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Causality constraints on multi-field EFTs

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“Anything goes” for EFT coefficients?

Lagrangian $$\mathscr{L}_{\text{EFT}} = \sum_i \Lambda^4 \, c_i \, \mathscr{O}_i \left(\frac{\text{boson}}{\Lambda}, \frac{\text{fermion}}{\Lambda^{3/2}}, \frac{\partial}{\Lambda} \right)$$

Λ: EFT cutoff c_i: Wilson coefficients

Amplitude $$\mathscr{A}_{\text{EFT}}(s,t) = \sum_{m,n} \frac{c_{m,n}}{\Lambda^{2m+2n}} s^m t^n$$

Question: Are Wilson coefficients $c_{m,n}$ allowed to take any values?

Answer: No!

Positivity bounds/Causality constraints

high energy UV theory
maybe unknown, but assume causality, unitarity, ...

Positivity bounds/
Causality constraints

"EFT bootstrap"

low energy EFT
constraints on Wilson coefficients

Simple example

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2 + \cdots$$

$$A(s, t=0) = \cdots + \frac{2\lambda s^2}{\Lambda^4} + \cdots$$

"First" positivity bound: $\lambda > 0$

$$\mathscr{L}_{\mathrm{DBI}} \sim -\sqrt{1 + (\partial\phi)^2}$$ ✓

$$\mathscr{L}_{\overline{\mathrm{DBI}}} \sim -\sqrt{1 - (\partial\phi)^2}$$ ✗

Significant advances recently:

Adams, Alberte, Aoki, Arkani-Hamed, Baumann, Bellazzini, Bern, Caron-Huot, Chandrasekaran, Cheung, Chiang, Creminelli, de Rham, Dubovsky, Elias Miro, Fuks, Grall, Green, Guerrieri, Hanada, Henriksson, Herrero-Valea, Hirano, Huang, Jaitly, Janssena, Jenkins, Kim, Kundu, Lee, Lewandowski, Li, Liu, McPeak, Melville, Momeni, Noller, Noumi, Nicolis, O'Connell, Penedones, Porto, Rattazzi, Remmen, Riembau, Riva, Rodd, Rodina, Russo, Rumbutis, Santos-Garcia, Senatore, Serra, Sgarlata, Shahbazi-Moghaddam, Shiu, Sinha, Timiryasov, Tokareva, Tokuda, Tolley, Trincherini, Trott, Van Duong, Vichi, Wang, Weng, Xu, Yamashita, Yang, Zahed, Zhang, Zhiboedov, Zhou, …

Snowmass White Paper: UV Constraints on IR Physics, de Rham, Kundu, Reece, Tolley & **SYZ**, 2203.06805

Similar to swampland idea

But positivity bounds take more conservative approach

Causality implies analyticity

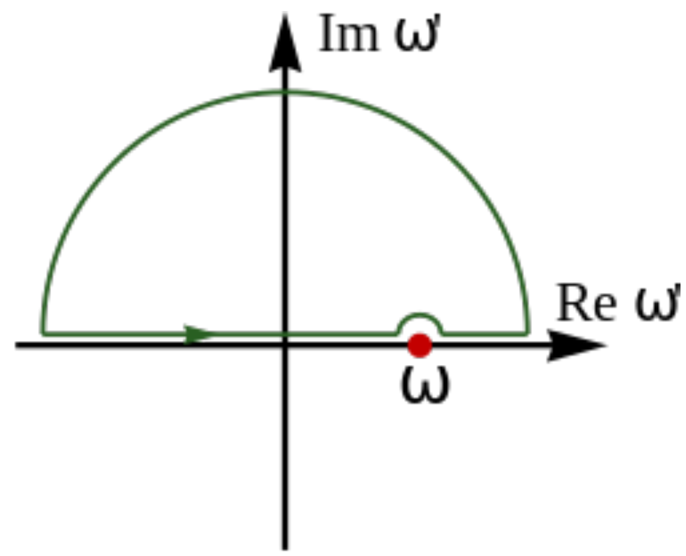
Kramers-Kronig dispersion relation

Titchmarsh's theorem

$$f(t<0)=0$$

$\tilde{f}(\omega)$ square-integrable

→ $\tilde{f}(\omega)$ analytic in upper ω plane



→
$$\tilde{f}(\omega)=\frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{\omega'-\omega}\tilde{f}(\omega')$$

eg, complex refractive index n

Relativistic version: response restricted with light-cone

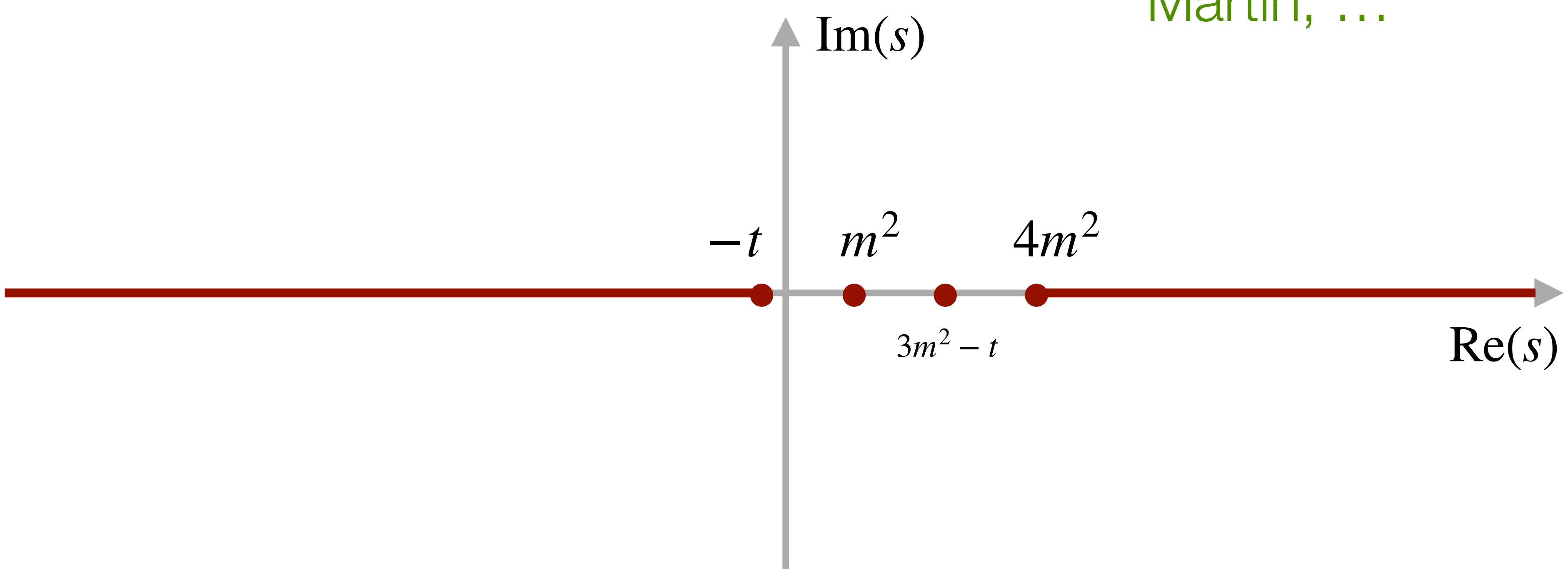
$$f(t,\boldsymbol{x})=\theta(t-\boldsymbol{\xi}\cdot\boldsymbol{x})f(t,\boldsymbol{x})$$
$$\boldsymbol{\xi}^2<1$$

→
$$\tilde{f}(\omega,\boldsymbol{k}_0+\omega\boldsymbol{\xi})=\frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{\omega'-\omega}\tilde{f}(\omega',\boldsymbol{k}_0+\omega'\boldsymbol{\xi})$$

Analyticity of scattering amplitude

$A(s, t)$ as analytic function

rigorously proven in 60'
Martin, …



Locality: $A(s, t)$ is polynomially bounded at high energies

Froissart(-Martin) bound: Froissart, 1961; Martin, 1962

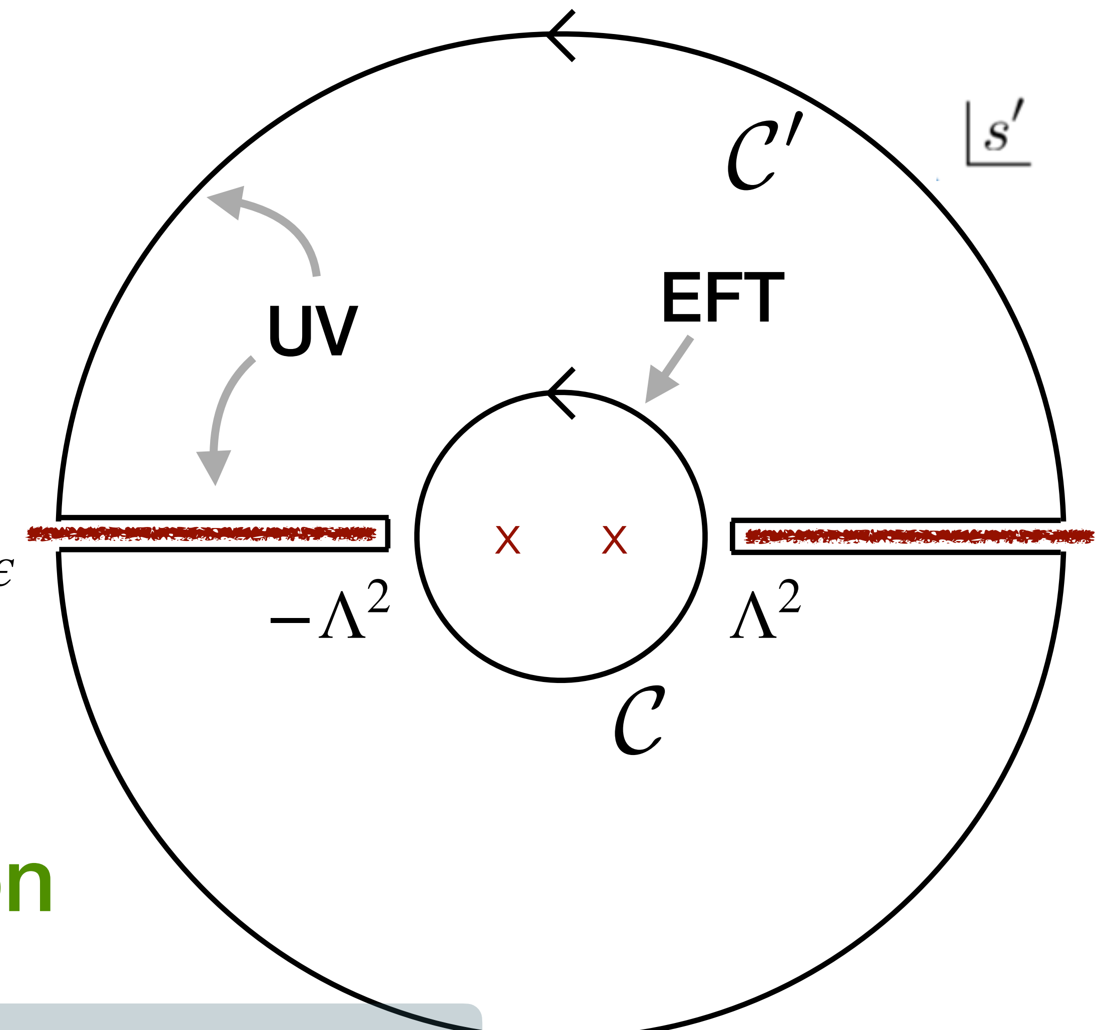
$$\lim_{s\to\infty} |A(s,t)| < C s^{1+\epsilon(t)}, \quad t < 4m^2, \quad 0 < \epsilon(t) < 1$$

Fixed t dispersion relation

- Analyticity in complex s plane (fixed t)

$$A(s,t) = \frac{1}{2\pi i}\oint_{\mathcal{C}} ds' \, \frac{A(s',t)}{s'-s}$$

- Froissart bound $|A(s' \to \infty, t)| < s'^{2-\epsilon}$
- su crossing symmetry



Twice subtracted dispersion relation

$$A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu-s}+\frac{u^2}{\mu-u}\right]\mathrm{Im}\,A(\mu,t)$$

EFT amplitude **IR/UV connection** **UV full amplitude**

Forward positivity bounds

Forward limit $t = 0$

$$A(s,0) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{s^2}{\mu + s}\right] \mathrm{Im}\, A(\mu, 0)$$

$$c_{2,0}s^2 + c_{4,0}s^4 + \cdots = \left(\int \frac{2\,\mathrm{d}\mu}{\pi\mu^3}\mathrm{Im}\, A(\mu,0)\right)s^2 + \left(\int \frac{2\,\mathrm{d}\mu}{\pi\mu^5}\mathrm{Im}\, A(\mu,0)\right)s^4 + \cdots$$

matching → **Sum rules:**

$$c_{2n,0} = \int \frac{2\,\mathrm{d}\mu}{\pi\mu^{1+2n}}\mathrm{Im}\, A(\mu,0)$$

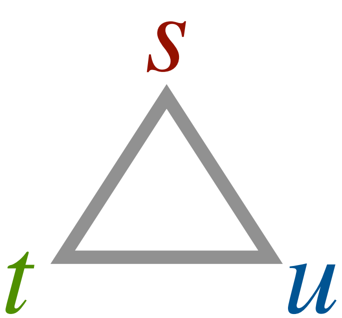
μ: scale of UV particles

"First" bounds

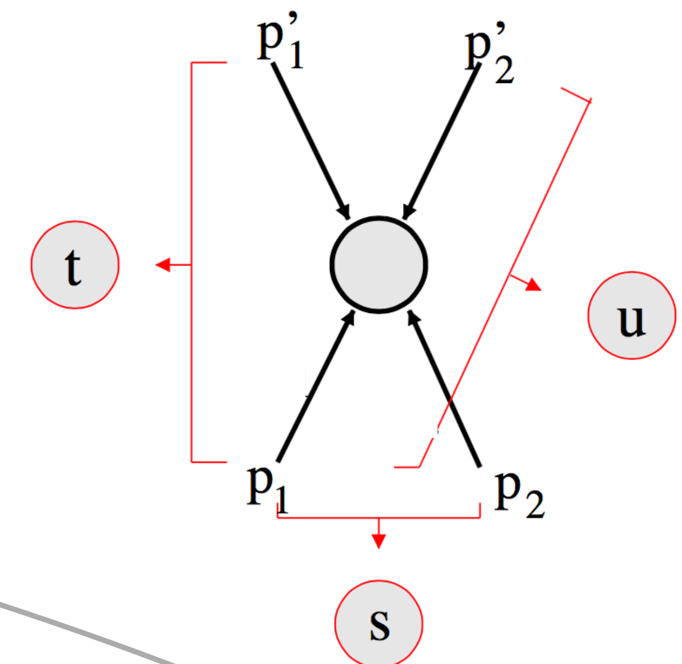
Optical theorem
$\mathrm{Im}[A(s,0)] \propto \sigma(s) > 0$ → $c_{2n,0} > 0$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006 + earlier works

Magic of crossing symmetry



We have not used *st* symmetry



$$A(u,t) = A(s,t) = A(t,s)$$

$$\int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{t^2}{\mu - t} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, s)$$

Null constraints

Tolley, Wang & **SYZ**, 2011.02400
Caron-Huot & Duong, 2011.02957

$$\sum_{\ell} \int d\mu \frac{\mathrm{Im} a_\ell(\mu)}{\mu^{i+j}} \Gamma_{i,j}^{(n)}(\ell) = 0$$

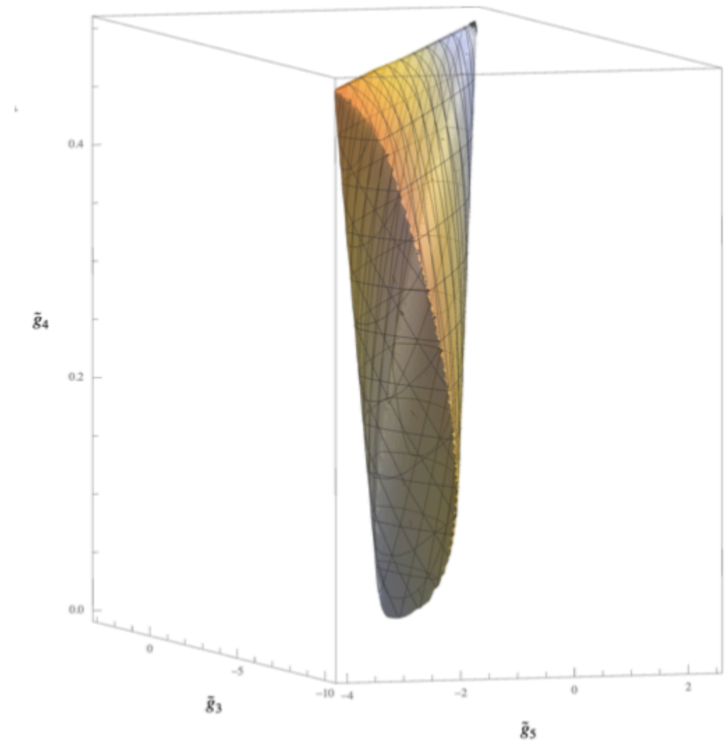
st crossing imposes constraints on $\mathrm{Im} a_\ell$

Two-sided bounds

$$A(s,t) \sim c_{2,0}s^2 + c_{2,1}s^2 t + c_{2,2}s^2 t^2 + \cdots$$

All Wilson coefficients have to be parametrically $O(1)$!

EFT coefficient	Lower bound	Upper bound
$\tilde{g}_3$	-10.346	3
$\tilde{g}_4$	0	0.5
$\tilde{g}_5$	-4.096	2.5
$\tilde{g}_6$	0	0.25
$\tilde{g}'_6$	-12.83	3
$\tilde{g}_7$	-1.548	1.75
$\tilde{g}_8$	0	0.125
$\tilde{g}'_8$	-10.03	4
$\tilde{g}_9$	-0.524	1.125
$\tilde{g}'_9$	-13.60	3
$\tilde{g}_{10}$	0	0.0625
$\tilde{g}'_{10}$	-6.32	3.75



used to be a folklore, called "naturalness/naive dimension analysis" but now a rigorous QFT theorem

Tolley, Wang & **SYZ**, 2011.02400
Caron-Huot & Duong, 2011.02957

Universe is more complex than just one identical scalar!

Multi-field bounds for s^2 coefficients

- lowest order positivity bounds — dim-8 ops
- phenomenologically more relevant

$$A(s, t=0) = \frac{c^{2,0} s^2}{\Lambda^4} + \cdots \quad \Longrightarrow \quad A_{ij\to kl}(s, t=0) = \frac{c^{2,0}_{ij\to kl} s^2}{\Lambda^4} + \cdots$$

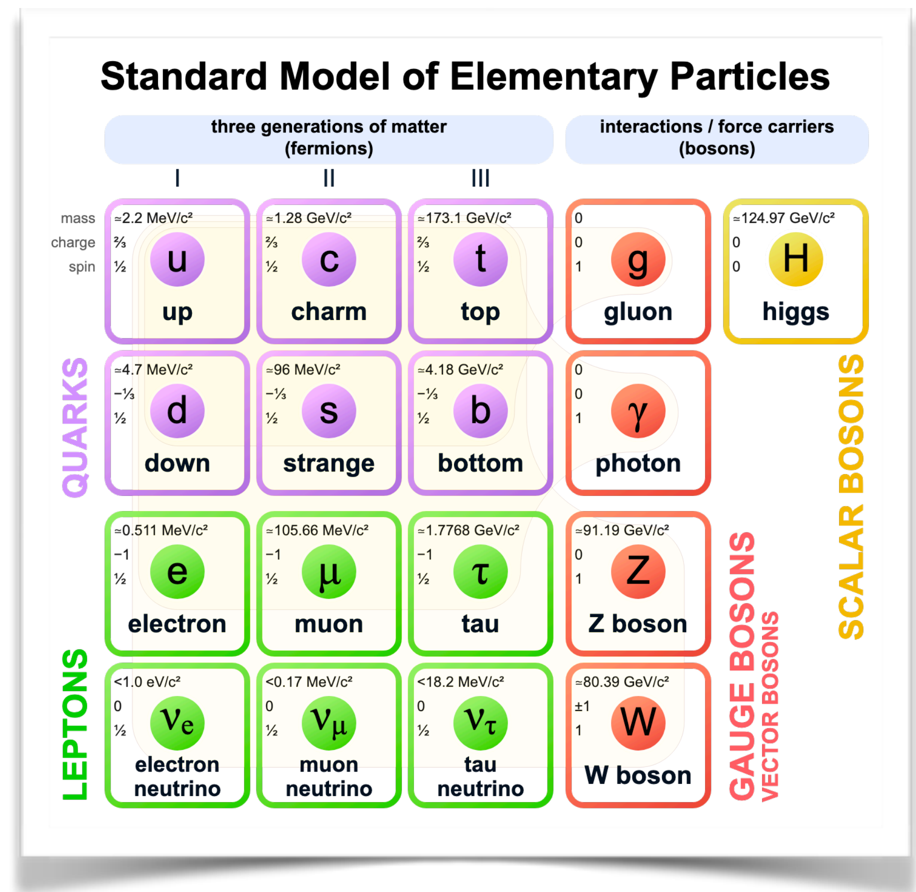
Motivation: SMEFT

SM Effective Field Theory (SMEFT)

$$\mathscr{L}_{\mathrm{SMEFT}} = \mathscr{L}_{\mathrm{SM}} + \sum_j \frac{c_j^{(6)} O_j^{(6)}}{\Lambda^2} + \sum_i \frac{c_i^{(8)} O_i^{(8)}}{\Lambda^4} + \cdots$$

- SM particle contents
- SM symmetries

- Parametrize new physics
- Popular current approach



if consider up to dim-8, or order s^2

still huge parameter space!

How much can positivity bounds reduce the parameter space?

Generalized elastic positivity bounds

previous positivity bounds valid for identical particle scattering

Elastic scattering: particle i + particle j $\to$ particle i + particle j

$$M^{ijij} = c_{2,0}^{ijij} > 0$$

proof still goes through

Generalized elastic scattering: $a + b \to a + b$

massless limit

superposed states $|a\rangle = \sum_i u_i |i\rangle, \quad |b\rangle = \sum_j v_j |j\rangle$

u_i, v_i: arbitrary constants

$$M^{abab} = \sum_{ijkl} u_i v_j u_k^* v_l^* M^{ijkl} = \sum_{ijkl} u_i v_j u_k^* v_l^* c_{2,0}^{ijkl} > 0$$

Application: Vector boson scattering (1)

$$V_1 + V_2 \to V_3 + V_4, \quad V_i \in \{Z, W^+, W^-, \gamma\}$$

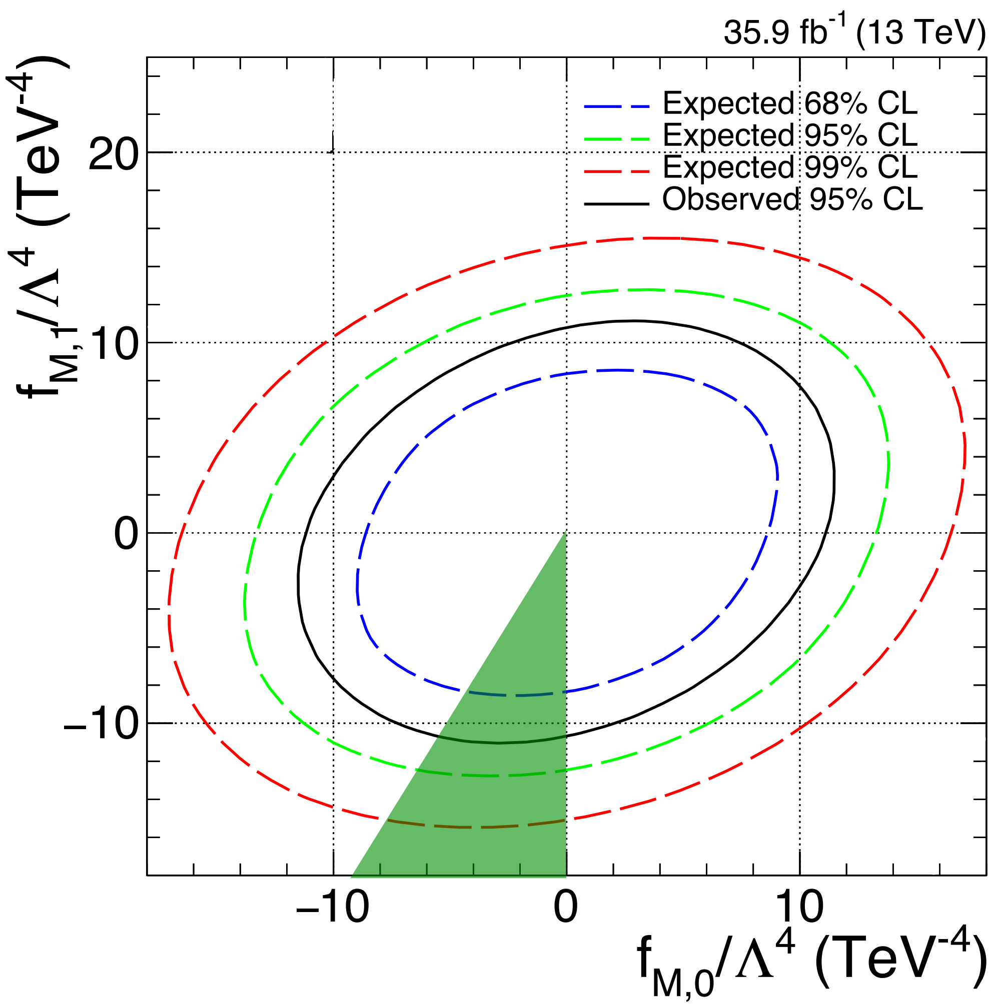
$$O_{S,0} = [(D_\mu\Phi)^\dagger D_\nu\Phi] \times [(D^\mu\Phi)^\dagger D^\nu\Phi]$$
$$O_{S,1} = [(D_\mu\Phi)^\dagger D^\mu\Phi] \times [(D_\nu\Phi)^\dagger D^\nu\Phi]$$
$$O_{S,2} = [(D_\mu\Phi)^\dagger D_\nu\Phi] \times [(D^\nu\Phi)^\dagger D^\mu\Phi]$$
$$O_{M,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right]$$
$$O_{M,1} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right]$$
$$O_{M,2} = \left[\hat{B}_{\mu\nu}\hat{B}^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right]$$
$$O_{M,3} = \left[\hat{B}_{\mu\nu}\hat{B}^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right]$$
$$O_{M,4} = \left[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}D^\mu\Phi\right] \times \hat{B}^{\beta\nu}$$
$$O_{M,5} = \left[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}D^\nu\Phi\right] \times \hat{B}^{\beta\mu}$$
$$O_{M,7} = \left[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}\hat{W}^{\beta\mu}D^\nu\Phi\right]$$

$$O_{T,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \mathrm{Tr}\left[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}\right]$$
$$O_{T,1} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}\right]$$
$$O_{T,2} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}\right]$$
$$O_{T,5} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta}$$
$$O_{T,6} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times \hat{B}_{\mu\beta}\hat{B}^{\alpha\nu}$$
$$O_{T,7} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times \hat{B}_{\beta\nu}\hat{B}^{\nu\alpha}$$
$$O_{T,8} = \hat{B}_{\mu\nu}\hat{B}^{\mu\nu} \times \hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta}$$
$$O_{T,9} = \hat{B}_{\alpha\mu}\hat{B}^{\mu\beta} \times \hat{B}_{\beta\nu}\hat{B}^{\nu\alpha},$$
$$O_{T,10} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\tilde{W}^{\alpha\beta}],$$
$$O_{T,11} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\tilde{B}^{\alpha\beta}.$$

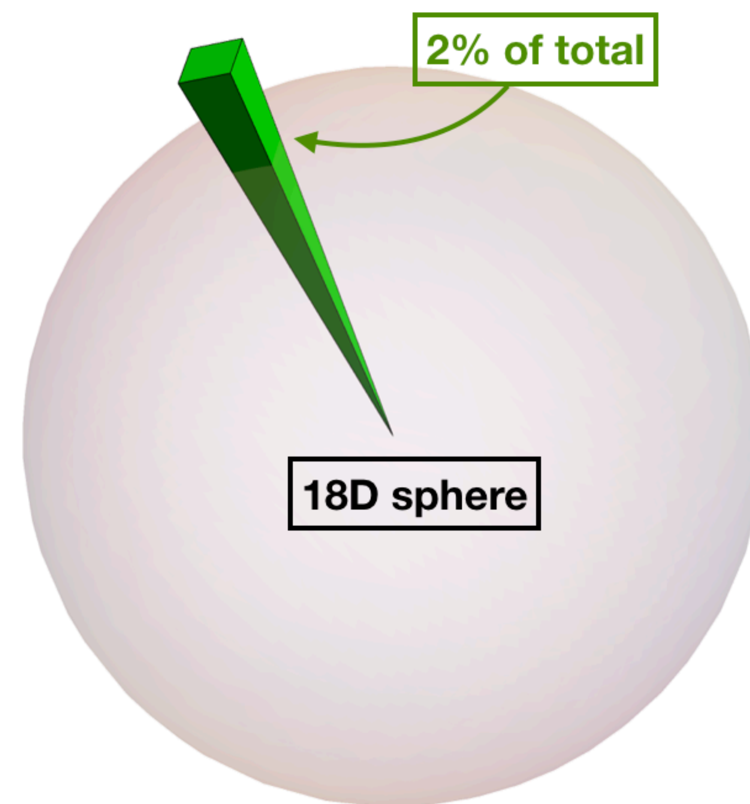
They lead to anomalous Quartic Gauge Couplings (aQGCs)

Application: Vector boson scattering (2)

O_{M0} and O_{M1}



Space of 18 Wilson coeffs for aQGCs



Only **~2%** of the total aQGC parameter space admits an analytic UV completion!

Zhang & **SYZ**,1808.00010

Stronger positivity bounds?

$$M^{abab} = \sum_{ijkl} u_i v_j u_k^* v_l^* M^{ijkl} = \sum_{ijkl} u_i v_j u_k^* v_l^* c_{2,0}^{ijkl} > 0$$

Is it possible such that

$$M^T = \sum_{ijkl} T_{ijkl} M^{ijkl} > 0, \text{ and } \{T_{ijkl}\} \supset \{u_i v_j u_k^* v_l^*\} ?$$

Yes, T_{ijkl} is more than $u_i v_j u_k^* v_l^*$!

Example: W-boson scatterings in SMEFT

$$F_{T,2} \geq 0, \quad 4F_{T,1} + F_{T,2} \geq 0$$
$$F_{T,2} + 8F_{T,10} \geq 0, \quad 8F_{T,0} + 4F_{T,1} + 3F_{T,2} \geq 0$$
$$12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10} \geq 0$$
$$4F_{T,0} + 4F_{T,1} + 3F_{T,2} + 12F_{T,10} \geq 0$$

Zhang & **SYZ**, 2005.03047

old: $|a\rangle|b\rangle \to |a\rangle|b\rangle$

new: $|U\rangle \to |U\rangle$

scatterings of entangled states

$$T_{ijkl} \sim \Sigma_n \lambda_n U_{ij}^n U_{kl}^n$$

Best bounds from ERs of $\mathscr{T}$ cone

generalized optical theorem

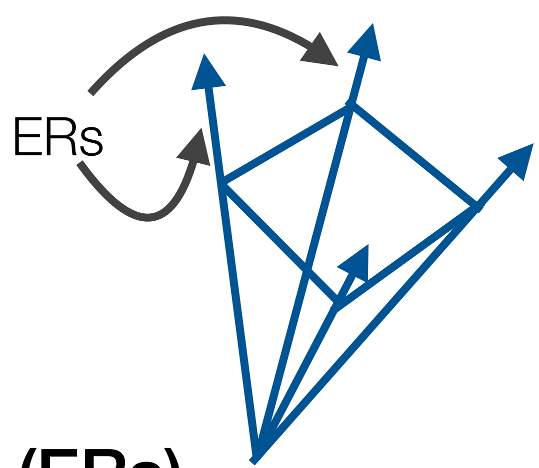
$$T_{ijkl} \in \mathscr{T} \equiv \mathscr{T}^+ \cap \overrightarrow{\mathbf{S}} \qquad \begin{cases} \mathscr{T}^+ \equiv \left\{ T_{ijkl} \mid T_{ij,kl} \succeq 0 \right\} \\ \overrightarrow{\mathbf{S}} \equiv \left\{ T_{ijkl} \mid T_{ijkl} = T_{ilkj} = T_{kjil} = T_{jilk} \right\} \end{cases}$$

$\mathscr{T}$ is a spectrahedron

Li, Xu, Yang, Zhang & **SYZ**, 2101.01191

(spectrahedron) = (convex cone of PSD matrices) $\cap$ affine-linear space

To get best bounds, find all ERs of $\mathscr{T}$



all elements of $\mathscr{T}$: $T_{ijkl} = \Sigma_p \alpha_p T^{(p)}_{ijkl}, \ \alpha_p > 0$

p enumerates all **Extreme Rays (ERs)**

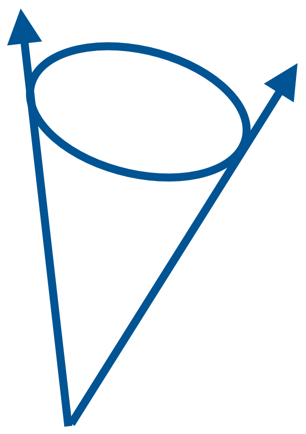
Best positivity bounds:

$$\sum_{ijkl} T^{(p)}_{ijkl} M^{ijkl} > 0$$

Semi-definite program (SDP)

spectrahedron is parameter space of a semi-definite program

Use SDP to find best positivity bounds



$$\text{minimize} \quad \sum_{ijkl} T_{ijkl} M^{ijkl}$$

$$\text{subject to} \quad T_{ijkl} \in \mathscr{T} \equiv \mathscr{T}^{+} \cap \overrightarrow{\mathbf{S}}$$

$\min(T \cdot M) > 0$, then M^{ijkl} is within positivity bounds

Compared to elastic approach ($uvuvM > 0$)

- stronger bounds
- **more efficient (polynomial complexity)**

Li, Xu, Yang, Zhang & **SYZ**, 2101.01191

Convex cone $\mathscr{C}$ of amplitudes

$$\mathscr{C} \equiv \{M^{ijkl}\} = \text{cone}\left(\left\{m^{i(j}m^{|k|l)}\right\}\right)$$

$m^{ij} \sim M^{ij\to X}$

X: intermediate state

$\mathscr{C}$ is dual cone of $\mathscr{T}$: $\mathscr{T} \equiv \left\{ T^{ijkl} \,\middle|\, T\cdot M \equiv \sum_{ijkl} T_{ijkl} M^{ijkl} > 0 \right\}$

For m^{ij} to be extremal, it can not be split to two amplitudes

$$\Rightarrow \quad m^{ij}_{(\text{ER})} \sim M^{ij\to X_{\text{irrep}}} \sim C^{r,\alpha}_{i,j}$$

CG coefficient

Get $\mathscr{C}$ cone by symmetries of EFT

$$\Rightarrow \quad \mathscr{C} = \text{cone}\left(\left\{P_r^{i(j|k|l)}\right\}\right)$$

$$P_r^{ijkl} \equiv \sum_\alpha C^{r,\alpha}_{i,j}\left(C^{r,\alpha}_{k,l}\right)^*$$

group projector

Zhang & **SYZ**, 2005.03047

The inverse problem

Structure of $\mathscr{C}$ cone implies

Zhang & **SYZ**, 2005.03047

Extremal Ray ⟷ **UV particle**

$P_r^{i(j|k|l)}$

Example: Higgs $\mathscr{C}$ cone in SMEFT

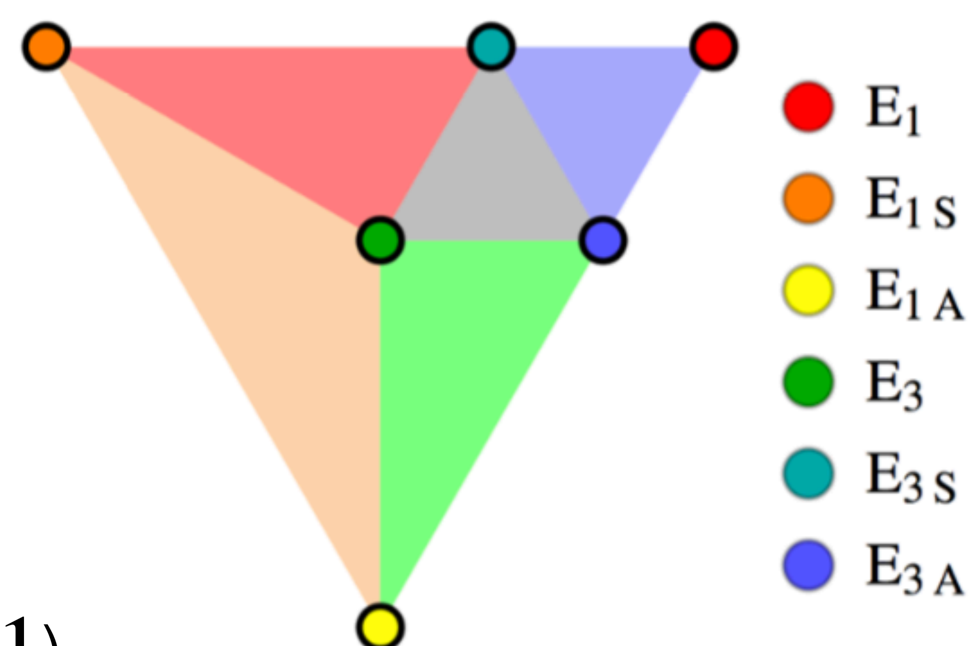
Wilson coeffs fall in blue region

↓

E_1 must exit

↓

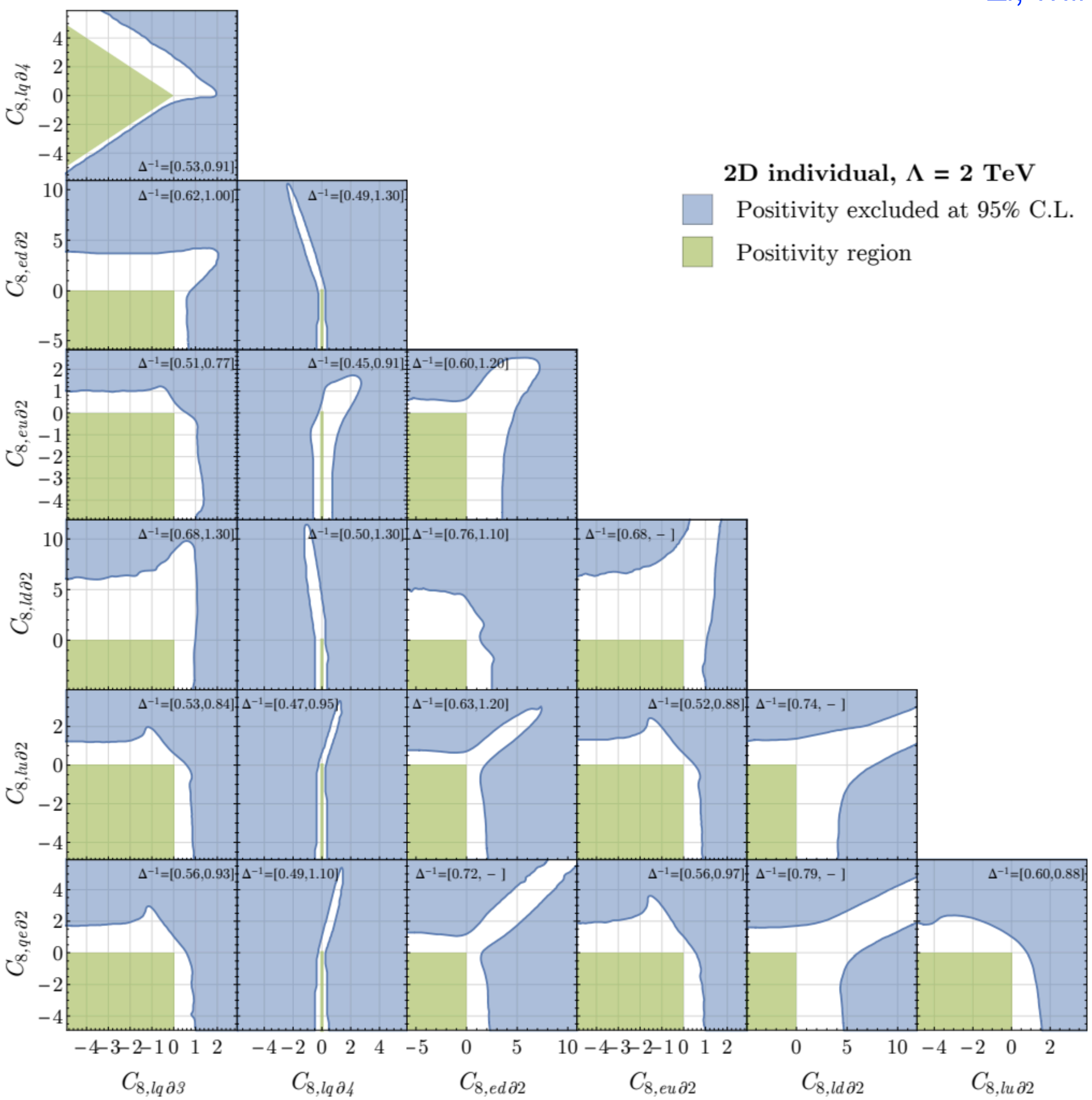
new UV state ($SU(2)_L$ singlet, $Y = 1$)

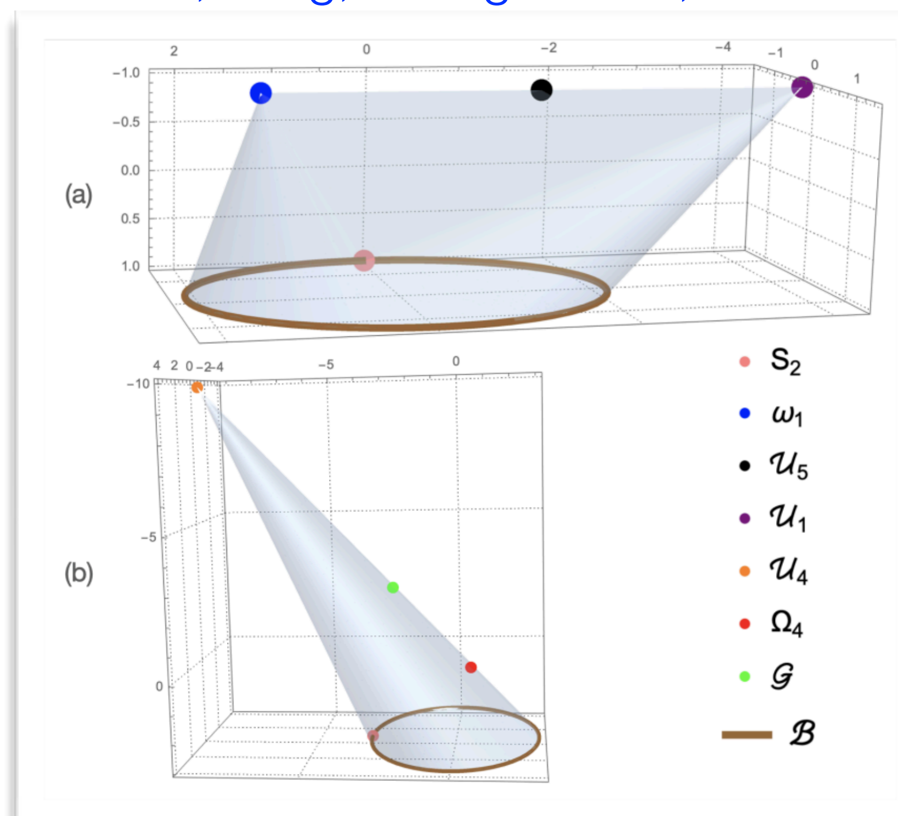


ERs of $\mathscr{C}$ (or dim-8 operators) are important to reverse-engineer UV physics!

Application: Test positivity with Drell-Yan at LHC

Li, Mimasu, Yamashita, Yang, Zhang & **SYZ**, 2204.13121





UV particle H	$\lambda_{\max}$ [TeV^{-4}]	$\frac{M_H}{\sqrt{g_H}}$ [TeV]
$\mathcal{S}_2$	0.0015	≥ 5.1
$\mathcal{U}_4$	1.2	≥ 0.95
Ω_4	1.1	≥ 0.97
ω_1	0.092	≥ 1.8
$\mathcal{U}_5$	0.046	≥ 2.2
$\mathcal{B}$	0.00075	≥ 6.1
$\mathcal{G}$	2.5	≥ 0.80
$\mathcal{U}_1$	0.092	≥ 1.8

Multi-field bounds for all coefficients

Single field vs multiple fields

Optical theorem (for identical particle)

$$\mathrm{Im}\, a_\ell^{iiii} = \sum_X a_\ell^{ii\to X} \left(a_\ell^{ii\to X}\right)^* = \sum_X |a_\ell^{ii\to X}|^2 > 0$$

use **linear programing** to obtain optimal bounds

Generalized optical theorem (for multiple fields)

$$\mathrm{Im}\, a_\ell^{ijkl} = \sum_X a_\ell^{ij\to X} \left(a_\ell^{kl\to X}\right)^*$$

not a positive number, but upgraded to a **positive matrix**

use **semi-definite programing** to obtain optimal bounds

SDP with a continuous decision variable

μ: the UV scale; solvable by SDPB

Du, Zhang & **SYZ**, 2111.01169

Sum rules for multi-fields

$$c^{m,n}_{ijkl} = \left\langle C^{m,n}_{ijkl} \right\rangle \equiv \left\langle \left[m^{ij}_\ell m^{kl}_\ell + (-1)^m m^{il}_\ell m^{kj}_\ell \right] \frac{C^{m,n}_\ell}{\mu^{m+n+1}} \right\rangle$$

$$m^{ij}_\ell \sim \operatorname{Im} a^{ij\to X}_\ell \qquad \langle \cdots \rangle \equiv \sum_{X',\ell} \int_{\Lambda^2}^{\infty} \mathrm{d}\mu (\cdots)$$

Null constraints

impose st symmetries on su dispersion relations

crossing is trivial for scalars and massless particles

$$\Bigg\langle \left[m^{ij}_\ell (m^{kl}_\ell)^* + m^{il}_\ell \left(m^{kj}_\ell\right)^* \right] \frac{E^+_{p,q}}{\mu^{p+q+1}} + \left[m^{ij}_\ell (m^{kl}_\ell)^* - m^{il}_\ell \left(m^{kj}_\ell\right)^* \right] \frac{E^-_{p,q}}{\mu^{p+q+1}}$$
$$+ \left[m^{ik}_\ell \left(m^{jl}_\ell\right)^* + m^{il}_\ell \left(m^{jk}_\ell\right)^* \right] \frac{F^+_{p,q}}{\mu^{p+q+1}} + \left[m^{ik}_\ell \left(m^{jl}_\ell\right)^* - m^{il}_\ell \left(m^{jk}_\ell\right)^* \right] \frac{F^-_{p,q}}{\mu^{p+q+1}} \Bigg\rangle = 0,$$

Semi-definite program with SDPB

if $$\sum_{i,j,k,l;m,n} \mathcal{Q}^{ijkl}_{m,n} C^{m,n}_{ijkl} = \sum_{i,j,k,l;m,n} \mathcal{Q}^{ijkl}_{m,n} \left[m^{ij}_\ell (m^{kl}_\ell)^* + (-1)^m m^{il}_\ell \left(m^{kj}_\ell \right)^* \right] \frac{C^{m,n}_\ell}{\mu^{m+n+1}}$$

$$\mathcal{Q}^{ijkl}_{m,n} = (-1)^m \mathcal{Q}^{ilkj}_{m,n} \qquad = \sum_{i,j,k,l} m^{ij}_\ell \left[\sum_{m,n} \mathcal{Q}^{ijkl}_{m,n} \frac{C^{m,n}_\ell}{\mu^{m+n+1}} \right] (m^{kl}_\ell)^* \geq 0, \quad \text{for all } \mu, \ell, m^{ij}_\ell(\mu),$$

then $$\sum_{i,j,k,l;m,n} \mathcal{Q}^{ijkl}_{m,n} c^{m,n}_{ijkl} = \left\langle \sum_{i,j,k,l;m,n} \mathcal{Q}^{ijkl}_{m,n} C^{m,n}_{ijkl} \right\rangle \geq 0$$

Positivity bounds:

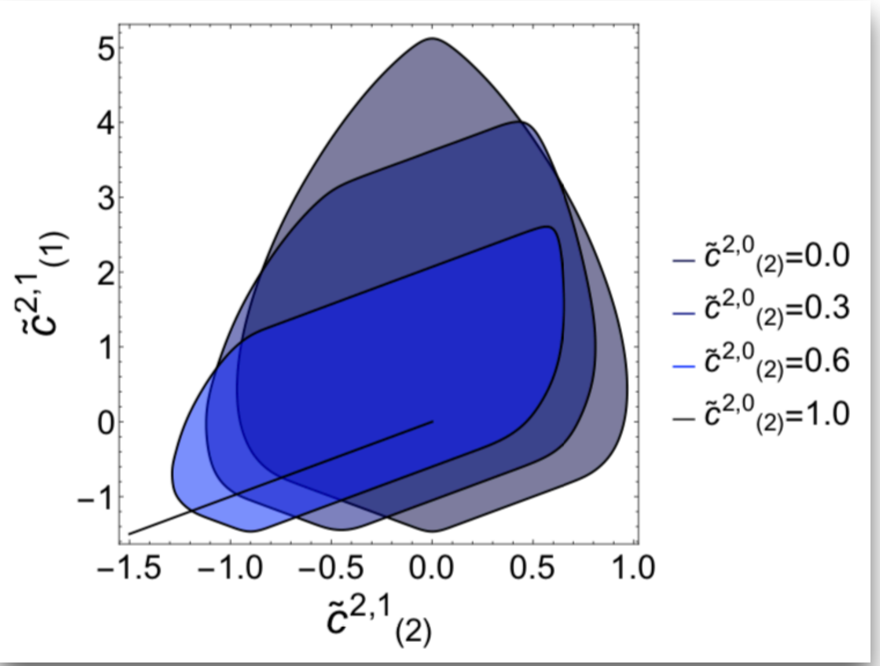
added null constraints

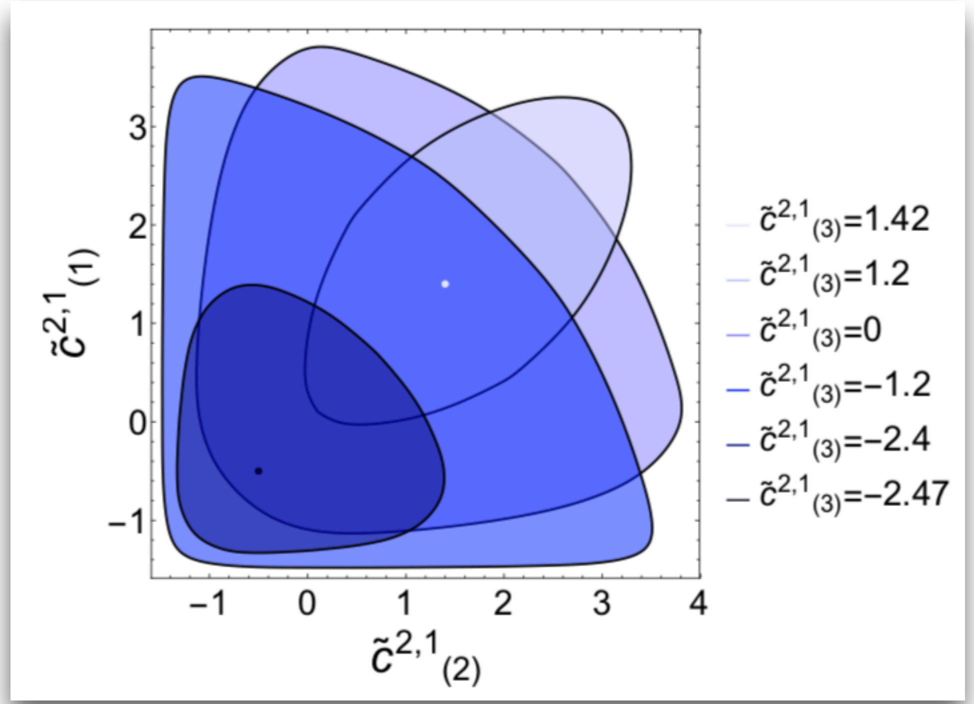
$$\text{minimize} \sum_{m,n} \mathcal{Q}_{m,n} \cdot c^{m,n} = 0$$

$$\text{subject to} \sum_{m,n} \mathcal{Q}_{m,n} \frac{C^{m,n}_\ell}{\mu^{m+n+1}} + \sum_{\mathcal{N},p,q} \frac{N^{(\mathcal{N})}_{p,q}}{\mu^{p+q+1}} \succeq 0 \qquad \text{for all } \mu, \ell$$

continuous decision variable μ

Example: Z_2 bi-scalar theory





All Wilson coefficients are again parametrically $O(1)$!

Du, Zhang & **SYZ**, 2111.01169

More applications:

- Cosmology

Tolley, Wang & **SYZ**, 2011.02400
de Rham, Melville, Tolley & **SYZ**,
1702.08577, 1804.10624
Wang, Zhang & **SYZ**, 2011.05190

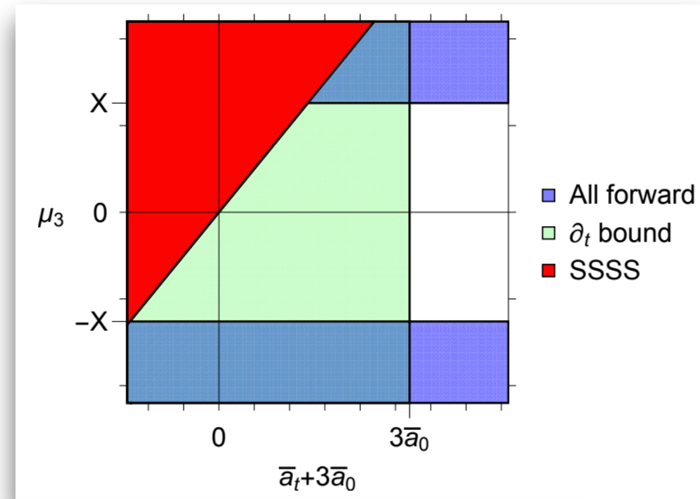
- Chiral PT

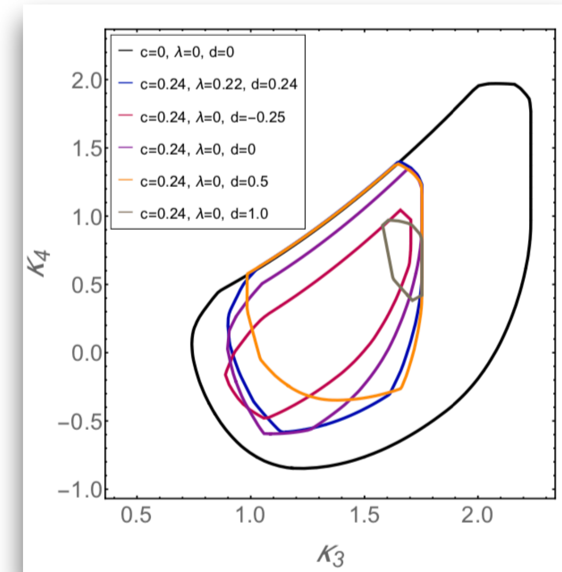
Wang, Feng, Zhang & **SYZ**, 2004.03992
Tolley, Wang & **SYZ**, 2011.02400

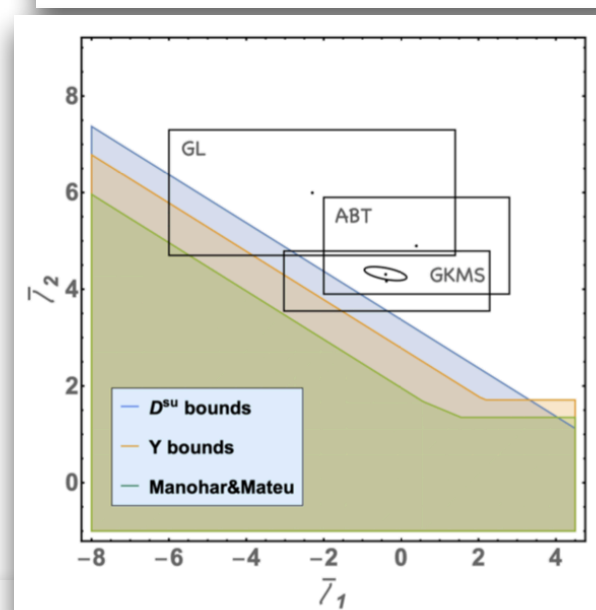
- SMEFT

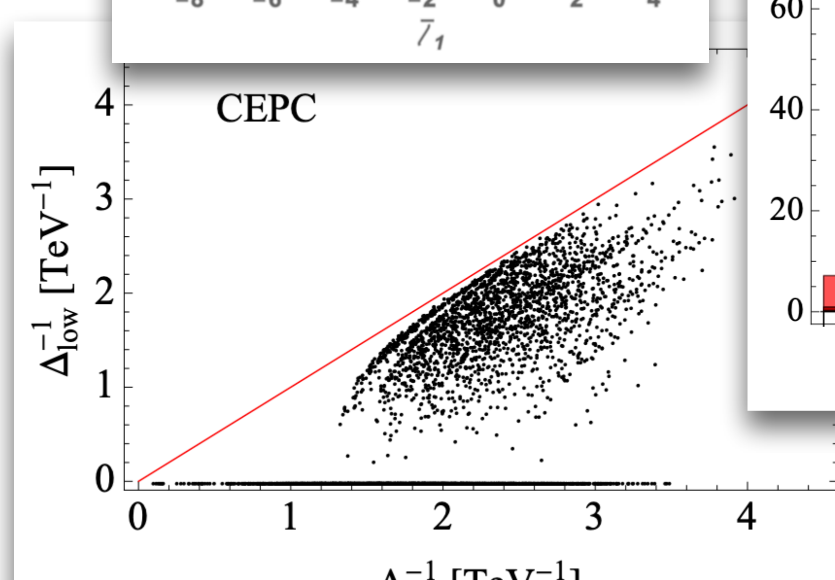
asu, Yamashita, Yang, Zhang & **SYZ**, 2204.13121
Li, Xu, Yang, Zhang & **SYZ**, 2101.01191
Yamashita, Zhang & **SYZ**, 2009.04490
Fuks, Liu, Zhang & **SYZ** 2009.02212
Bi, Zhang & **SYZ**, 1902.08977
Zhang & **SYZ**, 2005.03047, 1808.00010

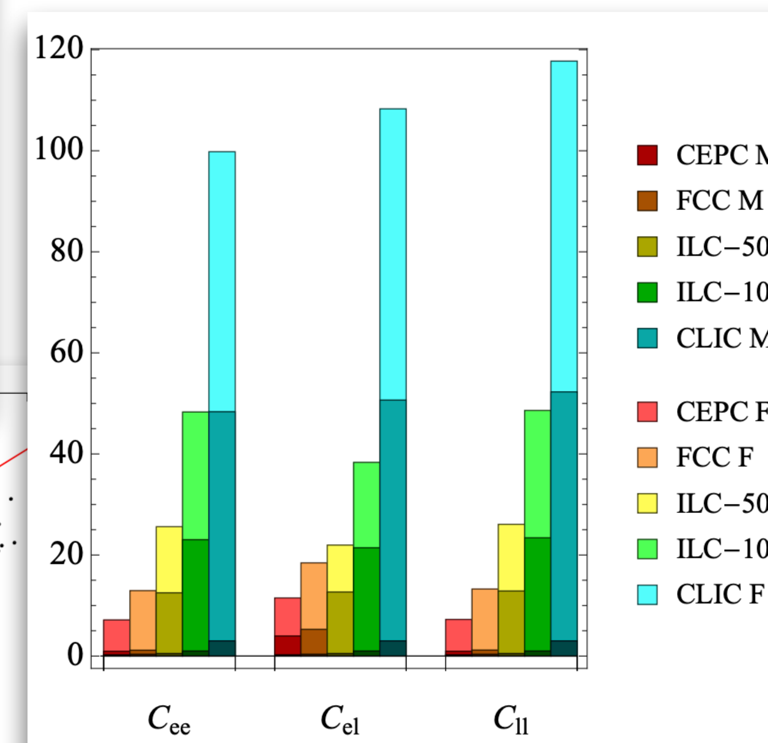
+ many other authors











Summary

- Positivity bounds are **robust** — from axioms of QFT
- Wilson coeffs are bounded to be:

$$c_i \sim O(1)$$

"naturalness" is a rigorous result!

- This can be generalized to multi-fields

  generalized optical theorem; linear program → semi-definitive program

- **Convex cone** approach gives **best** s^2 positivity bounds
- **dim-8** ops are important to **reverse engineer UV theory**

Thank you!

Backup slides

Unitarity

Unitarity: conservation of probabilities $S^\dagger S = 1 \Rightarrow T - T^\dagger = iT^\dagger T$

Generalized optical theorem

$$A(I \to F) - A^*(F \to I) = i \sum_X \int d\Pi_X (2\pi)^4 \delta^4(p_I - p_X) A(I \to X) A^*(F \to X)$$

optical theorem ($\theta = 0$): $\mathrm{Im}[A(I \to I)] \sim \sum_X \sigma(I \to X) > 0$

Partial wave expansion: $A(s,t) \sim \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) a_\ell(s)$

(2-2 scattering, for scalar)

Partial wave unitary bounds:

$$0 \leq |a_\ell(s)|^2 \leq \mathrm{Im}\, a_\ell(s) \leq 1$$

EFT-Hedron for $t = 0$

Arkani-Hamed, Huang & Huang, *talks in 2017*, 2012.15849
Bellazzini, Miro, Rattazzi, Riembau & Riva, 2011.00037

$$c_{2n,0} = \int_{\Lambda^2}^{\infty} \frac{2\,\mathrm{d}\mu}{\pi \mu^{1+2n}} \mathrm{Im}\, A(\mu, 0) \quad \xrightarrow{x \equiv \Lambda/\mu} \quad c_{2n,0} = \int_0^1 x^n d\rho(x)$$

This is a Hausdorff moment problem!

Solution:

Define Hankel matrix $H(c_{2n,0}) = \begin{pmatrix} c_{2,0} & c_{4,0} & c_{6,0} & \\ c_{4,0} & c_{6,0} & c_{8,0} & \cdots \\ c_{6,0} & c_{8,0} & c_{10,0} & \\ & \vdots & & \ddots \end{pmatrix}$

nonlinear positivity bounds

$$H(c_{2n,0}) \succeq 0 \quad \& \quad H^{\text{shift}}(c_{2n,0}) \equiv H(c_{2n,0})\Big|_{c_{2n,0} \to c_{2n+2,0}} \succeq 0$$

$$\& \quad H(c_{2n,0}) - H^{\text{shift}}(c_{2n,0}) \succeq 0$$

su symmetric sum rules

$$\sum_{i,j} c_{i,j} s^i t^j = A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, t)$$

Expand dispersion relation and match $s^i t^j$ on both sides

partial wave expansion: $A(s,t) \sim \Sigma_\ell P_\ell(1 + 2t/s) a_\ell(s)$

unitarity: $0 \leq |a_\ell(s)|^2 \leq \mathrm{Im}\, a_\ell(s) \leq 1$

Sum rules:

$$c_{i,j} \sim \sum_\ell \int d\tilde{\mu} \frac{D_{i,j}(\eta)}{\mu^{i+j}}$$

$$d\tilde{\mu} \equiv d\mu\, \mathrm{Im} a_\ell(s)$$

$$\eta \equiv \ell(\ell + 1)$$

$D_{i,j}$ is polynomial of η that is bounded below

su symmetric bounds $c_{i,j} \sim \sum_\ell \int d\tilde{\mu} \frac{D_{i,j}(\eta)}{\mu^{i+j}} > D^{\min}_{i,j} \sum_\ell \int d\tilde{\mu} \frac{1}{\mu^{i+j}} = D^{\min}_{i,j} c_{2,0}$

Tolley, Wang & **SYZ**, 2011.02400

$$D^{\min}_{i,j} = \min_\eta \left[D_{i,j}(\eta)\right]$$

Powerful two-sided bounds

$$\sum_\ell \int d\tilde{\mu} \frac{\Gamma^{(n)}_{i,j}(\eta)}{\mu^{i+j}} = 0$$

Add null constraints to sum rules:

$$c_{i,j} \sim \sum_\ell \int d\tilde{\mu} \frac{D_{i,j}(\eta) + \sum_n \alpha_n \Gamma^{(n)}_{i,j}(\eta)}{\mu^{i+j}}$$

can choose α_n to make $D_{i,j} + \Sigma_n \alpha_n \Gamma^{(n)}_{i,j}$ bounded from blow and above

before: $D_{i,j}$ only has **min**

now: $D_{i,j} + \Sigma_n \alpha_n \Gamma^{(n)}_{i,j}$ can have **min and max**

α_n can be positive or negative

→ **Wilson coeff's $c_{i,j}$ have two-sided bounds**

Tolley, Wang & **SYZ**, 2011.02400

Two-sided bounds

Tolley, Wang & **SYZ**, 2011.02400

(m,n)	**Lower bounds**	**Upper bounds**
$(1,1)$	$c_{1,1} > -\frac{3}{2}\sqrt{c_{1,0}c_{2,0}}$	$c_{1,1} < 8\sqrt{c_{1,0}c_{2,0}}$
$(2,1)$	$c_{2,1} > -\frac{5}{2}\sqrt{c_{2,0}c_{3,0}}$	$c_{2,1} < \frac{465}{38}\sqrt{c_{2,0}c_{3,0}}$
$(2,2)$	$c_{2,2} > -\frac{9}{2}c_{3,0}$	$c_{2,2} < \frac{2961}{58}c_{3,0}$
$(3,1)$	$c_{3,1} > -\frac{7}{2}\sqrt{c_{3,0}c_{4,0}}$	$c_{3,1} < \frac{1097}{58}\sqrt{c_{3,0}c_{4,0}}$
$(3,2)$	$c_{3,2} > -7c_{4,0}$	$c_{3,2} < \frac{10027}{59}c_{4,0}$
$(3,3)$	$c_{3,3} + \frac{3}{4}c_{4,1} > -\frac{147}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 8c_{4,1} > -154\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - \frac{481}{12}c_{4,1} > -\frac{7777}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 104c_{4,1} > -3369\sqrt{c_{4,0}c_{5,0}}$	$c_{3,3} - \frac{650}{41}c_{4,1} < -\frac{2310}{41}\sqrt{c_{4,0}c_{5,0}}$
$(4,2)$	$c_{4,2} > -\frac{17}{2}c_{5,0}$	$c_{4,2} < \frac{3923}{12}c_{5,0}$
$(4,3)$	$c_{4,3} + \frac{3}{4}c_{5,1} > -\frac{253}{8}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{180}{41}c_{5,1} > -\frac{8705}{82}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{325}{12}c_{5,1} > -\frac{16825}{24}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{169}{2}c_{5,1} > -\frac{11187}{4}\sqrt{c_{5,0}c_{6,0}}$ $c_{4,3} - \frac{743}{4}c_{5,1} > -\frac{63279}{8}\sqrt{c_{5,0}c_{6,0}}$	$c_{4,3} - \frac{73153}{1748}c_{5,1} < -\frac{708543}{3496}\sqrt{c_{5,0}c_{6,0}}$
$(4,4)$	$c_{4,4} + \frac{25}{24}c_{5,2} > -\frac{147}{8}c_{6,0}$, $c_{4,4} - \frac{125}{37}c_{5,2} > -\frac{71175}{74}c_{6,0}$, $c_{4,4} - \frac{785}{52}c_{5,2} > -\frac{83490}{13}c_{6,0}$, $c_{4,4} - \frac{2485}{69}c_{5,2} > -\frac{1144125}{46}c_{6,0}$	$c_{4,4} - 15c_{5,2} < -\frac{195}{2}c_{6,0}$, $c_{4,4} + \frac{368085}{36544}c_{5,2} < -\frac{2365845}{18272}c_{6,0}$

Some further developments

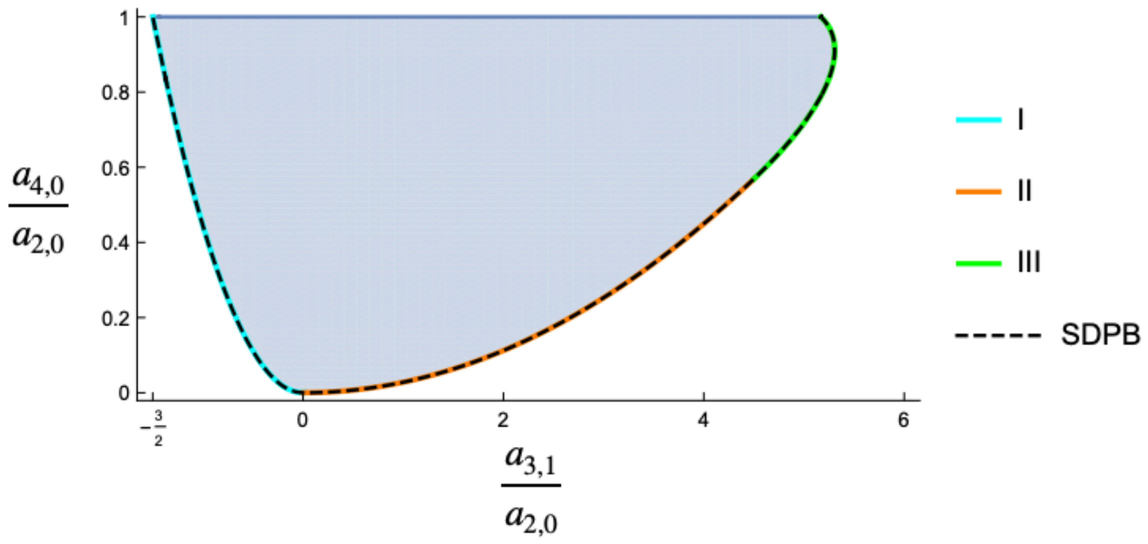
Fully crossing symmetric dispersion relation

Sinha & Zahed, 2012.04877

Analytical approach

reduce to bi-variate moment problem
(GL rotations + triple-crossing slices)

Chiang, Huang, Li, Rodina & Weng, 2105.02862



Application: Ruling out Galileon

$$\pi \to \pi + c + b_\mu x^\mu, \quad c, b_\mu = const$$

- linked to dRGT massive gravity
- applications in cosmology

original Galileon marginally ruled out by Adams et al, 2006

Weakly broken Galileon theories

$$\mathcal{L} \sim \mathcal{L}_{\text{galileon}} - \frac{m^2}{2}\pi^2$$

stu symmetric bounds

$\Lambda \sim m$ **not a valid EFT**

may also add $\alpha(\partial\phi)^4, \ |\alpha| \ll 1$ leads to same conclusion

Tolley, Wang & **SYZ**, 2011.02400

Convex geometry: 1-slide crash course

Convex cone C:

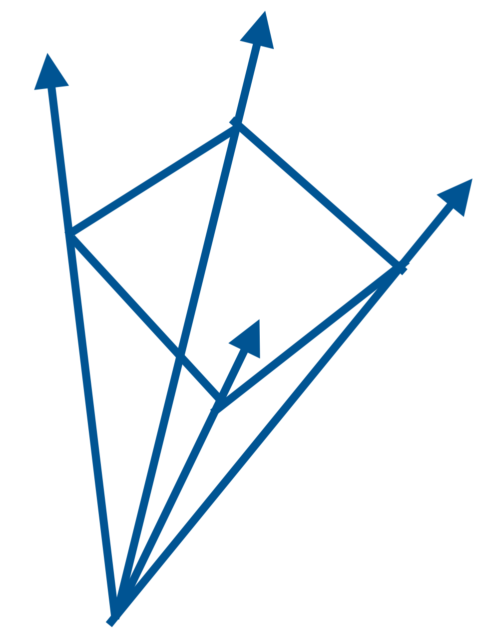
subset of linear space; closed under conical combinations

$$x \in C,\, y \in C,\, \alpha > 0,\, \beta > 0 \;\Rightarrow \alpha x + \beta y \in C$$

$\mathrm{cone}(Y)$: conical combination of set Y

Extremal ray (ER):

split into 2 other elements



Dual cone of C:

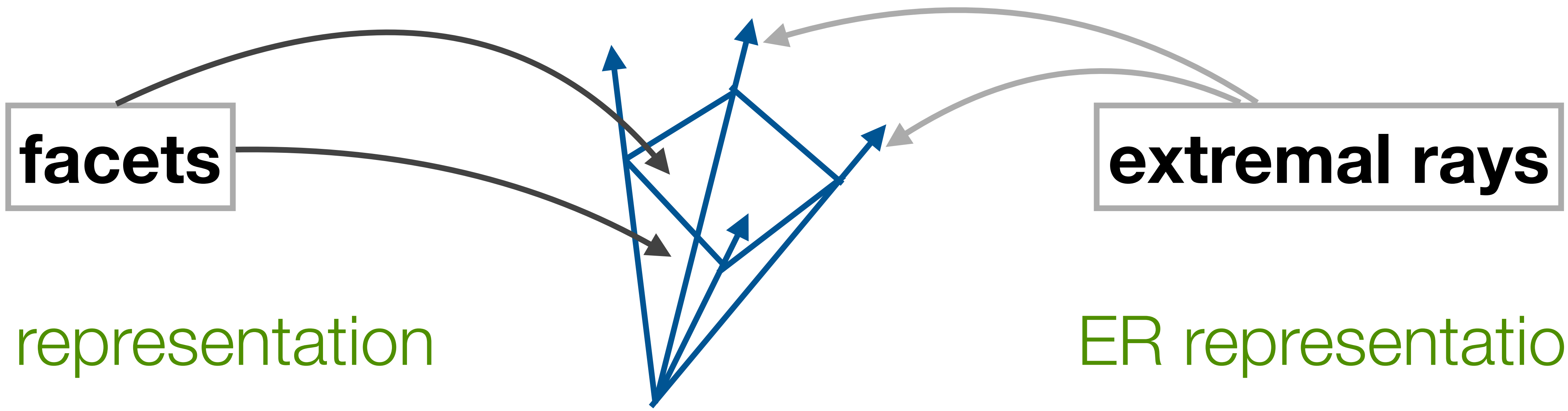
$$C^* = \{y \,|\, y \cdot x > 0,\, x \in C\}, \qquad (C^*)^* = C$$

Positive semi-definite matrices $\mathscr{P}_n$:

$$\mathscr{P}_n = \mathrm{cone}(\{m^I m^J \mid m^I \in \mathbb{R}\}), \qquad m^I m^J \text{ are ERs of } \mathscr{P}_n$$

$\mathscr{C}$ cone vs $\mathscr{T}$ cone

Two ways to describe one convex cone



bound representation

ER representation

facets of cone $\leftrightarrow$ ERs of dual cone

ERs of cone $\leftrightarrow$ facets of dual cone

Positivity bounds are ERs of $\mathscr{T}$ cone or facets of $\mathscr{C}$ cone