第三届全国场论与弦论
学术研讨会

A new analytic method and solutions for the quantum integrable models without U(1) symmetry

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Ref

References：

- Nucl. Phys. B 974 (2022) 115626;
- Phys. Rev. B 102 (2020) 085115;
- JHEP 11 (2021) 044;
- Phys. Rev. B 103 (2021) L220401;
- JHEP 07 (2021) 133.

Intro






Bethe	Onsager	Yang	Baxter	Faddeev	Wang
1931	1944	1967	1971	1979	2013
Bethe ansatz	2D Ising	Yang-Baxter Equation		ABA	ODBA

Exact solution problem for systems with U(1) symmetry solved.

Exact solution problem for systems without U(1) symmetry solved.

Intro

Homo BAEs：

$$\left(\frac{\lambda_j - \frac{i}{2}}{\lambda_j + \frac{i}{2}}\right)^N = -\prod_{l=1}^{M} \frac{\lambda_j - \lambda_l - i}{\lambda_j - \lambda_l + i}, \quad j = 1, \ldots, M.$$

with U(1)

Fredholm integral eq.：

$$\rho(\lambda) + \rho^h(\lambda) = \frac{dZ(\lambda)}{d\lambda} = a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\rho(\mu) d\mu,$$

without U(1)

ODBA method

inhomo terms

Inhomo BAEs:

$$e^{\lambda_j} a(\lambda_j) Q(\lambda_j - \eta) - e^{-\lambda_j - \eta} d(\lambda_j) Q(\lambda_j + \eta) - c(\lambda_j) a(\lambda_j) d(\lambda_j) = 0,$$
$$j = 1, \ldots, N.$$

Intro

Non-uniqueness

ODBA-1

$$\begin{cases} \Lambda(u)Q(u) = a(u)e^{u}Q(u-\eta) - e^{-u-\eta}d(u)Q(u+\eta) - c(u)a(u)d(u), \\ Q(u) = \prod_{j=1}^{N} \frac{\sinh(u-\lambda_j)}{\sinh\eta}, \end{cases}$$

ODBA-2

$$\begin{cases} \Lambda(u) = e^{u}a(u)\frac{Q_1(u-\eta)}{Q_2(u)} - e^{-u-\eta}d(u)\frac{Q_2(u+\eta)}{Q_1(u)} - c(u)\frac{a(u)d(u)}{Q_1(u)Q_2(u)}, \\ Q_1(u) = \prod_{j=1}^{M} \sinh(u-\mu_j), \quad Q_2(u) = \prod_{j=1}^{M} \sinh(u-\nu_j), \end{cases}$$

Roots patterns are complicated

Intro

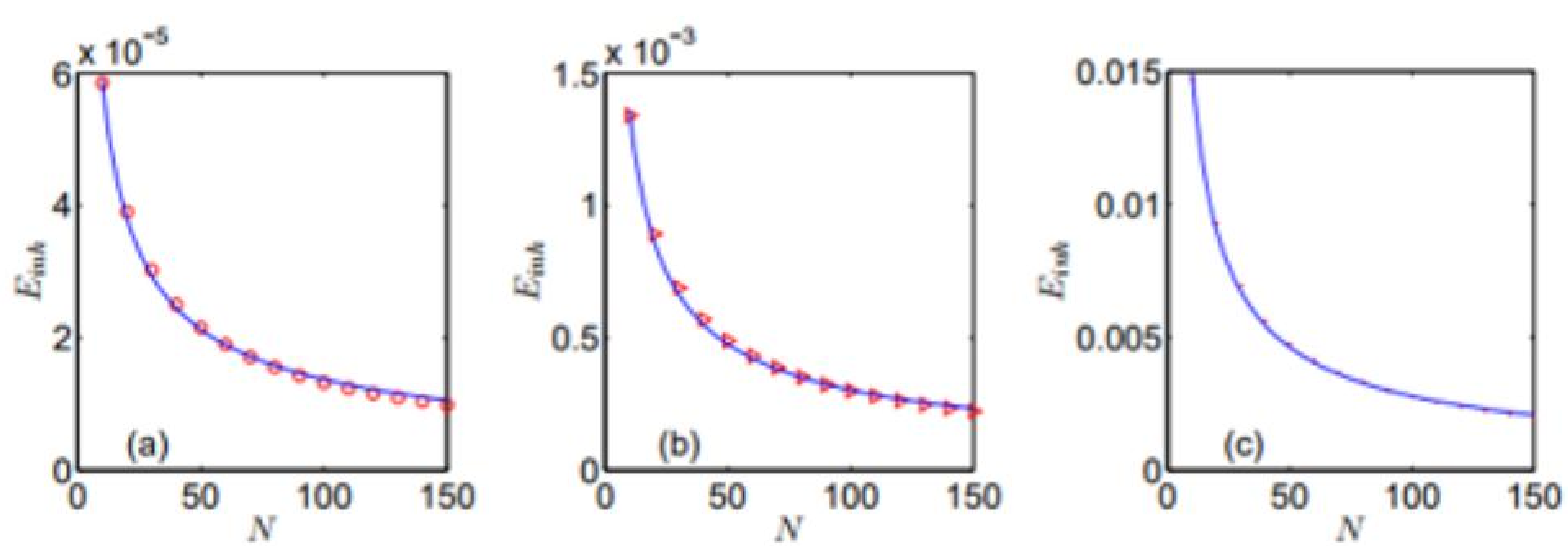
①Find the special point —— XXZ model with ABC

$$\eta_m = -\frac{\alpha_- + \alpha_+ \pm (\theta_- - \theta_+) + 2\pi i m}{N+1}.$$

$$\left[\frac{\sinh(\lambda_j - i\frac{\theta}{2})}{\sinh(\lambda_j + i\frac{\theta}{2})}\right]^{2N} \frac{\sinh(2\lambda_j - i\theta)}{\sinh(2\lambda_j + i\theta)} \frac{\sinh(\lambda_j + ia_+)}{\sinh(\lambda_j - ia_+)} \times \frac{\sinh(\lambda_j + ia_-)}{\sinh(\lambda_j - ia_-)} \frac{\cosh(\lambda_j + \beta + i\frac{\theta}{2})}{\cosh(\lambda_j + \beta - i\frac{\theta}{2})} \frac{\cosh(\lambda_j - \beta + i\frac{\theta}{2})}{\cosh(\lambda_j - \beta - i\frac{\theta}{2})} = -\prod_{l=1}^{N} \frac{\sinh(\lambda_j - \lambda_l - i\theta)\sinh(\lambda_j + \lambda_l - i\theta)}{\sinh(\lambda_j - \lambda_l + i\theta)\sinh(\lambda_j + \lambda_l + i\theta)},$$

Li Y.-Y, et al., Nucl. Phys. B 884, 2014, 884: 17-27.

②Finite size correction of inhomo terms——SU(3) model with ABC



Wen F, et al., Nucl. Phys. B, 2017, 915: 119-134.

η=π i/3

XXZ model Hamilton： $H = -\sum_{j=1}^{N}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cosh\eta\,\sigma_j^z\sigma_{j+1}^z\right],$

Boundary condition $\sigma_{N+1}^{\alpha} = \sigma_1^x\sigma_1^{\alpha}\sigma_1^x, \quad \text{for} \quad \alpha = x, y, z. \quad \eta = \frac{i\pi}{3}.$

R-matrix：

$$R_{0,j}(u) = \frac{1}{2}\left[\frac{\sinh(u+\eta)}{\sinh\eta}(1+\sigma_j^z\sigma_0^z) + \frac{\sinh u}{\sinh\eta}(1-\sigma_j^z\sigma_0^z)\right] + \frac{1}{2}(\sigma_j^x\sigma_0^x + \sigma_j^y\sigma_0^y),$$

Yang-Baxter equation

$$R_{1,2}(u_1-u_2)R_{1,3}(u_1-u_3)R_{2,3}(u_2-u_3) = R_{2,3}(u_2-u_3)R_{1,3}(u_1-u_3)R_{1,2}(u_1-u_2).$$

$\eta=\pi i/3$

Transfer matrix $t(u) = tr_0\{\sigma_0^x R_{0,N}(u-\theta_N)\ldots R_{0,1}(u-\theta_1)\},$

Using the YBE, we can prove

$$[t(u), t(v)] = 0.$$

Expanding t(u) in terms of u

$$t(u) = t^{(1)}e^{(N-1)u} + t^{(2)}e^{(N-3)u} + \cdots + t^{(N)}e^{-(N-1)u}.$$

We readily have that the coefficients are mutually commuting.

Hamiltonian can be expressed by：

integrability

$$H = -2\sinh\eta\frac{\partial \ln t(u)}{\partial u}\Big|_{u=0,\theta_j=0} + N\cosh\eta.$$

Hamiltonian ⟷ Transfer matrix

η=π i/3

Acting transfer matrix on an eigenstate：

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle.$$

We express the eigenvalue in terms of its N − 1 zero points and an overall coefficient constant

$$\Lambda(u) = \Lambda_0 \prod_{j=1}^{N-1} \sinh(u - z_j).$$

Let us introduce a 3N − 3 degree trigonometric polynomial

$$F_3(u) = \Lambda(u)\,\Lambda(u-\eta)\,\Lambda(u-2\eta).$$

η=π i/3

which enjoys the properties

$$
\begin{aligned}
F_3(u+\eta) &= (-1)^{N-1}\, F_3(u),\\
F_3(u) &= F_3^{(1)} e^{(3N-3)u} + F_3^{(2)} e^{(3N-5)u} + \cdots + F_3^{(3N-2)} e^{-(3N-3)u},\\
F_3(\theta_j) &= -a(\theta_j)d(\theta_j-\eta)\,\Lambda(\theta_j-2\eta), \quad j=1,\cdots,N,\\
F_3(\theta_j+\eta) &= -a(\theta_j)d(\theta_j-\eta)\,\Lambda(\theta_j+\eta), \quad j=1,\cdots,N,\\
F_3(\theta_j+2\eta) &= (-1)^N a(\theta_j)d(\theta_j-\eta)\,\Lambda(\theta_j+\eta), \quad j=1,\cdots,N.
\end{aligned}
$$

The above relations can uniquely determine the 3N−3 trigonometric polynomial F3(u), and the result is that the eigenvalue Λ(u) should satisfy

$$
\begin{aligned}
\Lambda(u)\Lambda(u-\eta)\Lambda(u-2\eta) = &-a(u)d(u-\eta)\Lambda(u-2\eta) - a(u-\eta)d(u-2\eta)\Lambda(u)\\
&+(-1)^N a(u+\eta)d(u)\Lambda(u-\eta).
\end{aligned}
$$

η=π i/3

The zero points of the eigenvalue Λ(u) must satisfy

$$(-1)^N \frac{d(z_j)}{a(z_j)} = \prod_{l=1}^{N} \frac{\sinh(z_j - \theta_l)}{\sinh(z_j - \theta_l - 2\eta)} = \prod_{k\neq j}^{N-1} \frac{\sinh(z_j - z_k + \eta)}{\sinh(z_j - z_k - \eta)}, \quad j = 1, \cdots, N-1.$$

where：

$$a(u) = \prod_{l=1}^{N} \frac{\sinh(u - \theta_l + \eta)}{\sinh \eta}, \quad d(u) = a(u - \eta).$$

The energy spectrum

$$E = 2\sinh\eta \sum_{j=1}^{N-1} \coth z_j + N\cosh\eta.$$

η=π i/3

Numerical solutions

λ_1	λ_2	λ_3	λ_4	λ_5	E_n	n
0	0.6969	−0.6969	0.2664	−0.2664	−6.4785	1
−0.6700	0.0310	0.3300	−0.2415	1.0477 − 1.5708*i*	−5.1280	2
0.6700	−0.0310	−0.3300	0.2415	−1.0477 + 1.5708*i*	−5.1280	3
−0.0804	−0.4734	0.2076	0.6357	−0.3695 − 1.5708*i*	−4.2182	4
0.0804	0.4734	−0.2076	−0.6357	0.3695 + 1.5708*i*	−4.2182	5
0.1528	−0.1528	−0.5836	0.5836	−1.5708*i*	−3.8607	6
−0.3047	0.3047	0	1.0124 − 1.5708*i*	−1.0124 + 1.5708*i*	−3.6496	7
−0.2687	0.4515	0.0516	0.3238 − 1.5708*i*	−0.9667 + 1.5708*i*	−2.6398	8
0.2687	−0.4515	−0.0516	−0.3238 + 1.5708*i*	0.9667 + 1.5708*i*	−2.6398	9
0.1301	−0.2030	0.5604	−0.0819 − 1.5708*i*	−0.8975 + 1.5708*i*	−2.2241	10
−0.1301	0.2030	−0.5604	0.0819 − 1.5708*i*	0.8975 + 1.5708*i*	−2.2241	11
0.6203	0.1974	−0.0862	−0.4537 + 1.0526*i*	−0.4537 − 1.0526*i*	−2.0000	12
−0.6203	−0.1974	0.0862	0.4537 + 1.0526*i*	0.4537 − 1.0526*i*	−2.0000	13
0	−0.4175	0.4175	0.2632 + 1.5708*i*	−0.2632 + 1.5708*i*	−1.6234	14
0.1817	−0.1817	1.5708*i*	−0.8587 − 1.5708*i*	0.8587 + 1.5708*i*	−0.4043	15
0.0541	−0.2606	0.4254 + 1.0539*i*	0.4254 − 1.0539*i*	−0.9414 − 1.5708*i*	−0.2277	16
−0.0541	0.2606	−0.4254 + 1.0539*i*	−0.4254 − 1.0539*i*	0.9414 + 1.5708*i*	−0.2277	17
−0.5749	−0.1613	0.5231	0.1269 + 1.0479*i*	0.1269 − 1.0479*i*	0.5222	18
0.5749	0.1613	−0.5231	−0.1269 + 1.0479*i*	−0.1269 − 1.0479*i*	0.5222	19
0.0067	0.4202	0.2161 − 1.5708*i*	−0.3803 + 1.0558*i*	−0.3803 − 1.0558*i*	1.0892	20
−0.0067	−0.4202	−0.2161 − 1.5708*i*	0.3803 + 1.0558*i*	0.3803 − 1.0558*i*	1.0892	21
−0.5380	−0.1223	0.2012 + 1.0510*i*	0.2012 − 1.0510*i*	0.6784 − 1.5708*i*	2.0000	22
0.5380	0.1223	−0.2012 + 1.0510*i*	−0.2012 − 1.0510*i*	−0.6784 + 1.5708*i*	2.0000	23
0.4900	−0.2257	0.0925 − 1.0486*i*	0.0925 + 1.0486*i*	−0.8593 − 1.5708*i*	2.3827	24
−0.4900	0.2257	−0.0925 − 1.0486*i*	−0.0925 + 1.0486*i*	0.8593 − 1.5708*i*	2.3827	25
0.4146	−0.4146	1.5708*i*	+1.0499*i*	−1.0499*i*	3.5137	26
0	0.3767 − 1.0590*i*	0.3767 + 1.0590*i*	−0.3767 − 1.0590*i*	−0.3767 + 1.0590*i*	3.7515	27
0.1845	−0.1721 + 1.0544*i*	−0.1721 − 1.0544*i*	−0.5975 − 1.5708*i*	0.7995 − 1.5708*i*	4.1565	28
−0.1845	0.1721 − 1.0544*i*	0.1721 + 1.0544*i*	0.5975 − 1.5708*i*	−0.7995 − 1.5708*i*	4.1565	29
−0.0279 − 1.0558*i*	−0.0279 + 1.0558*i*	0.3317 + 1.0832*i*	0.3317 − 1.0832*i*	−0.4288	6.2874	30
0.4288	0.0279 + 1.0558*i*	0.0279 − 1.0558*i*	−0.3317 + 1.0832*i*	−0.3317 − 1.0832*i*	6.2874	31
−1.5708*i*	0.1962 + 1.1268*i*	0.1962 − 1.1268*i*	−0.1962 − 1.1268*i*	−0.1962 + 1.1268*i*	8.7513	32

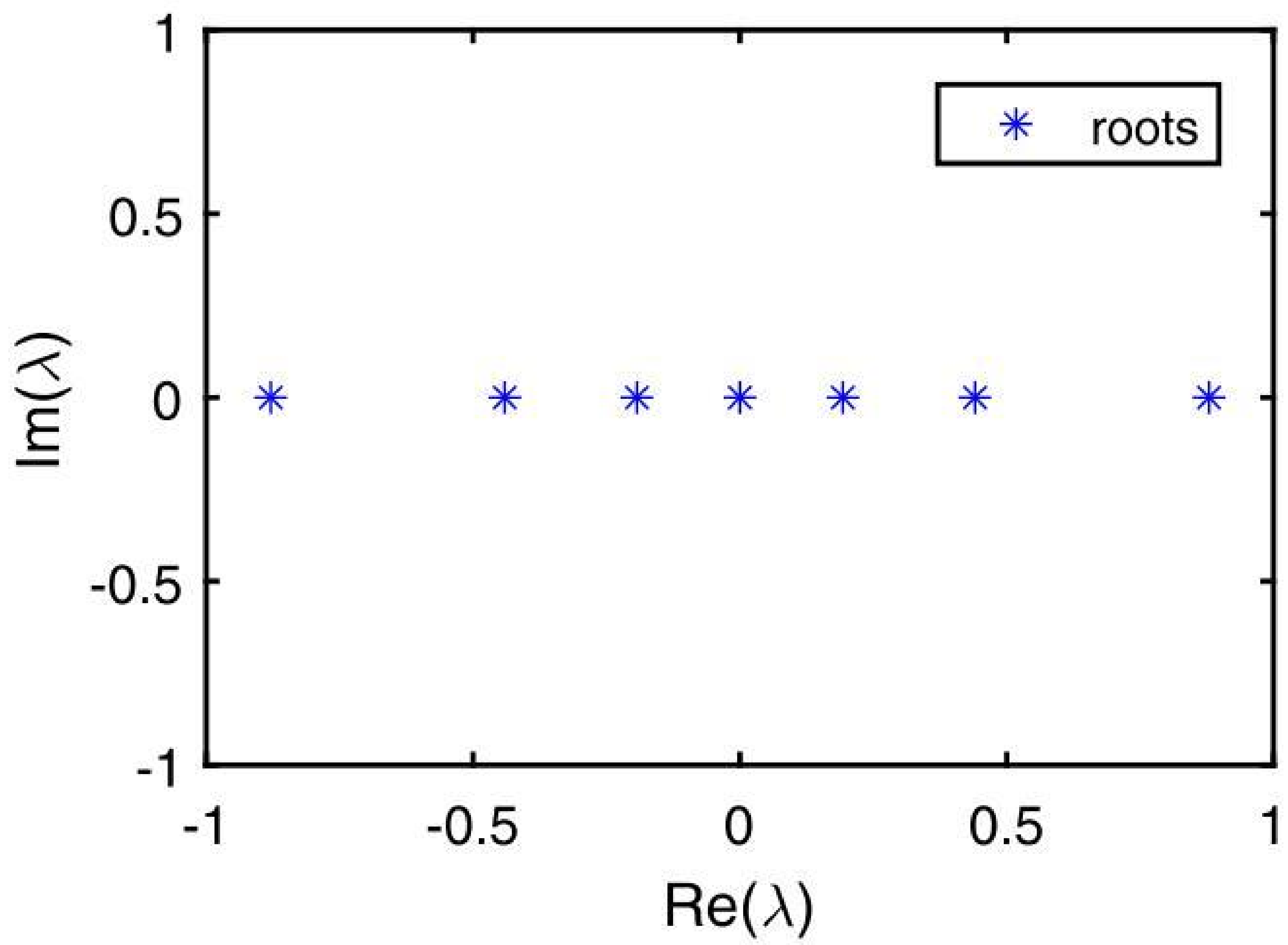
η=π i/3

Thermodynamic limit

Difference

BAEs
$$\left[\frac{\sinh(\lambda_j - \frac{i}{6}\pi)}{\sinh(\lambda_j + \frac{i}{6}\pi)}\right]^N = -\prod_{k=1}^{N-1} \frac{\sinh(\lambda_j - \lambda_k + \frac{i}{3}\pi)}{\sinh(\lambda_j - \lambda_k - \frac{i}{3}\pi)}, \qquad j = 1, \cdots, N-1,$$

roots for GS



Taking the logarithm
$$\theta_1(\lambda_j) = \frac{2\pi I_j}{N} - \frac{1}{N}\sum_{k=1}^{N-1} \theta_2(\lambda_j - \lambda_k). \qquad \theta_m(\lambda) = -i \ln \frac{\sinh(\frac{i\pi m}{6} - \lambda_j)}{\sinh(\frac{i\pi m}{6} + \lambda_j)}.$$

quantum number $\{I_j\} = \left\{ -\frac{N}{2}+1, -\frac{N}{2}+2, \cdots, \frac{N}{2}-2, \frac{N}{2}-1 \right\}.$

η=π i/3

Define the counting function $Z(\lambda)=\frac{1}{2\pi}\left[\theta_1(\lambda)+\frac{1}{N}\sum_{k=1}^{N-1}\theta_2(\lambda-\lambda_k)\right].$

The derivative $\frac{dZ(\lambda)}{d\lambda}\equiv\rho_g(\lambda)+\rho_g^h(\lambda),$

Taking the derivative of BAEs

$$\rho_g(\lambda)+\rho_g^h(\lambda)=a_1(\lambda)+\int_{-\infty}^{\infty}a_2(\lambda-\mu)\rho_g(\mu)d\mu,$$

where $\rho_g^h(\lambda)=\frac{1}{3N}\delta(\lambda-\lambda_0^h)+\frac{1}{3N}\delta(\lambda+\lambda_0^h).$

The number of zero points $\int_{-\infty}^{\infty}\rho_g(\lambda)d\lambda=\frac{N-1}{N}.$

η=π i/3

The solution of integral equation

$$\rho_g(\lambda) = \frac{3i}{4\pi}\Big[\mathrm{csch}(\frac{3}{2}\lambda + \frac{i\pi}{4}) - \mathrm{csch}(\frac{3}{2}\lambda - \frac{i\pi}{4})\Big] - \frac{1}{3N}\Big\{\delta(\lambda - \lambda_0^h) + \delta(\lambda + \lambda_0^h) + \frac{3}{4\pi}\mathrm{sech}[\frac{3}{2}(\lambda - \lambda_0^h)] + \frac{3}{4\pi}\mathrm{sech}[\frac{3}{2}(\lambda + \lambda_0^h)]\Big\}.$$

Thus the ground state energy is

$$E_g = 2N\sinh(\frac{i}{3}\pi)\int_{-\infty}^{\infty}\coth(\lambda - \frac{i}{6}\pi)\rho_g(\lambda)d\lambda + N\cosh(\frac{i}{3}\pi)$$
$$= \frac{3-3\sqrt{3}}{2}N + \triangle(\lambda_0^h) + \triangle(-\lambda_0^h),$$

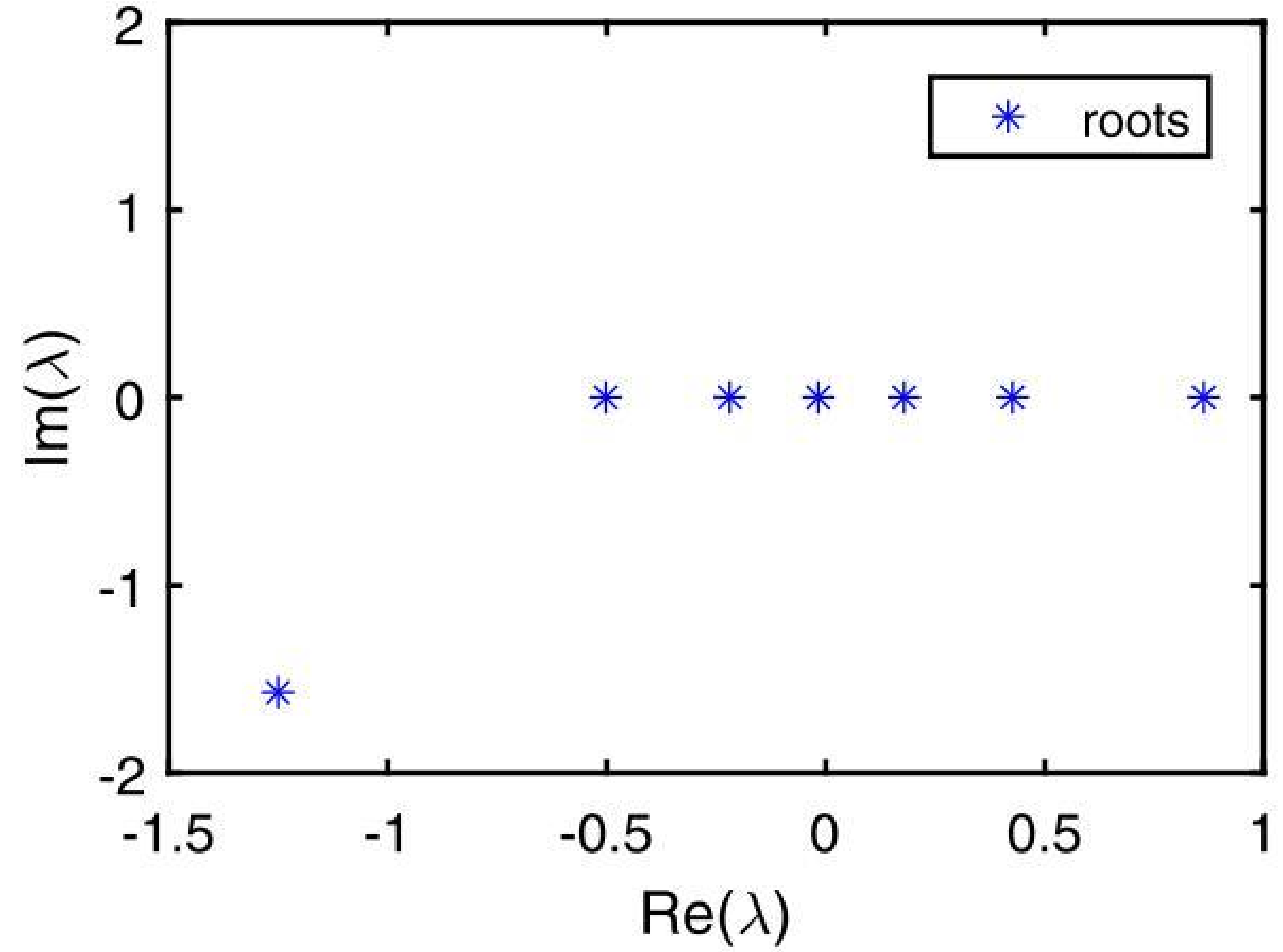
where $\triangle(\lambda_0^h) = \frac{\sqrt{3}}{4}\Big[\mathrm{sech}(\frac{3}{2}\lambda_0^h + \frac{i\pi}{4}) + \mathrm{sech}(\frac{3}{2}\lambda_0^h - \frac{i\pi}{4})\Big] + \frac{\sqrt{3}}{2}i\tanh(3\lambda_0^h).$

λ^h_0 turns to infinity to minimize the energy.

η=π i/3

Elementary excitation of type I

roots for EE I



We set $\lambda_{N-1} = \alpha - i\frac{\pi}{2}$.

The BAEs reads

$$\left[\frac{\sinh(\lambda_j - \frac{i}{6}\pi)}{\sinh(\lambda_j + \frac{i}{6}\pi)}\right]^N = -\prod_{k=1}^{N-2} \frac{\sinh(\lambda_j - \lambda_k + \frac{i}{3}\pi)}{\sinh(\lambda_j - \lambda_k - \frac{i}{3}\pi)} \frac{\cosh(\lambda_j - \alpha + \frac{i}{3}\pi)}{\cosh(\lambda_j - \alpha - \frac{i}{3}\pi)},$$

$j = 1, \cdots, N-2,$

$$\left[\frac{\cosh(\alpha - \frac{i}{6}\pi)}{\cosh(\alpha + \frac{i}{6}\pi)}\right]^N = \prod_{k=1}^{N-2} \frac{\cosh(\alpha - \lambda_k + \frac{i}{3}\pi)}{\cosh(\alpha - \lambda_k - \frac{i}{3}\pi)}.$$

η=π i/3

Taking the logarithm

$$\theta_1(\lambda_j) = \frac{2\pi I_j}{N} - \frac{1}{N}\sum_{k=1}^{N-2}\theta_2(\lambda_j - \lambda_k) + \frac{1}{N}\theta_1(\lambda_j - \alpha),$$

$$\theta_2(\alpha) = \frac{2\pi J}{N} - \frac{1}{N}\sum_{k=1}^{N-2}\theta_1(\alpha - \lambda_k),$$

consecutive

Quantum numbers

$$\{I_j\} = \Big\{-\frac{N-1}{2}+1, -\frac{N-1}{2}+2, \cdots, \frac{N-1}{2}-2, \frac{N-1}{2}-1\Big\},$$

$$J \in \Big\{-\frac{N}{2}+1, -\frac{N}{2}+2, \cdots, \frac{N}{2}-2, \frac{N}{2}-1\Big\}.$$

No hole

The density difference

$$\delta\rho_1(\lambda) = -\frac{3i}{4\pi N}\Big[\mathrm{csch}(\frac{3}{2}\lambda - \frac{3}{2}\alpha + \frac{i\pi}{4}) - \mathrm{csch}(\frac{3}{2}\lambda - \frac{3}{2}\alpha - \frac{i\pi}{4})\Big].$$

The EE energy

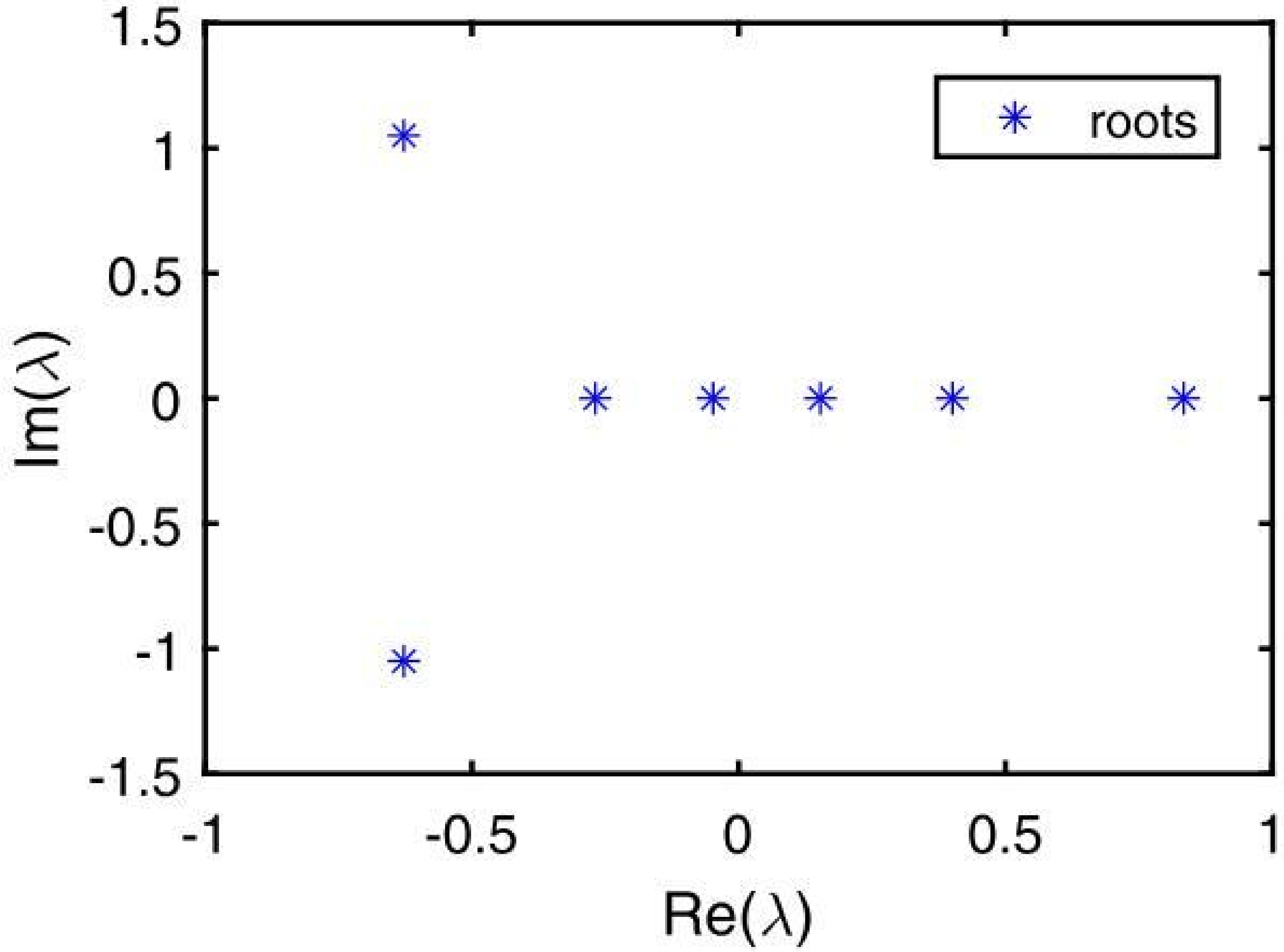
$$\delta e_1 = 2N\sinh\big(\frac{i}{3}\pi\big)\int_{-\infty}^{\infty}\coth\big(\lambda - \frac{i}{6}\pi\big)\delta\rho_1(\lambda)d\lambda + 2\sinh\big(\frac{i}{3}\pi\big)\coth\big(\alpha - \frac{2i}{3}\pi\big)$$

$$= \frac{3\sqrt{3}}{2}\mathrm{sech}(\frac{3\alpha}{2}).$$

η=π i/3

Elementary excitation of type II

roots for EE II



We set

$$\lambda_{N-1} = \alpha + i\frac{\pi}{3} \quad \lambda_{N-2} = \alpha - i\frac{\pi}{3}$$

The BAEs reads

$$\left[\frac{\sinh(\lambda_j - \frac{i}{6}\pi)}{\sinh(\lambda_j + \frac{i}{6}\pi)}\right]^N = -\prod_{k=1}^{N-3} \frac{\sinh(\lambda_j - \lambda_k + \frac{i}{3}\pi)}{\sinh(\lambda_j - \lambda_k - \frac{i}{3}\pi)} \frac{\sinh(\lambda_j - \alpha + \frac{2i}{3}\pi)}{\sinh(\lambda_j - \alpha - \frac{2i}{3}\pi)},$$

$$j = 1, \cdots, N-3,$$

$$\left[\frac{\sinh(\alpha + \frac{i}{6}\pi)}{\sinh(\alpha - \frac{i}{6}\pi)}\right]^N = \prod_{k=1}^{N-3} \frac{\sinh(\alpha - \lambda_k + \frac{2i}{3}\pi)}{\sinh(\alpha - \lambda_k - \frac{2i}{3}\pi)}.$$

η=π i/3

Taking the logarithm

$$\theta_1(\lambda_j) = \frac{2\pi I_j}{N} - \frac{1}{N}\sum_{k=1}^{N-3}\theta_2(\lambda_j - \lambda_k) + \frac{1}{N}\theta_2(\lambda_j - \alpha),$$

$$\theta_1(\alpha) = \frac{2\pi J}{N} - \frac{1}{N}\sum_{k=1}^{N-3}\theta_2(\alpha - \lambda_k),$$

inconsecutive

Quantum numbers

$$\{I_j\} = \Big\{ -\frac{N-1}{2}+1, -\frac{N-1}{2}+2, \cdots, \frac{N-1}{2}-j-1, \frac{N-1}{2}-j+1, \cdots, \frac{N-1}{2}-2, \frac{N-1}{2}-1\Big\},$$

$$J = \frac{N-1}{2} - j, \quad j = 1, \cdots, N-2.$$

Holes

The density difference

$$\delta\rho_2(\lambda) = -\frac{3}{4\pi N}\left[\operatorname{sech}(\frac{3}{2}\lambda - \frac{3}{2}\alpha) + \operatorname{sech}(\frac{3}{2}\lambda - \frac{3}{2}\lambda_2^h)\right] - \frac{1}{N}\delta(\lambda - \lambda_2^h).$$

The EE energy

$$\delta e_2 = 3\Delta(\alpha) + 3\Delta(-\alpha) = \frac{3\sqrt{6}\cosh(\frac{3\alpha}{2})}{\cosh(3\alpha)}.$$

$\eta \in i\,\mathbb{R}$

Arbitrary η

At inhomo para $\Lambda(\theta_j)\,\Lambda(\theta_j-\eta) = -a(\theta_j)\,d(\theta_j-\eta), \quad j = 1,\ldots,N.$

We focus on $\Lambda(u)\Lambda(u-\eta)$.

With the fusion techniques

$$\mathbf{t}(u)\mathbf{t}(u-\eta) = \mathrm{tr}_{1,2}\{P^{(-)}_{1,2}\sigma^x_1\sigma^x_2\mathbf{T}_2(u)\mathbf{T}_1(u-\eta)P^{(-)}_{1,2}\} + \mathrm{tr}_{1,2}\{P^{(+)}_{1,2}\sigma^x_1\sigma^x_2\mathbf{T}_2(u)\mathbf{T}_1(u-\eta)P^{(+)}_{1,2}\},$$

we have the following relation

$$\mathbf{t}(u)\mathbf{t}(u-\eta) = -a(u)d(u-\eta)\times\mathbf{id} + d(u)\mathbf{W}(u),$$

$\eta \in i\,R$

Acting on an eigenstate

$$\Lambda(u)\Lambda(u-\eta) = -a(u)d(u-\eta) + d(u)W(u),$$

where $W(u) = W_0 \sinh^{-N}\eta \prod_{l=1}^{N} \sinh(u - w_l),$

Leading terms

$$W_0 e^{\pm \sum_{l=1}^{N} w_l} = 1$$

$u = w_l$

$$\Lambda_0^2 \prod_{j=1}^{N-1} \sinh\left(w_l - z_j + \frac{\eta}{2}\right) \sinh\left(w_l - z_j - \frac{\eta}{2}\right) = -\sinh^{-2N}\eta \sinh^N(w_l + \eta) \sinh^N(w_l - \eta).$$

BAEs

$u = z_j - \eta/2$

$$\sinh^N\left(z_j - \frac{3\eta}{2}\right) \sinh^N\left(z_j + \frac{\eta}{2}\right) = W_0 \sinh^N\left(z_j - \frac{\eta}{2}\right) \prod_{l=1}^{N} \sinh\left(z_j - w_l - \frac{\eta}{2}\right).$$

$u = 0$

$$\Lambda_0^2 \prod_{j=1}^{N-1} \sinh\left(z_j + \frac{\eta}{2}\right) \sinh\left(z_j - \frac{\eta}{2}\right) = (-1)^{N-1}.$$

Topological momentum

shift operator $\mathbf{t}(0)=\sigma_1^x P_{1,N} P_{1,N-1}\cdots P_{1,2},$

”momentum” operator $\mathbf{P}_q=-i\ln \mathbf{t}(0).$

expression with z_j

$$k=-\frac{i}{2}\sum_{j=1}^{N-1}\ln\frac{\sinh\left(z_j+\frac{\eta}{2}\right)}{\sinh\left(z_j-\frac{\eta}{2}\right)}+(1-(-1)^{N-1})\frac{\pi}{4}.$$

we have $\mathbf{t}^{2N}(0)=1,$

eigenvalues of $\mathbf{P}_q$ take values

$$k=\frac{\pi l}{N}\ \mathrm{mod}\,\{\pi\},\qquad l=\{-N,-N+1,\cdots,N-1\}.$$

Conserved charge

Definition

$$\mathbf{M}_q = \frac{1}{2}(\mathbf{l}_q^+ + \mathbf{l}_q^-)$$
$$= \frac{1}{4} e^{-\frac{(N-1)\eta}{2}} \lim_{u\to\infty} (2\sinh\eta e^{-u})^{N-1} \mathbf{t}(u),$$

where

$$\mathbf{l}_q^\pm = \frac{1}{2}\sum_{j=1}^{N} e^{\mp\frac{\eta}{2}\sum_{k=j+1}^{N}\sigma_k^z}\sigma_j^\pm e^{\pm\frac{\eta}{2}\sum_{k=1}^{j-1}\sigma_k^z},$$

expression with z_j

$$M_q = \frac{1}{4}\sinh^{N-1}\eta\,\Lambda_0\, e^{-\sum_{k=1}^{N-1} z_k}.$$

Only when η → 0, the model tends to an isotropic spin chain and the U (1) symmetry recovers with $\mathbf{M}_q = \sum_{j=1}^{N}\sigma_j^x/2,$ which is just the U (1) charge.

Pattern of roots

Transfer matrix satisfies

$$\mathbf{t}^{\dagger}(u) = (-1)^{N-1}\mathbf{t}(u^{*} - \eta),$$
$$\Lambda(u) = (-1)^{N-1}\Lambda^{*}(u^{*} - \eta).$$

if z_j is a root, z^*_j must also be a root.

Therefore, z_j can be classified into 3 sets:
1.real z_j; 2.Im(z_j) = −iπ/2; 3.complex conjugate pairs.

From t-W relation $W^{*}(u^{*}) = (-1)^{N} W(u),$

if w_j is a root, w^*_j must also be a root.

$\eta \in i\,R$

Ground state. For the ground state, all roots z_j and w_l take real values around zero symmetrically.

Taking the logarithms of BAEs and its complex conjugate

$$2\theta_1(z_j) - \theta_3(z_j) = \frac{4\pi I_j}{N} - \frac{1}{N}\sum_{l=1}^{N}\theta_1(z_j - w_l),$$

$$\theta_n(x) = 2\cot^{-1}(\coth x \tan \tfrac{n\gamma}{2}). \quad I_j = \left\{-\frac{N-2}{2}, -\frac{N-4}{2}, \cdots, \frac{N-4}{2}, \frac{N-2}{2}\right\}.$$

and $$\ln\left|\Lambda_0 \sinh\left(z_j - \frac{3\eta}{2}\right)\right| = \frac{1}{N}\sum_{l=1}^{N}\ln\left|\sinh\left(z_j - w_l - \frac{\eta}{2}\right)\right|,$$

Taking the continuum limits

$$2a_1(z) - a_3(z) = 2\rho(z) + 2\rho^h(z) - a_1 * \sigma(z),$$

$$b_3(z) = b_1 * \sigma(z),$$

where $a_n(z) = \theta_n'(z)/(2\pi), \quad b_n(z) = \ln'|\sinh(z - n\eta/2)|/\pi.$

convolution

η∈i R

With Fourier transformation we readily have

$$\rho(z) + \rho^h(z) = \frac{2\cosh\left(\frac{\pi z}{\pi-\gamma}\right)\sin\left(\frac{\pi\gamma}{2\pi-2\gamma}\right)}{(\pi - \gamma)\left[\cosh\left(\frac{2\pi z}{\pi-\gamma}\right) + \cos\left(\frac{\pi(\pi-2\gamma)}{\pi-\gamma}\right)\right]}.$$

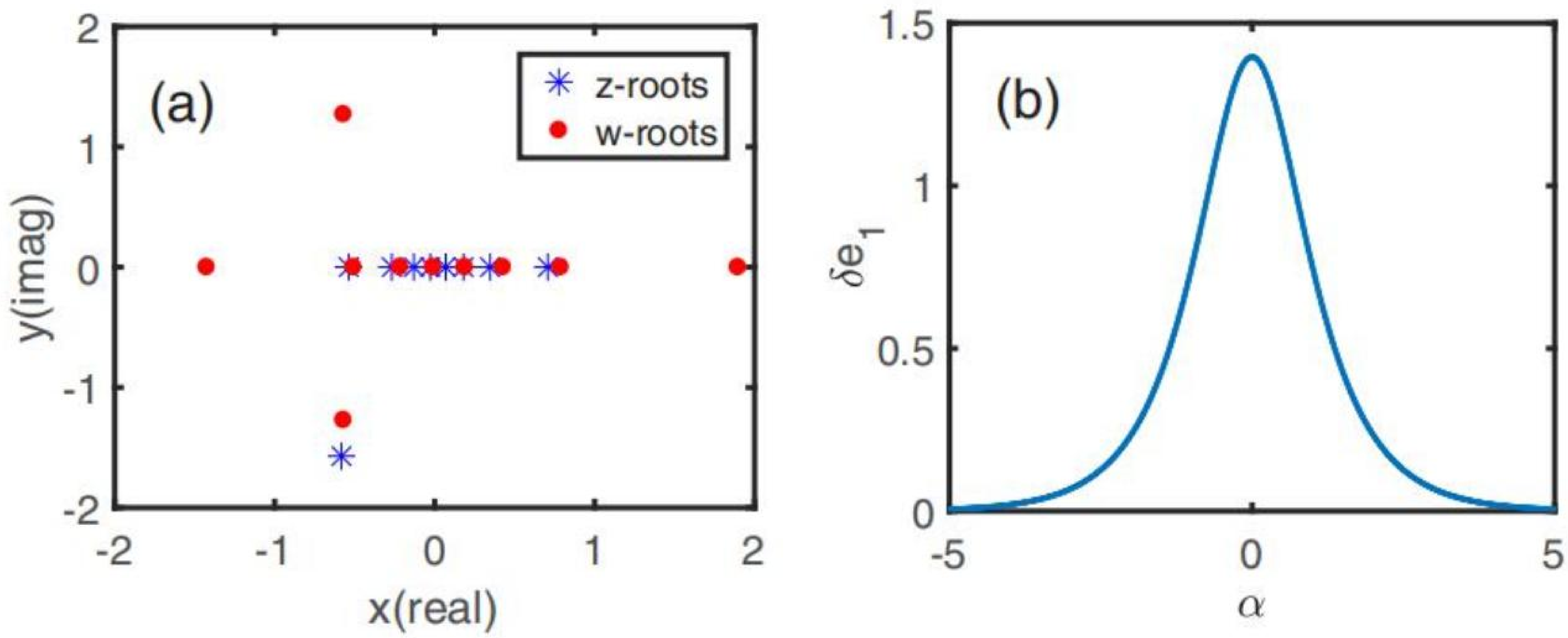
The ground state energy density reads

$$e_g = -\sin\gamma \int \frac{\cosh\left[\frac{(\pi-2\gamma)\tau}{2}\right]\tanh\left[\frac{(\pi-\gamma)\tau}{2}\right]}{\sinh\left(\frac{\pi\tau}{2}\right)} d\tau + \cos\gamma,$$

which is the same as that of η=π i/3 case.

η∈i R

Elementary excitations I



We set $z = \alpha - i\pi/2, \quad w_{\pm} = \beta \pm m\eta/2.$

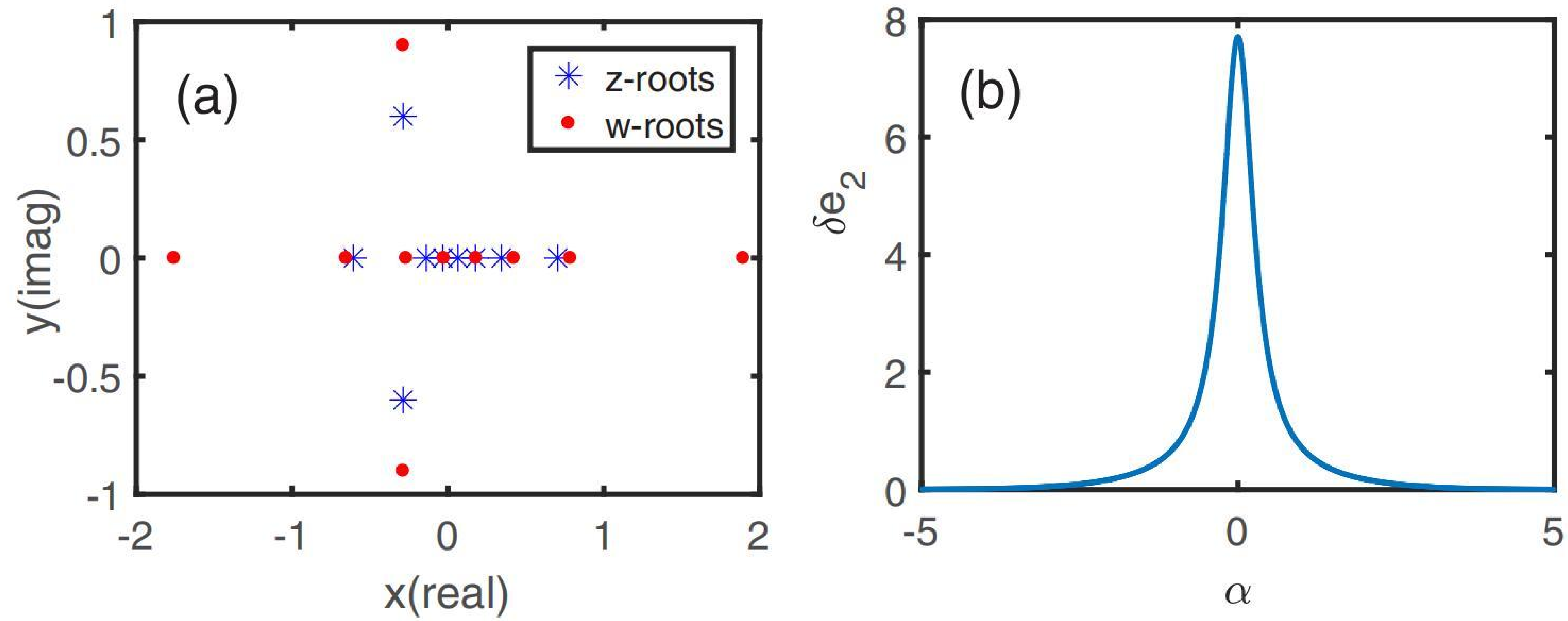
Convergence of the density function $m + 1 - \frac{\pi}{\gamma} = 0, \quad \beta = \alpha.$

Excitation energy

$$\delta e_1 = \sin\gamma \int \frac{\cos(\tau\alpha)\tanh\left[\frac{(\pi-\gamma)\tau}{2}\right]\cosh\left(\frac{\tau\gamma}{2}\right)}{\sinh\left(\frac{\pi\tau}{2}\right)} d\tau + \frac{2\sin^2\gamma}{\cosh(2\alpha) + \cos\gamma}.$$

$\eta \in i$ R

Elementary excitations II



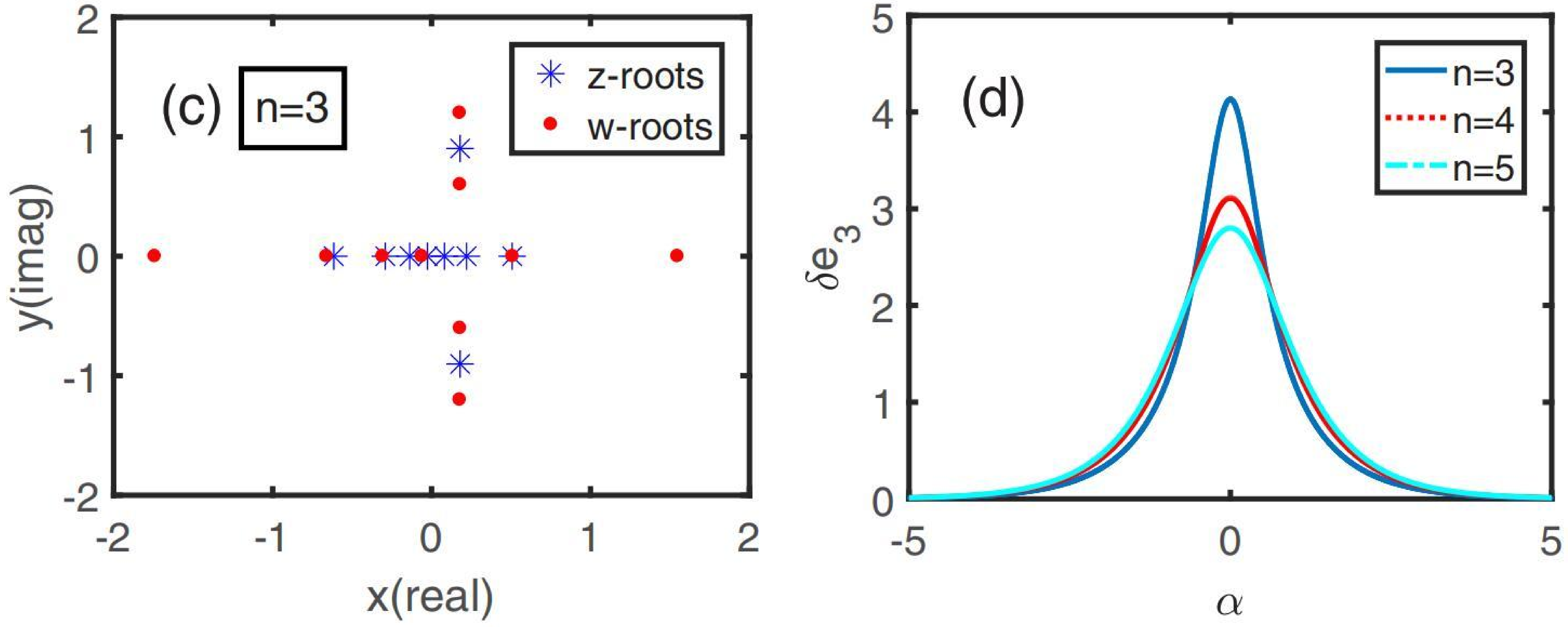
We set $z_{\pm} \sim \alpha \pm \eta,\ w_{\pm} \sim \alpha \pm 3\eta/2.$

Excitation energy

$$\delta e_2 = \sin\gamma \int \frac{\cos(\tau\alpha)\tanh\left[\frac{(\pi-\gamma)\tau}{2}\right]\cosh\left[\frac{(\pi-3\gamma)\tau}{2}\right]}{\sinh\left(\frac{\pi\tau}{2}\right)} d\tau + \frac{4\sin^2\gamma}{\cosh(2\alpha)-\cos\gamma} - \frac{2\sin\gamma\sin(3\gamma)}{\cosh(2\alpha)-\cos(3\gamma)}.$$

η∈i R

Elementary excitations III



we set $z_{\pm} \sim \alpha \pm n\eta/2, \quad w \sim \alpha \pm (n+1)\eta/2, \alpha \pm (n-1)\eta/2.$ $\boxed{n \geqslant 3}$

Excitation energy

$$\delta e_3 = 2\sin\gamma \int \frac{\cos(\tau\alpha)\tanh\left(\frac{\pi-\gamma}{2}\tau\right) f(\tau)}{\sinh\left(\frac{\pi\tau}{2}\right)} d\tau + \frac{2\sin\gamma\sin[(n-1)\gamma]}{\cosh(2\alpha) - \cos[(n-1)\gamma]} - \frac{2\sin\gamma\sin[(n+1)\gamma]}{\cosh(2\alpha) - \cos[(n+1)\gamma]},$$

$$\delta_m = m\gamma/(2\pi) - \lfloor m\gamma/(2\pi) \rfloor.$$

where $f(\tau) = \cosh[(1 - \delta_{n-1} - \delta_{n+1})\pi\tau/2]\cosh[(\delta_{n-1} - \delta_{n+1})\pi\tau/2]$.

η∈R

w_j to θ_j

Constrain equations

$$\Lambda_0^2 \prod_{l=1}^{N-1} \sinh\left(\theta_j - z_l + \frac{\eta}{2}\right) \sinh\left(\theta_j - z_l - \frac{\eta}{2}\right)$$
$$= -\sinh^{-2N}\eta \prod_{l=1}^{N} \sinh(\theta_j - \theta_l + \eta) \sinh(\theta_j - \theta_l - \eta).$$

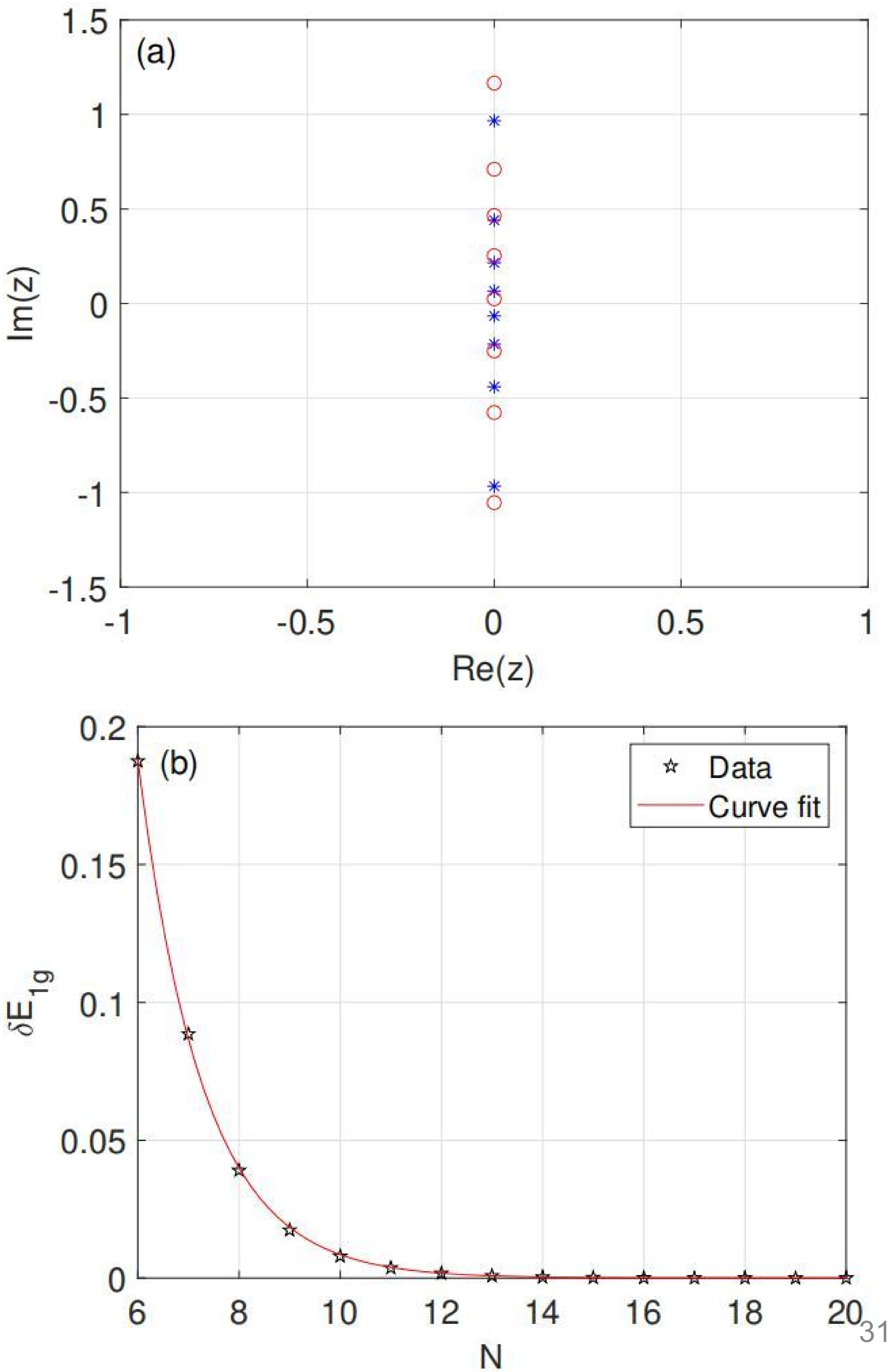
Density of zero roots

$$\rho_{1g}(x) = \frac{1}{\pi}\sum_{k=1}^{\infty} 2\cos(2kx)e^{-k\eta} + \frac{1}{\pi}\left(1 - \frac{1}{N}\right).$$

Energy for GS

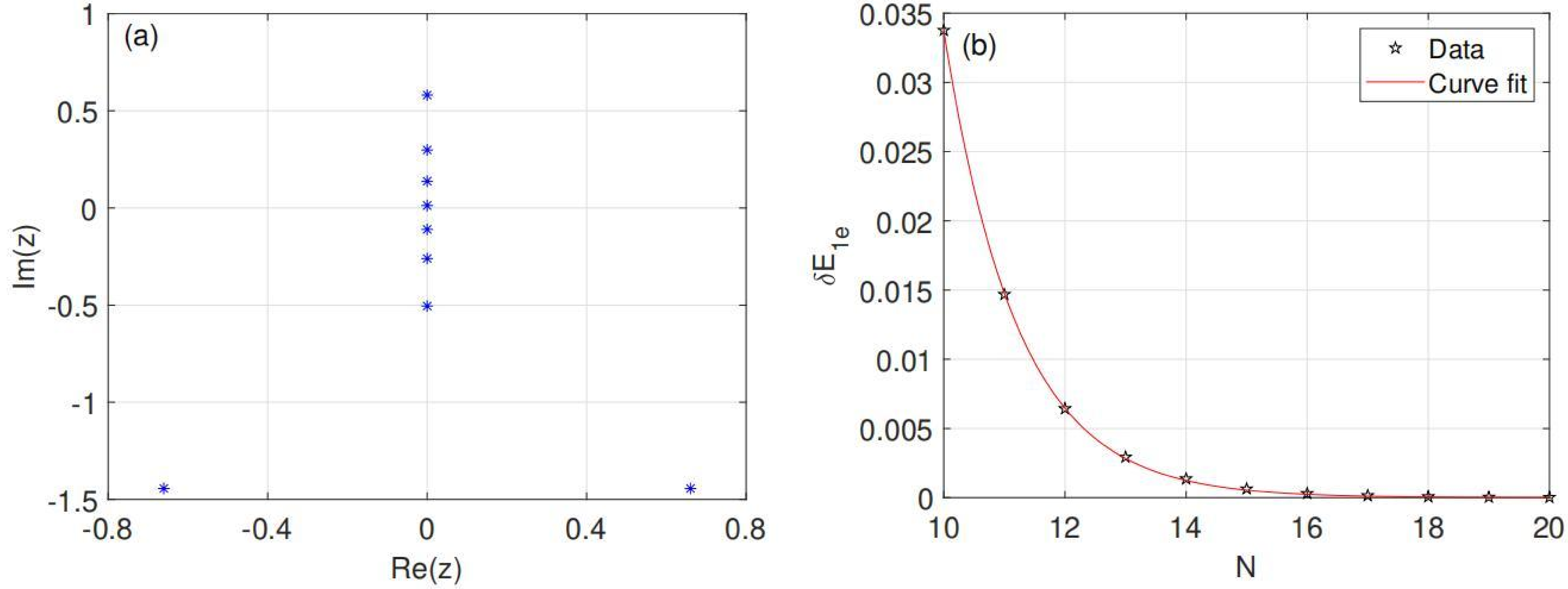
$$E_{1g} = 2N\sinh\eta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \coth\left(ix - \frac{\eta}{2}\right)\rho_{1g}(x)dx + N\cosh\eta$$
$$= -N\cosh\eta + 2\sinh\eta.$$

z_j roots for GS



η∈R

Elementary excitations



Density of zero roots

$$\rho_{1e}(x)=\frac{1}{\pi}\sum_{k=1}^{\infty}\left[2\cos(2kx)e^{-k\eta}-2\cos[2k(x-\alpha)]\frac{e^{-n\eta k}+e^{-(n-2)\eta k}}{N}\right]+\frac{1}{\pi}\left(1-\frac{3}{N}\right).$$

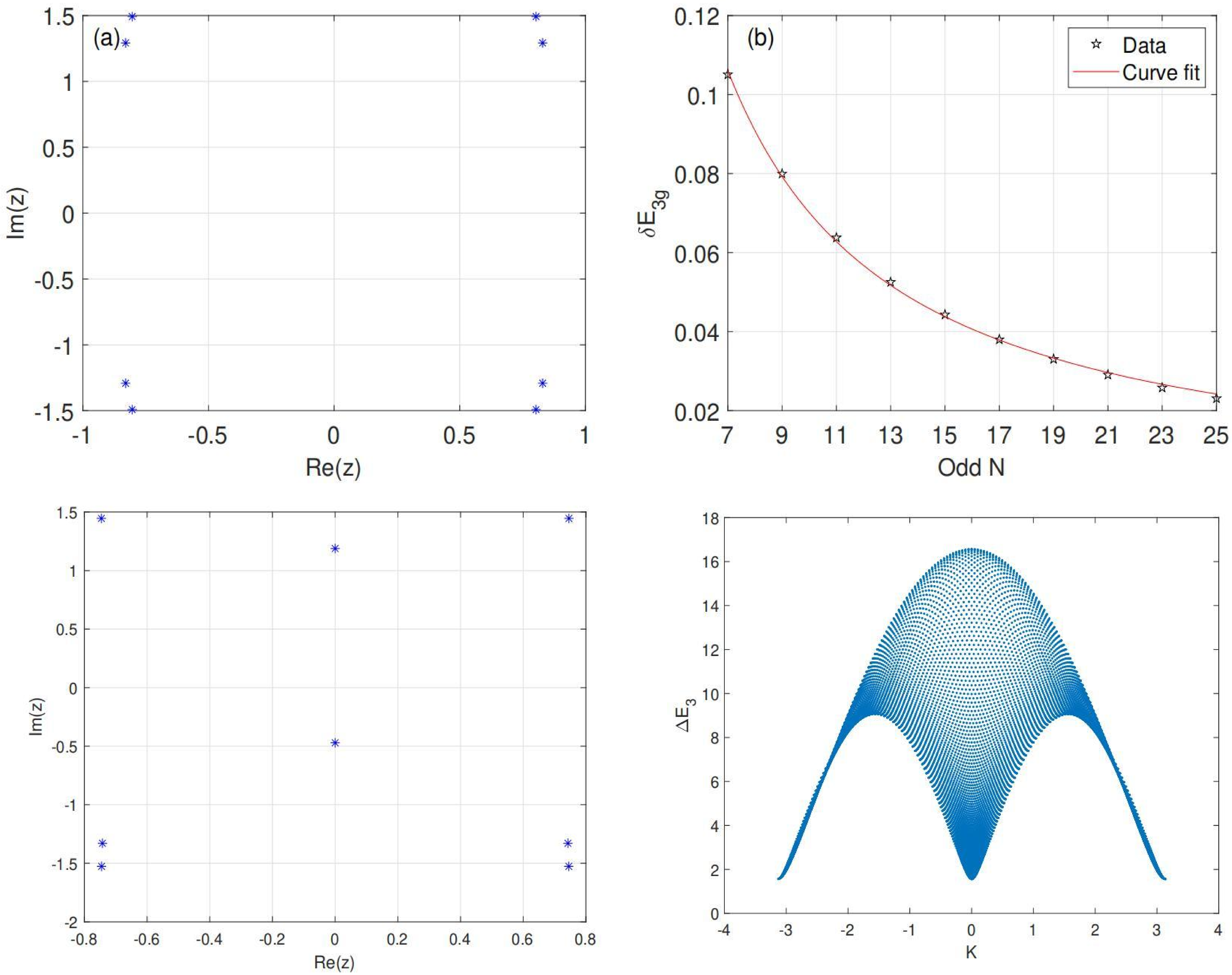
Energy for GS $$\Delta E_1 = 4\sinh\eta\frac{\sinh[(n-1)\eta]}{\cosh[(n-1)\eta]-2\cos(2\alpha)}.$$

η∈R+iπ

η∈R+iπ

GS

EE



η∈R+iπ

Energy for EE $\Delta E_3 = \epsilon(p) + \epsilon(q),$

where $\epsilon(t) = -4\sinh\eta_+ \sum_{k=1}^{\infty}(-1)^{k+1} e^{-\eta_+ k}\tanh(\eta_+ k)\cos(2kt) + 2\sinh\eta_+ \frac{\sinh\eta_+}{\cosh\eta_+ + \cos 2t}.$

Momentum $K = \zeta(p) + \zeta(q),$

where $\zeta(t) = \sum_{k=1}^{\infty}(-1)^k \frac{\sin(2kt)}{k} e^{-\eta_+ k}\tanh(\eta_+ k) - \frac{i}{2}\ln\left[-\frac{\cosh\left(it + \frac{\eta_+}{2}\right)}{\cosh\left(it - \frac{\eta_+}{2}\right)}\right].$

OBC

Hamiltonian
$$H=\sum_{j=1}^{N-1}\left\{\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\cosh\eta\,\sigma_j^z\sigma_{j+1}^z\right\}$$
$$+\vec{h}_-\cdot\vec{\sigma}_1+\vec{h}_+\cdot\vec{\sigma}_N,$$

boundary fields
$$h_\pm^z=\mp\frac{\sinh\eta\cosh\alpha_\pm\sinh\beta_\pm}{\sinh\alpha_\pm\cosh\beta_\pm},$$
$$h_\pm^x=\frac{\sinh\eta\cos\theta_\pm}{\sinh\alpha_\pm\cosh\beta_\pm},\qquad h_\pm^y=\frac{\sinh\eta\sin\theta_\pm}{\sinh\alpha_\pm\cosh\beta_\pm},$$

Integrability
$$H=\sinh\eta\frac{\partial\ln t(u)}{\partial u}\bigg|_{u=0,\{\theta_j=0\}}-c_0,$$
$$c_0=N\cosh\eta+\tanh\eta\sinh\eta,$$

OBC

transfer matrix

$$t(u) = tr_0\{K_0^+(u)R_{0N}(u-\theta_N)\cdots R_{01}(u-\theta_1)$$
$$\times K_0^-(u)R_{10}(u+\theta_1)\cdots R_{N0}(u+\theta_N)\}.$$

boundary reflection matrix

$$K^-(u) = \begin{pmatrix} K_{11}^-(u) & K_{12}^-(u) \\ K_{21}^-(u) & K_{22}^-(u) \end{pmatrix},$$

$$K_{11}^-(u) = 2\sinh\alpha_-\cosh\beta_-\cosh u$$
$$+2\cosh\alpha_-\sinh\beta_-\sinh u,$$

$$K_{12}^-(u) = e^{-i\theta_-}\sinh(2u), \quad K_{21}^-(u) = e^{i\theta_-}\sinh(2u),$$

$$K_{22}^-(u) = 2\sinh\alpha_-\cosh\beta_-\cosh u$$
$$-2\cosh\alpha_-\sinh\beta_-\sinh u,$$

dual boundary matrix

$$K^+(u) = K^-(-u-\eta)|_{(\alpha_-,\beta_-,\theta_-)\to(-\alpha_+,-\beta_+,\theta_+)}.$$

OBC

Transfer matrix satisfies the identities

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta)=a(\theta_j)a(-\theta_j),\qquad j=1,\ldots,N,$$

$$\Lambda(0)=a(0),\qquad \Lambda\left(\frac{i\pi}{2}\right)=a\left(\frac{i\pi}{2}\right),$$

where

$$\begin{aligned} a(u)=&-4\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\sinh(u-\alpha_-)\cosh(u-\beta_-)\\ &\times\sinh(u-\alpha_+)\cosh(u-\beta_+)\\ &\times\prod_{l=1}^{N}\frac{\sinh(u-\theta_l+\eta)\sinh(u+\theta_l+\eta)}{\sinh^2\eta}. \end{aligned}$$

homogeneous limit

$$[\Lambda(u)\Lambda(u-\eta)]^{(n)}|_{u=0}=[a(u)a(-u)]^{(n)}|_{u=0},$$

parametrization

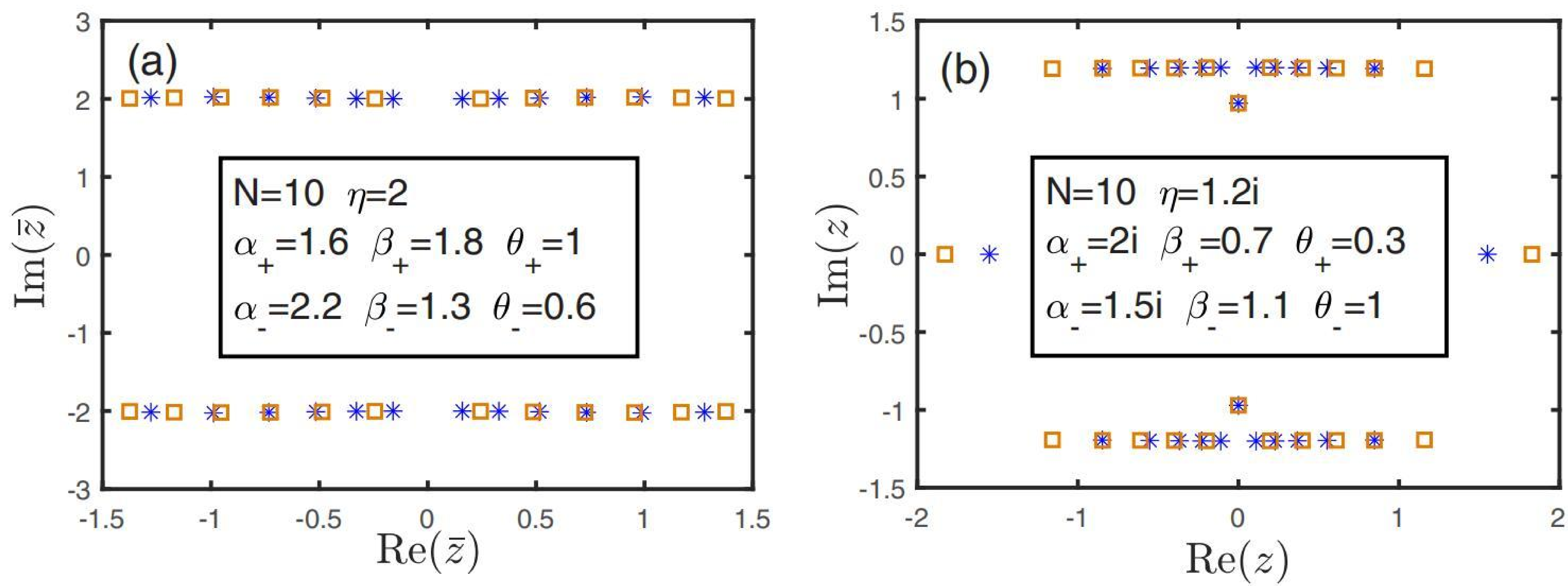
$$\Lambda(u)=\Lambda_0\prod_{j=1}^{N+2}\sinh\left(u-z_j+\frac{\eta}{2}\right)\sinh\left(u+z_j+\frac{\eta}{2}\right).$$

$$\Lambda_0=-8\cos(\theta_--\theta_+)\sinh^{-2N}\eta$$

OBC

Energy $$E = \sinh\eta \sum_{j=1}^{N+2} \left[\coth\left(z_j + \frac{\eta}{2}\right) - \coth\left(z_j - \frac{\eta}{2}\right) \right] - c_0.$$

By choosing a proper set of inhomogeneity parameters, the root distributions possess manageable patterns in the thermodynamic limit.



$\bar{z}_j \equiv -iz_j$

OBC

Energy $E(N, \eta, \alpha_+, \alpha_-, \beta_+, \beta_-, \theta_+, \theta_-)$.

The energy is invariant under the parameter changes:

(i) $\alpha_\pm \to -\alpha_\pm$, (ii) $\beta_\pm \to -\beta_\pm$, (iii) $\alpha_+ \to -\alpha_+, \beta_+ \to -\beta_+, \theta_+ \to \pi + \theta_+$,

(iv) $\alpha_- \to -\alpha_-, \beta_- \to -\beta_-, \theta_- \to \pi + \theta_-$, (v) $\beta_+ \to \beta_-, \beta_- \to \beta_+$.

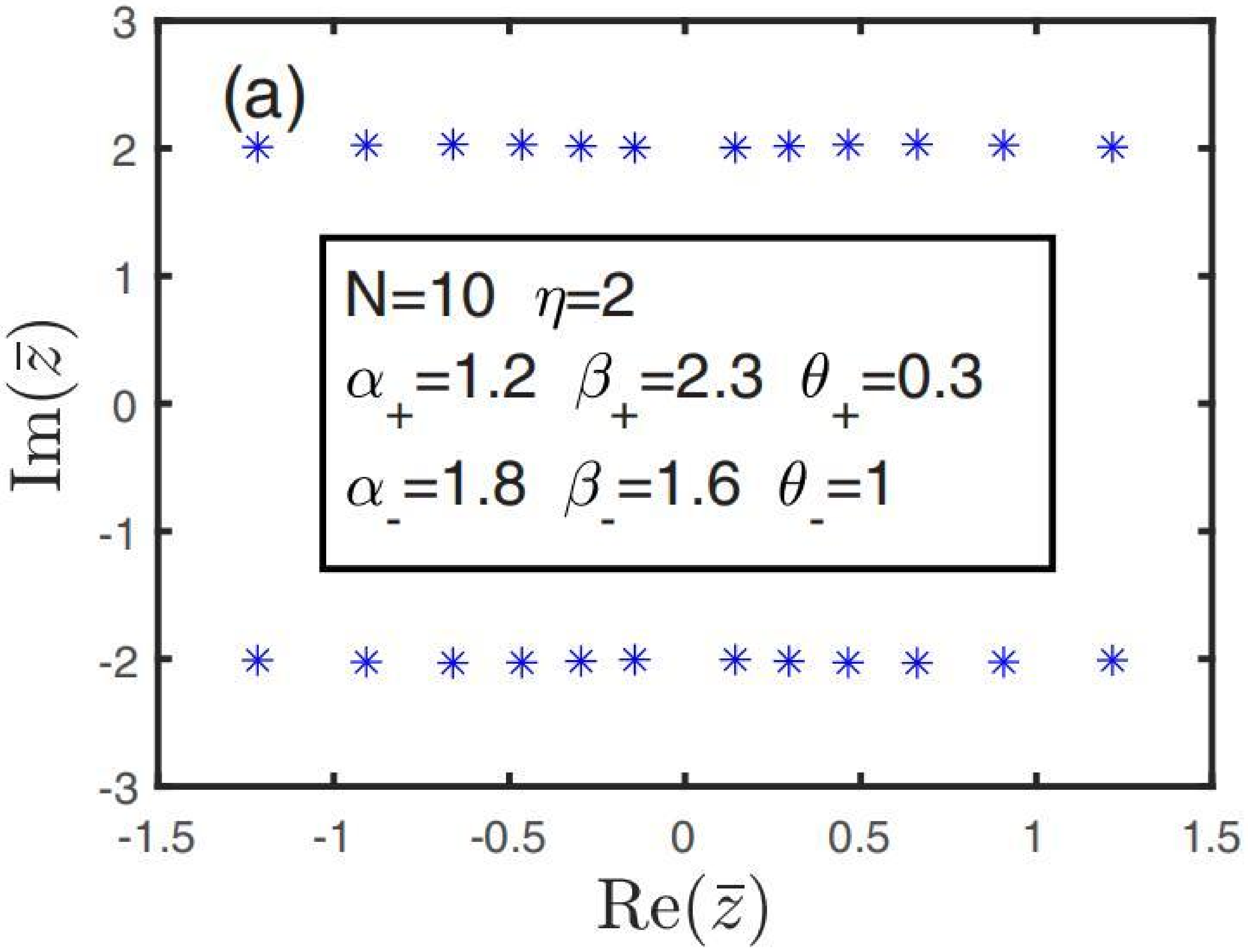
We consider only the case of $\alpha_\pm, \beta_+ > 0$ and $|\beta_+| \geqslant |\beta_-|$.

It is sufficient to quantify the boundary contributions by tuning β_- in four regimes:

(I) $\beta_+ > \beta_- > \eta/2$, (II) $\eta/2 \geqslant \beta_- \geqslant 0$, (III) $0 > \beta_- > -\eta/2$, and (IV) $-\eta/2 \geqslant \beta_- > -\infty$.

OBC

Zero roots for region I



BAEs

$$N\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[b_2(u-\bar\theta)+b_2(u+\bar\theta)]\sigma(\bar\theta)d\bar\theta+b_2\left(u-\frac{\pi}{2}\right)$$
$$+b_2(u)+b_{\frac{2\beta_-}{\eta}}\left(u-\frac{\pi}{2}\right)+b_{\frac{2\beta_+}{\eta}}\left(u-\frac{\pi}{2}\right)+b_{\frac{2\alpha_-}{\eta}}(u)$$
$$+b_{\frac{2\alpha_+}{\eta}}(u)=N\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[b_1(u-\tilde z)+b_3(u-\tilde z)]\rho(\tilde z)\,d\tilde z$$
$$+b_1(u)+b_1\left(u-\frac{\pi}{2}\right),$$

Density of zero roots

$$\tilde\rho(k)=\left[2N\tilde b_2\tilde\sigma(k)+[1+(-1)^k](\tilde b_2-\tilde b_1)+\tilde b_{\frac{2\alpha_+}{\eta}}\right.$$
$$\left.+\tilde b_{\frac{2\alpha_-}{\eta}}+(-1)^k\left(\tilde b_{\frac{2\beta_+}{\eta}}+\tilde b_{\frac{2\beta_-}{\eta}}\right)\right]/[N(\tilde b_1+\tilde b_3)],$$

OBC

Energy

$$E_{b1} = e_b(\alpha_+, \beta_+) + e_b(\alpha_-, \beta_-) + e_{b0},$$

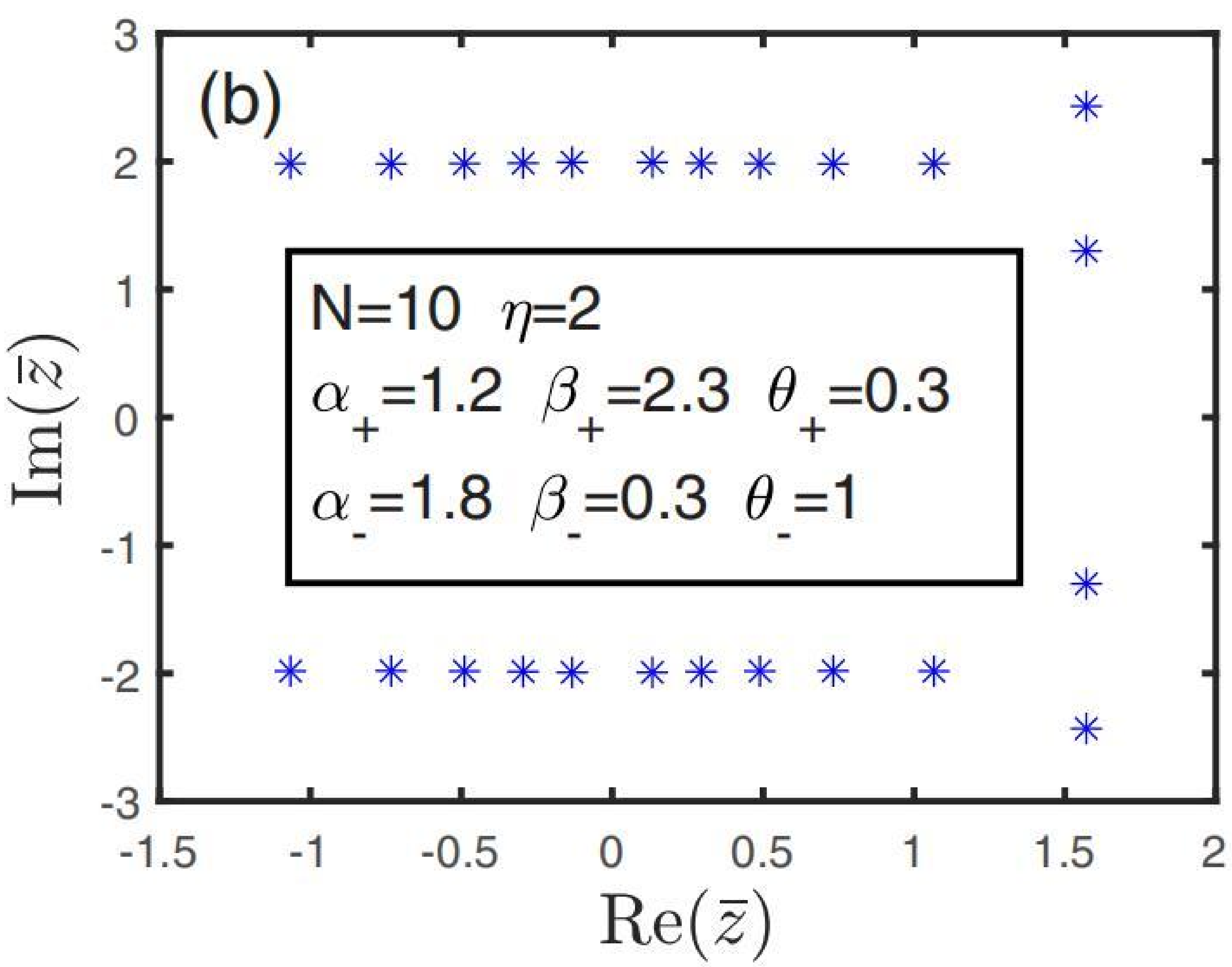
$$e_b(\alpha, \beta) = -2\sinh\eta \sum_{k=1}^{\infty} \tanh(k\eta)\{(-1)^k e^{-2k\eta} + e^{-2k|\alpha|} + (-1)^k e^{-2k|\beta|}\} - \tanh\eta \sinh\eta,$$

$$e_{b0} = -2\sinh\eta \sum_{k=1}^{\infty} \{\tanh(k\eta)[1-(-1)^k]e^{-2k\eta} - [1+(-1)^k]e^{-k\eta}\} + \tanh\eta \sinh\eta,$$

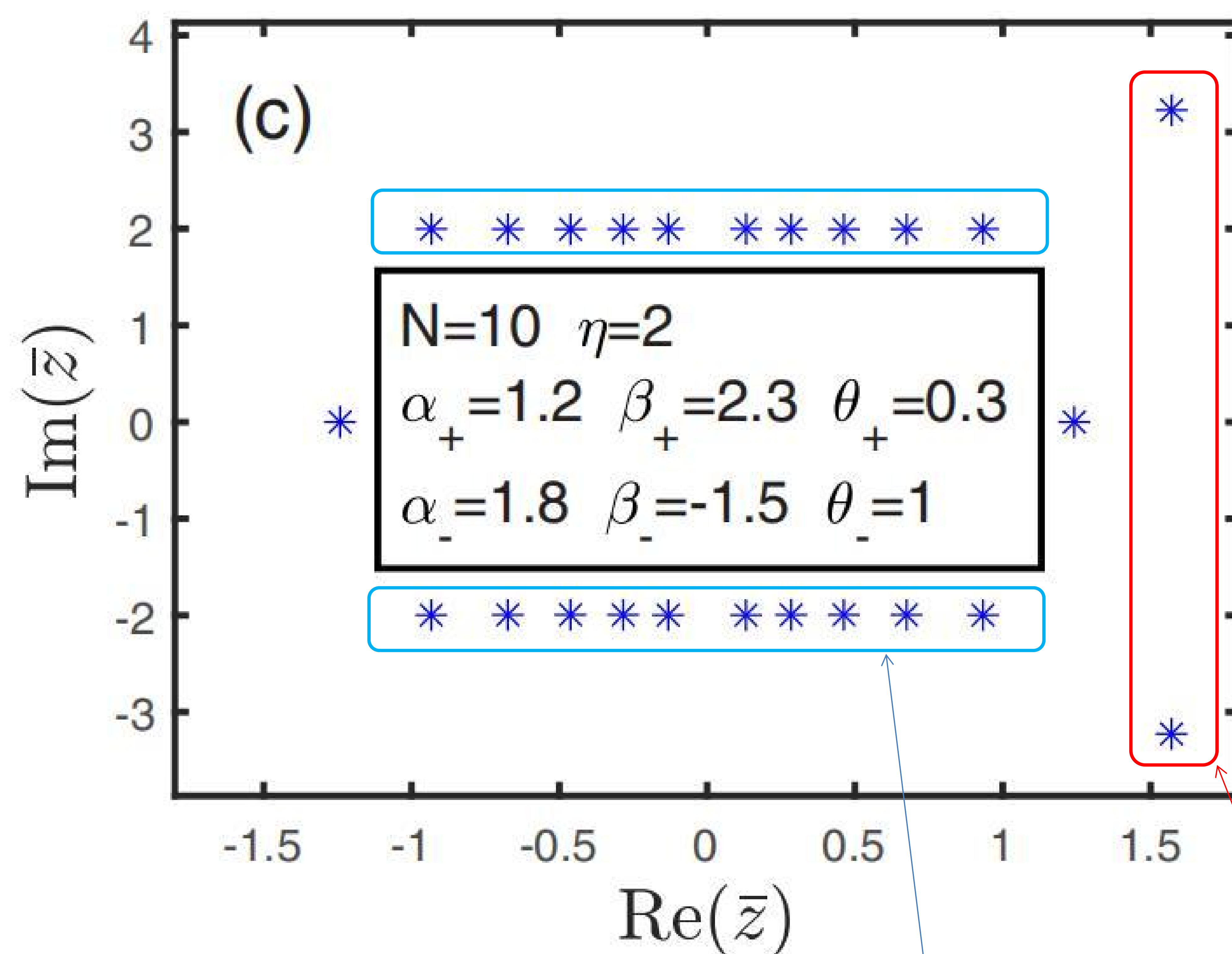
where $e_b(\alpha, \beta)$ indicates the contribution of one boundary field and e_{b0} is the surface energy induced by the free open boundary.

OBC

Zero roots for region II/III



Zero roots for region IV



Boundary string $\frac{\pi}{2} \pm (\beta_- + \frac{\eta}{2})i$ and $\frac{\pi}{2} \pm (\beta_+ + \frac{\eta}{2})i$.

bulk string

boundary string

OBC

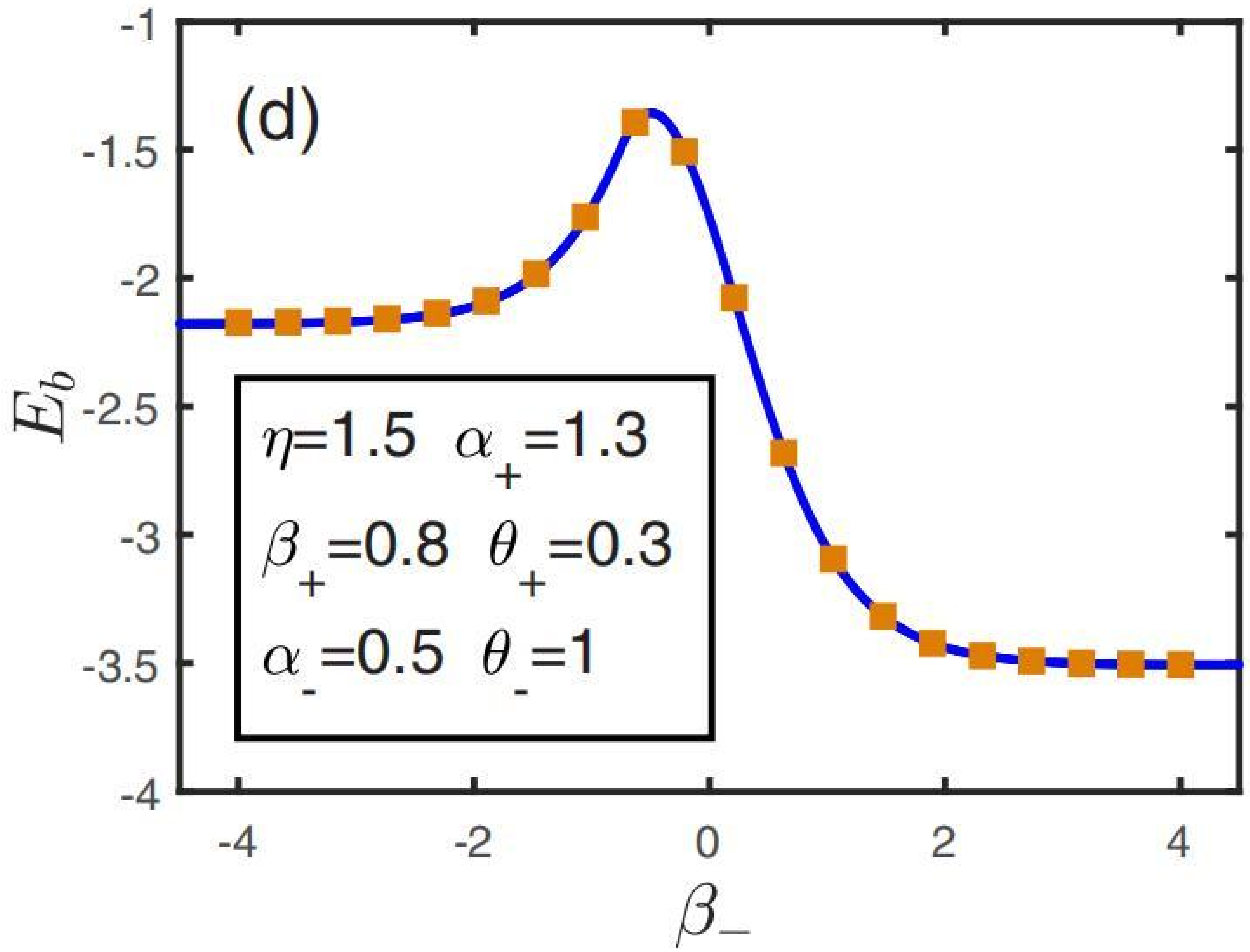
Energy

$$E_{b3} = 4\sinh\eta\sum_{k=1}^{\infty}(-1)^k e^{-k\eta}\tanh(k\eta)\cosh(2k\beta_- + k\eta) + E_{b1} + \sinh\eta[\tanh(\beta_- + \eta) - \tanh(\beta_-)].$$

$$E_{b4} = E_{b1} + E_h,$$

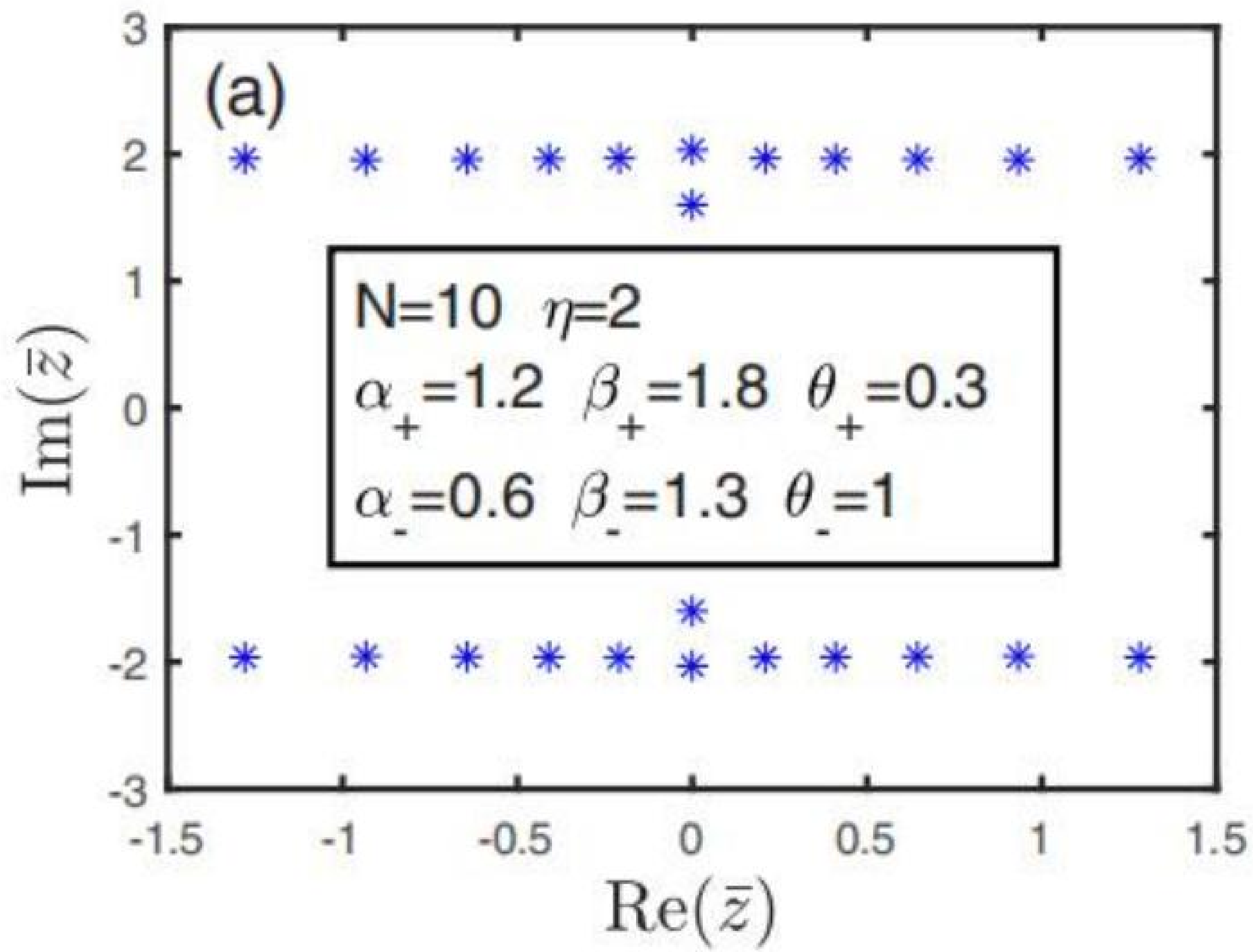
$$E_h = 2\sinh\eta\left[\sum_{k=1}^{\infty}\frac{(-1)^k 2\tanh(k\eta)}{e^{k\eta}} + \tanh\frac{\eta}{2}\right].$$

DMRG results

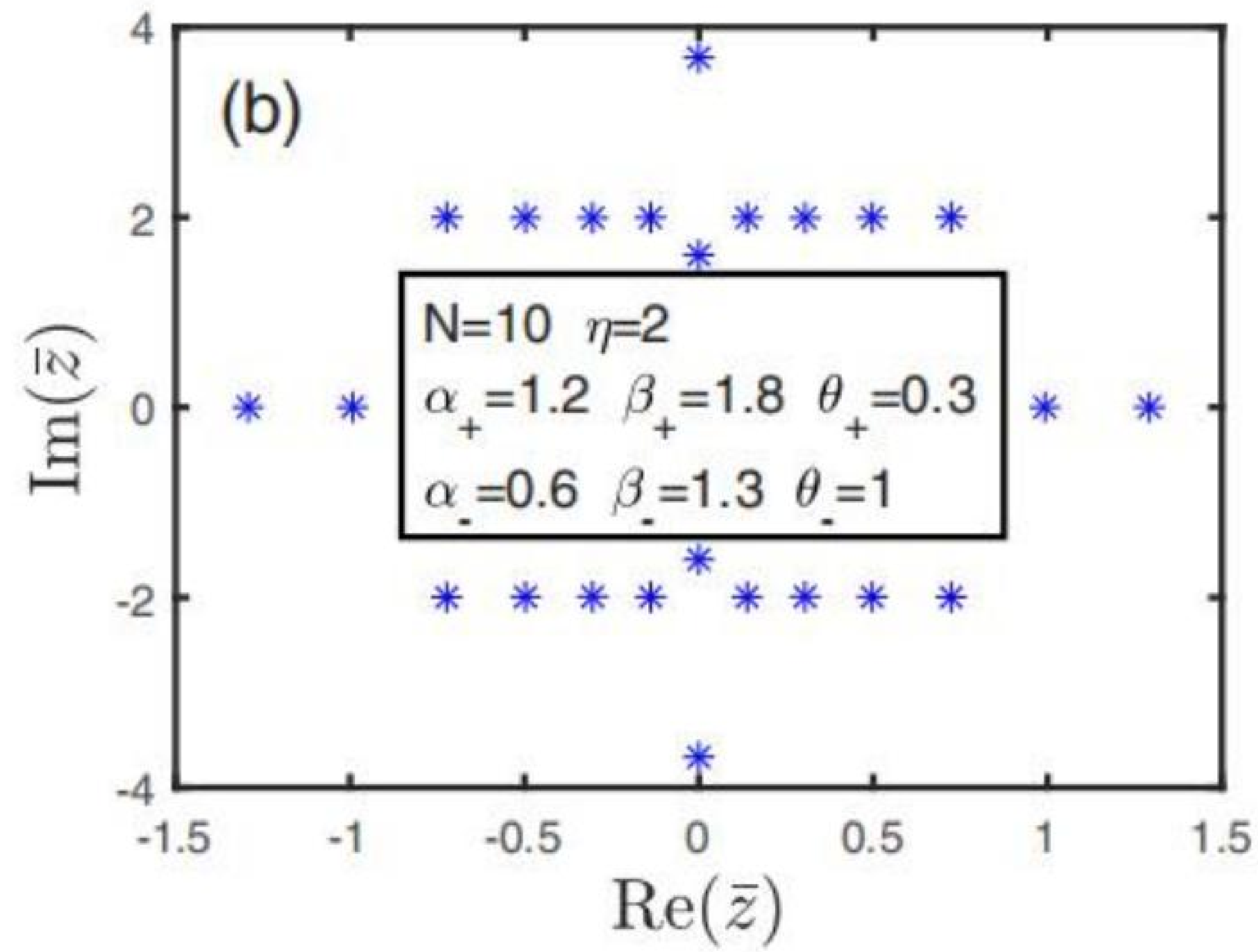


OBC

GS



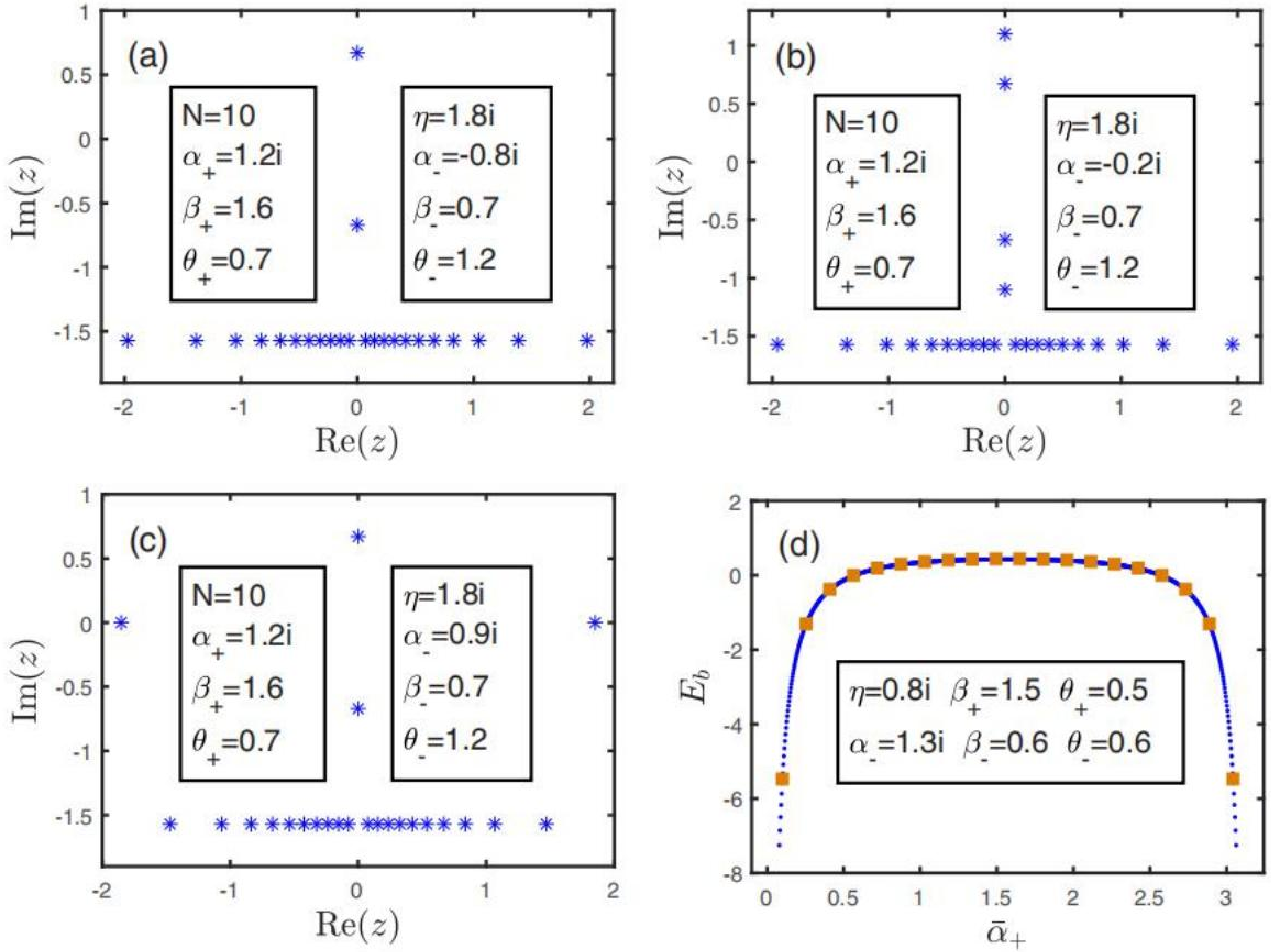
EE



energy for EE $E_e = \varepsilon(z_1) + \varepsilon(z_2), \quad \varepsilon(z) = 2\sinh\eta \sum_{k=-\infty}^{\infty} \frac{e^{-2ikz}}{\cosh(k\eta)}.$

OBC

η∈i R



Energy for GS

$$E_b = -\frac{\sin\gamma}{2}\int_{-\infty}^{\infty}\frac{\tanh(\frac{k\gamma}{2})}{\sinh(\frac{k\pi}{2})}\{\cosh\frac{k(\pi-2\gamma)}{2} - 1 + \cosh\frac{k(\pi-2\bar{\alpha}_+ + 2\pi\lfloor\frac{\bar{\alpha}_+}{\pi}\rfloor)}{2} + \cos\beta_+$$

$$+\cosh\frac{k(\pi-2\bar{\alpha}_- + 2\pi\lfloor\frac{\bar{\alpha}_-}{\pi}\rfloor)}{2} + \cos\beta_- - \cosh\frac{k\gamma}{2} - \cosh\frac{k(\pi-\gamma)}{2}\}dk.$$

Thermo-dynamics

thermodynamics

1969 The quantum Bose gas

— Thermal Bethe ansatz

1971 Heisenberg spin chain

infinitely many nonlinear integral equations (NLIEs)

1992 Heisenberg spin chain — quantum transfer matrix (QTM)

finite number of NLIEs, SU(n): 2^n-2

1995 numerical method — transfer-matrix renormalization group (TMRG)

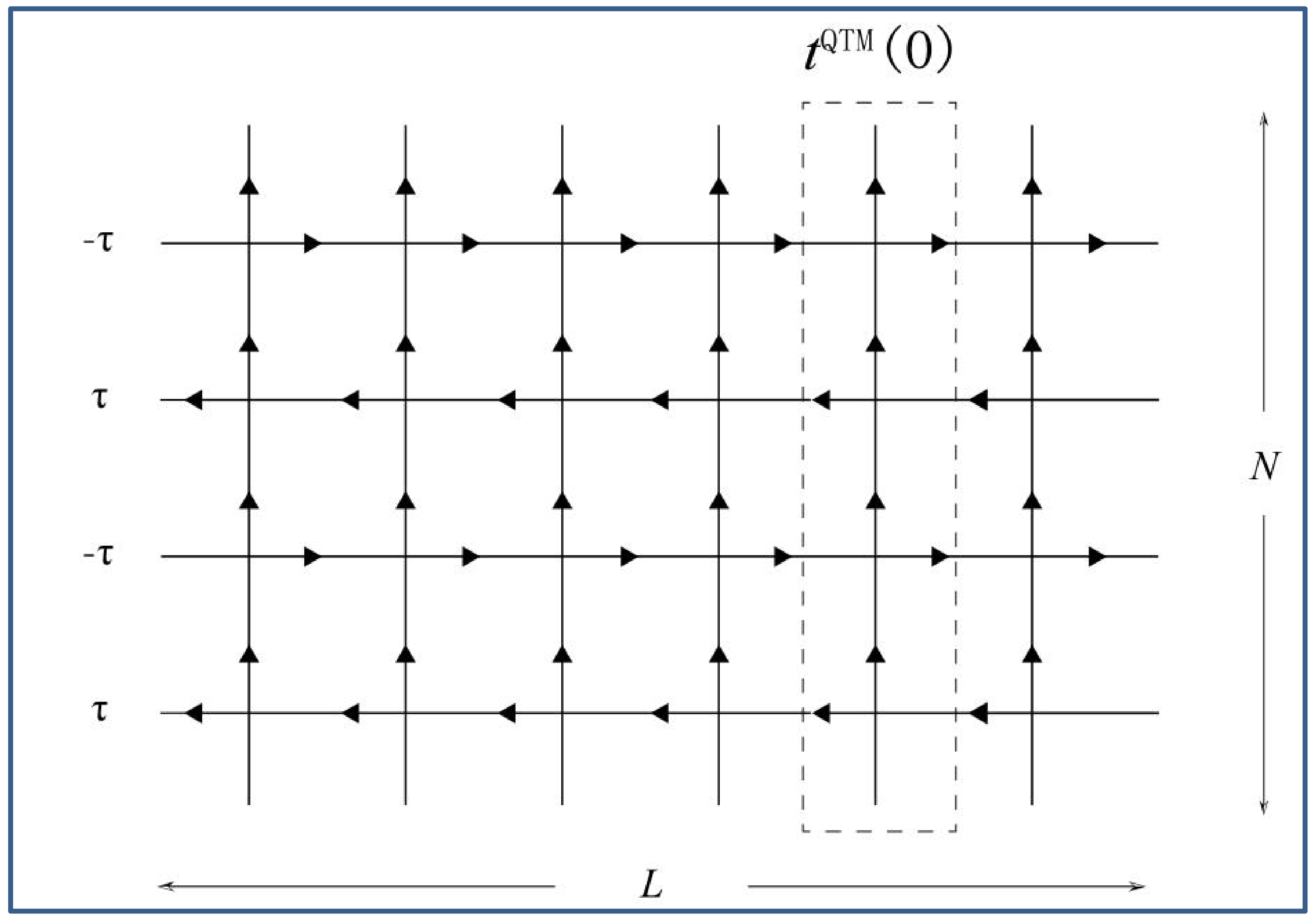
Thermo-dynamics

A one-dimensional quantum system at a finite temperature can be mapped into a classical system on two-dimensional inhomogeneous lattice by the Trotter-Suzuki mapping.

Suzuki M, Phys. Rev. B, 1985, 31(5): 2957.

QTM $t^{(Q)}(u)$

$$Z(\beta) = e^{\beta JL} \lim_{L\to\infty} \lim_{N\to\infty} tr_{1,\cdots,N}\left\{\left\{t^{(Q)}(0)\right\}^{L}\right\}$$



$\Lambda^{(Q)}(0)_{max}$? **t-W method**

Thermo-dynamics

Hamiltonian of the periodic Heisenberg spin chain in anti-ferromagnetic regime

$$H = 4J \sum_{j=1}^{L} \vec{S}_j \cdot \vec{S}_{j+1} + h \sum_{j=1}^{L} S_j^z$$

where $\vec{S}_{L+1} = \vec{S}_1$

R-matrix

$$R_{0j}(u) = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}$$

Thermo-dynamics

Quantum transfer matrix

$$t^{(Q)}(u) \quad = \quad tr_0\{e^{\frac{h\beta}{2}\sigma_0^z}\left(R_{0\,N}(u - \frac{2\eta J\beta}{N})\,R_{0\,N-1}(u + \frac{2\eta J\beta}{N} - \eta)\right)\ldots$$
$$\times(R_{0\,2}(u - \frac{2\eta J\beta}{N})\,R_{0\,1}(u + \frac{2\eta J\beta}{N} - \eta)\}$$

The transfer matrix t(u) satisfies the t − W relation, and the corresponding relation for their eigenvalues is

$$\boxed{\Lambda^{(Q)}(u)\,\Lambda^{(Q)}(u-\eta) = a(u)\,d(u-\eta) + e^{\frac{h\beta}{2}}\,d(u)\,W^{(Q)}(u)}$$

Thermo-dynamics

Express any eigenvalue Λ(u) of the transfer matrix (or W(u) of the fused one) in terms of its N zero points

$$\Lambda^{(Q)}(u) = 2\cosh\frac{h\beta}{2}\prod_{j=1}^{N}(u - z_j)$$

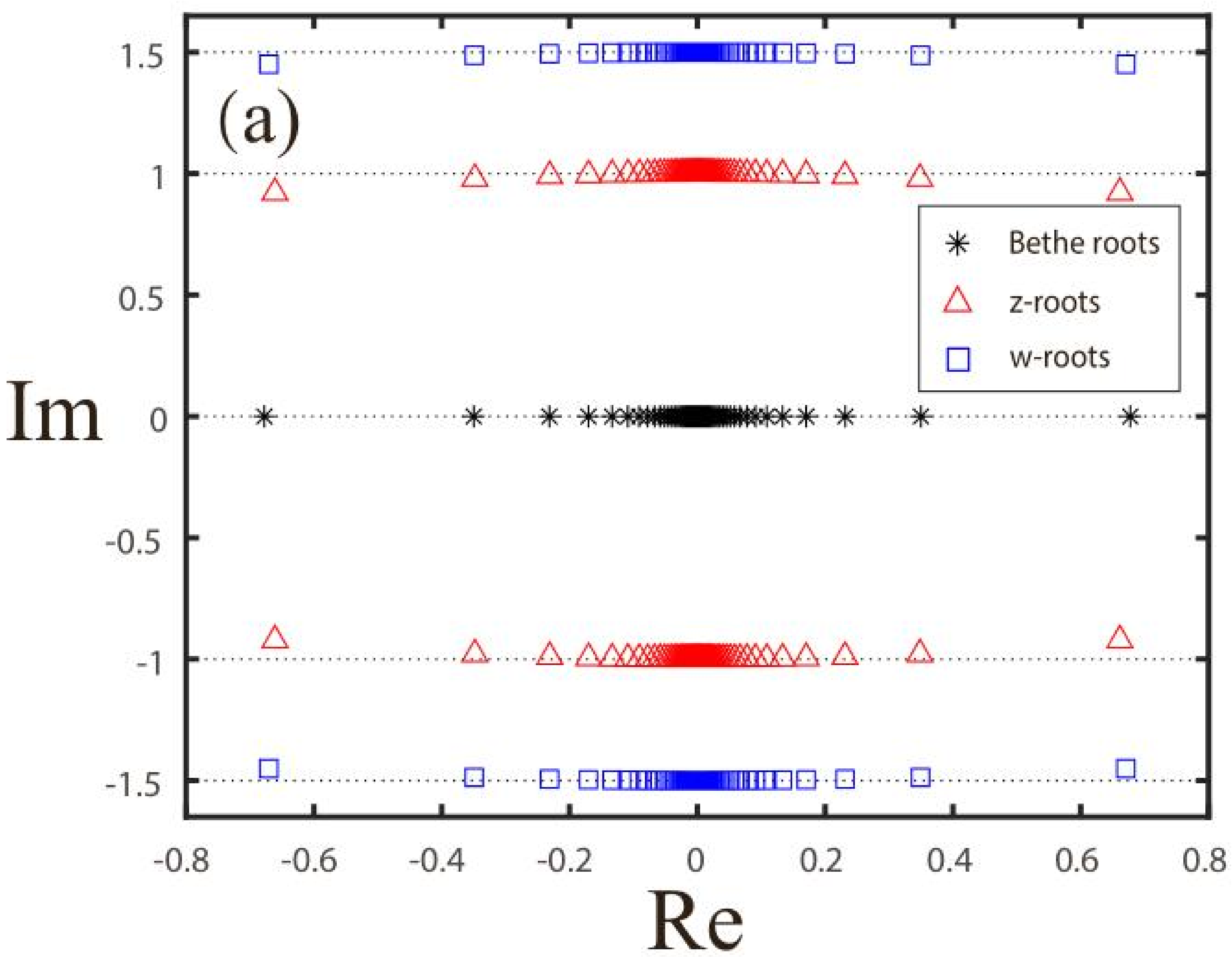
$$W^{(Q)}(u) = (4\cosh^2\frac{h\beta}{2} - 1)\prod_{j=1}^{N}(u - w_j)$$

BAEs

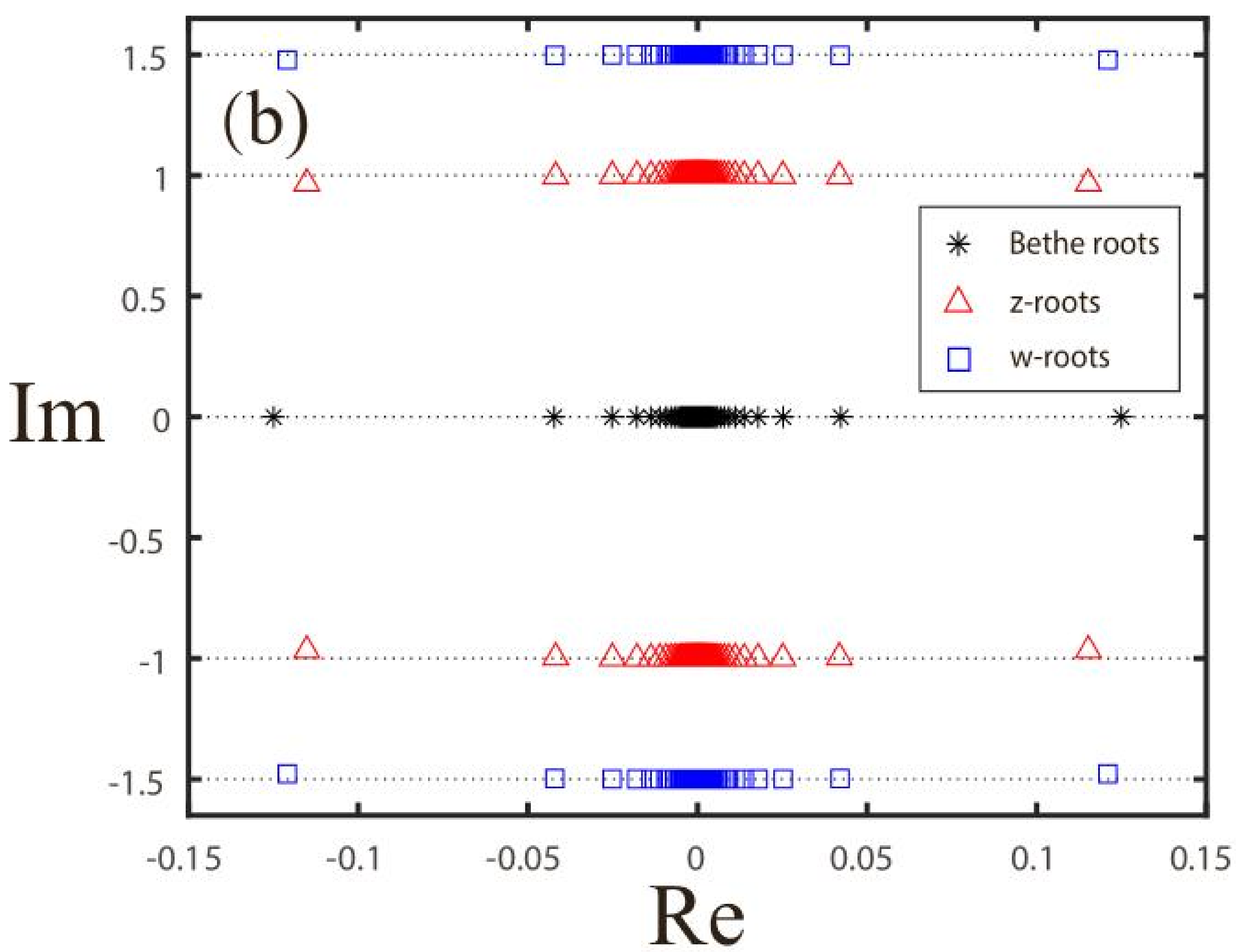
$$a(z_j)\,d(z_j-\eta) = -e^{\frac{h\beta}{2}}d(z_j)\,W^{(Q)}(z_j), \qquad j = 1,\cdots,N$$

$$a(w_j)\,d(w_j-\eta) = \Lambda^{(Q)}(w_j)\Lambda^{(Q)}(w_j-\eta), \quad j = 1,\cdots,N$$

The distributions of the Bethe roots and zeros



$T = 0.5$



$T = 5$

Thermo-dynamics

t − W relation：

$$
\begin{aligned}
&\bar{\Lambda}^{(Q)}(u+\frac{\eta}{2})\bar{\Lambda}^{(Q)}(u-\frac{\eta}{2}) \\
&= \frac{(u+\eta\tau+\frac{\eta}{2})^M(u-\eta\tau-\frac{\eta}{2})^M}{(u-\eta\tau+\frac{\eta}{2})^M(u+\eta\tau-\frac{\eta}{2})^M} \\
&\quad +(4\cosh^2\frac{h\beta}{2}-1)\frac{\prod_{j=1}^M(u-w_j^{(+)}-\frac{3}{2}\eta)(u-w_j^{(-)}+\frac{3}{2}\eta)}{(u+\eta\tau-\frac{3}{2}\eta)^M(u-\eta\tau+\frac{3}{2}\eta)^M} \\
&\overset{\text{def}}{=} q(u)+(4\cosh^2\frac{h\beta}{2}-1)\bar{w}(u)+O(\frac{1}{N})
\end{aligned}
$$

Integral representation

$$
\begin{aligned}
\ln\bar{\Lambda}^{(Q)}(u) \;=\; &\ln 2\cosh\frac{h\beta}{2}+\frac{1}{2\pi i}\oint_{\mathcal{C}_1}dv\,\frac{\ln\left((q(v)+(4\cosh^2\frac{h\beta}{2}-1)e^{-\beta\bar{\epsilon}(v)})/4\cosh^2\frac{h\beta}{2}\right)}{u-v-\frac{\eta}{2}} \\
&+\frac{1}{2\pi i}\oint_{\mathcal{C}_2}dv\,\frac{\ln\left((q(v)+(4\cosh^2\frac{h\beta}{2}-1)e^{-\beta\bar{\epsilon}(v)})/4\cosh^2\frac{h\beta}{2}\right)}{u-v+\frac{\eta}{2}}
\end{aligned}
$$

New NLIE

$$\ln(q(u)+(4\cosh^2\frac{h\beta}{2}-1)e^{-\beta\bar{\epsilon}(u)}) = 2\ln 2\cosh\frac{h\beta}{2}$$

$$+\frac{1}{2\pi i}\oint_{\mathcal{C}_1} dv(\frac{1}{u-v}+\frac{1}{u-v-\eta})\ln\left((q(v)+(4\cosh^2\frac{h\beta}{2}-1)e^{-\beta\bar{\epsilon}(v)})/4\cosh^2\frac{h\beta}{2}\right)$$

$$+\frac{1}{2\pi i}\oint_{\mathcal{C}_2} dv(\frac{1}{u-v+\eta}+\frac{1}{u-v})\ln\left((q(v)+(4\cosh^2\frac{h\beta}{2}-1)e^{-\beta\bar{\epsilon}(v)})/4\cosh^2\frac{h\beta}{2}\right)$$

Thermo-dynamics

Free energy

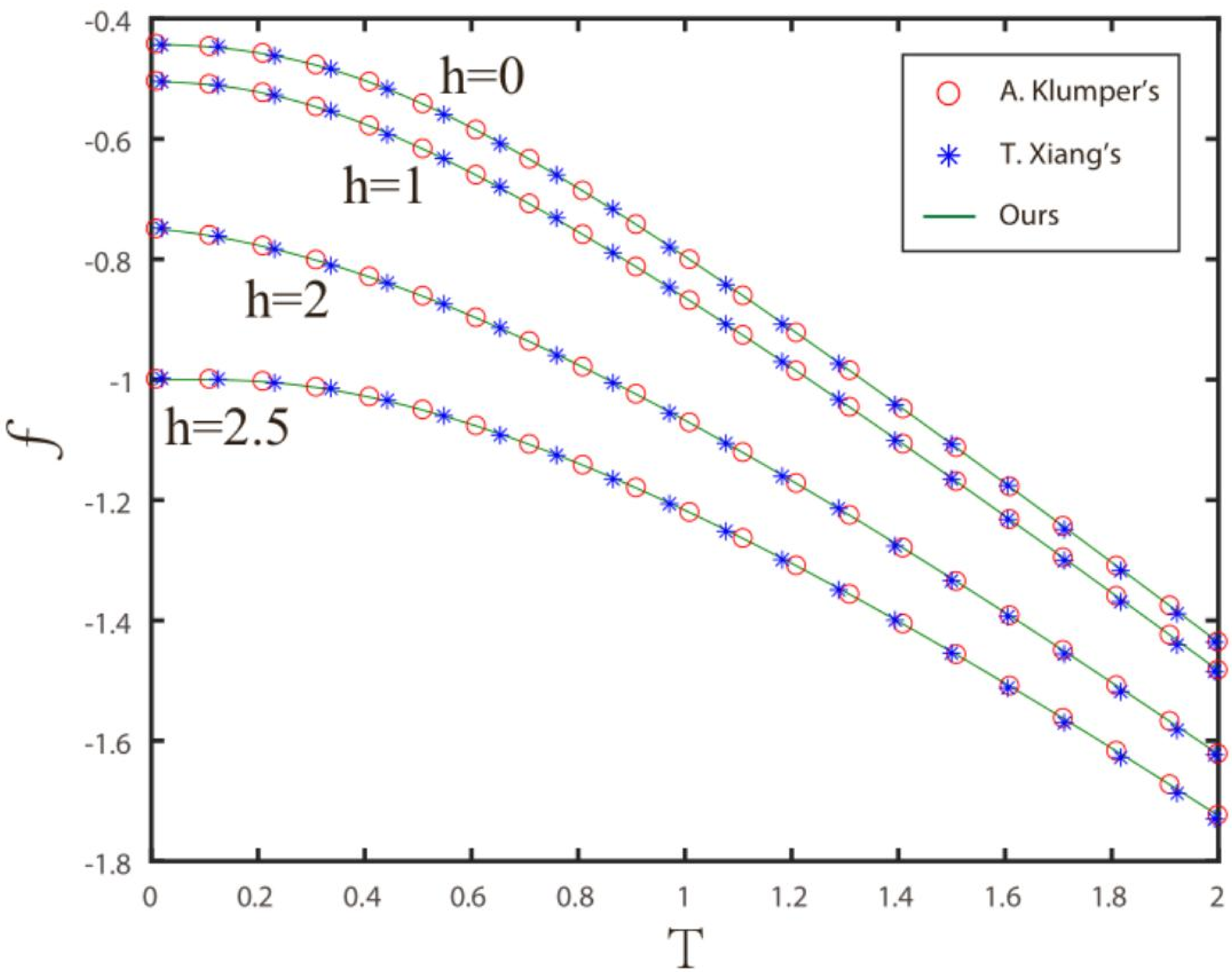
$$
\begin{aligned}
f(\beta) &= J - \frac{1}{\beta} \ln \bar{\Lambda}^{(Q)}(0) \\
&= e_g - \frac{1}{\beta} \int_{-\infty}^{+\infty} \frac{dv}{2\cosh \pi v} \ln\left(1 + (4\cosh^2 \frac{h\beta}{2} - 1) e^{-\beta \epsilon(v)}\right)
\end{aligned}
$$

High-temperature expansions

$$
f/T = -\ln(2\cosh(h/T)) - \frac{J}{T}\tanh^2(h/T) - \frac{3J^2}{2T^2}(1 - \tanh^4(h/T)) + \cdots
$$

Thermo-dynamics

Free energy f vs T for the closed XXX chain in different magnetic fields



热力学

t-W relation for SU(n) spin chain

$$t_m^{(Q)}(u)\, t_m^{(Q)}(u-\eta) = t_{m-1}^{(Q)}(u-\eta)\, t_{m+1}^{(Q)}(u) + a_m(u)\mathbb{W}_m^{(Q)}(u), \qquad m = 1, \cdots, n-1.$$

2n-2 auxiliary functions

Fujii A and Klumper A, Nucl. Phys B, 1999, 546(3): 751-764.

2n - 2 auxiliary functions

总结

So far, many typical U(1)-symmetry-broken models have been solved by the method:

- The spin-1/2 Heisenberg chain with arbitrary boundary fields.
- Integrable J_1-J_2 model.
- The t-J model with unparallel boundary fields.
- The Hubbard model with unparallel boundary fields.
- The open spin chains associated with the $D_2^{(1)}$ and $D_2^{(2)}$ algebras.

Thanks!