Two-Dimensional Turbulence and Conformal Field Theories

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Based on an upcoming paper with Xiaoquan Yu and Jinwu Ye

“Turbulence is the most important unsolved problem of classical physics.”

— Richard Feynman

Outline

Navier-Stokes Equation

- Navier-Stokes equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \frac{1}{\rho}\mathbf{f}$$

- Turbulence is a special kind of (weak) solution.
- It is believed that turbulence is a classical chaotic system.
- One of the Millennium Problems:

Navier-Stokes Equation

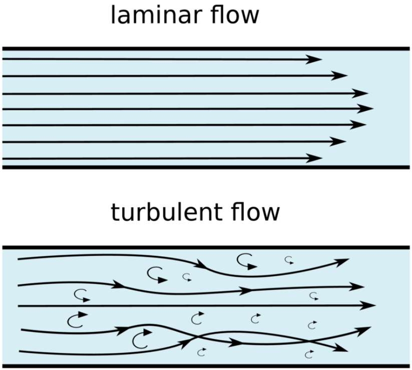
"Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations ..."

Laminar Flow & Turbulent Flow

Reynolds number:

$$Re = \frac{ud}{\nu} \quad (u\text{: velocity, } d\text{: diameter})$$

- $Re \lesssim 2300$: laminar flow
- $Re \gtrsim 2900$: turbulent flow



3D Kolmogorov Scaling

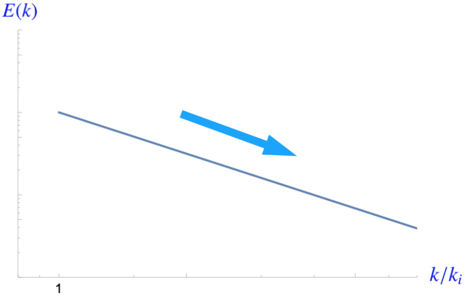
Energy and energy density: $(k \sim (\text{vortex size})^{-1})$

$$E = \frac{1}{2} \int d^2k \, \langle \mathbf{u}_\alpha(k) \, \mathbf{u}_\alpha(-k) \rangle = \int dk \, E(k)$$

Kolmogorov (1941):

- In the inertial range $k_o \ll k \ll k_c$:

$$E(k) \sim k^{-5/3}$$



2D Kraichnan Scalings

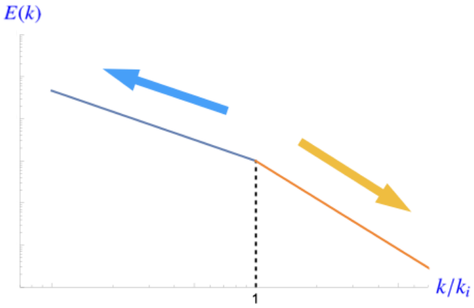
Kraichnan (1967):

- Inverse energy cascade in the range $k_0 \ll k \ll k_i$:

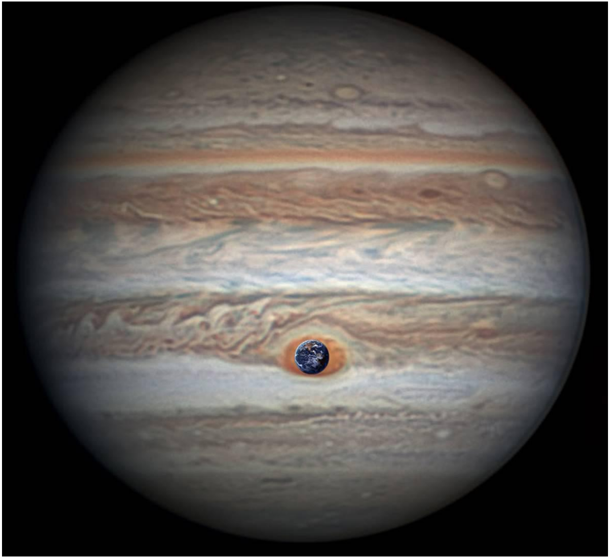
$$E(k) \sim k^{-5/3}$$

- Direct enstrophy cascade in the range $k_i \ll k \ll k_c$:

$$E(k) \sim k^{-3}$$



Jupiter's Big Red Spot



Polyakov's Idea

The Navier-Stokes equation (in velocity $\mathbf{u}$):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \frac{1}{\rho} \mathbf{f}$$

The Navier-Stokes equation (in vorticity ω):

$$\dot{\omega} + \epsilon_{\alpha\beta}\, \partial_\alpha \psi\, \partial_\beta \partial^2 \psi = \nu\, \partial^2 \omega + F$$

with

$$\omega \equiv \partial^2 \psi\,, \quad \mathbf{u}_\alpha = \epsilon_{\alpha\beta} \partial_\beta \psi$$

The turbulence emerges when $\nu \to 0$ and $\mathbf{f}/\rho \to 0$:

$$\dot{\omega} = -\epsilon_{\alpha\beta}\, \partial_\alpha \psi\, \partial_\beta \partial^2 \psi$$

Main idea: ('92 Polyakov)

- interpret this equation as an operator equation in a 2d CFT
- identify ψ with a primary field in the 2d CFT

Polyakov's Idea

For a steady flow: (reduce the time dependence)

$$\dot{\omega} = 0 \quad \Rightarrow \quad -\epsilon_{\alpha\beta}\, \partial_\alpha \psi\, \partial_\beta \partial^2 \psi = 0$$

Suppose that a CFT has the fusion:

$$[\psi] \times [\psi] = [\phi] + \cdots$$

then in terms of a small scale *UV* cutoff *a*:

$$-\epsilon_{\alpha\beta}\, \partial_\alpha \psi\, \partial_\beta \partial^2 \psi \;\sim\; (a\bar{a})^{h_\phi - 2h_\psi} \left(L_{-2} \bar{L}_{-1}^2 - \bar{L}_{-2} L_{-1}^2 \right) \phi\,,$$

Possible solutions to $-\epsilon_{\alpha\beta}\, \partial_\alpha \psi\, \partial_\beta \partial^2 \psi = 0$:

1. $h_\phi > 2 h_\psi$:

   the RHS vanishes in the limit $a \to 0$.

2. $h_\psi \leq 2 h_\psi$:

   require that ϕ is a degenerate field or the RHS is a symmetry.

2-Point and 3-Point Functions

- The energy and enstrophy are related to 2-point functions:

$$E = \frac{L^{2(h_{\widetilde{\alpha}}+h_{\widetilde{\beta}})}}{2} \int d^2k \, \langle \mathbf{u}(\mathbf{k}) \, \mathbf{u}(-\mathbf{k}) \rangle_{\widetilde{\alpha}\widetilde{\beta}}$$

$$H = \frac{L^{2(h_{\widetilde{\alpha}}+h_{\widetilde{\beta}})}}{2} \int d^2k \, \langle \omega(\mathbf{k}) \, \omega(-\mathbf{k}) \rangle_{\widetilde{\alpha}\widetilde{\beta}}$$

- The energy and enstrophy fluxes are related to 3-point functions:

$$\begin{aligned} J^{(E)}(q) &= -L^{2(h_{\widetilde{\alpha}}+h_{\widetilde{\beta}})} \int_{|\mathbf{k}|>q} d^2k \, \langle \dot{\mathbf{u}}(\mathbf{k}) \, \mathbf{u}(-\mathbf{k}) \rangle_{\widetilde{\alpha}\widetilde{\beta}} \\ &= L^{2(h_{\widetilde{\alpha}}+h_{\widetilde{\beta}})} \int_{|\mathbf{k}|<q} d^2k \, \langle \dot{\mathbf{u}}(\mathbf{k}) \, \mathbf{u}(-\mathbf{k}) \rangle_{\widetilde{\alpha}\widetilde{\beta}} \\ J^{(H)}(q) &= -L^{2(h_{\widetilde{\alpha}}+h_{\widetilde{\beta}})} \int_{|\mathbf{k}|>q} d^2k \, \langle \dot{\omega}(\mathbf{k}) \, \omega(-\mathbf{k}) \rangle_{\widetilde{\alpha}\widetilde{\beta}} \\ &= L^{2(h_{\widetilde{\alpha}}+h_{\widetilde{\beta}})} \int_{|\mathbf{k}|<q} d^2k \, \langle \dot{\omega}(\mathbf{k}) \, \omega(-\mathbf{k}) \rangle_{\widetilde{\alpha}\widetilde{\beta}} \end{aligned}$$

- The CFT can be a boundary CFT. Hence, there can be boundary operator insertions ($\widetilde{\alpha}$, $\widetilde{\beta}$).

Conformal Turbulence

- The scaling:

$$E(k) \sim \begin{cases} k^{4h_\psi + 1}\,, & \text{if } \langle \Phi \rangle = 0 \text{ for } \Phi \neq I\,; \\ k^{4h_\psi - 2h_\phi + 1}\,, & \text{if } \langle \Phi \rangle_{\tilde{\alpha}\,\tilde{\beta}} \neq 0 \text{ for any } \Phi\,. \end{cases}$$

- For the enstrophy cascade ($J^{(H)} = \text{const}$):

$$\begin{cases} h_\psi + h_\phi = -3\,, & \text{if } \langle \Phi \rangle = 0 \text{ for } \Phi \neq I\,; \\ h_\psi + h_\phi - h_\chi = \frac{1}{2}(h_{\tilde{\alpha}} + h_{\tilde{\beta}})\,, & \text{if } \langle \Phi \rangle_{\tilde{\alpha}\,\tilde{\beta}} \neq 0 \text{ for any } \Phi\,. \end{cases}$$

- For the energy cascade ($J^{(E)} = \text{const}$):

$$\begin{cases} h_\psi + h_\phi = -2\,, & \text{if } \langle \Phi \rangle = 0 \text{ for } \Phi \neq I\,; \\ h_\psi + h_\phi - h_\chi = \frac{1}{2}(h_{\tilde{\alpha}} + h_{\tilde{\beta}})\,, & \text{if } \langle \Phi \rangle_{\tilde{\alpha}\,\tilde{\beta}} \neq 0 \text{ for any } \Phi\,. \end{cases}$$

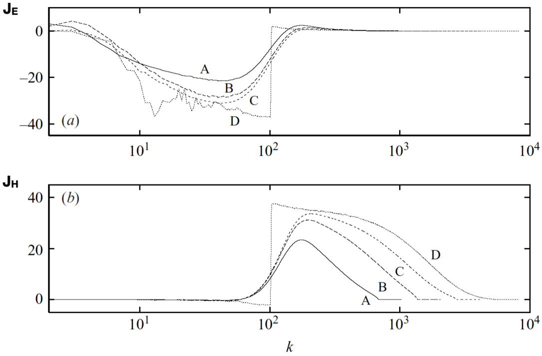
- **Polyakov's trial solution:**

  A solution is the $(21, 2)$ minimal model ($c = -354/7$) with $h_\psi = -\frac{8}{7}, \quad h_\phi = -\frac{13}{7} \quad \Rightarrow \quad E(k) \sim k^{-25/7}\,, \quad h_\phi > 2h_\psi$

Criteria for Candidate CFTs

For the candidate CFTs:

1. They should be Euclidean 2d non-unitary CFTs;
2. They should give the correct Kraichnan scalings;
3. They should give the correct fluxes and cascade directions. ('07 Boffetta)



The Status of the Problem

- **Enstrophy cascade**: (until Spring 2021)

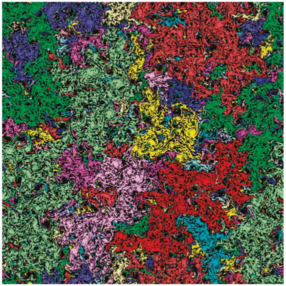
  Hundreds of CFT models have been constructed.

  No satisfactory answer has been found.

- **Energy cascade**: (until Spring 2021)

  There are some hints from numerical simulations.

  ('06 Bernard, Boffetta, Celani, Falkovich)



No satisfactory theoretical answer has been found.

CFT for the Enstrophy Cascade

- Recall that a minimal model $(p',\, p)$ has the central charge

$$c = 1 - 6\frac{(p-p')^2}{p\,p'}$$

and the primary operators labeled by $\phi_{r,\,s}$ $(1 \leq r \leq p'-1,$ $1 \leq s \leq p-1)$ with the conformal weights

$$h(r,\,s) = \frac{(rp - sp')^2 - (p-p')^2}{4p\,p'}$$

- CFT for the enstrophy cascade: ('22 JN, Yu, Ye)

minimal model $(p',\, p)$ **in the limit** $p' \to \infty$**,** p **finite**

$$c = -6p'/p + \mathcal{O}(1) \;\; \to \;\; -\infty$$

This is called the **classical limit** of minimal models.

Kraichnan Scaling from the CFT Model

- There is the operator fusion rule:

$$\phi_{3,1} \times \phi_{3,1} = I + \phi_{3,1} + \phi_{5,1}$$

- For the enstrophy cascade, we identify

$$\psi = \phi_{3,\,1}\,, \quad \phi = \phi_{5,\,1}$$

$$h_\psi = -1 + \frac{2p}{p'} + \mathcal{O}\left((p/p')^{-2}\right)\,, \; h_\phi = -2 + \frac{6p}{p'} + \mathcal{O}\left((p/p')^{-2}\right)$$

- The scaling:

$$E(k) \sim k^{4h_\psi + 1} = k^{-3+\mathcal{O}(c^{-1})}$$

- The $\mathcal{O}(c^{-1})$ correction is exactly needed for the convergence of the integral $E = \int dk\, E(k)$!

Fluxes from the CFT Model

- So far, we have not introduced viscosity in the N-S equation
  $\Rightarrow$ no fluxes
- In order to have a nonzero flux $J^{(H)}$, we turn on a small viscosity (still no stirring force):

$$\dot{\omega} + \epsilon_{\alpha\beta}\, \partial_\alpha \psi\, \partial_\beta \partial^2 \psi = \nu\, \partial^2 \omega + (F = 0)$$

- The enstrophy flux:

$$J^{(H)}(q) = -\int_{q<|\mathbf{k}|<a^{-1}} d^2k\, \langle \dot{\omega}(\mathbf{k})\, \omega(-\mathbf{k}) \rangle \simeq -\frac{\nu}{2\, a^2} \;\to\; \text{const}$$

The negative sign indicates that the direction of $J^{(H)}$ is direct.

- The energy flux:

$$J^{(E)}(q) = -\int_{q<|\mathbf{k}|<a^{-1}} d^2k\, \langle \dot{\mathbf{u}}(\mathbf{k})\, \mathbf{u}(-\mathbf{k}) \rangle \simeq -\nu \log \frac{1}{q\sqrt{\nu}} \;\to\; 0$$

General W_N CFT

The W_N CFT's are defined as the coset WZW models

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}$$

- It is a CFT with the W_N-algebra.
- It has the central charge

$$c = (N-1)\left[1 - \frac{N(N+1)}{(N+k)(N+k+1)}\right]$$

- Usually the 't Hooft limit of the W_N CFT is discussed in the literature: ('10 Gaberdiel, Gopakumar)

$$N \to \infty, \quad k \to \infty, \quad \lambda \equiv \frac{N}{N+k} \text{ finite}$$

- There exists another limit, the semiclassical limit, which makes $c \to \pm\infty$: ('12 Perlmutter, Procházka, Raeymaekers)

$$k \to -N-1, \quad N \text{ finite}$$

An Equivalent Formulation

minimal model (p', p) in the limit $p' \to \infty$, p finite
$=$ semiclassical W_2 CFT at $k = -3^-$ and $N = 2$

In this equivalent formulation:

the primary fields are denoted by (Λ_+, Λ_-),

where $\Lambda_\pm$ are representations of $\mathfrak{su}(2)$.

$$\psi = \left(\square\!\square\,,\ \bullet\right)\,,\quad \phi = \left(\square\!\square\!\square\!\square\,,\ \bullet\right)$$

$$\square\!\square \otimes \square\!\square = \bullet \oplus \square\!\square \oplus \square\!\square\!\square\!\square$$

$$h_\psi = -1 - \frac{12}{c} + \mathcal{O}(c^{-2})\,,\ h_\phi = -2 - \frac{36}{c} + \mathcal{O}(c^{-2})$$

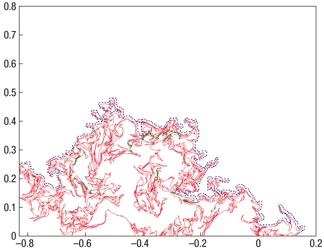
$$\Rightarrow \quad E(k) \sim k^{-3+\mathcal{O}(c^{-1})}\,,\quad h_\phi > 2h_\psi$$

SLE and CFT

- Use the Schramm-Loewner evolution (SLE) ('05 Cardy)

$$\langle (\xi(t) - \xi(0))^2 \rangle = \kappa t$$

to analyze 2d turbulence in energy cascade:
('06 Bernard, Boffetta, Celani, Falkovich)



From the fractal dimension:

$$D_\kappa = 1 + \frac{\kappa}{8}$$

$$\Rightarrow \quad \kappa = \frac{8}{3} \text{ and } 6$$

- A 2d SLE implies an underlying CFT with the central charge

$$c = \frac{(8 - 3\kappa)(\kappa - 6)}{2\kappa}$$

CFT for the Energy Cascade

- CFT for the energy cascade: ('22 JN, Yu, Ye)

$$((Q{=}1)\text{-Potts model}) \oplus (O(N{=}0)\text{ model}) \text{ with boundary}$$

- It is a boundary logarithmic CFT with $c = 0$.
- Two constituting CFTs have the partition functions:

$$Z_{N=0} = 1\,, \qquad \frac{dZ}{dN}\bigg|_{N=0} \neq 0$$

$$Z_{Q=1} = 1\,, \qquad \frac{dZ}{dN}\bigg|_{Q=1} \neq 0$$

- The direct sum CFT has

$$Z = 1\,, \qquad \frac{dZ}{dN}\bigg|_{c=0} \neq 0$$

$O(N)$ and Q-Potts Models

$O(N)$ model	dilute phase ($1 \leq g \leq 2$) (for $1 \leq g \leq \frac{3}{2}$ dual to $Q = N^2$ tricritical Potts model)	dense phase ($0 \leq g \leq 1$) (for $\frac{1}{2} \leq g \leq 1$ dual to $Q = N^2$ critical Potts model)
$N = 2$	XY model at the Kosterlitz-Thouless transition point ($g = 1$, $c = 1$)	
$N = \sqrt{2}$	$O(\sqrt{2})$ model ($g = \frac{5}{4}$, $c = \frac{7}{10}$)	(Q=2)-Potts model log CFT ($g = \frac{3}{4}$, $c = \frac{1}{2}$)
$N = 1$	Ising model ($g = \frac{4}{3}$, $c = \frac{1}{2}$)	(Q=1)-Potts model log CFT ($g = \frac{2}{3}$, $c = 0$)
$N = 0$	$O(0)$ model log CFT dilute polymer ($g = \frac{3}{2}$, $c = 0$)	(Q=0)-Potts model log CFT dense polymer ($g = \frac{1}{2}$, $c = -2$)
$N = -2$	$O(-2)$ model log CFT ($g = 2$, $c = -2$)	classical limit ($g \to 0^+$, $c \to -\infty$)

$O(N{=}0)$ and $(Q{=}1)$-Potts Models

('13 Cardy)

- There are several different log CFTs with $c = 0$.
- To distinguish them, we need to define a b-parameter.
- Assume that there exists an operator $\widetilde{T}$ with the conformal weights $(2+\delta,\, \delta)$.

$$b \equiv -\frac{1}{2} \lim_{c \to 0} \frac{c}{\delta}$$

- In the bulk:

$$b_{\text{bulk}} = \begin{cases} -5\,, & \text{for the 2d } O(N{=}0) \text{ model}\,; \\ -5\,, & \text{for the 2d } (Q{=}1)\text{-Potts model}\,. \end{cases}$$

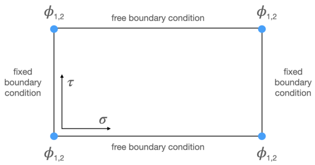
- On the boundary:

$$b_{\text{bdy}} = \begin{cases} \dfrac{5}{6}\,, & \text{for the 2d } O(N{=}0) \text{ model}\,; \\ -\dfrac{5}{8}\,, & \text{for the 2d } (Q{=}1)\text{-Potts model}\,. \end{cases}$$

A boundary condition changing operator $\phi_{2,1}$ in the $(Q{=}1)$-Potts model can again make $b = \frac{5}{6}$.

The CFT Realization

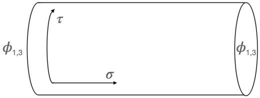
- Geometric setup is similar to 2d percolation: ('91 Cardy)



- Insert four $\phi_{1,2}$ at four corners of a rectangular region.

$$\phi_{1,2} \times \phi_{1,2} = I + \phi_{1,3}$$

- I and $\phi_{1,3}$ correspond to different boundary conditions:



- This is a boundary CFT with boundary operator insertions.

Kraichnan Scaling from Boundary Log CFT

- The have both Kac and non-Kac operators.
- The conformal weights for the Kac operators in the limit $c \to 0$:

$$h_{r,s}(c) = \frac{(3r-2s)^2 - 1}{24} + \mathcal{O}(c)$$

- Identify the operators with 2d energy cascade:

$$\phi_{1,3} \times \phi_{1,3} = I + \phi_{1,3} + \phi_{1,5}$$

$$\psi = \phi_{1,3}\,, \quad \phi = C\,\phi_{1,5} + T\overline{T}/\delta\,, \quad \chi = C'\,\phi_{1,5} + T\overline{T}/\delta$$

$$h_\psi = \frac{1}{3}\,, \qquad h_\phi = 2\,, \qquad h_\chi = 2$$

- The scaling:

$$E(k) \sim k^{4h_\psi - 2h_\phi + 1} = k^{-\frac{5}{3}}$$

Fluxes from Boundary Log CFT

- So far, we have not considered the stirring force in the N-S equation.
- In order to have a nonzero flux $J^{(E)}$, we turn on a small stirring force:

$$\widetilde{\mathbf{f}}(\mathbf{k}) = C_f \, \epsilon_{\alpha\beta} \, \frac{\mathbf{k}_\beta}{|\mathbf{k}|^2 + m^2}$$

- The enstrophy flux:

$$J^{(H)}(q) = -\int_{q<|\mathbf{k}|<a^{-1}} d^2k \, \langle \dot{\omega}(\mathbf{k}) \, \omega(-\mathbf{k}) \rangle \simeq \left(\frac{a}{\ell}\right)^2 \left(\frac{L}{\ell}\right)^{\frac{2}{3}} \to 0$$

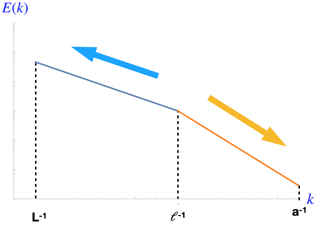
- The energy flux:

$$J^{(E)}(q) = -\int_{q<|\mathbf{k}|<a^{-1}} d^2k \, \langle \dot{\mathbf{u}}(\mathbf{k}) \, \mathbf{u}(-\mathbf{k}) \rangle \simeq \frac{C_f}{\rho} \left(\frac{L}{\ell}\right)^{2/3} \to \text{const}$$

The positive sign implies that the direction of $J^{(E)}$ is inverse.

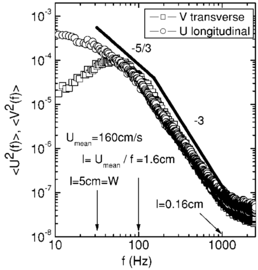
Test from Experimental Data

- The consistency of our CFT model requires a condition:



$$\left(\frac{L}{\ell}\right)^{\frac{2}{3}} \ll \left(\frac{\ell}{a}\right)^{2}$$

- Experimental data: ('02 Kellay, Goldburg; '05 Bruneau, Kellay; '20 Lee)



The condition is satisfied.

2D Turbulence and KdV

- There is a mysterious fact that 2d turbulence has infinitely many conserved quantities $H_n = \int d^2x\, \omega^n(x)$.
- In general, a CFT's energy-momentum tensor satisfies the quantum KdV equation:

$$\partial_t T = \frac{1}{6}(1-c)\,\partial_z^3 T - 3\,\partial_z(TT)$$

- In both limits $c \to -\infty$ and $c \to 0$, we obtain a classical KdV equation:

$$\partial_t T = \frac{1}{6}\partial_z^3 T - 3\,\partial_z(TT) \quad \Leftrightarrow \quad \partial_\tau u = \partial_z^3 u + 6\,u\,\partial_z u$$

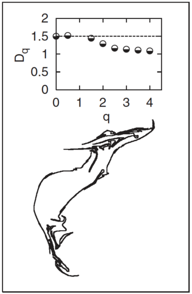
- A classical KdV equation naturally has infinitely many conserved quantities:

$$I_n \equiv \int_0^{2\pi} \frac{d\tau}{2\pi}\,[u^n(z,\tau) + \cdots]$$

- H_n and I_n are in one-to-one correspondence.

Deviation from CFT

- Multifactality: ('91 Sreenivasan; '07 Bernard, Boffetta, Celani, Falkovich)



- Recall the relation between a SLE$_\kappa$ and a CFT:

$$c = \frac{(8-3\kappa)(\kappa-6)}{2\kappa}\,, \qquad D_\kappa = 1 + \frac{\kappa}{8}$$

- The enstrophy cascade has a slight deviation from a CFT with $c \to -\infty$.

Summary and Prospect

Summary: We have found the CFTs for 2d turbulence.

- For the 2d direct enstrophy cascade:

  $$[(p', p) \text{ with } p' \to \infty, p \text{ finite}] = [W_2 \text{ CFT at } k = -3^-, N = 2]$$

- For the 2d inverse energy cascade:

  $$((Q{=}1)\text{-Potts model}) \oplus (O(N{=}0) \text{ model})$$

- Correct scalings
- Correct fluxes and cascade directions
- Explains the infinite conserved quantities

Prospect:

- 2d conformal turbulence from **non-unitary** AdS/CFT?
- Compute more quantities from the CFTs
- 3d conformal turbulence?

Thank you!