Coulomb branch, Affine Springer fiber and Double Affine Hecke Algebra

Wenbin Yan

2022. 8. 25

joint work with Peng Shan and Dan Xie

Motivation

4d $\mathcal{N}=2$ SCFTs — Higgs branch $\mathcal{M}_H$; Coulomb branch $\mathcal{M}_C$

4d Class-S theory $\mathcal{T}$

$\begin{pmatrix} \text{6d (2,0) SCFT} \\ \text{on } \mathbb{R}^3 \times S^1 \times \Sigma_{g,n} \end{pmatrix}$

$\mathcal{M}_H \simeq \text{Ass. Var}(V(\mathcal{T}))$

[Beem - Rastelli]
[Song - Xie - WY]
[Arakawa] ...

Observable ?

$\mathcal{M}_C =$ Hitchin system on $\Sigma_{g,n}$

[Gaiotto - Moore - Neitzke]

$$\begin{tikzcd}[column sep=large]
 & \text{Ass. Var } (W\text{-alg}) \\
\text{Generalised AD theories} \arrow[ur, "\mathcal{M}_H"] \arrow[dr, "\mathcal{M}_C"'] & \\
 & \text{Hitchin system on } \mathbb{P}^1 \text{ with 2 punctures}
\end{tikzcd}$$

Shared observables:

$$\left\{ \begin{array}{l} \mathbb{C}^\times (U(1)) \text{ - fixed points in} \\ \text{Certain Hitchin fiber} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \text{Irrep}(W\text{-alg}) \right\}$$

- [Fredrickson – Neitzke] (A_{N-1}, A_{M-1}) ($W_N(N, N+M)$ minimal model)
- [Fredrickson – Pei – Yan – Ye] (A_1, A_N), (A_1, D_N)

$\{ \hat{sl}_2, \text{Vir}, W_N, B_N \}$

$$
\begin{tikzcd}
& \text{Ass. Var.}(W\text{-alg}) \arrow[r] & W\text{-alg} \arrow[r] & \text{admissible rep}\,(\text{Adm}) \arrow[dd, leftrightarrow] \\
\text{Gen. AD theory} \arrow[ur, "M_H"] \arrow[dr, "M_C"'] & & & \\
& \text{Hitchin fiber} \arrow[r] & \text{Affine Springer fiber: } \mathcal{F}l_\gamma \arrow[r] & \mathcal{F}l_\gamma^{\mathbb{C}^\times}
\end{tikzcd}
$$

- $V_{Adm} = \mathrm{Span}\{\text{characters of } L \in Adm\}$

$SL(2,\mathbb{Z}) \circlearrowleft V_{Adm} \to$ Verlinde alg (Sub alg. of fusion alg)

$$K_{\mathbb{C}^*}(\mathcal{F}l_\gamma^{\mathbb{C}^*}) \overset{\text{as v.s}}{\simeq} V_{Adm}$$

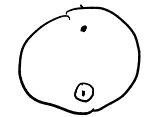
[Vasserot]
[Oblomkov - Yun]

$K_{\mathbb{C}^*}(\mathcal{F}l_\gamma^{\mathbb{C}^*}) \simeq V_{DAHA}$

$V_{Adm} \simeq V_{DAHA}$ as Verlinde alg.

$V_{DAHA} \circlearrowleft SL(2,\mathbb{Z})$ (perfect rep of DAHA) $\to$ Verlinde alg (polynomial alg)

From Coulomb branch to affine Springer fiber

gen. AD theories : 6d $\mathfrak{g}$ (2,0) SCFT on $\mathbb{R}^3 \times S^1 \times$  [Cecotti-Vafa] [Xie] ...

$\mathcal{M}_C$: moduli space of Hitchin system on $\mathbb{P}^1$ with two punctures

$$\simeq \left\{ \text{solutions of } \begin{array}{l} F_A + [\varphi, \varphi^\dagger] = 0 \\ \bar\partial_A \varphi = 0 \end{array} \right\} / \sim \qquad \left\{ \begin{array}{l} \varphi : \Sigma \to \mathrm{Ad}.\, \mathfrak{g} \\ A : \text{connection} \end{array} \right.$$

"unitary gauge" $\downarrow$

$$\simeq \left\{ \text{solutions of } \begin{array}{l} F_D + [\phi, \phi^{\dagger_h}] = 0 \\ \bar\partial_E \phi = 0 \end{array} \right\} \qquad \left\{ \begin{array}{l} (\Sigma, E) \to \text{complex v. b. with holo. str. } \bar\partial_E \\ D : \text{Chern connection} \\ \phi^{\dagger_h} = h^{-1} \phi^\dagger h \end{array} \right.$$

boundary condition $\qquad z = \dfrac{1}{w}$

regular at $z = 0$ $(w = \infty)$ $\qquad \phi = \dfrac{f}{z} + \cdots \qquad f$: nilpotent orbit in $\mathfrak{g}$

irregular at $w = 0$ $(z = \infty)$ $\qquad \phi = \dfrac{T}{w^{1 + \frac{u}{h^\vee}}} + \cdots \qquad T$: regular semisimple in $\mathfrak{g}$

gen. AD theories $(\mathfrak{g}, u, f)$

Hitchin map $\pi_H : \mathcal{M}_C \longrightarrow \mathcal{B}$ $\rightsquigarrow$ Hitchin base $\simeq$ CB on $\mathbb{R}^{3,1}$

$\phi(z) \longmapsto \det(x\,dz - \phi(z)) = 0$ $\rightsquigarrow$ SW curve

$$x^{h^\vee} = z^{u-h^\vee} + u_2 z^{u-h^\vee-2} + \cdots$$

$\mathbb{C}^\times$-action on $\mathcal{M}_C$ $\longrightarrow$ $U(1)_r$ symm. on Coulomb branch

$(z \mapsto \lambda^{h^\vee} z,\ \phi \mapsto \lambda^{u-h^\vee}\phi)$

$\mathbb{C}^\times$ fixed points in fiber of $x^{h^\vee} = z^{u-h^\vee}$ $(u_2 = \cdots = 0)$

$\pi_H^{-1}(x^{h^\vee} = z^{u-h^\vee})$: central / zero fiber

Description of central fiber : $\pi_H^{-1}(x^{h^\vee} = z^{u-h^\vee}) \ni \frac{1}{2}\gamma$

$\phi'(z) \equiv \phi(z) z$, model Solution $\gamma = f_\theta z^u + \sum_{i=1}^{r} e_{\alpha_i}$

θ : highest rt. α_i : simple root

- other solutions $\phi'(z) = g^{-1}\gamma g$ $g \in G((z))$
- Impose boundary condition at $z=0$: $\phi'(z) \in n_f + z\,\mathfrak{g}[[z]]$

n_f : nilradical of f

- gauge equivalence : $\phi'(z) \sim \phi''(z)$ if $\phi'(z) = g\,\phi''(z)\,g^{-1}$

$g \in P_f$ s.t. $P_f n_f P_f^{-1} \subset n_f$

$$\pi_H^{-1}(x^{h^\vee} = z^{u-h^\vee}) \simeq \underbrace{\{ g \in G((z))/P_f \mid g^{-1}\gamma g \subset n_f + z\,\mathfrak{g}[[z]] \}}$$

$\hookrightarrow$ affine Springer fiber

Affine Springer fiber / Affine Spaltenstein variety

Let $\mathfrak{g}$: simply-laced semisimple Lie algebra.

G : connected, simply-connected Lie group of $\mathfrak{g}$

$T \subset G$ max. torus

- Affine flag manifold $\mathcal{F}\ell = G((z))/I$. I = Iwahori subgroup
- Let $\gamma = \gamma_u = z^u f_\theta + \sum e_{\alpha_i} \in \mathfrak{g}[[z]]$,

is an homogeneous elliptic element if $(u, h^\vee) = 1$

The affine Springer fiber

$$\mathcal{F}\ell_\gamma = \{ g \in \mathcal{F}\ell \mid g^{-1}\gamma g \in \mathrm{Lie}(I) \} \subset \mathcal{F}\ell$$

is a finite dim. projective variety

- $\mathcal{F}\ell_\gamma$ carries a $\mathbb{C}^\times$-action give by $\rho \in X_*(T)$

$$\mathbb{C}^\times \longrightarrow (G((z)) \rtimes \mathbb{C}^\times_{rot}) \times \mathbb{C}^\times_{dil}, \quad z \longmapsto (z^{u\rho}, z^{h^\vee}, z^{-u})$$

$\mathcal{F}\ell_\gamma^{\mathbb{C}^*}$ is a finite set, fixed points are isolated

- $\forall g \in \mathcal{F}\ell_\gamma^{\mathbb{C}^*}$, $\frac{1}{z} g\gamma g^{-1}$ gives a f.p. of $\mathcal{M}_c$ with f = trivial

- for general f, let $P \subset G[\![z]\!]$ be a parabolic subgroup s.t f is regular in Levi of P

The affine Spaltenstein variety

$$\mathcal{F}\ell_{P,\gamma} = \{ g \in G((z))/P \mid g^{-1}\gamma g \in \mathrm{rad}(P) \} \circlearrowleft \mathbb{C}^\times$$

Surjection: $\mathcal{F}\ell_\gamma^{\mathbb{C}^*} \twoheadrightarrow \mathcal{F}\ell_{P,\gamma}^{\mathbb{C}^*}$

Higgs branch and W-algebra

gen. AD theory $(\mathfrak{g}, u, f) \xleftrightarrow[\text{BLLPRvR}]{\text{4d / VOA}} W_{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)$

[Cordova - Shao] [Buican et al]
[Song - Xie - Yan] [Xie - Yan]

$$\mathcal{M}_H \simeq \text{Ass. Var}(W\text{-alg})$$

f = trivial $\quad \hat{\mathfrak{g}}_{-h^\vee + \frac{h^\vee}{u}}, \quad k = -h^\vee + \frac{h^\vee}{u}$ boundary admissible

$(u, h^\vee) = 1$

[Kac - Wakimoto]

$\text{Adm}_k = \{$admissible weights of level $k\}$

Admissible rep: $\{L(\lambda) \mid \lambda \in \text{Adm}_k\}$: semi-simple in cat. $\mathcal{O}$

$V_{\text{Adm}} \equiv \text{Span}\{\chi_{L(\lambda)}(\tau, x) \mid \lambda \in \text{Adm}_k\}$: $SL(2,\mathbb{Z})$ rep.

f: even regular in a Levi

$$W_{-h^\vee+\frac{h^\vee}{u}}(\mathfrak{g}, f) = \text{quantum Drinfeld-Sokolov reduction of } \hat{\mathfrak{g}}_{-h^\vee+\frac{h^\vee}{u}}$$

$\Rightarrow$ BRST cohomology functor

$$H_f^{\frac{\infty}{2}+\bullet}(-) : \hat{\mathfrak{g}}_k\text{-mod} \to W_k(\mathfrak{g}, f)\text{-mod}$$

[Frenkel - Kac - Wakimoto], [Kac - Roan - Wakimoto], [Arakawa] ...

- Criterion for $H_f^{\frac{\infty}{2}+\bullet}(L(\lambda)) \neq 0$ for $\lambda \in Adm_k$
- $H_f^{\frac{\infty}{2}+\bullet}(L(\lambda))$ irreducible if $\neq 0$
- Criterion for $H_f^{\frac{\infty}{2}+\bullet}(L(\lambda)) = H_f^{\frac{\infty}{2}+\bullet}(L(\lambda'))$, $\lambda, \lambda' \in Adm_k$

$\mathcal{Fl}_\gamma^{\mathbb{C}^\times}$ vs $Adm_{-h^\vee + \frac{h^\vee}{u}}$

- VOA side

$$Adm_k = \{ t_b w\cdot (k\hat{\Lambda}_0) \mid b \in P^\vee,\ t_b w(\hat{\Delta}(k\Lambda_0)^{re}_+) \subset \hat{\Delta}^{re}_+ \}$$

$$(P^\vee / uQ^\vee)/\Omega \xrightarrow{\sim} Adm_k, \qquad b \longmapsto t_{-b} w\cdot (k\hat{\Lambda}_0)$$

- Fiber side

Similar description for $\mathcal{Fl}_\gamma^{\mathbb{C}^\times} \simeq (Q^\vee / uQ^\vee) \simeq (P^\vee/uQ^\vee)/\Omega$

Thm 1. For $k + h^\vee = \frac{h^\vee}{u}$, there is a natural bijection

$$Adm_k \simeq \mathcal{Fl}_\gamma^{\mathbb{C}^\times}$$

Verlinde algebra structure

- VOA side

$\lambda \in Adm_k$ $\qquad \chi_\lambda(\tau, x) = \mathrm{Tr}_{L(\lambda)} e^{2\pi i z x_0} q^{L_0 - c_k/24}$

$\tau \in \mathcal{H}$ = upper half plane in $\mathbb{C}$. $\quad q = e^{2\pi i \tau}$

[Kac - Wakimoto] $\quad V_{Adm} = \mathrm{span}\{\chi_\lambda(\tau, x) \mid \lambda \in Adm_k\} \circlearrowleft SL(2, \mathbb{Z})$

$\Rightarrow$ fusion coefficients for admissible reps. $\quad \lambda, \mu \in Adm_k$

$$\lambda \otimes \mu = \sum_{\nu \in Adm_k} N_{\lambda\mu}^{\nu}\, \nu + (\text{weight mod. log-mod}, \dots)$$

$\{Adm_k\}$: subalg. of the full fusion alg.

$N_{\lambda\mu}^{\nu}$: computed from S-mat by Verlinde formula.

- Fiber side.

$\mathbb{H}_{q,t}$ = Double affine Hecke algebra w/ W with $t^{h^\vee} \cdot q^{\mu} = 1$

$$\mathbb{H}_{q,t} \overset{[\text{Vasserot}]}{\curvearrowright} K_{\mathbb{C}^*}(\mathcal{F}\ell_\gamma)_{loc} = K_{\mathbb{C}^*}(\mathcal{F}\ell_\gamma^{\mathbb{C}^*})_{loc} := V_{DAHA}$$

V_{DAHA} is a perfect rep in the sense of Cherednik

$$\begin{array}{ccccc} \mathbb{H}_{q,t} & \curvearrowright & V_{DAHA} & \longrightarrow & \text{Verlinde alg.} \\ \circlearrowleft & & \circlearrowleft & & \\ SL(2,\mathbb{Z}) & \xrightarrow[\text{induce}]{} & SL(2,\mathbb{Z}) & & \end{array}$$

Thm 2 Specialize $q = e^{-2\pi i h^\vee/u}$, then the natural bijection in Thm 1 induces an isomorphism

$$V_{Adm} \simeq V_{DAHA}$$

which is compatible with $SL(2,\mathbb{Z})$-action

In particular, the fusion ring structures coincides

Rmk: 1. $\{\chi_\lambda \mid \lambda \in Adm_k\}$, $\{\mathcal{F}\ell^{\mathbb{C}^\times}_\gamma\}$ as basis.

$$S_{Adm} = S_{DAHA}, \qquad T_{Adm} = T_{DAHA}$$

2. $\{\mathcal{F}\ell^{\mathbb{C}^\times}_\gamma\} \xrightarrow{\sim} \{$non-sym. Macdonald poly$\}$

$(\mathcal{F}\ell_{p,\gamma})^{\mathbb{C}^\times}$ vs $\mathrm{Irr}(W_k(\mathfrak{g}, f))$

Compare :

$$\mathrm{Adm}_k \xrightarrow{H_f^{\frac{\infty}{2}+\bullet}(-)} \mathrm{Irr}(W_k(\mathfrak{g}, f))$$

$$\mathcal{F}\ell_\gamma^{\mathbb{C}^\times} \longrightarrow (\mathcal{F}\ell_{p,\gamma})^{\mathbb{C}^\times}$$

Thm 3 KRW conjecture is equivalent to a bijection

$$\mathrm{Irr}(W_k(\mathfrak{g}, f)) \xleftrightarrow{1-1} (\mathcal{F}\ell_{p,\gamma})^{\mathbb{C}^\times}$$

which is compatible with the bijection in Thm 1

Cor $$\#\,\mathrm{Irr}(W_k(\mathfrak{g}, f)) = \frac{1}{|W_f|} u^{r-j} \prod_{i=1}^{j} (u - e_j)$$

W_f = Weyl group for P, $r = rk(\mathfrak{g})$, $e_1 \cdots e_j$: exponents of W_f

Thm 4: There is also an identification of Verlinde algebras whenever the bijection holds

Rk: • Virasoro minimal model by [Koczae - Shakirov - Yan]

• $W_N(N, M)$ minimal model by [Gukov - Koroteev - Nawata - Pei - Saberi]

Rk: 1. $\mathfrak{g}$ can be non-simply laced. Fiber side, need to take ${}^L\hat{\mathfrak{g}}$. (e.g. ${}^L\hat{B}_2 = {}^2\hat{A}_3$)

2. Thm 1 can extend to non-adm. where $k+h^\vee = \frac{p}{u}$ where p being an elliptic regular number

(e.g. $SO(8)_{-2}$, $(E_6)_{-3}$)

Outlook

1. $\lambda \otimes \mu = \sum_{\nu \in \mathrm{Adm}_k} N_{\lambda\mu}^{\nu}\, \nu + \underbrace{(\text{weight mod. } \log\text{-mod}, \cdots)}$

geometry meaning?

2. When f = principal or $f \in \mathrm{Ass.\,Var}(\mathfrak{g}_k)$

$$H_\bullet(\mathcal{F}\ell_{p,\gamma}) \overset{?}{\simeq} C_2\text{-alg}(W_k(\mathfrak{g}, f))$$

3 More observables? Or even

$$\mathrm{Rep}(\underbrace{\mathrm{Alg}(\mathcal{F}\ell_{p,\gamma})}_{?}) \simeq \mathrm{Rep}(W_k(\mathfrak{g}, f))$$

Thank you !

A_1 DAHA

- A_1 Dynkin diagram $\bullet$
- A_1 Coxeter group s, $s^2 = 1$
- A_1 Hecke algebra T. $(T - t^{1/2})(T + t^{-1/2}) = 0$
- A_1 DAHA: $\mathbb{H}_{q,t} = \mathbb{C}_{q,t}[X^{\pm}, Y^{\pm}, T]/\sim$

$$\sim = \left\{ \begin{array}{ll} TXT = X^{-1} & TY^{-1}T = Y \\ Y^{-1}X^{-1}YXT^2 q^{1/2} = 1 & \\ (T - t^{1/2})(T + t^{-1/2}) = 0 & \end{array} \right\}$$

- $SL(2, \mathbb{Z})$ $\quad \tau_+ : X \mapsto X,\ T \mapsto T,\ Y \mapsto q^{-1/4} XY$

$\tau_- : Y \mapsto Y,\ T \mapsto T,\ X \mapsto q^{1/4} YX$

$\tau_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$