Modular Anomaly Equation for Schur Index of $\mathcal{N} = 4$ Super-Yang-Mills

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- In supersymmetric quantum mechanics (1d theory) with Hamiltonian H. We can compute

$$\text{partition function:} \quad \mathrm{Tr}e^{-\beta H}$$
$$\text{Witten index:} \quad \mathrm{Tr}(-1)^F e^{-\beta H} = \mathrm{Tr}(-1)^F$$

  Only ground states contribute to Witten index, encoding Euler number of the target space.
  Path integral formalism: compactify the Euclidean time on S^1.
  The $(-1)^F$ factor changes the fermionic boundary condition, from anti-periodic to periodic.

- In the original paper (1982), Witten considered compactification of 4d supersymmetric field theory on T^3. The Witten index gives a constrain on supersymmetry breaking. (some discussions on phenomenology)

- Superconformal index: radial quantization on $S^3\times$ time J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, arXiv:hep-th/0510251. Formal schematic definition

$$\mathcal{I} = \mathrm{Tr}\,(-1)^F e^{-\mu_i T_i} e^{-\beta\delta}, \qquad \delta = 2\{\mathcal{Q}, \mathcal{Q}^\dagger\},$$

  where $\mathcal{Q}$ is a supercharge and T_i a complete set of generators that commute with $\mathcal{Q}$ and with each other. μ_i are fugacities.
  By standard arguments, states with $\delta \neq 0$ cancel, so the index counts states with $\delta = 0$ (short multiplets) and is independent of β.
  The indices are a functions of 2, 3 and 4 continuous variables for $\mathcal{N} = 1, 2, 4$ superconformal symmetry $PSU(2,2|N)$.

- The case of $\mathcal{N} = 4$ super-Yang-Mills theory with $SU(N)$ gauge group is particularly interesting due to the holographic duality with type IIB string theory on $AdS_5 \times S^5$ background.

- The superconformal indices have many important applications. Most notably, they are essential for the understandings of the microscopic entropy of supersymmetric AdS_5 black holes. Their various expansions can be interpreted as the contributions of D-branes. (See the paper for references)

- For theories with a Lagrangian description (not necessarily at conformal fixed point), the d-dimensional superconformal index can be computed by path integral formalism as the supersymmetric partition function on $S^1 \times S^{d-1}$, which localizes to a matrix integral (which can be obtained alternatively by counting operators using state/operator correspondence).

- A particular specialization of the superconformal index, known as the Schur index A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, arXiv:1110. has some further nice mathematical properties.
  For example, in some cases it can be computed from the q-deformed 2d Yang-Mills A. Gadde et al , arXiv:1110.3740, or the vacuum character of a corresponding chiral algebra C. Beem et al, arXiv:1312.5344.

- Some remarkable (quasi)-modular properties of the index are studied recently in Y. Pan and W. Peelaers, arXiv:2112.09705; C. Beem, P. Singh, and S. S. Razamat, arXiv:2112.10715 in the context of a larger class of theories, based on some earlier works, e.g. J. Bourdier, N. Drukker, and J. Felix, arXiv:1507.08659.

- On the other hand, topological string theory on Calabi-Yau three-folds has been an active research area for decades, with many sophisticated available techniques.

- The goal of the present work is to apply one of these techniques to the calculations of Schur index. The relation between superconformal index and topological string amplitude has appeared before, in e.g. Kim, S.-S. Kim, and K. Lee, arXiv:1206.6781; A. Iqbal and C. Vafa, arXiv:1210.3605.

- In those cases, one has a 5d supersymmetric field theory from compactifying M-theory on a Calabi-Yau three-fold, and the 5d Nekrasov partition function on the Omega background $S^1 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ is simply equivalent to the refined topological string amplitude on the Calabi-Yau space.

- The 5d superconformal index at the fixed point of renormalization group flow can be computed by localization method as the partition function of the 5d field theory on $S^1 \times S^4$, and is written as an integral of a product of two complex conjugate refined topological string amplitudes (5d Nekrasov partition function).

- This is similar to Pestun's calculation of $\mathcal{N} = 2$ supersymmetric partition function on S^4, which localizes to a matrix integral in terms of 4d Nekrasov partition function.

- Similar relations appear also for 5d supersymmetric partition function on S^5 and 6d superconformal index, which are computed by an integral of a triple product of refined topological string amplitudes, e.g. G. Lockhart and C. Vafa, arXiv:1210.5909.

- Our setting is somewhat different, as the 4d superconformal index considered here seems much simpler than the 5d or 6d cases. We will directly apply topological string method of modular anomaly equation to the calculations of 4d Schur index, instead of writing it as an integral of topological string amplitudes.

- It is well known that the Eisenstein series E_4, E_6 freely generate the modular forms of $SL(2, \mathbb{Z})$. The second Eisenstein series E_2 is not exactly modular but transforms with a shift. The ring of polynomials of E_2, E_4, E_6, known as quasi-modular forms, is closed under the derivative action $q\frac{d}{dq}$.

- The quasi-modular forms appear in many studies in topological string theory, especially in geometries containing elliptic curves. In some cases there is a modular anomaly equation containing derivative with respect to the quasi-modular E_2, which is related to the celebrated BCOV holomorphic anomaly equation for general Calabi-Yau geometries without necessarily elliptic curves.

- For the case of $\mathcal{N}=4$ supersymmetry, besides a universal fugacity parameter denoted as q, the Schur index may have an extra flavor fugacity from the symmetry $SU(2)_F \subset SU(4)_R$. We will simply consider the unflavored index. As in the literature, it is convenient to treat the even and odd ranks of the gauge groups separately.

- We consider first the simpler $SU(2N+1)$ case. The formula is

$$\begin{aligned}\mathcal{I}_{2N+1}(q) = \frac{q^{\frac{N(N+1)}{2}}}{(2N+1)!}\prod_{n=1}^{\infty}(\frac{1-q^{n-\frac{1}{2}}}{1-q^n})^2\times \\ \oint \prod_{i=1}^{2N+1}\frac{dz_i}{2\pi i z_i}\prod_{i\neq j}(1-\frac{z_i}{z_j})\mathrm{PE}[i_V(q^{\frac{1}{2}})(\sum_{i,j=1}^{2N+1}\frac{z_i}{z_j})],\end{aligned} \tag{1}$$

where $i_V(q) = \frac{2q}{1+q}$ is the 1/8 BPS letter index, and $\mathrm{PE}[f(x_i)] = \exp[\sum_{k=1}^{\infty}\frac{f(x_i^k)}{k}]$ denotes the well known plethystic exponential applied to all variables q, z_i.
We have chosen the prefactors in the convention so that the results would have nice modular properties. For a finite N, it is not difficult to perform the contour integrals which are residues around $z_i \sim 0$ to obtain the q-expansion series to a finite order.

- Although the formula appears to have half integer powers in the q-expansion, the result actually has only integer powers. From the formula (1) it is obvious that the q-expansion starts at a high power as

$$\mathcal{I}_{2N+1}(q) = \mathcal{O}(q^{\frac{N(N+1)}{2}}). \tag{2}$$

- The exact calculations of (1) were first performed in J. Bourdier, N. Drukker, and J. Felix, arXiv:1507.08659 in terms of elliptic integrals and there is also an all order q-series formula

$$\begin{aligned}\mathcal{I}_{2N+1}(q) = \prod_{m=1}^{\infty}(1-q^m)^{-3}\sum_{n=0}^{\infty}(-1)^n \times \\ [\binom{2N+1+n}{2N+1} + \binom{2N+n}{2N+1}] q^{\frac{(n+N)(n+N+1)}{2}}.\end{aligned} \tag{3}$$

- The results were organized into nice formulas in terms of quasi-modular forms in Y. Pan and W. Peelaers, arXiv:2112.09705; C. Beem, P. Singh, and S. S. Razamat, arXiv:2112.10715 .

- We can list the formulas in term of Eisenstein series for the first few orders

$$
\begin{aligned}
&\mathcal{I}_1(q) = 1, \qquad \mathcal{I}_3(q) = \frac{E_2}{2} + \frac{1}{24}, \\
&\mathcal{I}_5(q) = \frac{E_2^2}{8} - \frac{E_4}{4} + \frac{E_2}{16} + \frac{3}{640}, \\
&\mathcal{I}_7(q) = \frac{E_2^3}{48} - \frac{E_2E_4}{8} + \frac{E_6}{6} + \frac{5E_2^2}{192} - \frac{5E_4}{96} + \frac{37E_2}{3840} + \frac{5}{7168}.
\end{aligned} \tag{4}
$$

- A general formula for all N's is also conjectured by Pan and Peelaers

$$
\mathcal{I}_{2N+1} = \sum_{k=0}^{N} \lambda_k^{(N)} \tilde{\mathbb{E}}_{2k}, \tag{5}
$$

where $\lambda_k^{(N)}$'s are constants determined by some rather complicated relations, and $\tilde{\mathbb{E}}_{2k}$ is a quasi-modular form of homogeneous weight $2k$ defined by

$$
\tilde{\mathbb{E}}_0 = 1, \qquad \tilde{\mathbb{E}}_{2k} = \sum_{\sum_{j\geq 1} j n_j = k} \prod_{p\geq 1} \frac{1}{n_p!}\left(-\frac{E_{2p}}{2p}\right)^{n_p} . \tag{6}
$$

- We use the following convention for the weight $2k$ Eisenstein series

$$E_{2k} = -\frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}. \tag{7}$$

There are well known relations e.g. $E_8 \sim E_4^2, E_{10} \sim E_4 E_6$.
The well known derivative formulas are due to ledendary Ramanujan

$$\begin{aligned} &q\frac{d}{dq}E_2 = -E_2^2 + 5E_4, \qquad q\frac{d}{dq}E_4 = -4E_2E_4 + 14E_6, \\ &q\frac{d}{dq}E_6 = -6E_2E_6 + \frac{60E_4^2}{7}. \end{aligned} \tag{8}$$

- Inspired particularly by the studies of the BPS partition functions of E-strings in J. A. Minahan, D. Nemeschansky, and N. P. Warner, arXiv:hep-th/9707149, we propose the following modular anomaly equation for the Schur index

$$\partial_{E_2} \mathcal{I}_{2N+1} = \sum_{k=1}^{N} c_k \mathcal{I}_{2N+1-2k}, \tag{9}$$

where c_k are some constants to be determined in a moment.

- We note that by string duality, the partition function in Minahan et al arXiv:hep-th/9707149 is equivalent to genus zero sector of topological string theory on a local half K3 Calabi-Yau space, and the modular anomaly equation has been subsequently generalized to higher genus Hosono et al 1999 and to refined theory Huang et al 2013.

- The modular anomaly equation in Minahan et al arXiv:hep-th/9707149 is recursive in the number of E-strings, which is identified with the rank of gauge group in another equivalent description in terms of $\mathcal{N} = 4$ topological Yang-Mills theories on a half K3 surface Minahan et al,1998. Therefore it is reasonable that we can also have an equation (9) recursive in the rank of the gauge group.

- There are certainly some notable differences with the usual form of modular anomaly equation familiar in topological string theory.
  (1) The right hand side of our equation (9) is purely linear in the lower rank indices, without the usual quadratic terms.
  (2) As seen from (4), the Schur index is inhomogeneous, i.e. a combination of quasi-modular forms of different weights, unlike the usual homogenous forms.

- The modular anomaly equation (9) determines the Schur index up to an E_2 independent term, a modular ambiguity which is polynomial of E_4, E_6. Since the index $\mathcal{I}_{2N+1}$ has a maximal weight of $2N$, the number of unknown coefficients in the ansatz for modular ambiguity can be easily counted.

- In general, the dimension of the space of modular forms of weight $2N$ is no more than $[\frac{N}{6}]+1$. So in our case we can estimate the number of unknown coefficients $\sum_{k=0}^{N}([\frac{k}{6}]+1) \sim \frac{N^2}{12}$ for large N.

- On the other hand, for a generic modular ambiguity, the q-expansion of the Schur index starts from the lowest constant q^0 term. Similar to the case of E-string, the vanishing condition (2) imposes very strong constrains, generically fixing $\frac{N(N+1)}{2}$ unknown coefficients, always overdetermining the ansatz.

- Staring from a very simple initial condition $\mathcal{I}_1(q) = 1, c_1 = \frac{1}{2}$, we can recursively efficiently compute all Schur indices $\mathcal{I}_{2N+1}$ and also determine the constants c_k's in (9), which are $\frac{1}{2}, \frac{1}{24}, \frac{1}{180}, \frac{1}{1120}, \frac{1}{6300}, \cdots$. We then observe a general formula for the constants

$$c_k = \frac{(k-1)!^2}{(2k)!}. \tag{10}$$

- Our anomaly equation (9) is compatible with the general formula (5). It is easy to see that $\partial_{E_2} \tilde{\mathbb{E}}_{2k+2} = -\frac{1}{2} \tilde{\mathbb{E}}_{2k}$, so the weight $2k$ components of each term in (9) are always proportional to $\tilde{\mathbb{E}}_{2k}$. More precisely, comparing the coefficients in (9) and (5) we find the relation

$$\lambda_{k+1}^{(N)} = -2 \sum_{l=1}^{N} c_l \lambda_k^{(N-l)}, \quad k \geq 0. \tag{11}$$

- There is also another interesting method to compute the Schur index. It is pointed out in M. J. Kang, C. Lawrie, and J. Song, arXiv:2106.12579 that in this case, the Schur index is simply a MacMahon's generalized sum-of-divisors function

$$\mathcal{I}_{2N+1}(q) = \sum_{0<m_1<\cdots<m_N} \frac{q^{m_1+\cdots+m_N}}{(1-q^{m_1})^2\cdots(1-q^{m_N})^2}. \tag{12}$$

- In G. Andrews and S. Rose, arXiv:1010.5769, a recursion relation for the MacMahon's function is derived

$$\mathcal{I}_{2N+1}(q) = \frac{1}{2N(2N+1)}[(6\mathcal{I}_3(q)+N(N-1))\mathcal{I}_{2N-1}(q)-2q\frac{d}{dq}\mathcal{I}_{2N-1}(q)]. \tag{13}$$

Using the derivative relations of quasi-modular forms (8), it is the clear that $\mathcal{I}_{2N+1}$ is a inhomogeneous quasi-modular form of weight $2N$, and it can be also easily computed recursively. The q-series formula (3) was also proved, therefore the equivalence of Schur index and MacMahon's function in this case is clear.

- The structure of formula (5) of Schur index is preserved by the recursion (13) due to the following derivative formula

$$q\frac{d}{dq}\tilde{\mathbb{E}}_{2k-2} = k(2k+1)\tilde{\mathbb{E}}_{2k} - 3\tilde{\mathbb{E}}_2\tilde{\mathbb{E}}_{2k-2}, \tag{14}$$

  which is a generalization of Ramanujan formulas (8) and can be certainly checked for any finite k. It should be derivable from the differential equations of the twisted Eisenstein series used by Pan and Peelaers.

- The relation (13) is then equivalent to a recursion for the coefficients

$$\lambda_k^{(N)} = \frac{1}{8N(2N+1)}[(2N-1)^2\lambda_k^{(N-1)} - 8k(2k+1)\lambda_{k-1}^{(N-1)}], \tag{15}$$

  where in the derivation we only need to look at the E_2 monomial term in $\tilde{\mathbb{E}}_{2k}$ in (6).
  For $k < 0$ or $k > N$ the coefficients are defined as $\lambda_k^{(N)} = 0$.
  From a simple initial condition $\lambda_0^{(0)} = 1$ we can then use the recursion (15) to compute all coefficients.

- For the special cases $k = 0$ or $k = N$, simple formulas $\lambda_0^{(N)} = \frac{(2N)!}{2^{4N}(2N+1)N!^2}$ and $\lambda_N^{(N)} = (-1)^N$ can be easily derived from the recursion. The recursion (15) looks much simpler than those given by Pan and Peelaers, but they should certainly give the same result.

- In the paper arXiv:1506.04963, Rose further considered more general MacMahon's sum-of-divisors functions, and provide formulas for the generating functions in terms of Jacobi forms. This turns out to provide a proof of the anomaly equation (9).

- In our case, the generating function for Schur index can be written in terms of the Jacobi theta function as

$$F(q,x) := \sum_{N=0}^{\infty} (-1)^N \mathcal{I}_{2N+1}(q) x^{2N+1} = \frac{i\theta_1(q,z)}{\eta(q)^3}, \tag{16}$$

with identification of parameters $x = e^{\pi i z} - e^{-\pi i z}$.

- It is known that a Jacobi forms ϕ_m of index m satisfies a modular anomaly equation

$$(\partial_{E_2} - m(2\pi z)^2)\phi_m = 0. \tag{17}$$

This has been applied successfully in topological string theory for making ansatz, see e.g. Huang et al, 2015.
In our context, the generating function is not exactly a Jacobo form of $SL(2,\mathbb{Z})$, but of a subgroup with index $\frac{1}{2}$. The modular anomaly equation can be still applied similarly

$$(\partial_{E_2} - \frac{1}{2}(2\pi z)^2)F(q,x) = 0. \tag{18}$$

- Denoting $f(x) := (\pi z)^2$, we can solve it and show that $f(x)$ has a series expansion

$$f(x) = -\log^2[\frac{1}{2}(x+\sqrt{4+x^2})] = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(n-1)!^2}{(2n)!}x^{2n}, \tag{19}$$

since $f(x)$ satisfies a differential equation $(x^2+4)f''(x)+xf'(x)+2 = 0$. Thus we have derived the modular anomaly equation (9) with the formulas (10) for the coefficients.

- We can define a generating function

$$G(x,y) := \sum_{N=0}^{\infty} \sum_{k=0}^{N} \lambda_k^{(N)} x^{2N+1} y^{2k+1}. \tag{20}$$

Using the relation (11), we have

$$G(x,y) + 2y^2 G(x,y) f(ix) = \sum_{N=0}^{\infty} \frac{(2N)!}{2^{4N}(2N+1)N!^2} x^{2N+1} y = 2y f(ix)^{\frac{1}{2}}. \tag{21}$$

So we can also get a solution in terms of elementary functions

$$G(x,y) = \frac{2y f(ix)^{\frac{1}{2}}}{1 + 4y^2 f(ix)}. \tag{22}$$

- One can check the recursion (15) is satisfied due to the differential equation

$$[4\partial_x^2 - (x\partial_x)^2 + 4y^2\partial_y^2 y^2] G(x,y) = 0. \tag{23}$$

The $SU(2N)$ case

- This is a little more complicated but similar. The Schur index formula in our convention is

$$\mathcal{I}_{2N}(q) = \frac{q^{\frac{N^2}{2}}}{(2N)!} \oint \prod_{i=1}^{2N} \frac{dz_i}{2\pi i z_i} \prod_{i \neq j} (1 - \frac{z_i}{z_j}) \mathrm{PE}[i_V(q^{\frac{1}{2}})(\sum_{i,j=1}^{2N} \frac{z_i}{z_j})], \tag{24}$$

  similar to (1) but with a different prefactor.

- The vanishing constrains for the index is

$$\mathcal{I}_{2N}(q) = \mathcal{O}(q^{\frac{N^2}{2}}). \tag{25}$$

  In this case, the q-expansion has half integer powers, so this generically will impose N^2 constrains on the ansatz.

- The modular group is now $\Gamma^0(2)$, whose modular forms are generated by

$$\Theta_{r,s}(q) = \theta_2(q)^{4r}\theta_3(q)^{4s} + \theta_2(q)^{4s}\theta_3(q)^{4r}, \tag{26}$$

which has weight $2(r+s)$.

- Some low order formulas for the Schur indices are also available in the literature

$$\begin{aligned}
\mathcal{I}_2(q) &= \frac{E_2}{2} + \frac{\Theta_{0,1}}{24}, \\
\mathcal{I}_4(q) &= \frac{E_2^2}{8} + \frac{E_2\Theta_{0,1}}{48} + \frac{\Theta_{0,2}}{1152} - \frac{\Theta_{1,1}}{576} + \frac{E_2}{24} + \frac{\Theta_{0,1}}{288}.
\end{aligned} \tag{27}$$

- The number of unknown coefficients in the modular ambiguity in $\mathcal{I}_{2N}$ is counted by $\Theta_{r,s}(q)$'s with $r+s \leq N, r \leq s$, and goes like $\frac{N^2}{4}$ for large N, much smaller than the number of constrains N^2.

- It also turns out that there is no weight zero constant term in the modular ambiguity, as can be seen from the examples in (27). So starting also from the simple initial condition $\mathcal{I}_0 = 1, c_1 = \frac{1}{2}$, we can compute all Schur indices and fix the constants c_k's which turn out to be the same as in the $SU(2N+1)$ case (10).

- Of course we can also include the constant term in the ansatz for modular ambiguity, then we simply require the extra initial conditions for $\mathcal{I}_2, c_2$ to start the recursive algorithm.

- The Schur index can be represented by another MacMahon's generalized sum-of-divisors function. The procedures and results are similar to $SU(2N+1)$ case. We skip the details here.

Discussions

- Although the results for Schur index in the current study have been available in the literature, we find our method of using the anomaly equation (9) and the vanishing conditions (2) provides so far the simplest approach with minimal assumptions.

- The vanishing conditions are in fact highly redundant, providing consistency checks by themselves and automatically giving the coefficients (10) in the anomaly equation.

- Furthermore, using the anomaly equation we are able to solve the generating functions for the coefficients in the general formulas conjectured by Pan and Peelaers.

- It is would be interesting to check whether the more general MacMahon's sum-of-divisors functions studied by Rose 2015 have connections with Schur indices of some other superconformal field theories.

- Our anomaly equation (9) seems universally simple that it should have a wider applicability. It would be interesting to apply our proposal to more general superconformal indices, including more flavor fugacities. More results in a subsequent paper Y. Hatsuda and T. Okazaki, "$\mathcal{N} = 2^*$ Schur indices," arXiv:2208.01426 .

- A better understanding of the modular property would help the analysis of the asymptotic behavior of the index, which is essential for accounting for the black hole entropy in holographic duality, e.g. a subsequent paper G. Eleftheriou, arXiv:2207.14271 .

Thank You