TsT, black holes, and irrelevant deformations

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based on work with Luis Apolo, Stephane Detournay, Penxiang Hao, Wenxin Lai and Boyang Yu

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丘成桐数学科学中心
YAU MATHEMATICAL SCIENCES CENTER

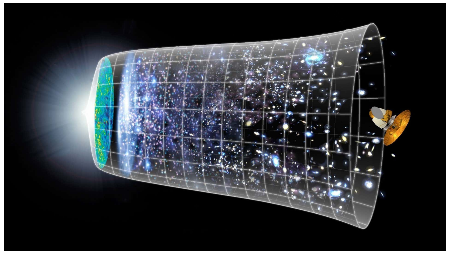
The AdS/CFT correspondence

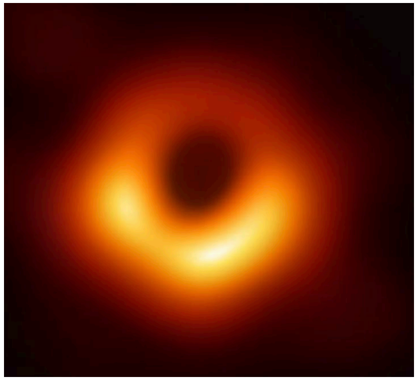
Quantum gravity in asymptotically AdS_{d+1} ⟷	CFT_d on the conformal boundary
empty AdS_{d+1}	vacuum of CFT_d
asymptotic symmetry	conformal symmetry
spectrum	spectrum
GKPW	correlation functions
black hole entropy	growth of states
minimal surface	entanglement entropy

Motivation

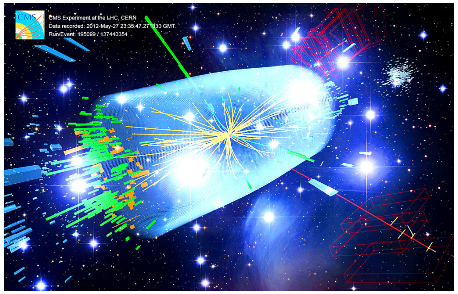
The study of the AdS/CFT correspondence has been very fruitful. But more steps are necessary to understand our real world.

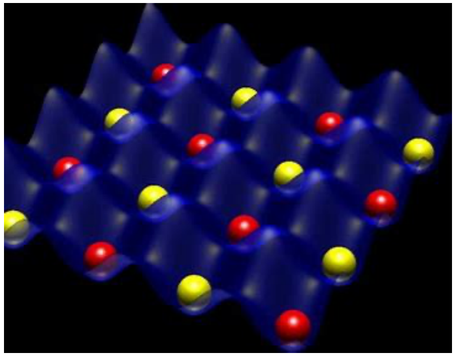
Non-AdS geometries





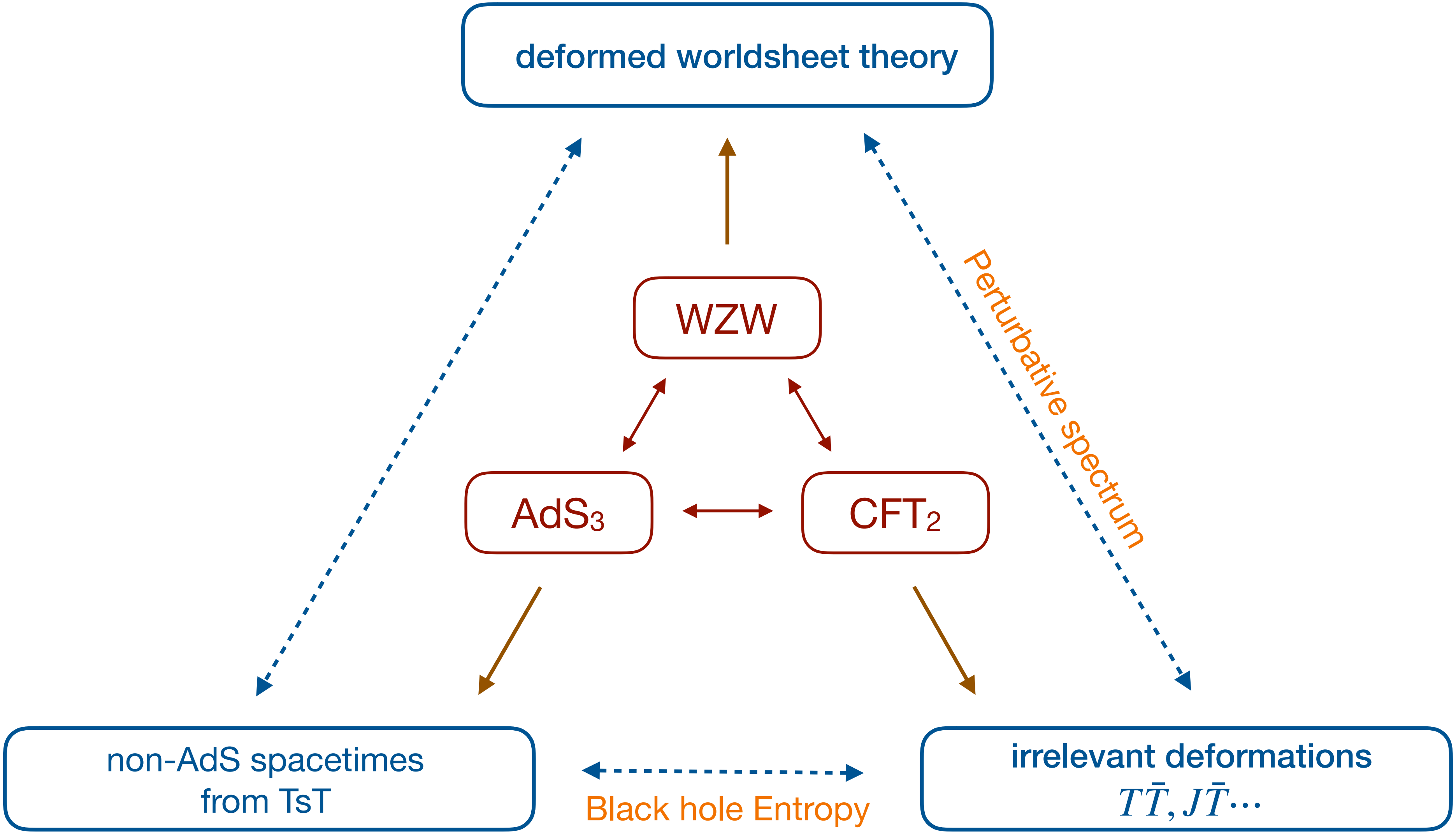
Non-CFT quantum systems





Question: holographic dualities beyond AdS/CFT?

This talk: a class of toy models of non-AdS holography
by deforming AdS/CFT



deformed worldsheet theory

WZW

AdS$_3$

CFT$_2$

Perturbative spectrum

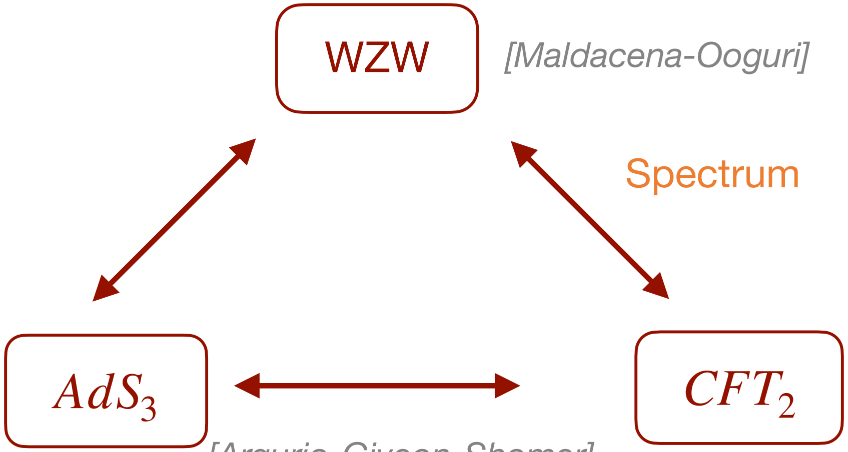
non-AdS spacetimes
from TsT

Black hole Entropy

irrelevant deformations
$T\bar{T}, J\bar{T}\cdots$

The inner triangle is a specific model of AdS$_3$/CFT$_2$ in string theory

weakly coupled string worldsheet
SL(2,R) WZW model with level k



WZW

[Maldacena-Ooguri]

Spectrum

AdS_3

CFT_2

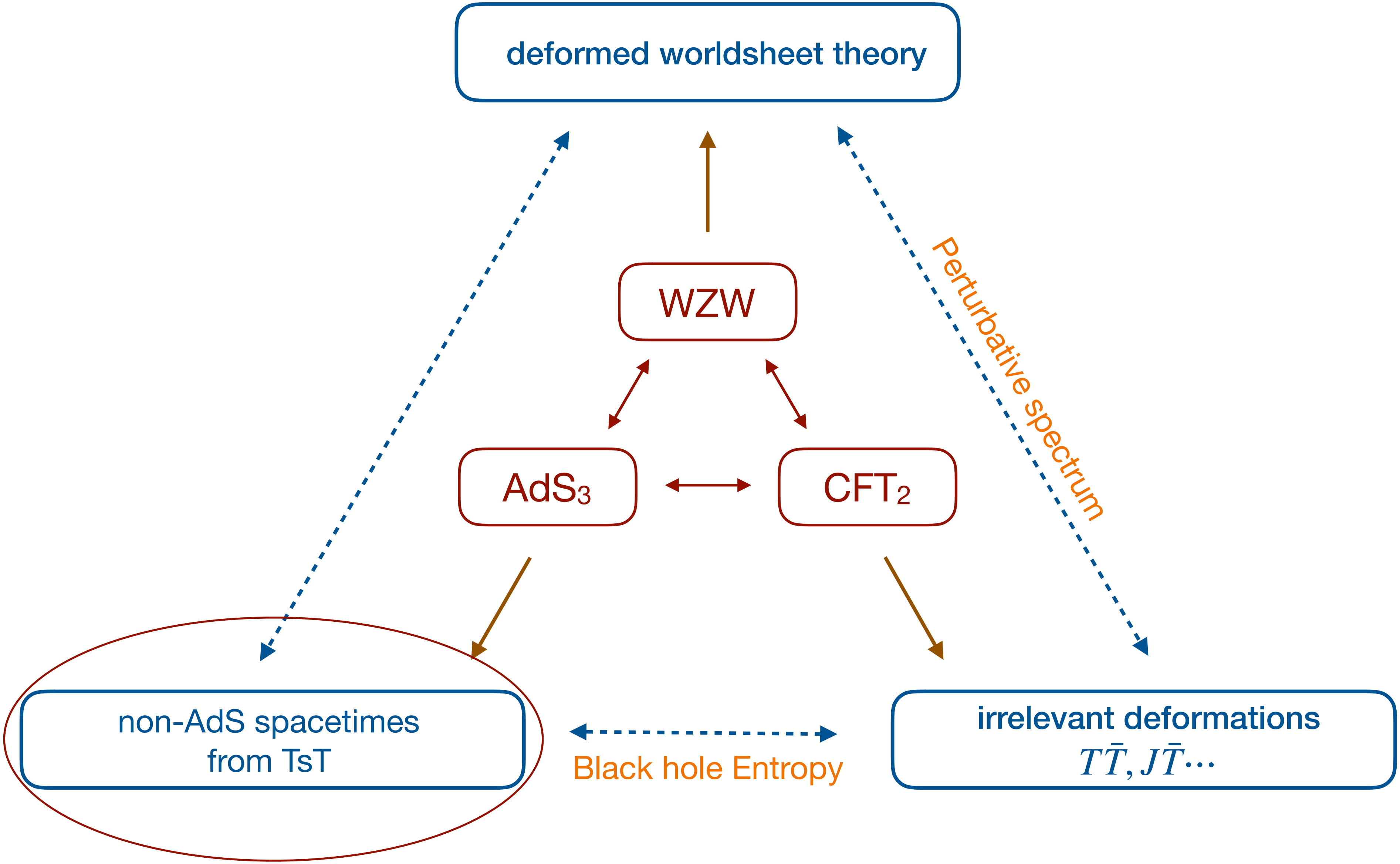
[Argurio-Giveon-Shomer]
[Eberhardt-Gaberdiel]
[Eberhardt]

II B string theory on $AdS_3 \times \mathcal{N}$
with k NS5, p NS1 charges
AdS_3 radius $\ell^2 = k\ell_s^2$

$\left(\mathscr{M}_{6k}\right)^p / S_p$

$c = 6kp$

$S_{BTZ} = S_{Cardy}$



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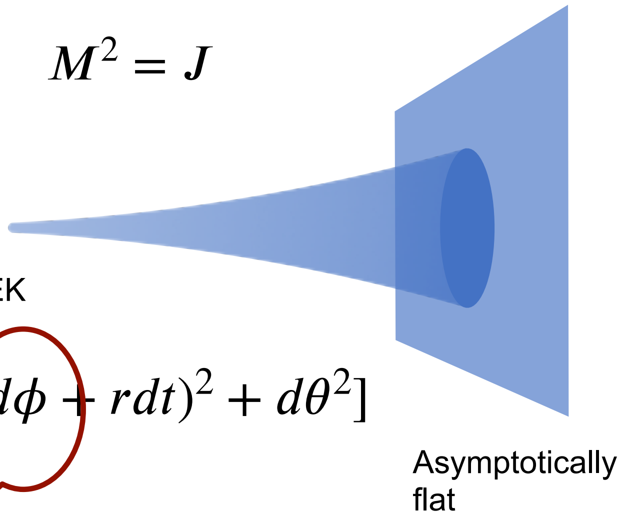
NHEK (Near Horizon Extremal Kerr) $M^2 = J$

[Bardeen, Horowitz '99]



$$ds^2 = 2J\Omega^2(\theta)[\underbrace{-r^2dt^2 + \frac{dr^2}{r^2}}_{\text{SL(2,R)}} + \frac{4\sin^2\theta}{(1+\cos^2\theta)^2}(\underbrace{d\phi}_{\text{U(1)}} + rdt)^2 + d\theta^2]$$

- NHEK geometry is *neither* asymptotically AdS3 nor flat
- (Universal) SL(2,R) x U(1) isometry group
- The Kerr/CFT correspondence *[Guica-Hartman-WS-Strominger 08']*

Warped AdS (WAdS): a toy model for extremal Kerr

Anninos-Padi-Li-WS-Strominger 08'

Isometry:

SL(2,R) x U(1)

- Null WAdS(Schrodinger space in 3d)

$$\frac{ds^2}{\ell^2} = r du dv + \frac{dr^2}{4r^2} + \lambda r^2 du^2$$

- Warped BTZ

$$\frac{\mathrm{ds}^2}{\ell^2} = -\left(\frac{r^2}{4T_v^2} - T_u^2\right) du^2 + \frac{dr^2}{4(r^2 - 4T_u^2 T_v^2)} + (1 - \alpha^2)\left(T_v dv + \frac{r du}{2T_v}\right)^2$$

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WAdS can be obtained from AdS_3 by a TsT transformation

TsT(T-duality, shift, T-duality) transformations are solution-generating techniques in SUGRA.

Starting from a background with two $U(1)$ isometries along X^m, X^n,

$$TsT_{(X^m,X^n,\lambda)} : T_{X^m} \to shift(X^n \to X^n - 2\lambda X^m) \to T_{X^m}$$

- TsT transformations change the local geometry
- depends on the two $U(1)$ directions and a deformation parameter

String solution before TsT

A two parameter family of classical solutions in IIB SUGRA with NSNS fluxes

$$d\tilde{s}_3^2 = \ell^2 \left\{ \frac{dr^2}{4\left(r^2 - 4T_u^2 T_v^2\right)} + r\,du\,dv + T_u^2 du^2 + T_v^2 dv^2 \right\}, \qquad (u, v) \sim (u + 2\pi, v + 2\pi)$$

$$e^{2\tilde{\Phi}} = \frac{k}{p}$$

→ *# of NS5 branes, magnetic charge*

→ *# of NS1 branes, electric charge*

- $T_u^2 \geq 0, T_v^2 \geq 0$: BTZ black holes
- $T_u = T_v = 0$: massless BTZ
- $T_u^2 < 0, T_v^2 < 0$: conical defect
- $T_u^2 = T_v^2 = -\frac{1}{4}$: global AdS$_3$

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WAdS can be obtained from $AdS_3 \times \mathcal{N}$ by a TsT transformation with one U(1) direction from AdS_3 and another from the internal manifold

[Azeyanagi-Hofman-WS-Strominger 13']

$$\begin{tikzcd}
AdS_3 \times S^3 \times M^4 \arrow[r, "TsT_{u,\psi}"] \arrow[d, "\text{Heating up}"'] & WAdS_3 \times_w S^3 \times M^4 \arrow[d, "\text{Heating up}"] \\
\text{BTZ} \times S^3 \times M^4 \arrow[r, "TsT_{u,\psi}"] & WBTZ \times_w S^3 \times M^4
\end{tikzcd}$$

(null warped)

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k → *# of NS5 branes, magnetic charge*

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TsT along two U(1)s in AdS_3

$$\frac{ds_3^2}{\ell^2} = \frac{dr^2}{4\left(r^2 - 4T_u^2 T_v^2\right)} + \frac{r\,du\,dv + T_u^2\,du^2 + T_v^2\,dv^2}{1 + 2\lambda r + 4\lambda^2 T_u^2 T_v^2} \qquad (u, v) \sim (u + 2\pi, v + 2\pi)$$

$$e^{2\Phi} = \frac{k}{p}\left(\frac{1 - 4\lambda^2 T_u^2 T_v^2}{1 + 2\lambda r + 4\lambda^2 T_u^2 T_v^2}\right) e^{-2\phi_0}$$

TsT along two U(1)s in AdS_3

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$$e^{2\Phi} = \frac{k}{p}\left(\frac{1 - 4\lambda^2 T_u^2 T_v^2}{1 + 2\lambda r + 4\lambda^2 T_u^2 T_v^2}\right)$$

- λ is the deformation parameter

- T_u, T_v parameterize the phase space of the theory

- The resulting geometry interpolates between

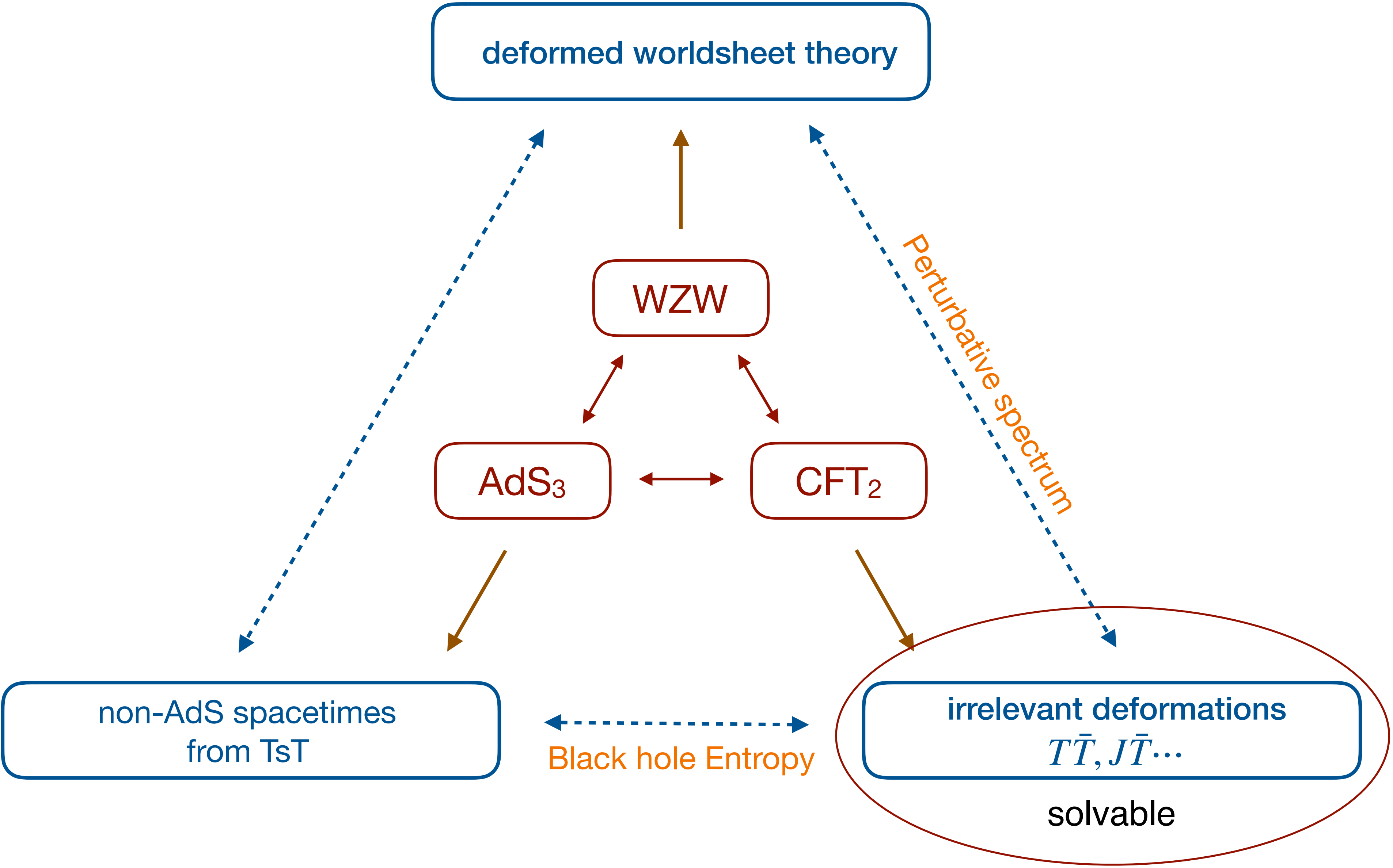
IR: locally AdS
UV: asymptotically flat spacetime in the string frame, with a linear dilaton

- $T_u = T_v = 0,$ the LST background of *[Giveon, Itzhaki, Kutasov]*

- $T_u = T_v = 0,\ \lambda = 1/2,$ *Horne-Horowitz* black string *[Horne-Horowitz]*

We have learned that TsT transformations can generate new backgrounds of string theory which are not asymptotically locally AdS spacetime.

What does TsT transformation corresponds to in the holographically dual theory?



deformed worldsheet theory

WZW

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CFT_2

Perturbative spectrum

non-AdS spacetimes from TsT

Black hole Entropy

irrelevant deformations $T\bar{T}, J\bar{T}\cdots$

solvable

A simple model of $T\bar{T}$

Free scalar CFT: $\mathscr{L}_0 = \partial\phi\bar{\partial}\phi$

stress tensor $T \propto \partial\phi\partial\phi, \bar{T} \propto \bar{\partial}\phi\bar{\partial}\phi$

Infinitesimal $T\bar{T}$ deformation: $\delta\mathscr{L} = \mu T\bar{T}$

This deformation can be integrated and the full deformed Lagragian is given by

$$\mathscr{L}_\mu = \frac{1}{2\mu}\left(\sqrt{4\mu\partial\phi\bar{\partial}\phi + 1} - 1\right) = -\frac{1}{2\mu} + \mathscr{L}_{NG}$$

(double-trace) $T\bar{T}$ deformations on the cylinder

[Zamolodchikov;Smirnov, Zamolodchikov; Cavaglia, Negro, Szecsenyi, Tateo; Cardy; Dubovsky, Flauger, Gorbenko; Dubovsky, Gorbenko, Mirbabayi; Conti, Iannella, Negro, Tateo; Frolov; …]

$$\frac{\partial S_\mu}{\partial \mu} = \int dx^2 \, \det T^\mu{}_\nu = \int dx^2 \, (T_{xx} T_{\bar{x}\bar{x}} - T_{x\bar{x}} T_{\bar{x}x})$$

$$x = \phi + t, \ \bar{x} = \phi - t, \ \phi \sim \phi + 2\pi R$$

$T_{\mu\nu}$: *stress tensor of the deformed theory at* μ

- solvable irrelevant deformation
- spectrum on a cylinder with $(x, \bar{x}) \sim (x + 2\pi R, \bar{x} + 2\pi R)$

$$E(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu E}{R} + \frac{4\mu^2 J^2}{R^2}}\right), \quad J(\mu) = J$$

- modular invariance, connections to random metric, Nambu-Goto action, JT gravity, string theory, CDD factors…*[Review: Jiang]*

Modular invariance and density of states

$T\bar{T}$ deformed CFTs are shown to be modular invariant *[Datta-Jiang]*

Assuming large c and sparceness condition, the entropy in some parameter regime can be written as *[Apolo-WS-Yu, WIP]*

$$S_{T\bar{T}} = 2\pi[\sqrt{\frac{c}{6}E_L(\mu)\left[1+2\mu E_R(\mu)\right]} + \sqrt{\frac{c}{6}E_R(\mu)\left[1+2\mu E_L(\mu)\right]}\,], \quad E_{L/R} = \frac{1}{2R}(E \pm J)$$

Alternative argument of the entropy formula: no level crossing in the spectrum

$$S_{T\bar{T}}(E_L(\mu), E_R(\mu)) = S_{Cardy}(E_L, E_R), \quad E_{L/R} = \frac{1}{2R}(E \pm J)$$

$T\bar{T}$ with $\mu < 0$

The spectrum on a cylinder with $(x, \bar{x}) \sim (x + 2\pi R, \bar{x} + 2\pi R)$

$$E(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu E}{R} + \frac{4\mu^2 J^2}{R^2}}\right), \quad J(\mu) = J, \qquad \mu < 0$$

- real energy for the ground state
- complex spectrum at very high energy
- high energy (but not too high) $S_{T\bar{T}}(E_L(\mu), E_R(\mu)) = S_{Cardy}(E_L, E_R), \quad E_{L/R} = \frac{1}{2R}(E \pm J)$

$$S = 2\pi[\sqrt{\frac{c}{6}E_L(\mu)\left[1 + 2\mu E_R(\mu)\right]} + \sqrt{\frac{c}{6}E_R(\mu)\left[1 + 2\mu E_L(\mu)\right]}]$$

c is the central charge before the deformation.

$T\bar{T}$ with $\mu > 0$

The spectrum on a cylinder with $(x, \bar{x}) \sim (x + 2\pi R, \bar{x} + 2\pi R)$

$$E(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu E}{R} + \frac{4\mu^2 J^2}{R^2}}\right), \quad J(\mu) = J, \qquad \mu > 0$$

$6k$: *central charge of the CFT*

- ground state $E^{vac}(\mu) = -\frac{1}{2\mu}(1 - \sqrt{1 - \frac{c\mu}{3R^2}})$, complex if $\lambda \equiv \frac{c\mu}{6R^2} > \frac{1}{2}$

critical value: $\lambda_c = 1/2$

- high energy states always have real energies
- entropy $S = 2\pi[\sqrt{\frac{c}{6}E_L(\mu)\left[1 + \frac{2\mu}{R}E_R(\mu)\right]} + \sqrt{\frac{c}{6}E_R(\mu)\left[1 + \frac{2\mu}{R}E_L(\mu)\right]}\,]$
- Hagedorn growth at very high energy $E(\mu) \gg \frac{1}{\mu}, \quad S_{T\bar{T}} \sim 2\pi\sqrt{\frac{c\mu}{3}}E(\mu)$
- temperatures $T_{L/R} \equiv \left(\partial S_{T\bar{T}}/\partial E_{L/R}\right)^{-1}$, have an upper bound $T_L T_R \leq \frac{3}{4\pi^2 c\mu}$

[Apolo-Detournay-WS]

A single trace version of $T\bar{T}$ deformation

A single trace version of $T\bar{T}$ deformation can be defined for as a symmetric product $(\mathscr{M}_\mu)^p/S_p$, where the seed theory is a (double trace) deformed CFT$_2$.

- The spectrum in the twisted sector is given by

[Apolo-WS-Yu, WIP]

$$E_L^{(n)}(0) = E_L^{(n)}(\mu) + \frac{2\mu}{nR} R E_L^{(n)}(\mu) E_R^{(n)}(\mu)$$

- The entropy is

$$S_{T\bar{T}}^{single\,trace}\left(E_L, E_R\right) = 2\pi\left[\sqrt{\frac{c}{6}RE_L(\mu)\left[1+\frac{2\mu}{Rp}E_R(\mu)\right]} + \sqrt{\frac{c}{6}RE_R(\mu)\left[1+\frac{2\mu}{Rp}E_L(\mu)\right]}\right]$$

TsT $\leftrightarrow T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$

A conjecture

[Apolo-Detournay-WS]

Starting from IIB string theory on locally $AdS_3 \times \mathcal{N}$ with NSNS background flux,

$$\mathrm{TsT}_{(X^m, X^{\bar{m}}; \hat{\mu})} \Longleftrightarrow \frac{\partial S_{\mathcal{M}_\mu}}{\partial \mu} = -4 \int J_{(m)} \wedge J_{(\bar{m})}$$

LHS: T-duality along X^m, then a shift $X^{\bar{n}} = X^{\bar{n}} - 2\hat{\mu}\, X^m$, and finally T-duality along X^m.

RHS: deformed symmetric product theory

Examples:

TsT with two $U(1)$s both in AdS_3 / one in AdS_3 and the other in $\mathcal{N}$ / both in $\mathcal{N}$

[Apolo-Detournay-WS] *[Chakraborty-Giveon-Kutasov; Apolo-WS]*

$\Updownarrow$

single trace $T\bar{T}$ / $J\bar{T}(T\bar{J})$ / $J\bar{J}$ deformations

[Giveon-Itzhaki-Kutasov, Giribet] *[Guica]*

evidence for TsT $\leftrightarrow T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$?

- solution space
- black hole thermodynamics

Stationary solutions with two parameters

$$\frac{ds_3^2}{\ell^2} = \frac{dr^2}{4\left(r^2 - 4T_u^2T_v^2\right)} + \frac{r\,du\,dv + T_u^2du^2 + T_v^2dv^2}{1 + 2\lambda r + 4\lambda^2T_u^2T_v^2} \qquad (u,v) \sim (u+2\pi, v+2\pi)$$

$$e^{2\Phi} = \frac{k}{p}\left(\frac{1-4\lambda^2T_u^2T_v^2}{1+2\lambda r + 4\lambda^2T_u^2T_v^2}\right)$$

- $T_u = T_v = 0 \leftrightarrow$ Ramond vacuum *[Giveon-Itzhaki-Kutasov]*
- $T_u = T_v = \frac{i}{2\lambda}(1-\sqrt{1-2\lambda}) \leftrightarrow$ ground state, NS vacuum *[Apolo-Detournay-WS]*
- Range of parameters from the bulk:
  - $0 < \lambda < \frac{1}{2}$, smooth and real solution
  - $\lambda < 0$, CTC and curvature singularities
  - upper bound for the temperatures by requiring real dilaton
- Bekenstein Hawking entropy ↔ entropy of single trace $T\bar{T}$

  $S_{TsT} = S_{T\bar{T}}^{single\,trace}\left(E_L, E_R\right)$

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  $S_{TsT} = S_{T\bar{T}}^{single\ trace}\left(E_L, E_R\right)$

Further evidence for TsT $\leftrightarrow T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$?

- the action
- the deformed spectrum

A useful rewriting of $T\bar{T}$ deformation

$$\frac{\partial S_\mu}{\partial \mu} = \int dx^2 \, (T_{xx}T_{\bar{x}\bar{x}} - T_{x\bar{x}}T_{\bar{x}x})$$

$x = \phi + t,\ \bar{x} = \phi - t$

$T_{\mu\nu}$: *stress tensor of the deformed theory at* μ

$$\boxed{\frac{\partial S_\mu}{\partial \mu} = -4 \int J_{(x)} \wedge J_{(\bar{x})}}$$

$J_{(x)}$: *Noether current that generates translation in* x

$J_{(\bar{x})}$: *Noether current that generates translation in* $\bar{x}$

$$T\bar{T}: \quad J_{(x)} = T_{xx}dx + T_{x\bar{x}}d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x}dx + T_{\bar{x}\bar{x}}d\bar{x}\,.$$

Solvable irrelevant deformations

- 1-parameter deformations:

$$\boxed{\frac{\partial S_\mu}{\partial \mu} = -4 \int J_{(m)} \wedge J_{(\bar{m})}}$$

$J_{(m)}/J_{(\bar{m})}$: *Noether currents*

chiral/anti-chiral at conformal point

$$T\bar{T}: \quad J_{(x)} = T_{xx}dx + T_{x\bar{x}}d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x}dx + T_{\bar{x}\bar{x}}d\bar{x}\,.$$

$$J\bar{T}: \quad J_{(n)} = J_x dx + J_{\bar{x}}d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x}dx + T_{\bar{x}\bar{x}}d\bar{x}\,.$$

$$T\bar{J}: \quad J_{(x)} = T_{xx}dx + T_{x\bar{x}}d\bar{x}, \quad J_{(\bar{n})} = J_x dx + J_{\bar{x}}d\bar{x}\,.$$

- 3-parameter deformations:

$$T\bar{T} + J\bar{T} + T\bar{J}: \quad \frac{\partial S_{\mu_0,\mu_+,\mu_-}}{\partial \mu_0} = -4 \int J_{(x)} \wedge J_{(\bar{x})}, \quad \frac{\partial S_{\mu_0,\mu_+,\mu_-}}{\partial \mu_+} = -4 \int J_{(n)} \wedge J_{(\bar{x})}$$

$$\frac{\partial S_{\mu_0,\mu_+,\mu_-}}{\partial \mu_-} = -4 \int J_{(x)} \wedge J_{(\bar{n})}$$

Solvable irrelevant deformations

- 1-parameter deformations:

$$\boxed{\frac{\partial S_\mu}{\partial \mu} = -4\int J_{(m)} \wedge J_{(\bar{m})}}$$

$J_{(m)}/J_{(\bar{m})}$: Noether currents

chiral/anti-chiral at conformal point

$$T\bar{T}: \quad J_{(x)} = T_{xx}dx + T_{x\bar{x}}d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x}dx + T_{\bar{x}\bar{x}}d\bar{x}\,.$$

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- 3-parameter deformations:

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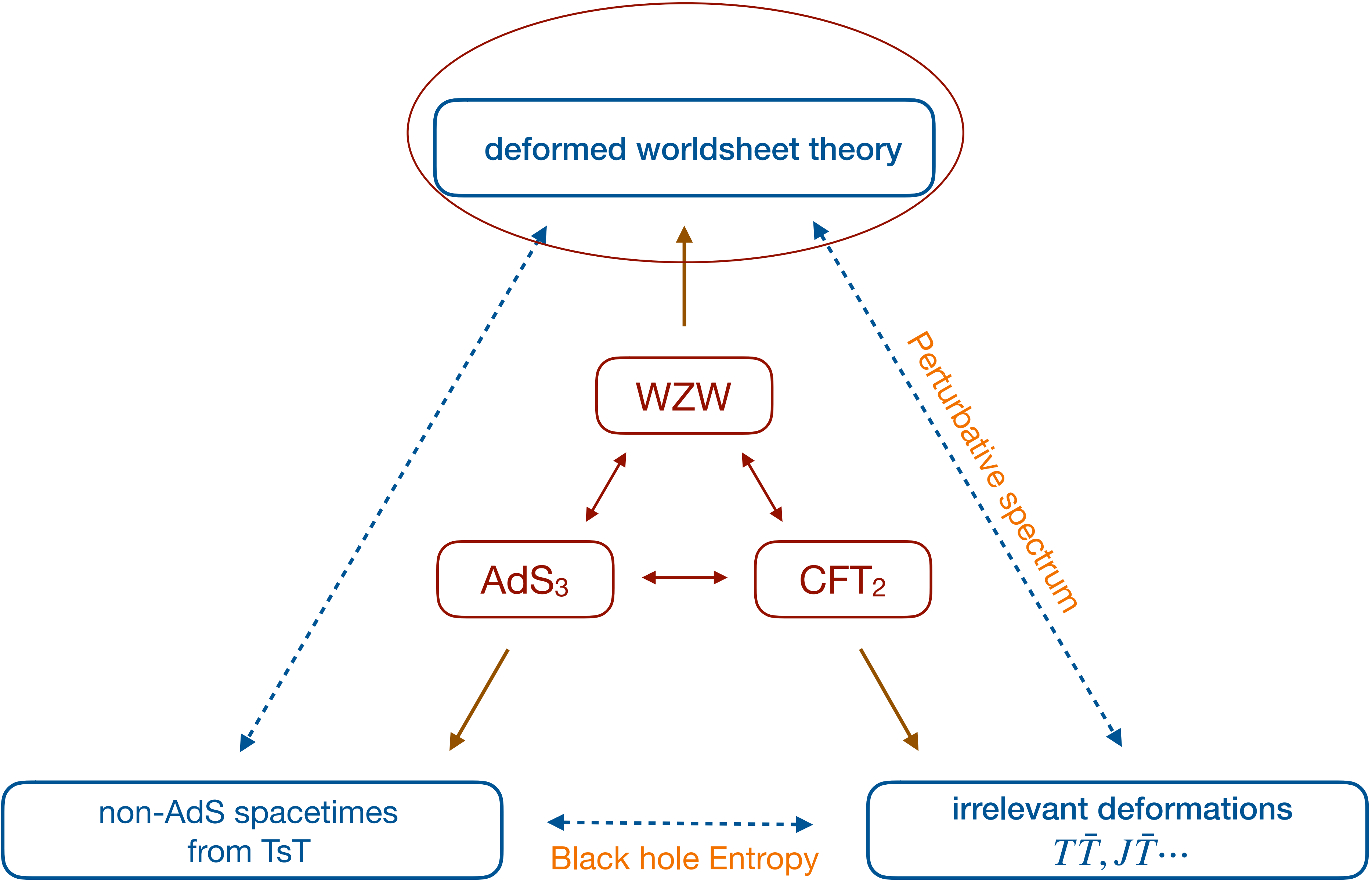
deformed worldsheet theory
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$T\bar{T}, J\bar{T}\cdots$

TsT $\leftrightarrow$ $T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$: the action

The string worldsheet action $S_{WS} = -\ell_s^{-2}\int d^2z\, M_{\mu\nu}\partial X^\mu\, \bar{\partial}X^\nu,$

$$M_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu}$$

$G_{\mu\nu}$: *Target space metric*
$B_{\mu\nu}$: *NS-NS potential*

TsT ↔ $T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$: the action

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TsT along $X^m, X^{\bar{m}}$: $M = \tilde{M}\left(I + 2\hat{\mu}\Gamma\tilde{M}\right)^{-1}, \; \Gamma_{\mu\nu} = \delta_\mu^m \delta_\nu^{\overline{m}} - \delta_\mu^{\overline{m}}\delta_\nu^m$

$X^m, X^{\bar{m}}$ are isometries

satisfies the differential equation: $\dfrac{\partial M}{\partial \hat{\mu}} = -2\ell_s^{-2} M\Gamma M$

TsT on string worldsheet can be formulated as : $\dfrac{\partial S_{WS}}{\partial \hat{\mu}} = -4\int \boldsymbol{j}_{(m)} \wedge \boldsymbol{j}_{(\overline{n})}$

$\boldsymbol{j}_{(m)}$, $\boldsymbol{j}_{(\overline{n})}$ *are* ***worldsheet*** *Noether 1-forms associated to ∂_{X^m}, and $\partial_{X^{\bar{m}}}$*

Noether charges $p_{(m)} \propto \oint \boldsymbol{j}_m$

marginal deformation on the worldsheet

TsT ↔ $T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$: the action

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marginal deformation on the worldsheet

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irrelavent deformation on the dual theory

$T\bar{T}$ on the dual field theory $(\mathcal{M}_\mu)^p/S_p$: $\dfrac{\partial S_{\mathcal{M}_\mu}}{\partial \mu} = -4\int J_{(m)} \wedge J_{(\bar{m})}$

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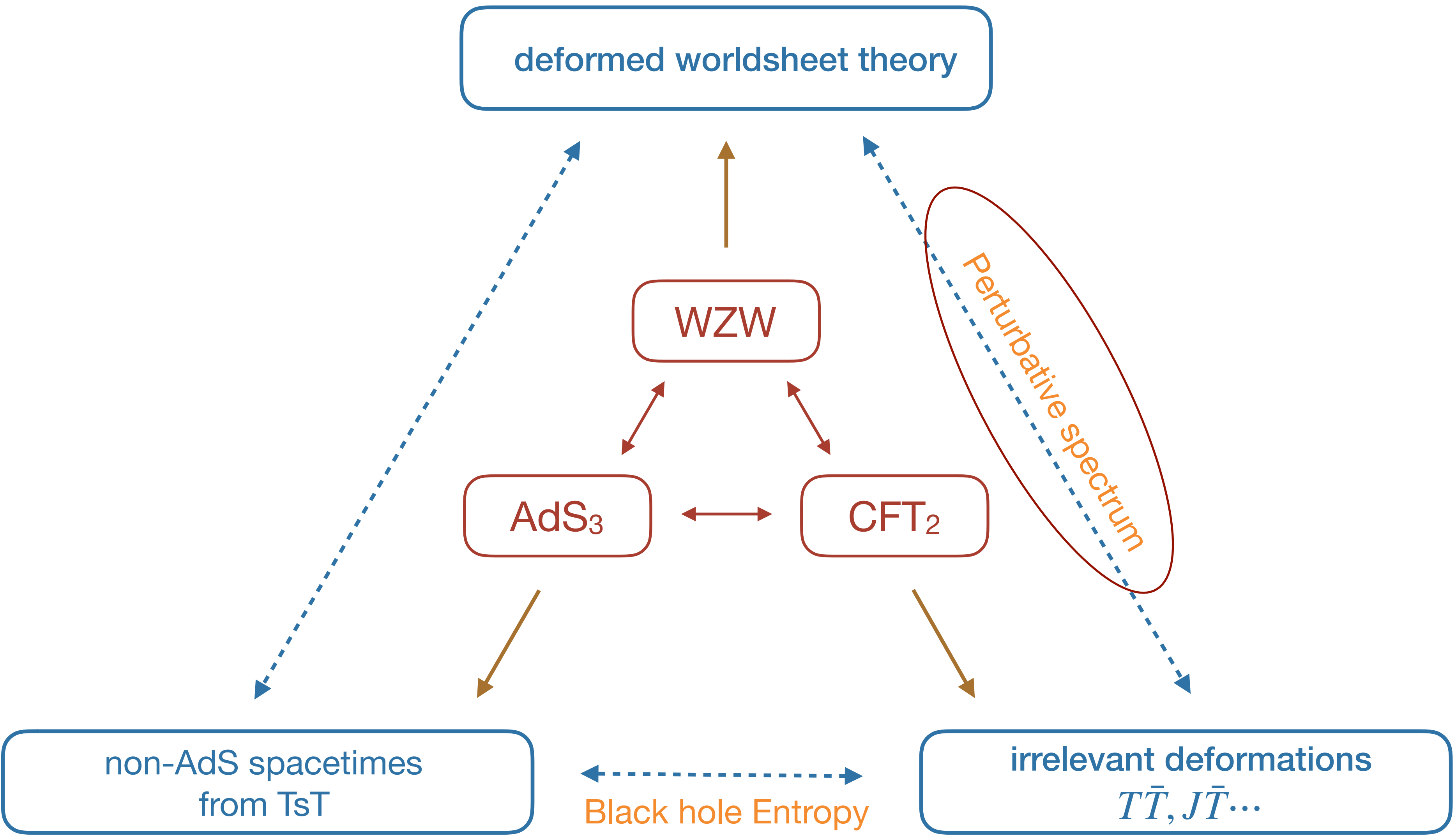
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Evidence for TsT ↔ $T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$: the spectrum

- The Virasoro constraints on AdS$_3$ background with winding w

$$X^1(\sigma+2\pi) = X^1(\sigma+2\pi) + 2\pi w, X^{\bar{2}}(\sigma+2\pi) = X^{\bar{2}}(\sigma+2\pi) + 2\pi w$$

is related to those of on AdS$_3$ background without winding

$$\hat{L}_0 = \tilde{L}_0 + wRp$$

Relation?

- The Virasoro constraints on the TsT background with winding

is related to those of on AdS$_3$ background by spectral flow transformations

$$\hat{L}_0 = \tilde{L}_0 + wRp(\hat{\mu}) + 2\hat{\mu}p(\hat{\mu})\bar{p}(\hat{\mu})$$

TsT on the worldsheet

Relation between string spectra with winding w before and after the TsT transformations

$$= wRp(0)$$

$$\hat{L}_0 = \tilde{L}_0 + wRp(\hat{\mu}) + 2\hat{\mu}p(\hat{\mu})\bar{p}(\hat{\mu})$$

$$\hat{\bar{L}}_0 = \tilde{\bar{L}}_0 - wR\bar{p}(\hat{\mu}) - 2\hat{\mu}p(\hat{\mu})\bar{p}(\hat{\mu})$$

keep fixed

$$= -wR\bar{p}(0)$$

The dictionary $\mu = \ell^2\hat{\mu}$

- $p \leftrightarrow \ell E_L, \quad \bar{p} \leftrightarrow -\ell E_R$
- w=-1: untwisted sector
- w<-1: twisted sector

String spectrum matches that of the single trace $T\bar{T}$ deformation

$$E_L(0) = E_L(\mu) - \frac{2\mu}{w\ell}E_L(\mu)E_R(\mu),$$

$$E_R(0) = E_R(\mu) - \frac{2\mu}{w\ell}E_L(\mu)E_R(\mu)$$

Another example

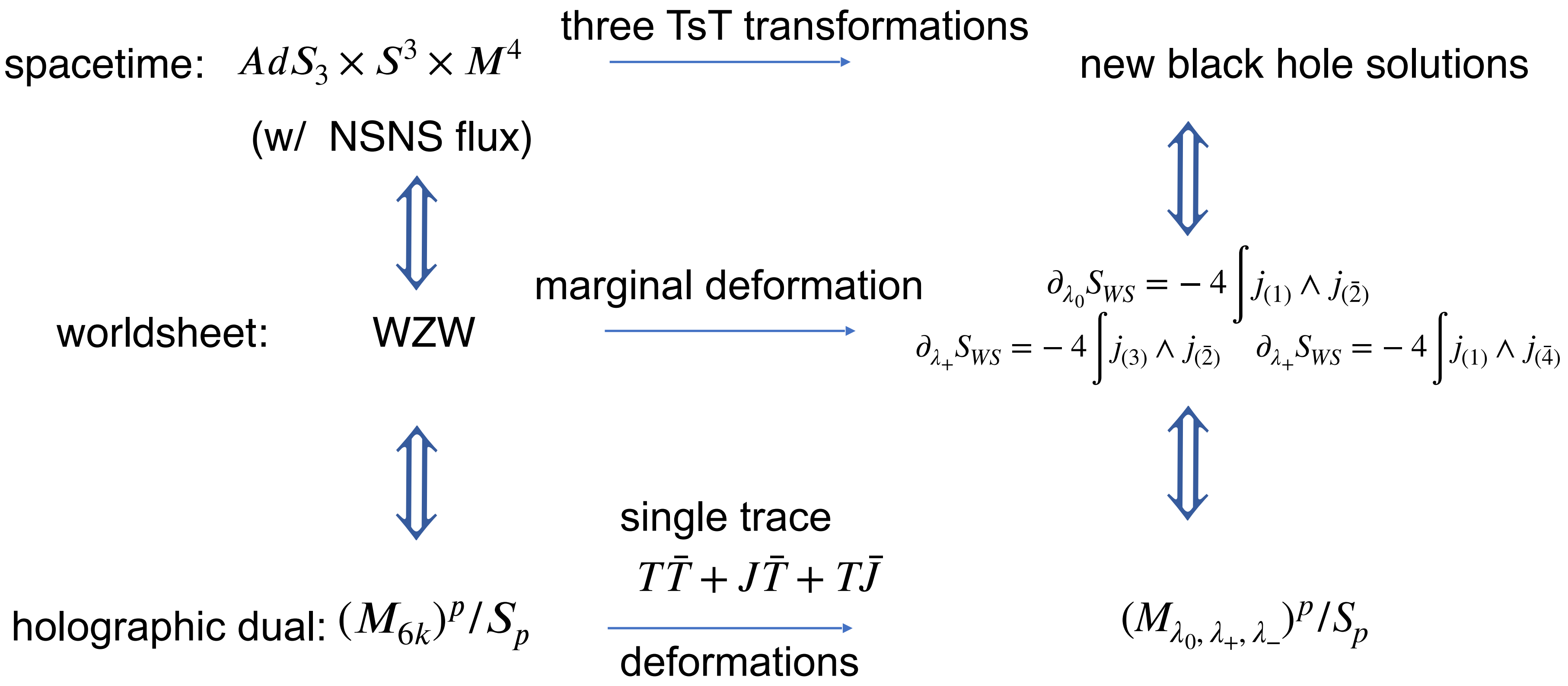
An explicit and tractable toy model for Kerr/CFT in string theory

[Azeyanagi-Hofman-WS-Strominger 13'
Apolo-WS 18', 19', Chakraborty-Giveon-Kutasov 18']

$$\begin{tikzcd}
\text{spacetime:} & AdS_3 \times S^3 \times M^4 \ \text{(w/ NSNS flux)} \arrow[r, "\text{TsT along}"] \arrow[d, Leftrightarrow] & WAdS_3 \times_w S^3 \times M^4 \arrow[d, Leftrightarrow] \\
\text{worldsheet:} & \text{WZW} \arrow[r, "\text{marginal deformation}"] \arrow[d, Leftrightarrow] & \partial_\lambda S_{WS} \propto \int d\sigma^2 j^- \bar{k}^3 \arrow[d, Leftrightarrow] \\
\text{holographic dual:} & (M_{6k})^p / S_p \arrow[r, "\text{single trace } J\bar{T} \text{ deformation}"] & (M_\mu)^p / S_p, \quad \partial_\mu S^i \propto \int dx^2 J_i \bar{T}_i
\end{tikzcd}$$

More general examples

[Chakraborty-Giveon-Kutasov 19’, Apolo-WS, 21’]



spacetime: $AdS_3 \times S^3 \times M^4$ (w/ NSNS flux)

three TsT transformations

new black hole solutions

worldsheet: WZW

marginal deformation

$$\partial_{\lambda_0} S_{WS} = -4\int j_{(1)} \wedge j_{(\bar{2})}$$

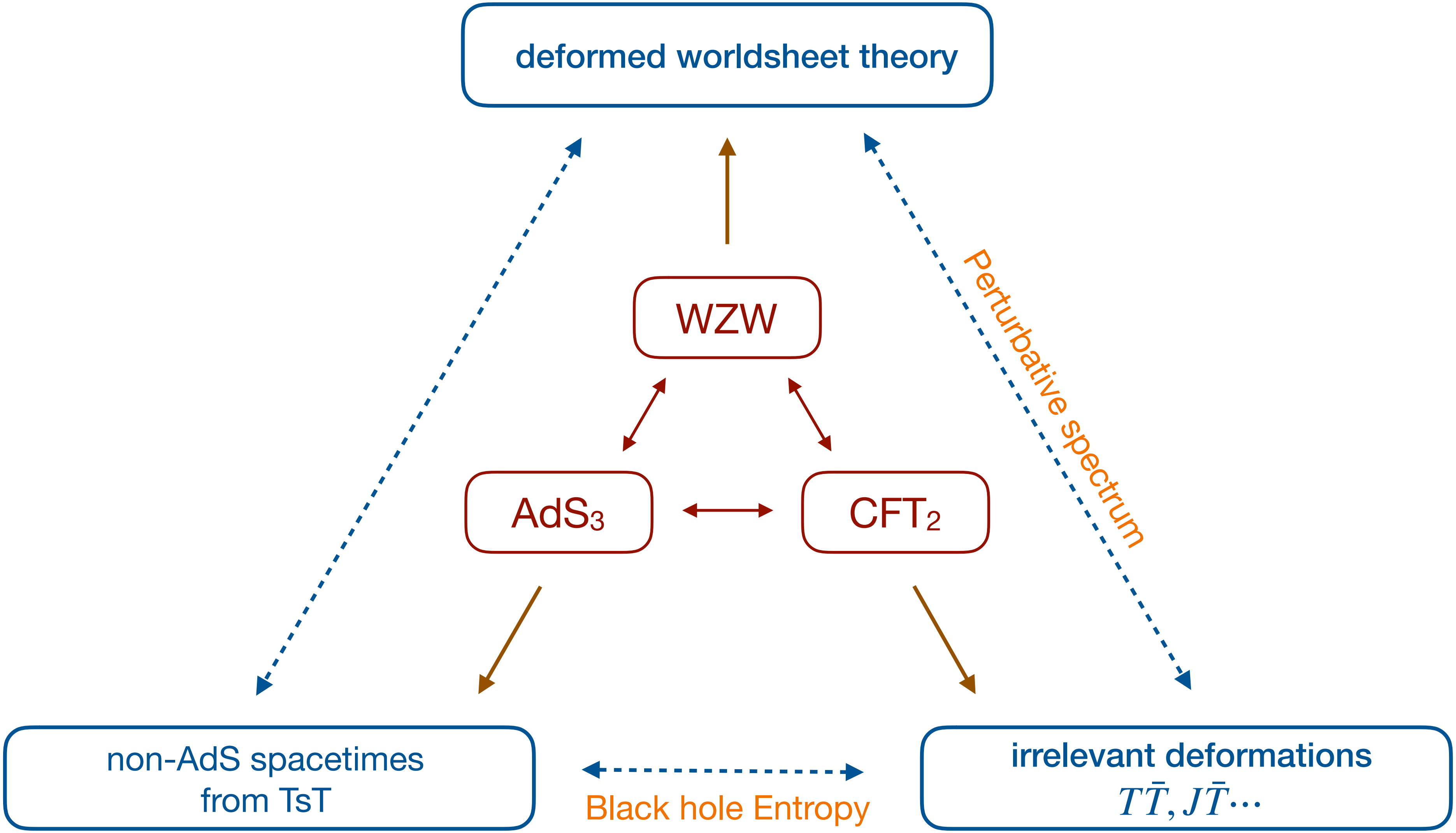
$$\partial_{\lambda_+} S_{WS} = -4\int j_{(3)} \wedge j_{(\bar{2})} \quad \partial_{\lambda_+} S_{WS} = -4\int j_{(1)} \wedge j_{(\bar{4})}$$

holographic dual: $(M_{6k})^p/S_p$

single trace $T\bar{T} + J\bar{T} + T\bar{J}$ deformations

$(M_{\lambda_0, \lambda_+, \lambda_-})^p/S_p$

Summary



deformed worldsheet theory

WZW

AdS$_3$

CFT$_2$

Perturbative spectrum

non-AdS spacetimes
from TsT

Black hole Entropy

irrelevant deformations
$T\bar{T}, J\bar{T}\cdots$

A class of holographic dualities beyond AdS/CFT

IIB string theory on TsT background	↔	solvable, single-trace irrelevant deformations
✓ smooth solution w/o horizon		vacuum state
• asymptotic symmetry		conformal symmetry
✓ spectrum		spectrum
• GKPW		correlation functions
✓ black hole entropy		growth of states
• minimal surface		entanglement entropy

Thank you!

Holographic dualities for irrelevant deformations

- *'double trace'*
- *Universal*
- *local geometry unchanged*
- *changes the boundary condition*

Holography for ``double trace" deformations

- $d = 2,\ T\bar{T}(with\ \mu < 0) \leftrightarrow$ cut-off AdS$_3$ in Einstein gravity *[McGough-Mezei-Verlinde]*
- $d = 2,\ T\bar{T}(with\ \mu > 0) \leftrightarrow$ glue-on AdS$_3$ in Einstein gravity *[Apolo-Hao-Lai-WS, WIP]*
- $d = 2,\ \ T\bar{T} + \Lambda_2 \leftrightarrow$ patch of dS *[Gorbenko-Silverstein-Torroba]*
- $d > 2,\ T\bar{T} \leftrightarrow$ cutoff AdS$_{d+1}$ in Einstein gravity *[Hartman-Kruthoff-Shaghoulian-Tajdini,Taylor]*
- $d = 1,\ T\bar{T} \leftrightarrow$ cutoff JT gravity *[Gross-Kruthoff-Rolph-Shaghoulian, Iliesiu-Kruthoff-Turiaci-Verlinde, Stanford-Yang]*
- $d = 2,\ \ J\bar{T} \leftrightarrow$ AdS$_3$ in Einstein gravity+Chern-Simons gauge theory *[Bzowski-Guica]*

- *'single trace'*
- *embedded in string theory*
- *asymptotic geometry changed*

Holography for ``single trace" deformations

- $d = 2,\ \ T\bar{T} \leftrightarrow$ *linear dilaton background [Giveon-Itzhaki-Kutasov, Apolo-Detournay-WS]*
- $d = 2,\ \ J\bar{T} \leftrightarrow$ WAdS$_3$ in string theory [Chakraborty-Giveon-Kutasov; Apolo-WS]
- $d = 2,\ \ T\bar{T} + J\bar{T} + T\bar{J} \leftrightarrow$ *three TsT transformations in string theory* [Chakraborty-Giveon-Kutasov; Apolo-WS]