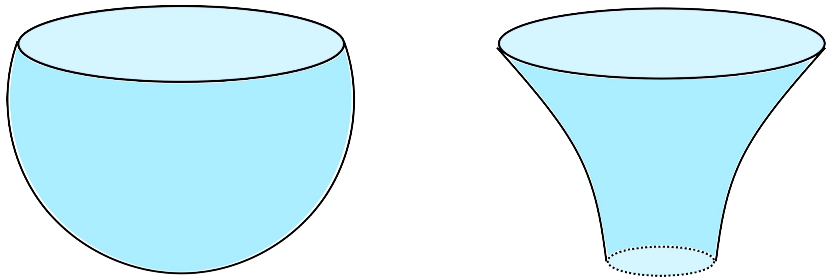
On Half-wormhole



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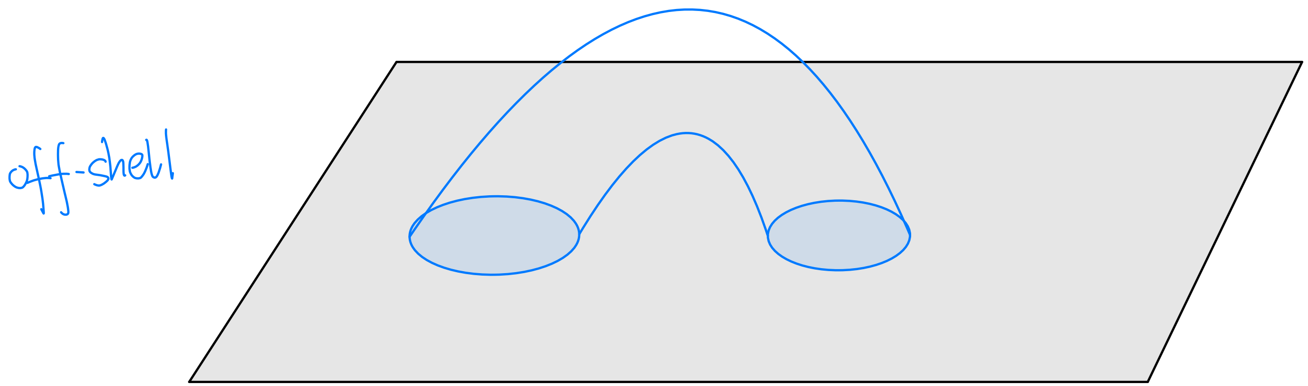
Based on arXiv: 2111.14856, 2205.01288 and oncoming
with Cheng Peng, Yingyu Yang and Jianghui Yu

Wormholes & Superselection sectors



Ordinary wormholes

(Coleman, Giddings and Strominger '88 '89)

non-local interaction :

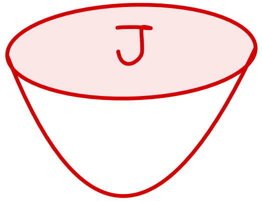
$$S_{WH} = -\frac{1}{2} \sum_{I,J} \int d^D x \, d^D y \, \mathcal{O}_I(x) \, \mathcal{O}_J(y)$$

$$\sim \int [d\alpha] \; e^{-\frac{1}{2}\alpha_I C_{IJ} \alpha_J} \; e^{-\int d^D x \sum_I \alpha_I \mathcal{O}_I(x)}$$

"localized" by introducing random couplings α, free para.

Wormholes in AdS/CFT

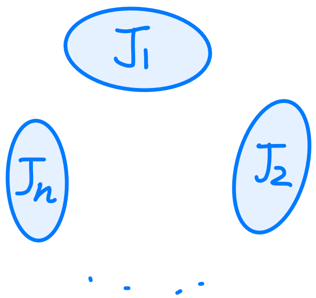
'04 Maldacena, Maoz
'20 Marolf Maxfield

AdS/CFT : $\int_{\Phi\sim J} D\Phi$ 

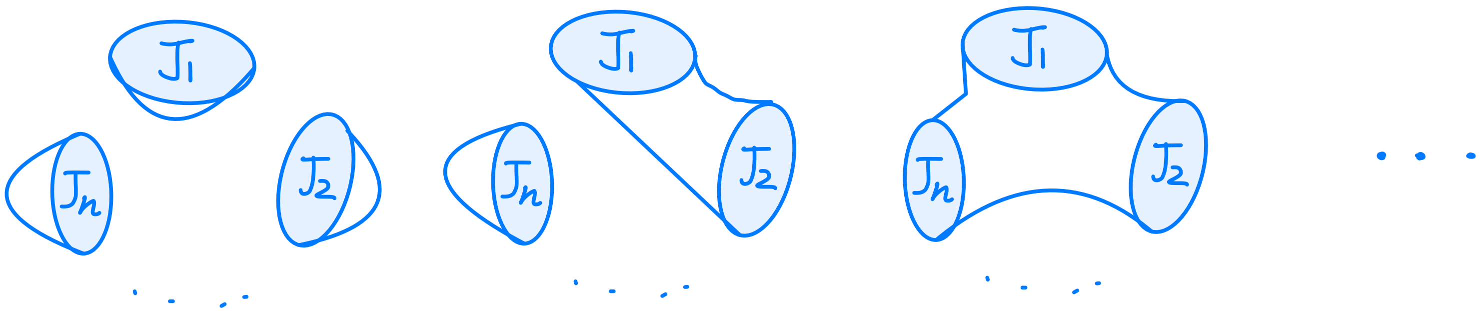
$= Z[J]$

gravitational path integral with bdy J ~ a partion function of dual CFT

$\Downarrow$ generalize the dictionary

$\int_{\Phi\sim J_i} D\Phi$ 

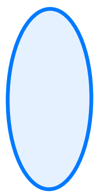
$= \langle Z[J_1] \cdots Z[J_N] \rangle$ ensemble average



on-shell

2D Examples

- Jakiw-Teiteboim (JT) gravity as a matrix integral
  ('19 Saad, Shenker, Stanford)



β

$\Longleftrightarrow$ operator in a RMT

$\mathrm{Tr}\, e^{-\beta H}$

$$\int_{\beta} [dg_{ij}] = \int dH$$

- This correspondence may be true for all 2d dilaton gravities
  ('20 Witten)

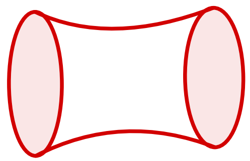
- Both perturbative and non-perturbative results match!
  AN EXACT CORRESPONDENCE!

- **Factorization Problem**

AdS / CFT or AdS / ENSEMBLE ?

$Z(\beta_1) Z(\beta_2) = Z(\beta_1) \times Z(\beta_2)$

$Z(\beta_1) Z(\beta_2) \neq Z(\beta_1) \times Z(\beta_2)$

 $\neq 0$

- If both AdS / CFT and AdS / ENSEMBLE are CORRECT, then there is the so called factorization problem.

- Of course AdS / CFT is correct and AdS / ENSEMBLE is just some approximation. The question is how factorization is restored!

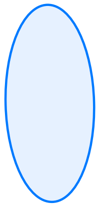
Outline

- Factorization in CGS model.
- Factorization in 0-SYK model.
- Their relation and generalization.

• Factorization in CGS model

'21 Saad, Shenker and Yao

• Marolf & Maxfield's topological surface theory

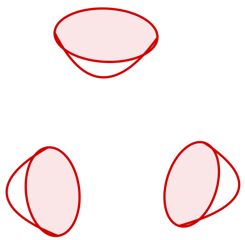
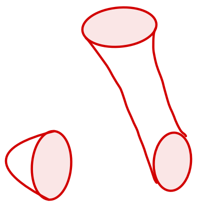
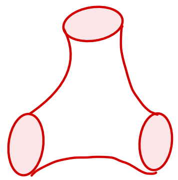
 $\equiv Z$: the bdy condition of a circular bdy.

The general gravitational amplitude is

$$\langle Z^n \rangle = \sum_{\substack{M \text{ with} \\ n\text{-bdy}}} \mu(M)\, e^{S_0 \chi(M)} \Rightarrow \# \sum_{\substack{\text{connected} \\ M'}}$$

$$\sum_{\substack{\text{connected} \\ M'}} \times = \sum_{g} e^{-S_0(2g-2)} \equiv \lambda$$

some constant for any connected surface

$\langle Z^3 \rangle =$  $+\ 3$  $+$  $= \lambda^3 + 3\lambda^2 + \lambda = B_3(\lambda)$

Bell polynomial

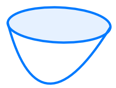
The dual ensemble theory: ensemble of theories with $H=0$

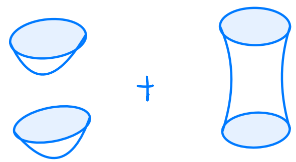
$$\langle Z^n \rangle = \sum_{d=1}^{\infty} e^{-\lambda}\frac{\lambda^d}{d!}\, d^n = \sum_{d=1}^{\infty} \mathrm{Pois}_\lambda(d)\, Z_d^n(d)$$

$\mathrm{Tr}(1) = \dim \mathcal{H}$

- The CGS model: disk-and-cylinder approximation of MM
  (Coleman-Giddings-Strominger)

$$\begin{cases} Z_1 = \mathrm{Disk} = e^{S_0} = \\ Z_{2,c} = \mathrm{Cyl} = 1, \quad Z_{k,c} = 0 \\ Z_2 = \mathrm{Disk}^2 + \mathrm{Cyl} = e^{2S_0} + 1 = \end{cases}$$





- The Hilbert space of closed universe

Different basis

1) Z basis

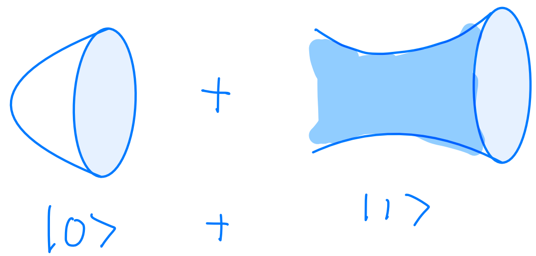
$$\begin{cases} \langle Z^n \rangle = \langle NB | Z^n | NB \rangle \\ |Z^k\rangle = \hat{Z}^k |NB\rangle \end{cases}$$

2) N basis

From $\langle Z \rangle = \langle NB | Z \rangle$ we can define one-closed universe state

$|1\rangle \sim (\hat{Z} - \text{Disk}) |NB\rangle$

$|Z\rangle \sim$



In general, $\hat{Z} - \text{Disk} = a + a^{+}$

$|n\rangle \sim (a^{+})^n |NB\rangle$

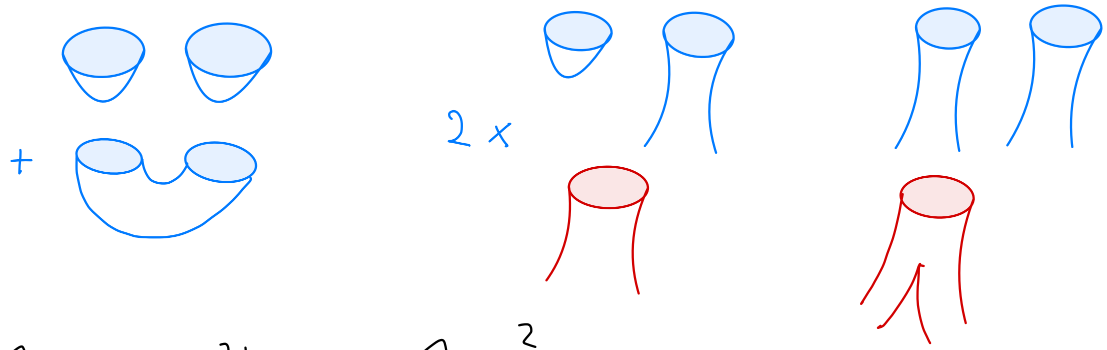
3) α basis

$\hat{Z}|\alpha\rangle = Z_\alpha |\alpha\rangle$, the eigenstates.

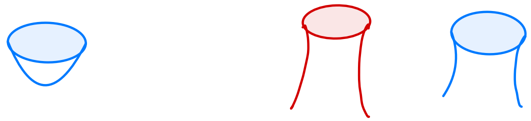
Proposal of factorization

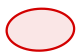
Note that $\langle \alpha | \hat{Z}^2 | \alpha \rangle = Z_\alpha^2 = \langle \alpha | \hat{Z} | \alpha \rangle^2$

and $\langle \alpha | \hat{Z}^2 | \alpha \rangle = \langle \alpha^2 | Z^2 \rangle = \langle \alpha^2 | 0 \rangle \langle 0 | Z^2 \rangle + \langle \alpha^2 | 1 \rangle \langle 1 | Z^2 \rangle + \langle \alpha^2 | 2 \rangle \langle 2 | Z^2 \rangle$

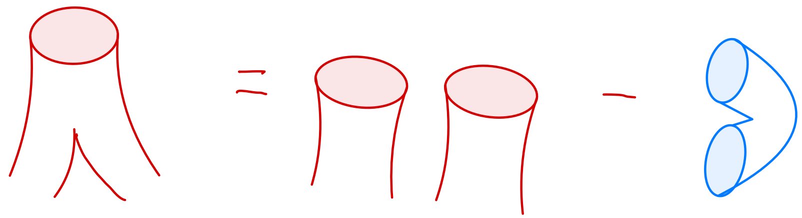


$= \langle \alpha | \hat{Z} | \alpha \rangle^2 = \left(\langle \alpha^2 | 0 \rangle \langle 0 | \hat{Z} \rangle + \langle \alpha^2 | 1 \rangle \langle 1 | \hat{Z} \rangle \right)^2$



$\Rightarrow$ { adding  (approximated by a random bdy condition)

$\langle \alpha^2 | 2 \rangle = \langle \alpha^2 | 1 \rangle^2 - \mathrm{Cyl}$



Factorization in 0-SYK

- 1- SYK model

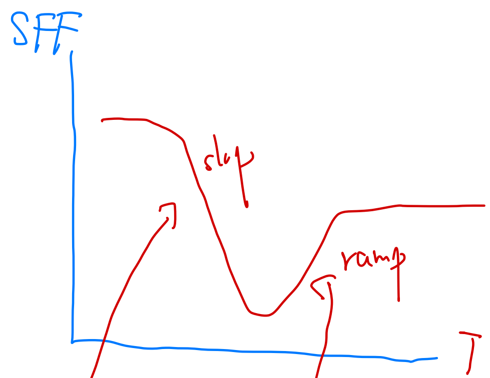
$$H = \sum J_{i_1 \cdots i_q} \psi^{i_1} \psi^{i_2} \psi^{i_3} \cdots \psi^{i_q} = \sum J_A \psi^A \qquad A \equiv i_1 \cdots i_q$$

$$\langle J_A \rangle = 0 \,, \quad \langle J_A J_B \rangle = \delta_{AB} \frac{(q-1)!}{N^{q-1}}$$

$$\begin{cases} Z = \int DG D\Sigma \, e^{-IN} \qquad I = -\frac{1}{2} \log \det (\partial_\tau - \Sigma) + \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 \left(\Sigma G - \frac{1}{q} G^q \right) \\ G(\tau_1 \tau_2) = \frac{1}{N} \sum_{i=1}^N \psi_i(\tau_1) \psi_i(\tau_2) \end{cases}$$

Spectral form factor

$$Z_L(\beta + iT) Z_R(\beta - iT) = \int DG D\Sigma \, e^{-N I(G_{LL}, G_{RR}, G_{LR})}$$



Two saddle solutions
$\begin{cases} \text{disconnected} & G_{LR} = 0, \quad G_{LL} = G_{RR} \\ \text{connected (wormhole)} & G_{LL} = G_{RR} = 0, \quad G_{LR} \neq 0 \end{cases}$

('19 Saad, Shenker & Stanford)

· 0-SYK (without averaging) '21 Saad, Shenker, Stanford & Yao

$$Z = \int d^N\psi \; e^{i^{q/2} \sum J_A \psi_A}$$ ψ_{i_1} is a Grassman number

$$= \mathrm{PF}(J)$$

$$Z_L Z_R = \mathrm{PF}(J)^2 = \int d^{2N}\psi \; e^{i^{q/2} \sum J_A (\psi_A^L + \psi_A^R)}$$

a trick motivated by averaged theory

$$= \int d^{2N}\psi \int dG_{LR}\, \underbrace{\delta\left(G_{LR} - \frac{1}{N}\sum \psi^L\psi^R\right)}_{\int d\Sigma_{LR}\, e^{i\Sigma_{LR}\left(G_{LR} - \frac{1}{N}\sum\psi^L\psi^R\right)}} e^{i^{q/2}\sum J_A(\psi_A^L + \psi_A^R)} \; e^{\frac{N}{q}\left(G_{LR} - \frac{1}{N}\sum\psi^L\psi^R\right)}$$

$$= \underbrace{\int d\Sigma \; dG \; e^{-NI(G,\Sigma)}}_{\langle Z^2\rangle} \; \Phi(\Sigma, J_A, \psi_A) \; e^{-N\log\Sigma}$$

$$\langle Z^2 \rangle \approx Z^2(G_{LR}^*, \Sigma_{LR}^*)$$ $(G_{LR}^*, \Sigma_{LR}^*)$ is the wormhole saddle

① check $\Phi(\Sigma_{LR}^*) \sim \langle \Phi(\Sigma_{LR}^*) \rangle_J = e^{N\log\Sigma^*}$

$\Rightarrow Z_L Z_R \approx \langle Z^2 \rangle + \cdots$ wormhole persists!

② check $\int dG\, e^{-N I(G,\Sigma)}\, e^{-N\log\Sigma}$ is peaked at $\Sigma = 0$

$\Rightarrow \quad Z_L Z_R \approx \Phi(0, J_A, \psi_A)$ half-wormhole exists!

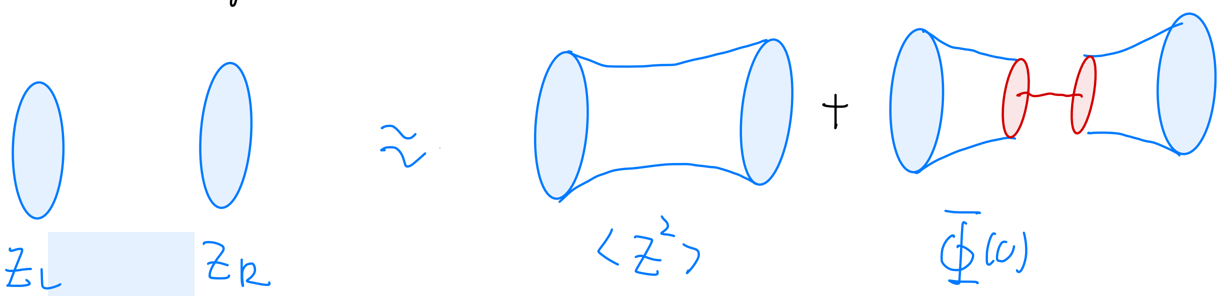
(Linked)

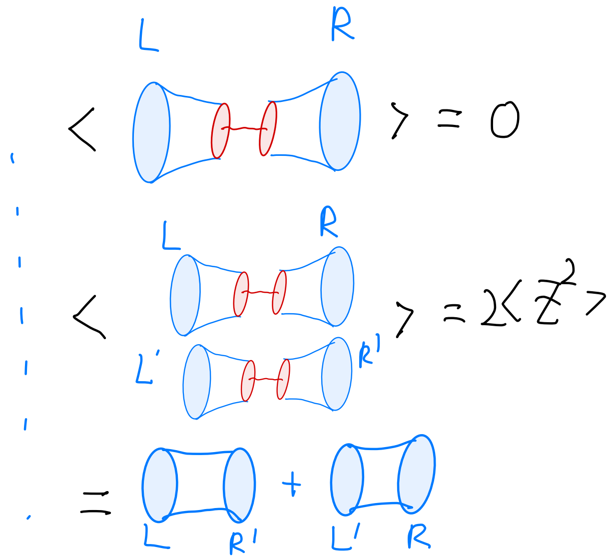
③ combine and check

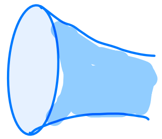
$Z_L Z_R \approx \langle Z^2 \rangle + \Phi(0)$ is good in the sense of the error

$$\frac{\langle (Z_L Z_R - \langle Z^2\rangle - \Phi(0))^2 \rangle}{\langle Z^4 \rangle} \ll 1$$

In summary:





Q_1: Can we find  (the single half-WH)

Yes! ('21 Cheng Peng, JT & Yingyu Yang)

But to find non-trivial result we require $\langle J_A \rangle = \mu$.

We find the new $\boxed{G_{h(D)} = \frac{1}{N} \sum_{i<j} \psi_i \psi_j}$ and $\Phi_h \equiv \Theta(\Sigma_h, J_A, \psi)$

such that

$$Z \simeq \langle Z \rangle + \Theta(0, J_A, \psi)$$

Q_2: Is the half-WH always given by $\Phi(0)$?

No! When $\langle J_A \rangle \neq 0$, we have to add new contributions to $Z_L Z_R$ which correspond to a saddle point with

$$\begin{cases} G_{LR} \neq 0 \\ G_{h(D)} \neq 0 \end{cases}$$

- **Other results**

  - O-SYK with J_A satisfying other distribution
  - Brownian-SYK
  - Modified Brownian-SYK

- **The upshot**

$$Z \text{ or } Z_L Z_R \approx \text{self-averging saddle} + \underbrace{\text{non-self-averging ``saddle''}}_{\text{to restore factorization we do not need all the information}}$$

- **The shortcoming**
  - Heavily rely on introducing the proper "GI" and the trick of inserting a suitable "I" but in general we do not know how to do.

- **A new proposal** (oncoming)

  The key idea is to generalize the analysis of CGS model

- A simple statistical model

Let us consider a function $Y(X_i)$ of a large number N independent Gaussian random variables X_i.

Each X_i can be thought of as a boundary operator, so this kind of model can be thought of as the CGS model with species. ('21 Saad, Shenker & Yao)

Then we have the following decomposition:

$$Y^n = \sum_{k=\sum_i n_i} \Gamma_k \qquad \Gamma_k : \text{the } k\text{-universe sector}$$

In particular Γ_0 is the self-averaging sector $\langle Y^n \rangle$ and $\Gamma_{k\neq 0}$ are the non-self-averaging sectors.

The Q: In the large N limit, which sectors will survive?

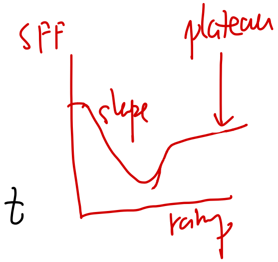
A example

$$Y = \sum_i e^{\beta \chi_i} , \quad \langle \chi_i \rangle = H \quad \langle \chi_i^2 \rangle - \langle \chi_i \rangle^2 = t^2 .$$

The result depends on the parameter βt:

① $\beta t \ll 1$. $\quad Y \approx \langle Y \rangle$, self-averging, hWH can be ignored

② $\beta t \gg 1$. $\quad Y \approx \Gamma_{\beta^2 t^2 \pm \beta t}$, non-self-avergrng



For Y^2

① $\beta t \ll 1 \quad Y^2 \approx \langle Y^2 \rangle$ ② $2\beta^2 t^2 \gg \log N \quad Y^2 \approx \Gamma_{2\beta^2 t^2 \pm \beta t}$

③ $1 \ll 2\beta^2 t^2 \ll \log N \quad Y^2 \approx \Gamma_{\beta^2 t^2 \pm \beta t}$ ④ $2\beta^2 t^2 \approx \log N \quad Y^2 \approx \Gamma_{2\beta^2 t^2 \pm \beta t} + \Gamma_{\beta^2 t^2 \pm \beta t}$

- **Results**

1. When $t^2/\kappa^2 \ll N^{\#}$, Y^N is self-averging and WH can be ignored

2. Only when $t^2/\kappa^2 \gg N^{\#}$, WH will be important. When only few non-self-averging sectors survive, we may identify them as "half-wormhole saddles"

3. Sometimes when $t, \kappa \sim O(1)$, all the sectors are important. It means that to restore factorization we need all the details of the couplings. $\longmapsto$ may not have a bulk dual.

- Revisit 0-SYK model.

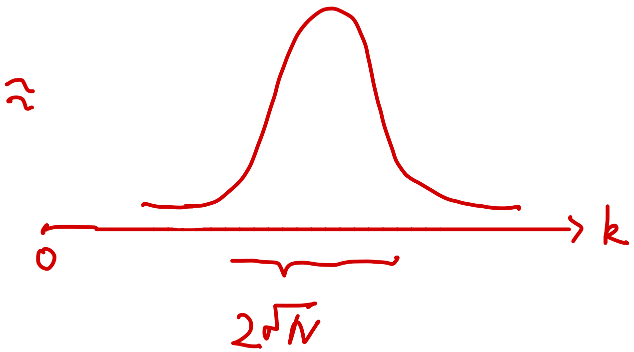
① When $\mu = 0$

$$Z^2 \approx \Gamma_0 + \Gamma_{max}$$

indeed only two sectors are left over and $\Gamma_{max} = \Phi(\omega)$

② When $\mu \neq 0$

$Z^2 \approx$



that's why the trick and the proposal $\Phi(\omega)$ fail.

- New hWH proposal for 1-SYK : Γ_{max}
- hWH for RMT : similar to the statistical model.

Thanks for attention !