + \text{perms} \\
& \left(\varepsilon_1 \cdot \ell_1\right) t_8\left(f_2, f_{[3,4]}, f_5, f_6\right) + \text{perms} \\
& t_8\left(f_1, f_{[2,3]}, f_{[4,5]}, f_6\right) + \text{perms} \\
& t_8\left(f_1, f_{[2,[3,4]]}, f_5, f_6\right) + \text{perms} \\
& t_{12}\left(f_1, f_2, f_3, f_4, f_5, f_6\right) \Big\}
\end{aligned}
$$



$$
f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu \qquad f_{[i,j]} = f_i f_j - f_j f_i
$$

$$
t_8\left(f_w, f_x, f_y, f_z\right) = \operatorname{tr}\left(f_w f_x f_y f_z\right) - \frac{1}{4}\operatorname{tr}\left(f_w f_x\right)\operatorname{tr}\left(f_y f_z\right) + \operatorname{cyc}(x, y, z)
$$

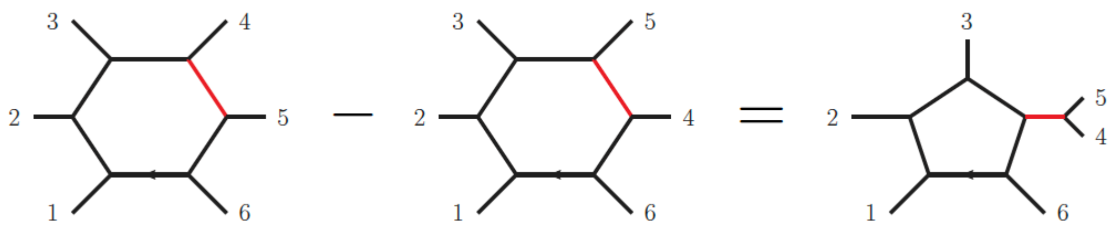
- Indeed contact terms are needed (e.g. to match pentagon/box cuts)

$$
\begin{aligned}
N_{123456} \supset \Big\{ & \varepsilon_1 \cdot \varepsilon_2 t_8\left(f_3, f_4, f_5, f_6\right)\left\{\ell_6^2, \ell_1^2, \cdots, \ell_5^2\right\} + \varepsilon_1 \cdot \varepsilon_3 t_8\left(f_2, f_4, f_5, f_6\right)\left\{\ell_6^2, \ell_1^2, \cdots, \ell_5^2\right\}, \\
& \varepsilon_1 \cdot \varepsilon_4 t_8\left(f_2, f_3, f_5, f_6\right)\left\{\ell_6^2, \ell_1^2, \ell_2^2\right\} \Big\} + \text{cyc}
\end{aligned}
$$

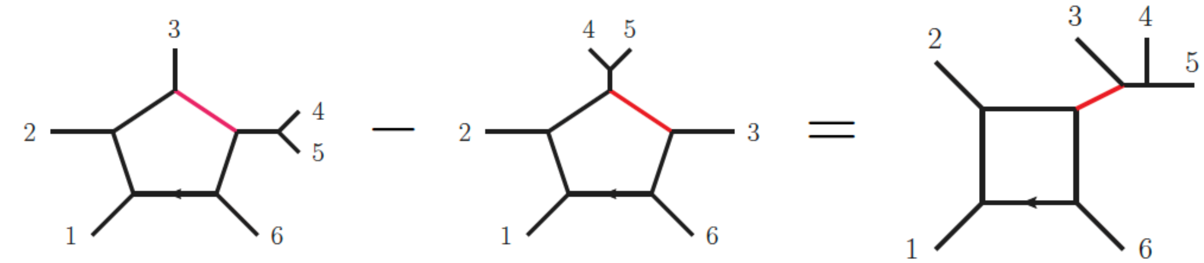
Integrands from BCJ + gauge invariance

- Ansatz: n-gon cut + BCJ (kinematic Jacobi) + power-counting

Figures from Edison's talk @ Amplitudes 2022



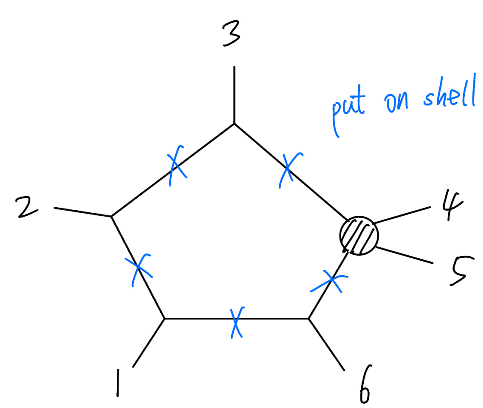
$$N_{123[45]6} = N_{123456} - N_{123546}$$

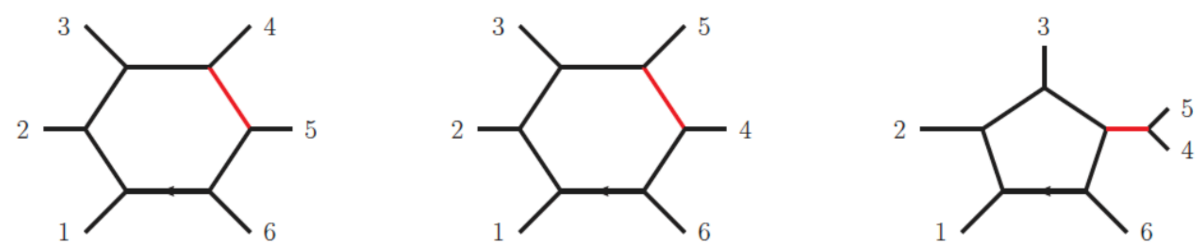


$$N_{12[3[45]]6} = N_{123[45]6} - N_{12[45]36}$$

$$\cdots \; N_{1[2[3[45]]]6} = N_{[12][3[45]]6} = N_{[12][34][56]} = 0 \text{ (vanishing triangle)}$$

- Lower cuts get contributions from various diagrams, e.g. 1-2-3-45-6 cut



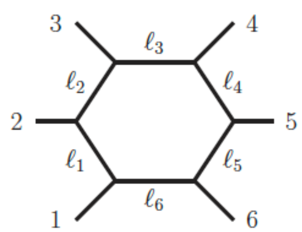


$$\left.\frac{N_{123456}}{(\ell_3+k_4)^2} + \frac{N_{123[45]6}}{(k_4+k_5)^2}\right|_{\text{N1 cut}}, \qquad \left.\frac{N_{123546}}{(\ell_3+k_5)^2} + \frac{N_{123[54]6}}{(k_4+k_5)^2}\right|_{\text{N1 cut}}$$

these cuts (product of trees) are gauge invariant, which puts extremely strong constraints on possible contact terms, e.g. completely fix n=6 ansatz!

Results [w. Edison, Johansson, Schlotterer, Teng, Zhang]

- Hexagon numerator

$N\Big[$  $\Big] =$

$$
\begin{aligned}
&(\varepsilon_1\cdot\ell_1)(\varepsilon_2\cdot\ell_2)\,t_8(f_3,f_4,f_5,f_6)-\frac{1}{2}(\varepsilon_1\cdot\ell_1)\,t_8(f_2,f_{[3,4]},f_5,f_6)\\
&+\frac{1}{4}t_8(f_1,f_{[2,3]},f_{[4,5]},f_6)+\frac{1}{6}t_8(f_1,f_{[2,[3,4]]},f_5,f_6)+\text{perms}\\
&+t_{12}(f_1,f_2,f_3,f_4,f_5,f_6)\\
&+\frac{1}{40}\Big[\varepsilon_1\cdot\varepsilon_2\,(3\ell_6^2-10\ell_1^2+3\ell_2^2)\,t_8(f_3,f_4,f_5,f_6)\\
&+\varepsilon_1\cdot\varepsilon_3\,(\ell_6^2-3\ell_1^2-3\ell_2^2+\ell_3^2)\,t_8(f_2,f_4,f_5,f_6)\\
&-\varepsilon_1\cdot\varepsilon_4\,(\ell_6^2+\ell_1^2)\,t_8(f_2,f_3,f_5,f_6)+\text{cyc}(1,2,3,4,5,6)\Big]
\end{aligned}
$$

+parity-odd (related to hexagon anomaly)

=> UV behavior of SYM/SG in D=8

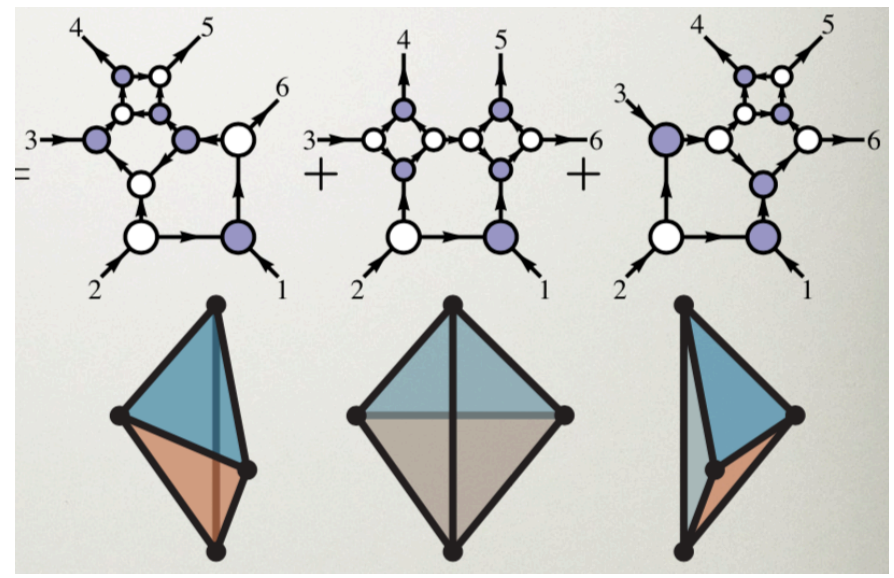
- n-gon cut determined for arbitrary multiplicity (similarly for half-SUSY)

$$
N_{12\ldots n}=\sum_{k=0}^{n-4}(-1)^{n-k}\sum_{1\le i_1\le i_2\le\ldots\le i_k}\epsilon_{i_1}\cdot\ell_{i_1}\epsilon_{i_2}\cdot\ell_{i_2}\ldots\ldots\epsilon_{i_k}\cdot\ell_{i_k}
$$
$$
\times\,\mathrm{tr}_{(\max)}(f_1\ldots\widehat{f_{i_1}}\ldots\widehat{f_{i_2}}\ldots\ldots\widehat{f_{i_k}}\ldots f_n)\ \mathrm{mod}\ \ell_j^2
$$

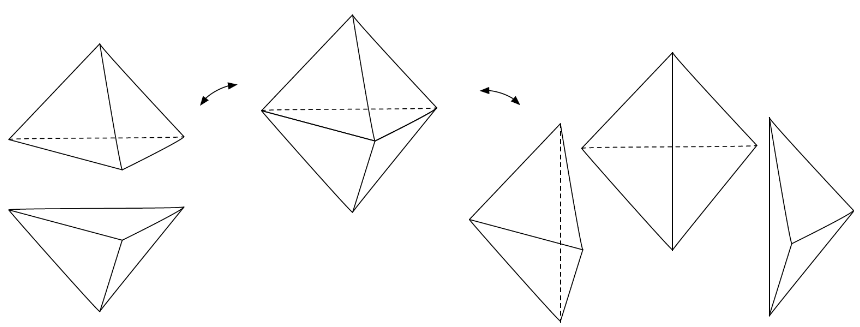
- Work in progress: determine n-pt numerator using n-gon cut +gauge inv. e.g. for 7-pt with fixed up to 30 parameters (contact terms), all drop in loop integrand!

Amplituhedron

Planar N=4 SYM: hydrogen atom of QFT + Ising model of 21st century
all-loop integrands from on-shell diagrams,
with infinite-dim symmetry <-> integrability



Amplitudes are volume of some “polytopes”!
—> geometry encoding QM & relativity!



$$Y = C \cdot Z$$

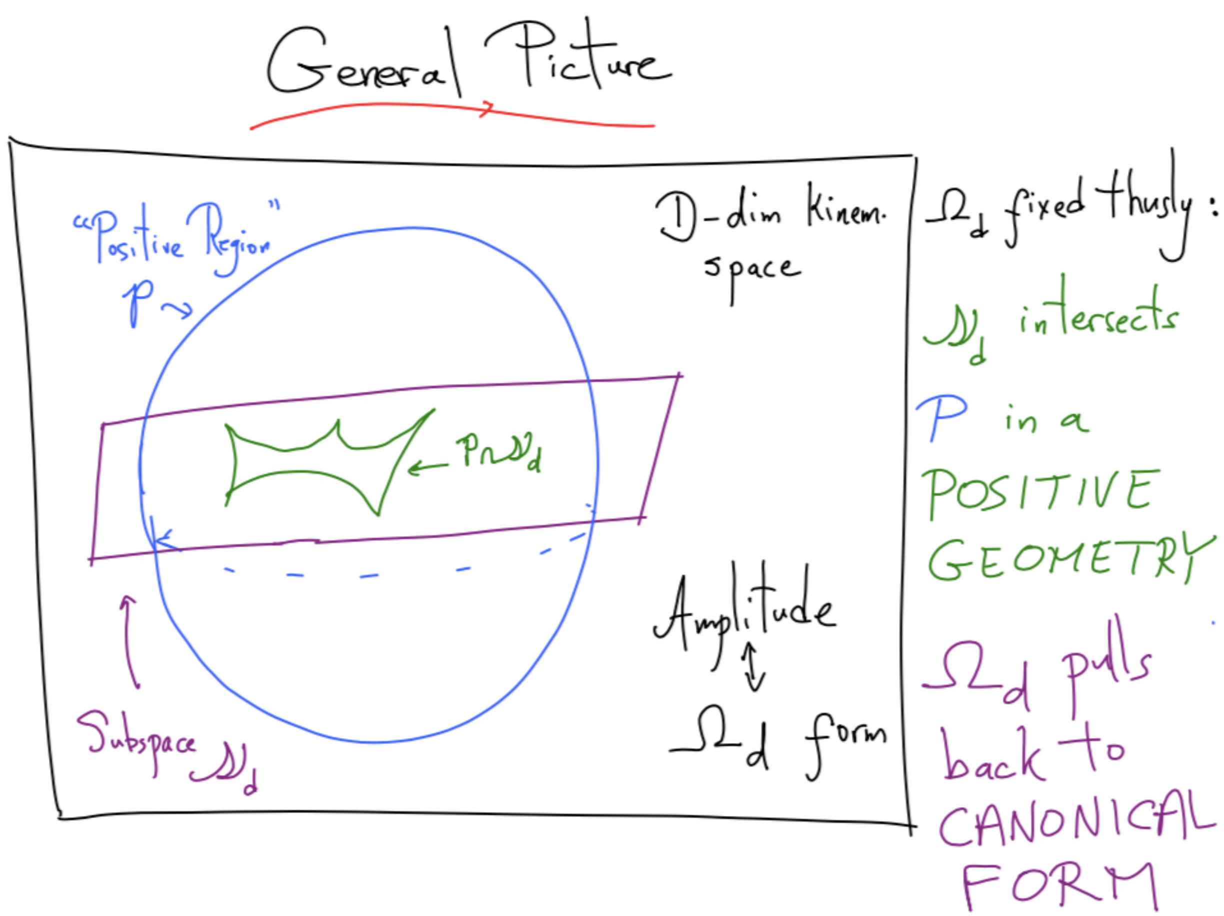
external data:
twistors

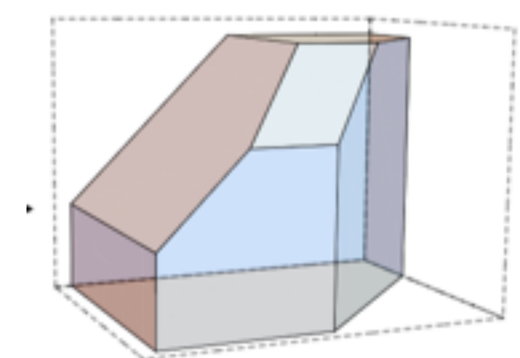
tree amplituhedron: gen. Grassmannian

New maths: positive geometry (real)
with a unique canonical form (complex):
only logarithmic singularities
@ boundaries (residues recursively defined)

differential forms in kinematic space for
any helicity amplitudes in any gauge theories
[SH, C. Zhang, 18]

Amplitudes as differential forms





Generalize amplituhedron to general theories in any dimensions (even phi^3)!

Bi-adjoint scalar: Amp (form)="volume" of associahedron in kinematic space

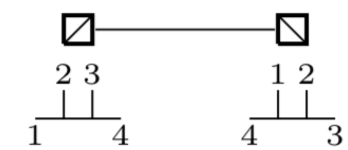
Geometrize color & its duality to kinematics, forms for gluon/pion amps etc.

Locality & unitarity emerges purely from geometries @ infinity of spacetime!

Kinematic associahedron

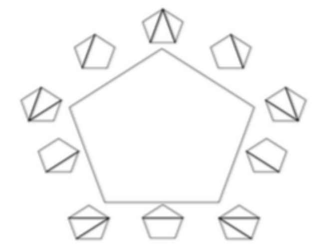
[Arkani-Hamed, Bai, SH, Yan, 2018]

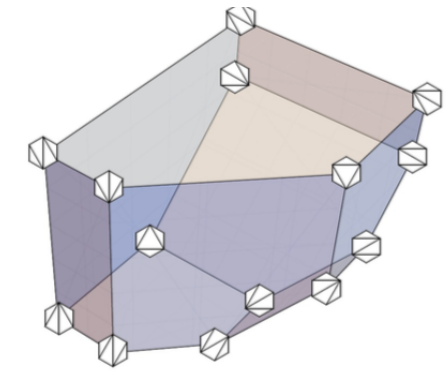
Associahedron of dim. (n-3): faces 1:1 corresp. with triangulation of n-gon



Positive region Δ_n: all planar variables $s_{i,i+1,\cdots,j} \geq 0$ (top-dimension)

Subspace H_n: $-s_{i\,j} = c_{i,j}$ as *positive constants*, for all non-adjacent pairs $1 \leq i, j < n$; we have $\frac{(n-2)(n-3)}{2}$ conditions $\Longrightarrow$ dim $H_n = n-3$

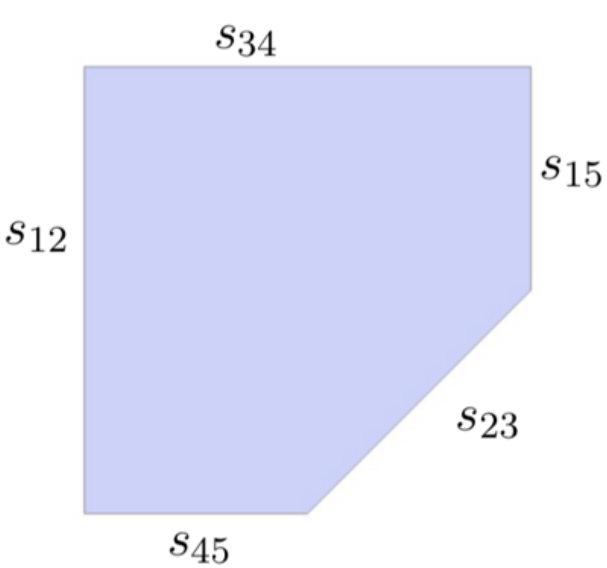


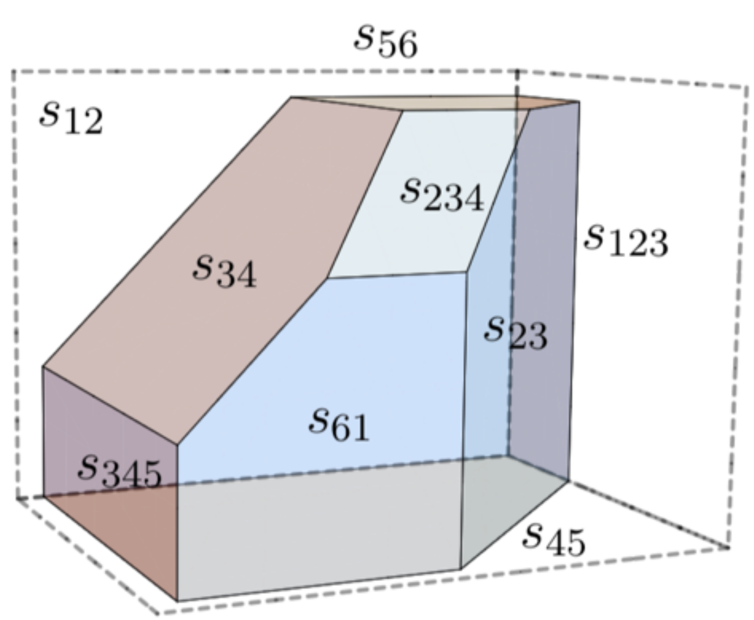


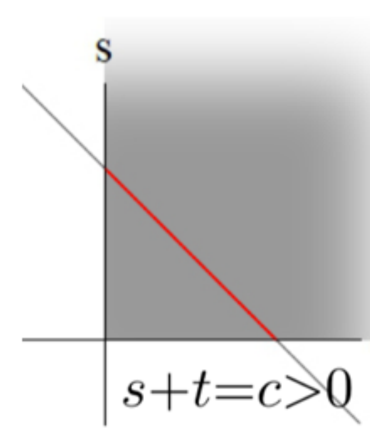
Kinematic Associahedron is their intersection! $\mathcal{A}_n := \Delta_n \,\cap\, H_n$

$e.g.\ \mathcal{A}_4 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$

encode singularities of any (colored) massless amplitudes at tree level: gluons, pions, etc.

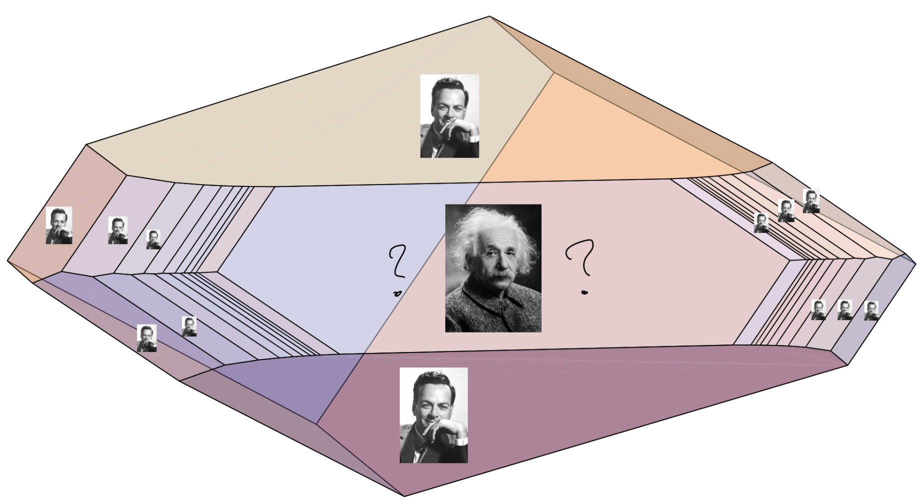


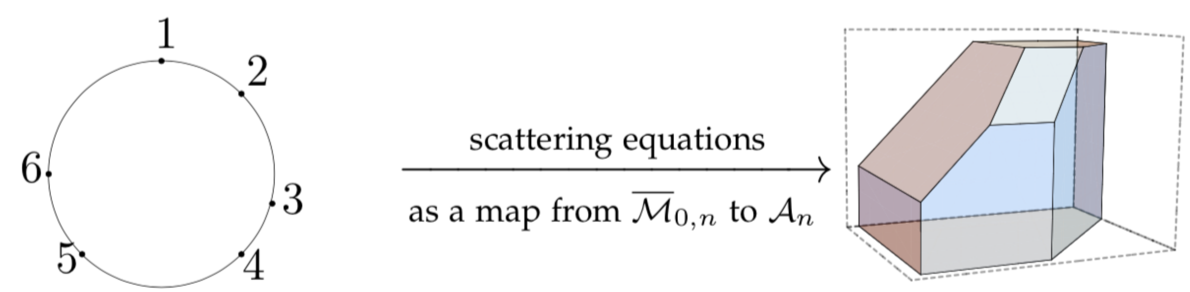




Particles & strings from geometries [Arkani-Hamed, SH, Lam; + Thomas + Salvatori, 2019 …]

- phi^3 amplituhedron: Amps="volume" (form) of polytope (positive geometry) Feynman-diagram expansion=special triangulation
- Hidden symmetry for even scalars manifest by geometry => deep connections between physics, geometry & cluster algebras!
- Surfacehedra [Arkani-Hamed et al]: all-loop phi^3 amplitudes from surfaces!



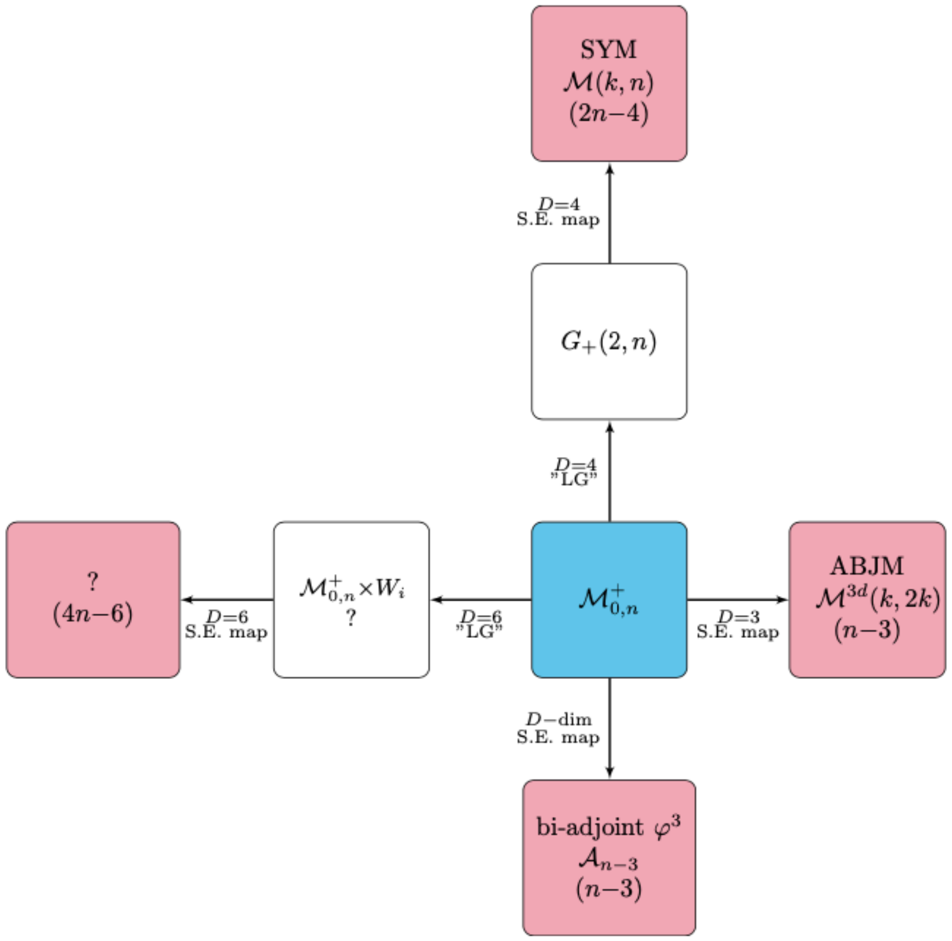


- Natural string integrals for surfacehedra: string without string!
- Field-theory (particles) $\alpha' \to 0$ = CHY formula with $\alpha' \to \infty$ (saddle points)
- Emergence of strings & particles from new geometries

Universality of positive geometries

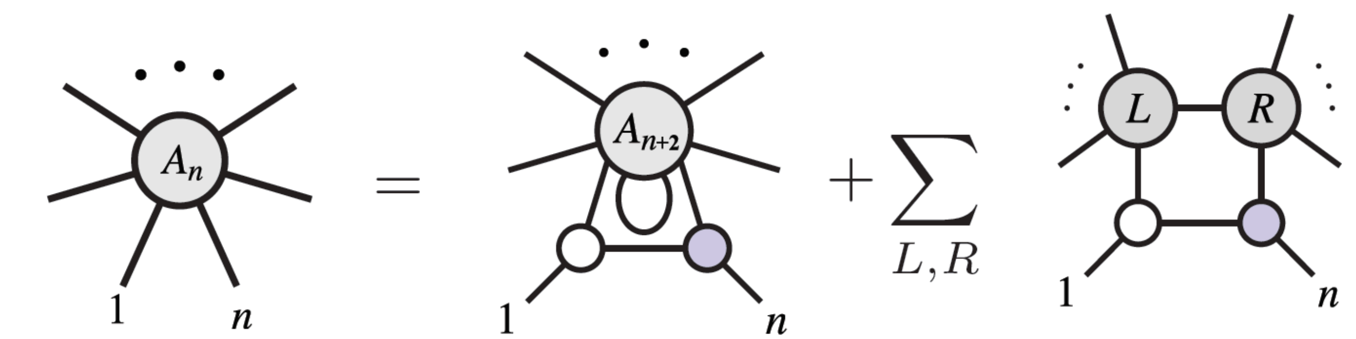
- Remarkably, such polytopes also appear for cosmology Ψ_{universe} , stringy & Feynman integrals, even EFThedron, CFThedron etc.
- Also evidence of positive geometries for non-planar N=4 SYM, but any example for all-loop integrands in e.g. ABJM (N=6 super-conformal Chern-Simons)?
- AdS5 strings <—> resum gluons of N=4 amplituhedron, could we see hints of M2 branes via some D=3 all-loop ABJM amplituhedron ???

- beautiful D=3 tree formula [Huang et al; CHY 14] provides a hint: reduced from Witten's twistor-string tree formula of N=4 SYM
- Pushforward from moduli space to tree amplituhedron of ABJM [w. Kuo, Zhang, 21]
- The unified picture for D=3,4 tree amplituhedra => ABJM geometries seem to be related to dim. red. of SYM ones!



Loops in SYM & ABJM

SYM: duality to (super) Wilson Loops (even w. correlators in null limits) -> integrand=WL with Lagrangian insertions -> satisfy the same all-loop recursion/amplituhedron!



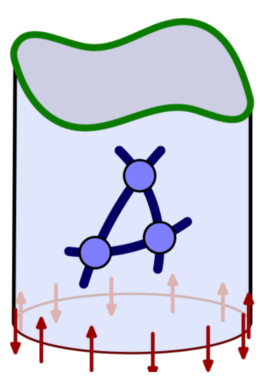
e.g. computed explicitly to L=3 for all n,k;

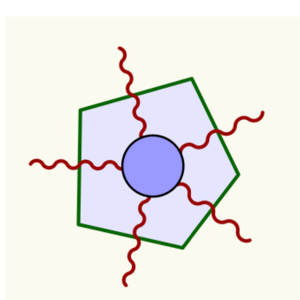
L=10 for n=4 & L=6 for n=5

though difficult using amplituhedron, it gives

Highly-nontrivial all-loop cuts!

ABJM: no duality to possible WL beyond n=4 (k=n/2 sector only), no def. of integrand for WL!
Even for n=4: duality checked to L=2, integrands conjectured up to L=3 [Bianchi et al]
amplitudes computed to L=2, n=6,8 (no known bootstrap, Qbar …) [Caron-Huot, Huang; w. Kuo, Huang, Li 22]





In both theories: extremely rich structure from strong coupling (AdS/CFT) & integrability, especially for 2,3-pt function etc.

Q: can we improve perturbative (amps/WL) side of ABJM?

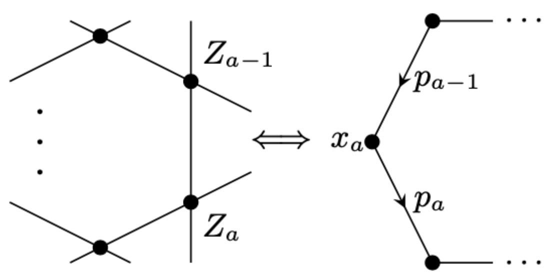
Are there ABJM amplituhedron (even for n=4)? Any connections?

Reduced amplituhedron for ABJM

[SH, Kuo, Li, Zhang, 2022]

The simplest guess works! Reducing external & loop momenta to D=3 gives a new geometry reduced amplituhedron => all-loop (at least n=4) integrands in ABJM!

Momentum twistors [Hodges]: "light rays" of dual spacetime, inspired by duality of $\mathcal{N}=4$ SYM planar amplitudes with Wilson loops [Alday et al; Brandhuber et al; ...]



- $Z^I=(\lambda^\alpha,\mu^{\dot\alpha}:=x^{\alpha,\dot\alpha}\lambda_\alpha)$: manifest dual conformal symmetry [Drummond et al]
- null polygon: $\lambda_a\tilde\lambda_a=p_a=x_{a+1}-x_a\leftrightarrow\{Z_1,\cdots,Z_n\}$ for n edges; $x_a:=(Z_{a-1},Z_a)$ is a line in twistor space
- (dual) loop momentum $x_0\leftrightarrow$ a line (AB) in twistor space

The n-point L-loop amplituhedron: $Z_{a=1,\cdots,n}$ for external kinematics and $(AB)_{i=1,\cdots,L}$ for loop momenta

For $n=4$ (only $k=0$): a $4L$-dim geometry in $(AB)_i$ space (Z's fixed):

$$\langle(AB)_i12\rangle>0,\quad\langle(AB)_i23\rangle>0,\quad\langle(AB)_i34\rangle>0,\quad\langle(AB)_i14\rangle>0,$$
$$\langle(AB)_i13\rangle<0,\quad\langle(AB)_i24\rangle<0$$

as well as mutual positivity: $\langle(AB)_i(AB)_j\rangle>0$ [Arkani-Hamed, Trnka 13]

External & loop momenta in $D=3$: twistor-space lines with symplectic conditions (in momentum space: $\lambda=\tilde{\lambda}$) [Elvang et al 14]:

$$\boldsymbol{\Omega}_{IJ} Z_a^I Z_{a+1}^J = \boldsymbol{\Omega}_{IJ} A_i^I B_i^J = 0\,, \quad \text{with } \boldsymbol{\Omega} = \begin{pmatrix} 0 & \epsilon_{2\times 2} \\ \epsilon_{2\times 2} & 0 \end{pmatrix}.$$

($a=1,2,\cdots,n$ and $i=1,\cdots,L$) $\to$ reduced amplituhedron

Focusing on $n=4$: a $3L$-dim geometry in constrained $(AB)_i$ for $D=3$

With parametrization $Z_{A_i}=Z_1+x_iZ_2-w_iZ_4,\quad Z_{B_i}=y_iZ_2+Z_3+z_iZ_4$
$\Longrightarrow$ def. of $n=4$ reduced amplituhedron:

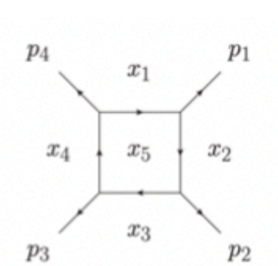
$$\forall i: x_i,y_i,z_i,w_i>0,\quad x_iz_i+y_iw_i=1,$$
$$\forall i,j: (x_i-x_j)(z_i-z_j)+(y_i-y_j)(w_i-w_j)<0$$

First look at $L=1$: the *canonical form* in $D=4$ = box integral

$$\Omega_1^{(D=4)} = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w} = \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$

Dim. reduction $\to$ $D=3$ box with ϵ num. = one-loop ABJM integrand [Chen, Huang 11]:

$$\Omega_1 = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w}\delta(xz+yw-1) = \frac{d^3(AB)\langle 1234\rangle^{3/2}(\langle AB13\rangle\langle AB24\rangle)^{1/2}}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$



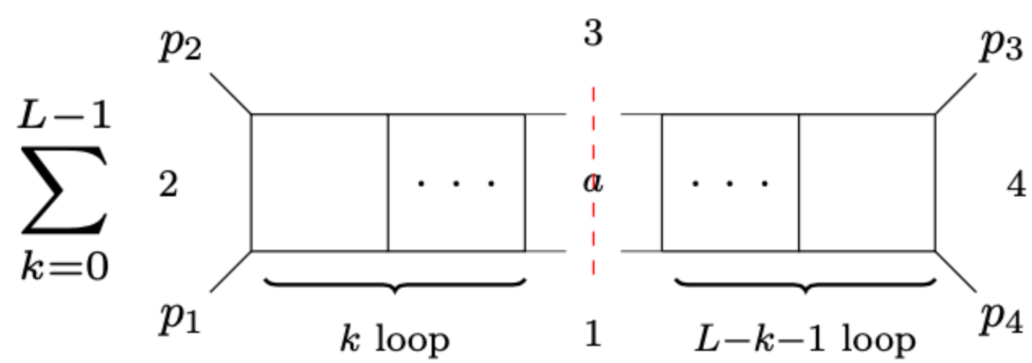
$$= \int \mathrm{d}^3x_5 \frac{\epsilon(5,1,2,3,4)}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}, \qquad \epsilon(i,j,k,l,m)\equiv \epsilon_{\mu\nu\rho\sigma\tau}x_i^\mu x_j^\nu x_k^\rho x_l^\sigma x_m^\tau$$

All-loop cuts from geometries

[SH, Kuo, Li, Zhang, 2022]

Beautifully manifest some all-loop cuts of ABJM: unitarity (QM) from geometries!

- Soft cut: *e.g.* $\langle \ell_i 12\rangle = \langle \ell_i 23\rangle = \langle \ell_i 34\rangle = 0$ or $y_i = z_i = w_i = 0 \implies$ manifest mutual positivity $D_{i,j} > 0$ for any j, residue $= (L-1)$-loop
- Vanishing cut: any cut isolating odd-point amplitude, *e.g.* $w_i = w_j = D_{i,j} = 0$ (triple cut) $\implies D_{i,j} \leq 0$, the residue vanishes; similarly, five-point cut $w_i = y_j = D_{i,j} = 0$ vanishes
- Double (unitarity) cut: $\langle \ell 14\rangle = \langle \ell 23\rangle = 0$



Explicitly checked up to $L = 5$. Can we prove it from the geometry?

Shorthand notation: *e.g.* $\underline{\Omega}_1 = \frac{c\epsilon_1}{s_1 t_1}$ (strip off $d^3\ell$)

$$\ell_i \equiv (AB)_i, \qquad c \equiv \langle 1234\rangle, \qquad \epsilon_i \equiv (c\langle \ell_i 13\rangle\langle \ell_i 24\rangle)^{1/2};$$

$$s_i \equiv \langle \ell_i 12\rangle\langle \ell_i 34\rangle \sim y_i w_i, \qquad t_i \equiv \langle \ell_i 23\rangle\langle \ell_i 14\rangle \sim z_i w_i, \qquad D_{ij} \equiv -\langle \ell_i \ell_j\rangle.$$

Log of (n=4) amps & inserted Wilson Loops

The simplest, finite object (1 var.) from log of (n=4) amps => Γ_{cusp} (up to L=3) [Chicherin et al]

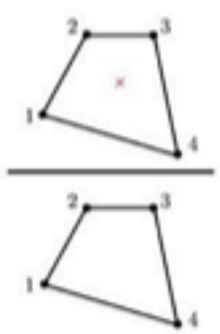
- IR divergence of log of amplitude

$$\log M = -\sum_{L\geq 1} g^{2L} \frac{\Gamma^{(L)}_{\text{cusp}}}{(L\epsilon)^2} + O(1/\epsilon)$$

- With one loop frozen and integrate over others

$$F_\Gamma\left(AB_0\right) = \int d\mu_{AB_1} \ldots d\mu_{AB_{L-1}} \Omega_\Gamma \text{ is IR finite}$$

This object is related to the Wilson loop with Lagrangian insertion



$$= \frac{1}{\pi^2} \frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} F(g; z), \qquad z = \frac{x_{20}^2 x_{40}^2 x_{13}^2}{x_{10}^2 x_{30}^2 x_{24}^2}$$

- Extract Γ-Cusp from $F(g, z)$

$$g \frac{\partial}{\partial_g} \Gamma_{\text{cusp}}\left(g\right) = -2I[F(g, z)] \quad \text{where} \quad I\left[z^p\right] = \frac{\sin(\pi p)}{\pi p}$$

Q: How to see these from (integrating) n=4 amplituhedron?

Higher-loop integrand extremely complicated, simplify them from geometries?

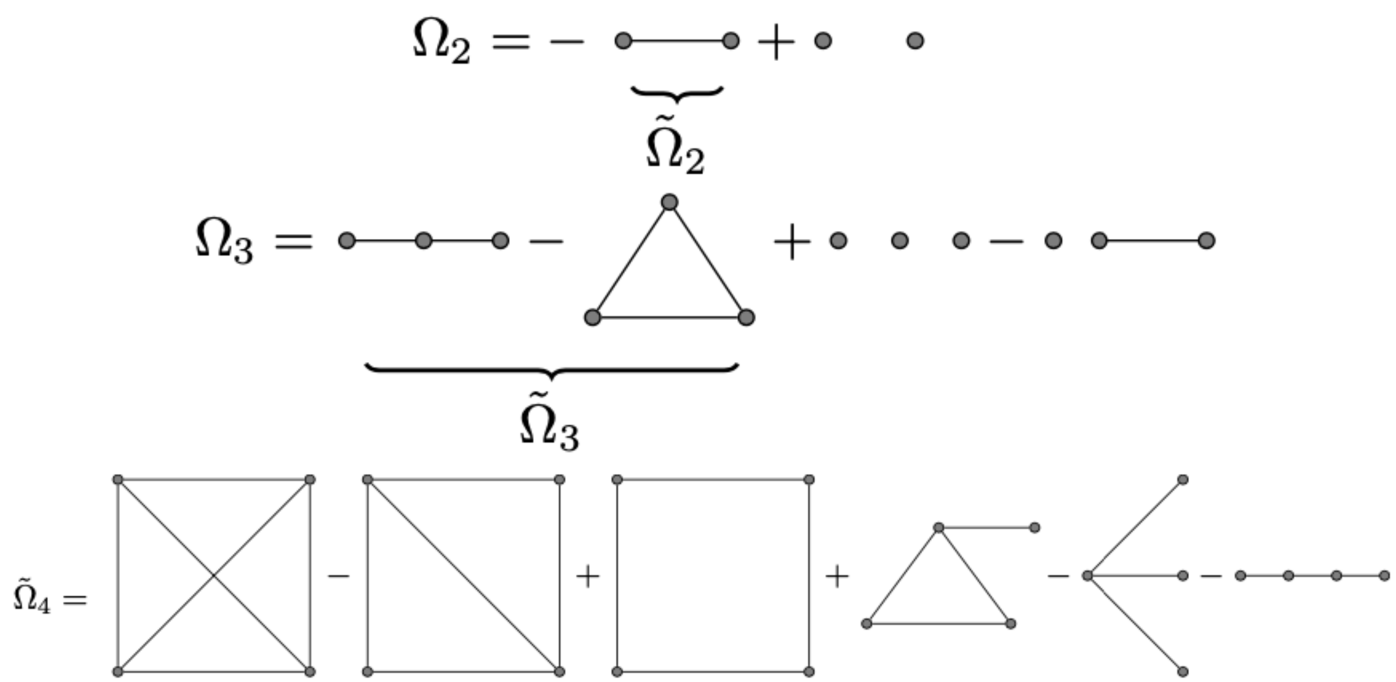
Could we even resum some integrated results =>non-pertubative info?

Negative geometries (n=4)

mutual positivity = no constraint - mutual negativity => “building blocks”

Decomposition into sum of negative geometries: 

The sum of connected graphs gives logarithm of amplitudes [Arkani-Hamed et al], *e.g.*



much easier to compute form/integrand of such negative geometries, especially “ladders” or even “trees” to all loops!

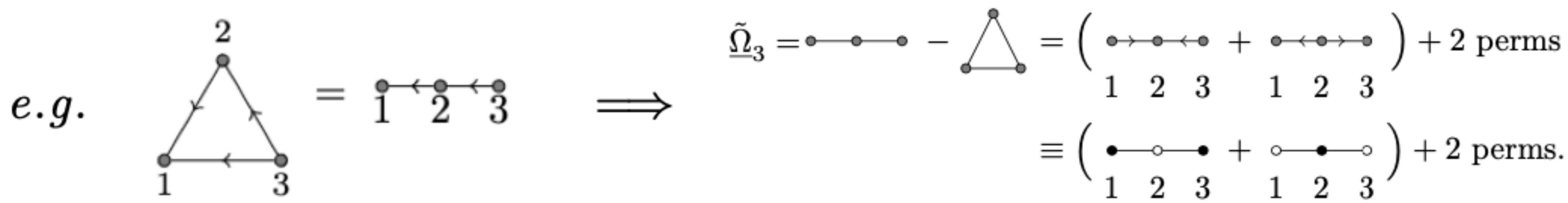
$$\Gamma_{\text{tree}}(g) = A\left(\frac{4}{\pi}\tan\frac{\pi A}{2} - A\right) \qquad \text{where } \frac{A}{2g\cos\frac{\pi A}{2}} = 1$$

$$\Gamma_{\text{ladder}}(g) = \frac{4}{\pi}\log\cosh\left(\sqrt{2}\pi g\right).$$

much easier to integrate (only 1-loop divergence), possible to resum to give “tree”/“ladder” contribution of Γ_{cusp}!

D=3 amplituhedron = bipartite geometries

Huge reduction in D=3: only bipartite graphs (with “arrows”) survive! e.g. no triangle for L=3 only 2 trees & box for L=4 (with source & sinks)



A tiny fraction ($\to 0$ as $L \to \infty$) of graphs remain (relatively simple ones):

L	top. of G	top. of g	directed acyclic graphs	bipartite g
2	1	1	2	1
3	2	1	18	3
4	6	3	446	19
5	21	5	26430	195
6	112	17	3596762	3031
7	853	44	1111506858	67263

Transitive reduction: $D = 3$ mutual negativity $\Longrightarrow$ (time) ordering, no closed loop $\to$ all non-bipartite (directed) graphs cancel (theorem [He et al 22]);

$$\sum_{\mathscr{G}} (-)^E \mathcal{A}_{\mathscr{G}} \overset{\text{time}}{\underset{\text{order}}{=\!=}} \sum_{\text{directed acyclic } G} (-)^E \mathcal{A}_G \overset{\text{trans.}}{\underset{\text{red.}}{=\!=}} \sum_{\text{bipartite } g} (-)^E \mathcal{A}_g$$

Canonical forms = ABJM integrands

Bipartite geometries give simple forms!

$$\underset{\displaystyle \frac{1}{s_i}}{\overset{i}{\bullet}} \qquad \underset{\displaystyle \frac{1}{D_{ij}}}{\overset{i}{\bullet}\!\!-\!\!\!-\!\!\overset{j}{\circ}} \qquad \underset{\displaystyle \frac{1}{t_j}}{\overset{j}{\circ}}$$

$$s_i \equiv \langle \ell_i 12\rangle\langle \ell_i 34\rangle \sim y_i w_i, \qquad t_i \equiv \langle \ell_i 23\rangle\langle \ell_i 14\rangle \sim z_i w_i, \qquad D_{ij} \equiv -\langle \ell_i \ell_j\rangle.$$

singularity structure: black (white) node has only poles $y_i w_i$ ($x_i z_i = 1 - y_i w_i$) !

Canonical forms for bipartite geometries: fix numerators given the poles

The log of amps for $L = 2, 3$: only (non-planar) ladders!

$$\underline{\tilde{\Omega}}_2 = \underset{1}{\bullet}\!-\!\!-\!\underset{2}{\circ} + \underset{1}{\circ}\!-\!\!-\!\underset{2}{\bullet} = \frac{2c^2}{D_{12}}\left(\frac{1}{s_1 t_2} + \frac{1}{t_1 s_2}\right)$$

$$= -2\frac{\langle 1234\rangle^2}{\langle \ell_1 12\rangle\,\langle \ell_1 34\rangle\,\langle \ell_1 \ell_2\rangle\,\langle \ell_2 23\rangle\,\langle \ell_2 14\rangle} + (\ell_1 \leftrightarrow \ell_2)$$

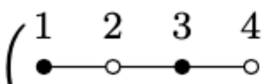
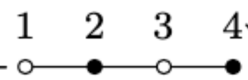
$$\underline{\tilde{\Omega}}_3 = \underset{1}{\bullet}\!-\!\underset{2}{\circ}\!-\!\underset{3}{\bullet} + \underset{1}{\circ}\!-\!\underset{2}{\bullet}\!-\!\underset{3}{\circ} + \underset{2}{\bullet}\!-\!\underset{1}{\circ}\!-\!\underset{3}{\bullet} + \underset{2}{\circ}\!-\!\underset{1}{\bullet}\!-\!\underset{3}{\circ} + \underset{3}{\bullet}\!-\!\underset{1}{\circ}\!-\!\underset{2}{\bullet} + \underset{3}{\circ}\!-\!\underset{1}{\bullet}\!-\!\underset{2}{\circ}$$

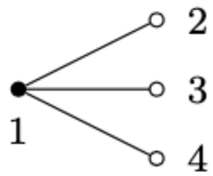
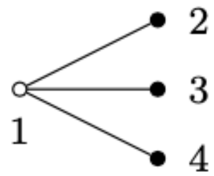
$$= \frac{4c^2\epsilon_2}{s_1 t_2 s_3 D_{12} D_{23}} + (s \leftrightarrow t) + 2 \text{ perms.},$$

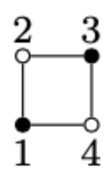
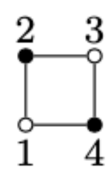
nicely confirm $n = 4$ 3-loop ABJM integrand conjectured in [Bianchi et al 11]

L=4,5,(6) & all-loop “tree” forms

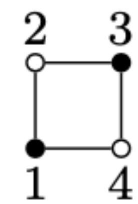
$L=4$: only chain, star, box graphs $\Longrightarrow \tilde{\Omega}_4 = -C - S + B$:

$C = \Big($  $+$  $\Big) + 11$ perms

$S = \Big($  $+$  $\Big) + 3$ perms

$B = \Big($  $+$  $\Big) + 2$ perms

,

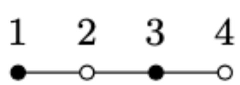


$$= 4\frac{4\epsilon_1\epsilon_2\epsilon_3\epsilon_4 - c(\epsilon_1\epsilon_3 N^t_{24} + \epsilon_2\epsilon_4 N^s_{13}) - c^2 N^{\rm cyc}_{1,2,3,4}}{D_{12}D_{23}D_{34}D_{41}s_1t_2s_3t_4}$$

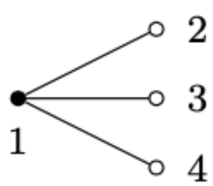
$$N^s_{ij} := \langle \ell_i 12\rangle\langle \ell_j 34\rangle + \langle \ell_j 12\rangle\langle \ell_i 34\rangle$$

$$N^t_{ij} := \langle \ell_i 14\rangle\langle \ell_j 23\rangle + \langle \ell_j 14\rangle\langle \ell_i 23\rangle$$

$$N^{\rm cyc}_{i,j,k,l} := \langle \ell_i 12\rangle\langle \ell_j 34\rangle\langle \ell_k 12\rangle\langle \ell_l 34\rangle + {\rm cyc}(1,2,3,4)$$



$$= 8c^2\frac{\epsilon_2\epsilon_3}{D_{12}D_{23}D_{34}s_1t_2s_3t_4}$$

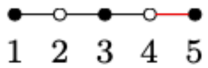


$$= 8c^3\frac{t_1}{D_{12}D_{13}D_{14}s_1t_2t_3t_4}$$

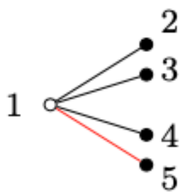
- Inverse soft construction of General Tree Formula
  Consider a L-tree obtained by connecting the added white vertex j to vertex i from a $L-1$-tree:

$$\underline{\Omega}^{\rm tree}_L(j\to i) = \underline{\Omega}^{\rm tree}_{L-1} \times \mathcal{T}_{j\to i}$$

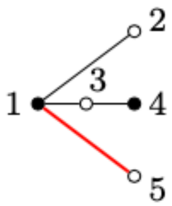
where $\mathcal{T}_{j\to i} = \frac{2\epsilon_i}{c^{1/2}D_{i,j}t_j}$ for valency v_i odd, and $\frac{2c^{1/2}t_i}{\epsilon_i D_{i,j}t_j}$ for v_i even. Similarly with $t\to s$ if we have black j attached to white i.



$$= \frac{8c^3\epsilon_2}{s_1t_2s_3t_4D_{12}D_{23}D_{34}} \times \frac{2\epsilon_4}{s_5D_{4,5}}$$



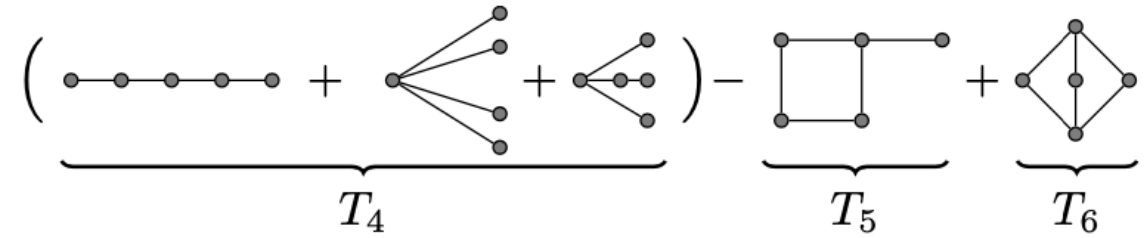
$$= \frac{8c^3 s_1}{t_1s_2s_3s_4D_{12}D_{13}D_{14}} \times \frac{2\epsilon_1}{s_5D_{15}}$$



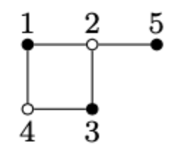
$$= \frac{8c^3\epsilon_3\epsilon_1}{s_1t_2t_3s_4D_{12}D_{13}D_{34}} \times \frac{2t_1}{\epsilon_1t_5D_{15}}$$

$L = 5$: contributions from 5 topologies $(4, 5, 6$ edges):

$\tilde{\Omega}_5 =$

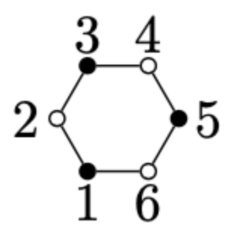


- More non-trivial forms for T_5 (box + edge) and T_6 ($K_{2,3}$ graph) *e.g.*



$$= 8c\frac{4\epsilon_1\epsilon_3\epsilon_4s_2 - \epsilon_1\epsilon_2\epsilon_3N^t_{24} - c(-\epsilon_1t_2N^t_{34} - \epsilon_3t_2N^t_{14} + \epsilon_4s_2N^s_{13} + \epsilon_2N^{\text{cyc}}_{1,2,3,4})}{s_1t_2s_3t_4s_5D_{12}D_{23}D_{34}D_{41}D_{25}}$$

- For higher loop, the hexagon is already known(fixed by L-cut)



$$=8\frac{8\prod_{i=1}^{6}\epsilon_i - c^2EP_2 - c^3N^{\text{cyc}}_{1,2,3,4,5,6}}{s_1t_2s_3t_4s_5t_6D_{12}D_{23}D_{34}D_{45}D_{56}D_{61}},$$

where

$$EP_2 := \epsilon_1\epsilon_3P^t_{13} + \text{cyc. by } 2\ (\ell_1, \cdots, \ell_6) + \epsilon_2\epsilon_4P^s_{24} + \text{cyc. by } 2(\ell_1, \cdots, \ell_6)$$

$$P^t_{13} := \langle\ell_5 23\rangle\langle\ell_2 23\rangle\langle\ell_4 14\rangle\langle\ell_6 14\rangle + (14) \leftrightarrow (23)$$

$$P^s_{24} := \langle\ell_6 12\rangle\langle\ell_3 12\rangle\langle\ell_5 34\rangle\langle\ell_1 34\rangle + (12) \leftrightarrow (34)$$

$$N^{\text{cyc}}_{1,2,3,4,5,6} := \prod_{i=1,3,5}\langle\ell_i 12\rangle \prod_{j=2,4,6}\langle\ell_j 34\rangle + \text{cyclic } (1,2,3,4).$$

- Other topologies at $L = 6$ are still in progress.

parity: odd/even # of ϵ for odd/even L

similar to N=4: only 1-loop divergence for even L (vanish for odd L)!

How to integrate them e.g. for L=4 => Γ_{cusp} at NNLO

any hint of resumming trees?

Summary & outlook

Scattering Amplitudes: applications to pheno/formal QFT, gravity, strings, math etc.

Double-copy, particles & strings: a web of relations for *gluons*, *pions*, *gravitons* …
double copy beyond amps: *classical solutions, gravity waves, …… ?*

massless S-matrix via punctured Riemann spheres (explains & extends double-copy);
higher loops? A (weak-weak) duality for S-matrix?

New geometries: "polytopes" in kinematic space & amps as differential forms
"theory at infinity": geometry/combinatorics → Lorentz inv. + unitarity

AdS/CFT & amps: higher-point ABJM amplituhedron? Wilson Loops?
other theories (D=6)? Integrations & resum => all-loop $\Gamma_{\rm cusp}$?
WHY such a relation at all ??? SYM -> ABJM fixed by (Yangian) symmetry?

"Marble statues in the Forest beyond Quantum Mechanics & Spacetime"
What will we see next?

Thank You!

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