On Carrollian Conformal Field Theory

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Collaborators: Reiko Liu, Hao-wei Sun and Yu-fan Zheng
Based on the papers: 2112.10514 and work in progress

Galilei vs Carroll

Let us start from the Lorentz boost

$$\begin{aligned} \vec{x}\,' &= \vec{x} + (\gamma - 1)\frac{(\vec{\beta}\cdot\vec{x})\vec{\beta}}{|\vec{\beta}|^2} + \vec{\beta}\gamma x^0 \\ x^{0\prime} &= \gamma(x^0 + \vec{\beta}\cdot\vec{x}) \end{aligned}$$

with $\gamma = (1 - \vec{\beta}^2)^{-1/2}$.

Limit 1: Introduce $t = x^0/c, \vec{b} = c\vec{\beta}$ and consider $c \to \infty$ limit, leading to the Galilean transformation

$$t' = t, \qquad \vec{x}\,' = \vec{x} + \vec{b}t$$

There is a notion of absolute time.

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There is a notion of absolute time.
Limit 2: Introduce another “time” $s = Cx^0, \vec{b} = -C\vec{\beta}$ and consider $C \to \infty$ limit, where $\vec{x}, s, \vec{b}$ are fixed, leading to the Carrollian boosts

$$
\vec{x}\,' = \vec{x}, \qquad s' = s - \vec{b}\cdot\vec{x}.
$$

There is a notion of absolute space.

With the translations and the rotations among spacial directions, they generate the Galilei group and Carroll group respectively.

Carroll group

The Galilei group could be produced by considering the $c \to \infty$ limit of the Poincaré group.
On the contrary, the Carroll group was found by considering the $C \to \infty (c \to 0)$ limit. J. Lévy-Leblond (1965), N.D. Sen Gupta (1966)

Intuitively, under the Carrollian limit, the lightcones collapse.

"since absence of causality as well as arbitrarinesses in the length of time intervals is especially clear in Alice's adventures (in particular in the Mad Tea-Party) this did not seem out of place to associate Lewis Carroll's name" (Lévy-Leblond (1965))

Carrollian particle

To study the motion of a free Carrollian particle, we may start from the massive particle moving in AdS/dS spacetime and then take the Carrollian limit. In the end, we find the action

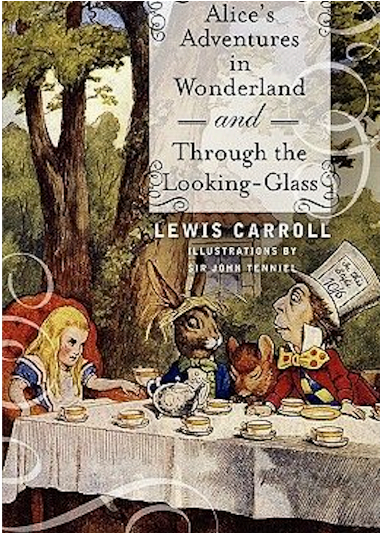
$$S_C = \int d\tau \left(-E\dot{s} + \dot{\vec{x}} \cdot \vec{p} - \frac{e}{2}(E^2 - M^2)\right)$$

which is invariant under the Carrollian transformation

$$\begin{aligned} s' &= s - \vec{b} \cdot R\vec{x} + a_s, \qquad & \vec{x}\,' &= R\vec{x} + \vec{a}, \\ \vec{p}\,' &= R\vec{p} - \vec{b}E, \qquad & E' &= E. \end{aligned}$$

The free Carrollian particle is at rest and does not move! C. Duval et.al 1402.0657, E. Bergshoeff et.al. 1405.2264

However, for two-particle system, there is non-trivial dynamics!



The Red Queen effect: running without moving, "ultralocal"

... *The most curious part of the thing was, that the trees and the other things round them never changed their places at all: however fast they went, they never seemed to pass anything.*

Carroll structure

Both the Galilei group and Carroll group are kinematical groups. H. Bacry and J. Lévy-Leblond (1968)

They don't need to be understood from the contractions of Poincare group.

The Carroll group is the automorphisms of the flat Carroll structure $(\mathcal{C}, g, \xi)$:

$$\mathcal{C}^{d+1} = R \times R^d, \qquad g = \delta_{ij} dx^i dx^j, \qquad \xi = \frac{\partial}{\partial s}.$$

Note that the metric is degenerate.

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More generally, one may consider the Carroll manifold and New-Cartan manifold, leading to new kinds of gravity.

Carrollian physics

The Carrollian boosts appears as the isometry group of plane-gravitational wave. J. M. Souriau (1973)...

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Celestial holography L. Donnay et.al. 2202.04702, Bagchi et.al. 2202.08438

Motivation

We have been studying 2D Galilean/Carrollian conformal bootstrap in the past few years. BC, P.X. Hao, R. Liu and Z.F. Yu, 2011.11092, 2207.01474, 2203.10490

1. Multiplet structure
2. Galilean conformal blocks for multiplets
3. Harmonic analysis of GCA: GCPW
4. Shadow formalism ($\xi \neq 0$)
5. Four-point function in GGFT and BMS free scalar in different ways
6. Spectral density by using Hardy-Littlewood tauberian theorem.

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Our original idea was to extend the bootstrap program to higher dimensional Carrollian conformal field theories (CCFT).

Higher dimensional CCFT

Carrollian conformal algebra (CCA)

One can obtain CCA_d by taking the Carrollian limit of the usual d-dim. conformal algebra. The generators are labeled by $\{D, P^\mu, K^\mu, B^i, J^{ij}\}$ with $\mu = 0, 1, \ldots, d-1$, $i, j = 1, \ldots, d-1$, where the Carrollian boost generators B^i come from the rotation generators: $J^{i0} \xrightarrow{c \to 0} B^i$.

$$
\begin{aligned}
&[D, P^\mu] = P^\mu, \quad [D, K^\mu] = -K^\mu, \quad [D, B^i] = [D, J^{ij}] = 0, \\
&[J^{ij}, G^k] = \delta^{ik} G^j - \delta^{jk} G^i, \quad G \in \{P, K, B\} \\
&[J^{ij}, P^0] = [J^{ij}, K^0] = 0, \\
&[J^{ij}, J^{kl}] = \delta^{ik} J^{jl} - \delta^{il} J^{jk} + \delta^{jl} J^{ik} - \delta^{jk} J^{il}, \\
&[B^i, P^j] = \delta^{ij} P^0, \quad [B^i, K^j] = \delta^{ij} K^0, \quad [B^i, B^j] = [B^i, P^0] = [B^i, K^0] = 0, \\
&[K^0, P^0] = 0, \quad [K^0, P^i] = -2B^i, \quad [K^i, P^0] = 2B^i, \quad [K^i, P^j] = 2\delta^{ij} D + 2J^{ij}.
\end{aligned}
$$

Stabilizer algebra and highest weight representations

The stabilizer algebra g_0 is generated by dilation D, generalized rotations $M = \{J, B\}$ and special conformal transformations (SCTs) K

$$[D, M] = 0, \quad [D, K] \subset K, \; [M, K] \subset K$$

i.e., K is a representation of D and M.
The commutativity of the dilatation and the rotations implies that the local operators $\mathcal{O}^a$ can be diagonalized into the eigenstates of the dilation , $[D, \mathcal{O}] = \Delta_{\mathcal{O}} \mathcal{O}$, and simultaneously into a representation of the rotations, $[M, \mathcal{O}^a] = M^a_b \mathcal{O}^b$.

Highest weight repr.: $[K, \mathcal{O}^a] = 0$.
This is often referred to as the primary condition.

Multiplet

For $d \geq 3$ CCFT, the generalized rotation group, CCA rotation group, is the Euclidean group $ISO(d-1)$. It is not semi-simple, and its finite dimensional representations are generally reducible but indecomposable, and can be organized as multiplet representations.

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Example: vector representation $\mathcal{O}^\mu$ of CCA$_4$

$$[J^{ij}, \mathcal{O}^k] = \delta^{ik}\mathcal{O}^j - \delta^{jk}\mathcal{O}^i, \quad [B^i, \mathcal{O}^j] = \delta^{ij}\mathcal{O}^0, \quad [J^{ij}, \mathcal{O}^0] = [B^i, \mathcal{O}^0] = 0.$$

$$\begin{tikzcd} \mathcal{O}^+ \arrow[r, leftrightarrow, "J^\pm"] \arrow[dr, "B^-"'] & \mathcal{O}^3 \arrow[r, leftrightarrow, "J^\pm"] \arrow[d, "B^3"] & \mathcal{O}^- \arrow[dl, "B^+"] \\ & \mathcal{O}^0 & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \boxed{3} \arrow[d] \\ \boxed{1} \end{tikzcd}$$

Here

$$J = -iJ^{12}, \qquad J^\pm = \frac{1}{\sqrt{2}}(\mp J^{23} + iJ^{31}), \qquad B^\pm = \frac{1}{\sqrt{2}}(iB^1 \pm B^2)$$

Tensor representation

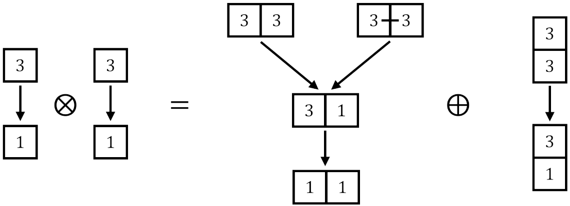

Figure: The rank-2 tensor representation of CCA. It is decomposed into a 10-dimensional representation and a 6-dimensional representation

The multiplet representations for $d > 2$ case have much more complicated structures since there is a non-trivial $ISO(d-1)$ part, leading to net representations rather than just chain-like ones in logCFT or $2d$ CCFT.

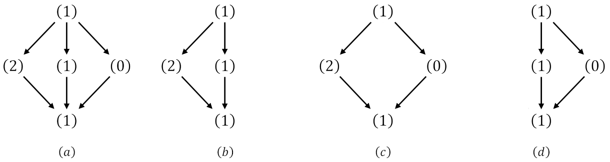

Figure: All the four net representations are legal although the middle level of the representations are different.

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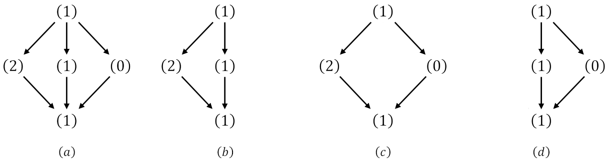

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Nevertheless, the finite dimensional representation of the CCA rotations are all multiplet representations with every sub-sector being irreducible representation of $SO(d-1)$, due to a theorem by Jakobsen (2011).

Notations: the number in the bracket indicates the irr. representation w.r.t. $SO(d-1)$, the arrows stand for the actions of the generators B_i.

Chain representations

The possible chain representations must take the following patterns:
rank 2

$$
\begin{aligned}
&(j) \to (j+1), \\
&(j) \to (j), \quad j \neq 0, \\
&(j) \to (j-1).
\end{aligned}
$$

rank 3 or higher

$$
\begin{gathered}
(0) \to (1) \to (0), \\
\cdots \to (j) \to (j+1) \to (j+2) \to \cdots\ , \\
\cdots \to (j) \to (j-1) \to (j-2) \to \cdots\ ,
\end{gathered}
$$

where the patterns works for all possible values of $j \in \{0\} \cup \mathbb{Z}_+/2$.

Correlators of singlets

In principle, the 2-pt and 3-pt functions of the operators in CCFT can be determined by using the Ward identities. However, due to complicated structure in representations, it is hard to discuss the most general case. We discussed the correlators of the operators in chain representations carefully. BC, Reiko Liu and Yu-fan Zheng, 2112.10514

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For a singlet in CCFT$_4$, there is

$$\langle \mathcal{O}_1(t_1,\vec{x}_1)\mathcal{O}_2(t_2,\vec{x}_2)\rangle = c_1 \frac{1}{r^{\Delta_1+\Delta_2}} + c_2\delta^{(3)}(\vec{x}_{12})\frac{1}{t^{\Delta_1+\Delta_2-3}},$$

- If $c_1 \neq 0$, $c_2 = 0$, the Ward identities of K^i will force $\Delta_1 = \Delta_2$, and the resulting 2-pt function coincides with the scalar 2-pt function in CFT$_3$.
- If $c_1 = 0$, $c_2 \neq 0$, it can be understood in an concrete model: the Carrollian free scalar with the action Bagchi et.al. 2019

$$S = \int d^3\vec{x}dt\,\phi\partial_t^2\phi.$$

Close relation between 3D Carrollian CFT and celestial holography!

L. Donnay et.al. 2202.04702, Bagchi et.al. 2202.08438...

Correlators of chain representations I: trivial one

Generic structure of 2-pt correlators:

$$\left\langle \mathcal{O}_1^{(m_1,q_1)}(x_1)\mathcal{O}_2^{(m_2,q_2)}(x_2) \right\rangle = f\,{}^{m_1,m_2}_{q_1,q_2}(x_{12})$$

In the following discussion on correlators, we focus on the one with only spatial dependence.

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For the trivial case that $\mathcal{O}_1, \mathcal{O}_2 \in (1) \to (0)$,

$$\begin{array}{llll} \text{Level 3:} & f\,^{m_1,m_2}_{2,2} = \dfrac{C\, I^{m_1,m_2}_{1,1}}{|\vec{x}_{12}|^{2\Delta}}, & \quad \mathcal{O}_1 \in \begin{array}{c}(1)\\ \downarrow \\ (0)\end{array} & \mathcal{O}_2 \in \begin{array}{c}(1)\\ \downarrow \\ (0)\end{array} \\ \text{Level 2:} & f\,^{0,m_2}_{1,2} = 0, \qquad f\,^{m_1,0}_{2,1} = 0, & & \\ \text{Level 1:} & f\,^{0,0}_{1,1} = 0, & \quad \text{with } \Delta_1 = \Delta_2 = \Delta. & \end{array}$$

Correlators of chain representations I: the simplest nontrivial case

For the simplest nontrivial case,

$$\mathcal{O}_1 \in (1) \to (0), \qquad \mathcal{O}_2 \in (0) \to (1).$$

Level 3: $$f_{2,2}^{\,m_1,0} = \frac{C\ t_{12}/|\vec{x}_{12}|\ I_{1,0}^{m_1}}{|\vec{x}_{12}|^{2\Delta}},$$

Level 2: $$f_{1,2}^{\,0,0} = \frac{C}{|\vec{x}_{12}|^{2\Delta}}, \qquad f_{2,1}^{\,m_1,m_2} = \frac{C\ I_{1,1}^{m_1,m_2}}{|\vec{x}_{12}|^{2\Delta}},$$

Level 1: $$f_{1,1}^{\,0,m_2} = 0.$$

$$\mathcal{O}_1 \in \begin{array}{c}(1)\\ \downarrow\\ (0)\end{array} \qquad \mathcal{O}_2 \in \begin{array}{c}(0)\\ \downarrow\\ (1)\end{array}$$

with $\Delta_1 = \Delta_2 = \Delta$.

Here $I_{j_1,j_2}^{\,m_1,m_2}$ is the 2-point tensor structure.

Longer chains

There are three kinds of long chains:

$$
\begin{gathered}
(0) \to (1) \to (0), \\
\cdots \to (j) \to (j+1) \to (j+2) \to \cdots\ , \\
\cdots \to (j) \to (j-1) \to (j-2) \to \cdots\ .
\end{gathered}
$$

Nontrivial 2-pt correlators:

- Case 1: Two operators whose representations are of entirely *inverse* pattern;
- Case 2: Two operators whose representations are at least partially *inverse* in the sense that the representation of one operator have the *inverse* pattern to the *leading* sub-sector of the representations of the other operator.

Remarks on correlators

- Due to the multiplet structure of the representations, the correlators present multi-level structures. At each level, there are more than one 2-pt coefficients. Even if considering the basis change and renormalization of the operators, not all 2-pt coefficients can be fixed by the Ward identities;

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- In spite of the multi-level structure, the structure of the 2-pt correlators at each level in CCFT is quite similar with the one in CFT. It consists of a scaling factor $|\vec{x}_{12}|^{-(\Delta_1+\Delta_2)}$ representing the scaling behavior, a tensor structure $I(\vec{x})$ representing the behavior under spacial rotations $\{J^{ij}\}$ and a factor being of powers of $t_{12}/|\vec{x}_{12}|$ representing the behavior under the Carrollian boosts $\{B^i\}$

$$f_{\text{2-pt}}^{(\text{CCFT})} \propto \frac{(t_{12}/|\vec{x}_{12}|)^n \; I}{|\vec{x}_{12}|^{(\Delta_1+\Delta_2)}}, \quad \text{with } \Delta_1 = \Delta_2.$$

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- Not all the correlators have time-dependence. The non-trivial correlators with time dependence appear only for the representations of certain structure.

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- As the representations are reducible, there is short of selection rule on the representations. This means that the 2-pt correlators of the operators in different representations could be nonvanishing.
- We explored the 2-pt correlators of net representations and the 3-pt correlators of chain representations. It turns out that the constraints from the Ward identities are quite loose, and we had to compute them case by case.

Construction of Carrollian field theories

Constructions of Carrollian field theories

The study of Carrollian field theories got revived in the past few years. There are two existing ways

- Hamiltonian formalism M. Henneaux et.al.
- Taking Carrollian limit of the usual QFT Duval et.al., Bagchi et.al.

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We propose a novel way to construct Carrollian (conformal) field theories, starting from the Bargmann field theories. We have successfully reproduced all Carrollian field theories in the literatures. BC. H.W. Sun and Y.F. Zheng, in progress

Bargmann symmetry

A Bargmann manifold has three ingredients, $(\mathcal{B}, G, \xi)$, where $\mathcal{B}$ is a $(d+1)$-dimensional manifold with metric G of Lorentz signature and a vertical vector ξ, a nowhere vanishing null vector.

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$$\mathcal{B} = \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}, \qquad G = 2dudv + \delta_{ij}dx^i dx^j, \qquad \xi = \partial_u,$$

where u, v are the lightcone coordinates.

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$$Barg(d,1) = ISO(d,1)/\{J^{\ 0}{}_{d+1}, 1/\sqrt{2}\left(J^{\ i}{}_{0} - J^{\ i}{}_{d+1}\right)\}$$

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that keep the null vector ξ invariant. The Bargmann generators are $\{P_\alpha, J^i{}_j, B_i^{\mathcal{B}}\}$, where $B_i^{\mathcal{B}}$ is the Bargmann boost. The actions on point $(u, \vec{x}, v)$ in the manifold are shown in the following Table

generator	vector field	finite transformation
p_α	∂_α	$x^\alpha + x_0^\alpha$
$m^i{}_j$	$x^i\partial_j - x_j\partial^i$	$(u, \mathbf{M}\vec{x}, v)$
$b_i^{\mathcal{B}}$	$v\partial_i - x_i\partial_u$	$\left(u - \vec{\nu}\cdot\vec{x} - \frac{1}{2}\vec{\nu}^2 v, \vec{x} + \vec{\nu}v, v\right)$

Carrollian symmetry from Bargmann symmetry

Restricting the Bargmann group on the null hyper-surface $v = 0$, we can immediately see the Bargmann structure reduce to Carroll structure $(\mathcal{C}, g, \xi)$ with the coordinates $(t = u, \vec{x})$, the degenerated metric

$$g_{\mu\nu} = G_{\mu\nu} = \delta^i_\mu \delta^j_\nu \delta_{ij}$$

while ξ^μ being the timelike vector, and the Carroll group is subgroup of Bargmann group $Carr(d) = Barg(d, 1)/\{P_v\}$.

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This motivates us to construct Carrollian field theories by restricting Bargmann field theories on the null hyper-surface.

However, trivially restricting Bargmann fields with configuration $\Phi(u, \vec{x}, v) = \Phi(u, \vec{x})\delta(v)$ on $v = 0$ causes many difficulty from the Dirac delta function. The trick is to introduce an uniformly distributed function over small interval of v.

Bargmann scalar field theories

The building blocks of Bargmann field theories are geometric invariants $G^{\alpha\beta}$ and ξ^α. For a free scalar Φ, the Bargman invariant action could be

$$S_E^{\mathcal{B}} = \frac{1}{2}\int d^{d+1}x\, \xi^\alpha \xi^\beta \partial_\alpha \Phi \partial_\beta \Phi, \qquad S_M^{\mathcal{B}} = -\frac{1}{2}\int d^{d+1}x\, G^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi.$$

The subscript M is for magnetic sector and E for electric sector, corresponding to magnetic/electric Carrollian field theories. M. Henneaux and P. Salgado-Rebolledo, 2109.06708

Electric sector

$$S_E^{\mathcal{B}} = \frac{1}{2}\int d^{d+1}x\, \xi^\alpha \xi^\beta \partial_\alpha \Phi \partial_\beta \Phi.$$

Expand Φ near $v = 0$, we have

$$\Phi(u, \vec{x}, v) = \phi(u, \vec{x}) + v\chi(u, \vec{x}) + \mathcal{O}(v^2).$$

Inserting this in the action, and noticing $\xi^\alpha = (1, \vec{0}, 0)$, we have

$$S_E^{\mathcal{B}} = -\frac{1}{2}\int d^{d+1}x\, \partial_u \Phi \partial_u \Phi = -\frac{1}{2}\int d^{d+1}x\, \partial_u \phi \partial_u \phi + 2v\partial_u \chi \partial_u \phi + \mathcal{O}(v^2),$$

and thus we have the Carrollian action

$$S_E^{\mathcal{C}} = \lim_{\epsilon \to 0} S_{E,\epsilon}^{\mathcal{B}} = -\frac{1}{2}\int d^d x\, \partial_0 \phi \partial_0 \phi.$$

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Actually, it is not only Carrollian invariant, but even Carrollian conformal invariant. From

$$\langle \phi(x)\phi(y)\rangle = \frac{i|t|}{2}\delta^{(d-1)}(\vec{x}),$$

we see that ϕ is a primary operator when $d > 2$.

Magnetic sector

$$S_M^{\mathcal{B}} = -\frac{1}{2}\int d^{d+1}x\; G^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi.$$

Insert the expansion of Φ, we get:

$$S_M^{\mathcal{B}} = -\frac{1}{2}\int d^{d+1}x\, 2\partial_u\Phi\partial_v\Phi + \partial_i\Phi\partial_i\Phi = -\frac{1}{2}\int d^{d+1}x\, 2\chi\partial_u\phi + \partial_i\phi\partial_i\phi + \mathcal{O}(v).$$

Thus we reproduce the action of magnetic Carrollian scalar theory M. Henneaux and P. Salgado-Rebolledo, 2109.06708.

$$S_M^{\mathcal{C}} = -\frac{1}{2}\int d^dx\; 2\chi\partial_0\phi + \partial_i\phi\partial_i\phi$$

The fundamental fields in this theory are ϕ and χ.

The above action is Carrollian conformal invariant as well. The scalar ϕ is still a primary fields, and the field χ appears as part of staggered module of ϕ's conformal family.

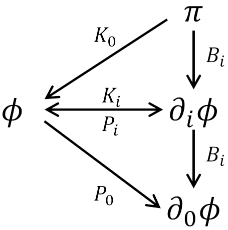

Figure: The staggered structure of fields ϕ, $\partial_\mu\phi$ and χ.

$$\langle\phi(\vec{x}_1,t_1)\phi(\vec{x}_2,t_2)\rangle = 0$$

$$\langle\phi(\vec{x}_1,t_1)\chi(\vec{x}_2,t_2)\rangle = -\frac{i\ sign(t)}{2}\delta^{(d-1)}(\vec{x})$$

$$\langle\chi(\vec{x}_1,t_1)\chi(\vec{x}_2,t_2)\rangle = \frac{i|t|}{2}\vec{\partial}^2\delta^{(d-1)}(\vec{x})$$

where $t = t_1 - t_2$ and $\vec{x} = \vec{x}_1 - \vec{x}_2$. They indeed satisfy the Ward identities of CCA.

The similar construction can be applied to other field theories: BC, H.W. Sun and Y.F. Zheng, work in progress

1. Carrollian p-form field theories, including electromagnetic theory
2. Carrollian Yang-Mills theory
3. Carrollian scalar QED
4. ······

Summary

1. We tried to study the higher dimensional ($d \geq 3$) Carrollian conformal invariant theories in a systematic way. As the conformal algebra is not semi-simple, the finite dimensional h.w.r. present some novel features: multiplet structure, staggered module, chain-like and even net-like representations.

- We classified all the chain representations
- We discussed the 2-pt and 3-pt correlators of chain operators

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1. We tried to study the higher dimensional ($d \geq 3$) Carrollian conformal invariant theories in a systematic way. As the conformal algebra is not semi-simple, the finite dimensional h.w.r. present some novel features: multiplet structure, staggered module, chain-like and even net-like representations.

- We classified all the chain representations
- We discussed the 2-pt and 3-pt correlators of chain operators

2. We proposed a novel way to construct Carrollian field theories from Bargmann field theories.

Thanks for your attention!