Yangian symmetry *in* holographic correlators

Xinan Zhou (*KITS-UCAS*)

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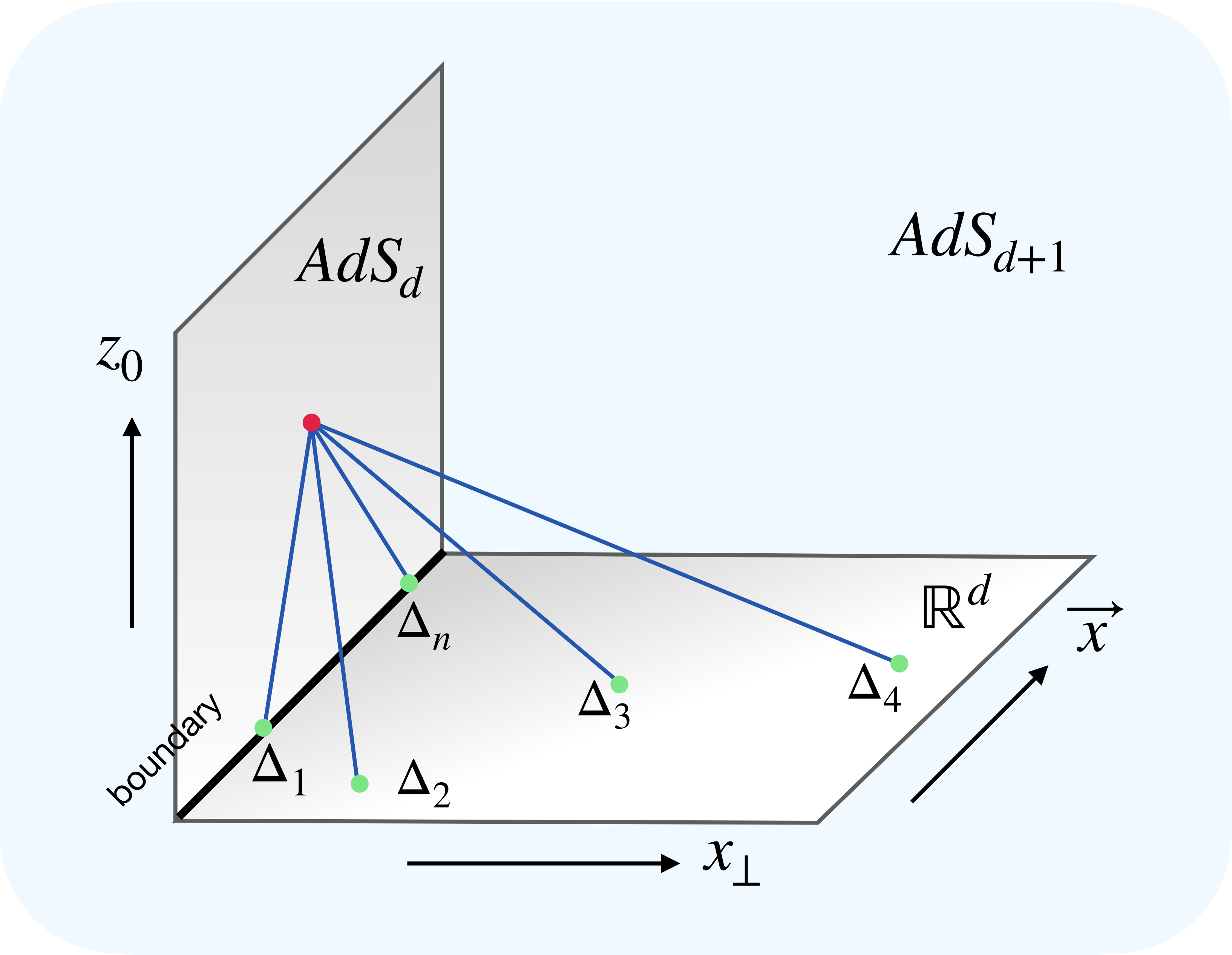
Introduction

- **Holographic correlators**
  - Basic observables for exploring and exploiting the AdS/CFT correspondence.
  - They encode a wealth of theory data which can be extracted using conformal block decompositions.
  - They can also be viewed as scattering amplitudes in curved spacetime, and therefore can be used to explore generalizations of various properties of flat-space amplitudes.
  - Current status: Two decades into studying AdS/CFT, our knowledge of these objects is still very limited! Compared with flat-space scattering amplitudes, we certainly know much less about them.

- **Difficult to compute**
  - The *traditional diagrammatic method* is a mess: all quartic vertices for $AdS_5 \times S^5$ IIB SUGRA take 15 pages to write down [Arutyunov, Frolov '99].
  - A better strategy is *bootstrap*: so far all tree-level 4-pt functions of all KK modes are known in all maximally susic theories ($AdS_4 \times S^7$, $AdS_5 \times S^5$, $AdS_7 \times S^4$) [Rastelli, Zhou '16; Alday, Zhou '20, '21], and in half maximally susic theories for those correspond to super gluons [Alday, Behan, Ferrero, Zhou '21]. Also results for higher-pt [Goncalves, Pereira, Zhou '19, Alday, Goncalves, Zhou '22] and loops [Alday, Bissi; Aprile, Drummond, Heslop, Paul; Alday Zhou; Alday, Bissi, Zhou; Huang, Yuan; Drummond, Paul…].
  - But this strategy also faces increasing technical challenges. It's very possible that superconformal symm and consistency conditions will stop fixing everything at some point.
- *Other independent guiding principles?*

- **Integrability?**
  - The paradigmatic example: IIB string theory $AdS_5 \times S^5$ dual to *4d N=4 SYM*, known to be *integrable* at the planar level. Can integrability be of help?
  - Not clear at the moment: the standard integrability techniques for studying N=4 SYM *have difficulties* at the *SUGRA limit*.
  - Currently there are no results for correlators of short 1/2-BPS operators.
- **In this talk…**
  - We do not offer a solution.
  - However, I will point out *a new connection* with integrability which may hopefully shed some light on it.

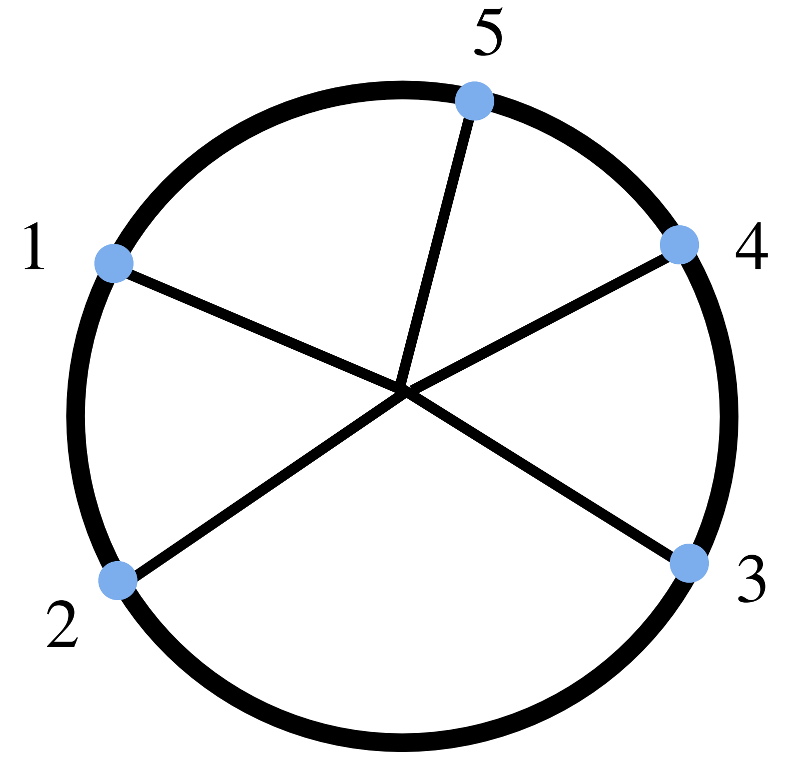
Main characters: On the one hand, we consider the following infinite family of *Witten diagrams* in a simple setup.



$$W_n = \int \frac{dz_0 d^{d-1}\vec{z}}{z_0^d} \prod_{i=1}^{n} G_{B\partial}^{\Delta_i}(z, \vec{x}_i, m_i)$$

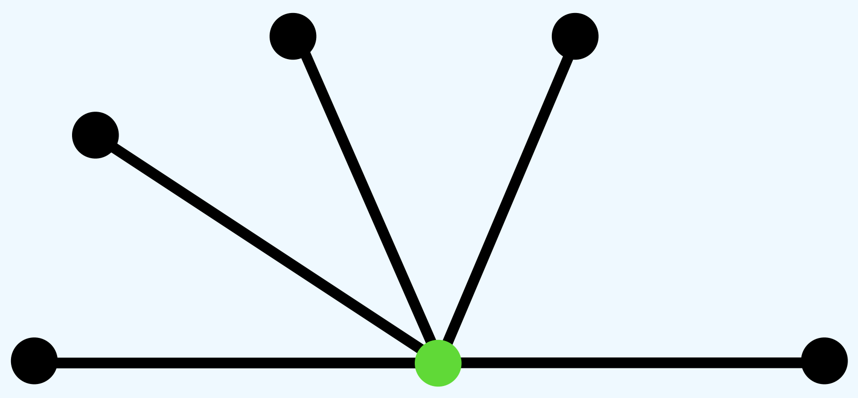
$$G_{B\partial}^{\Delta_i}(z, \vec{x}_i, m_i) = \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x}_i)^2 + m_i^2} \right)^{\Delta_i}$$

- Simple holographic models of boundary (or interface) CFTs, e.g., probe branes.
- When all operators are inserted at the boundary these are just standard *D-functions*.



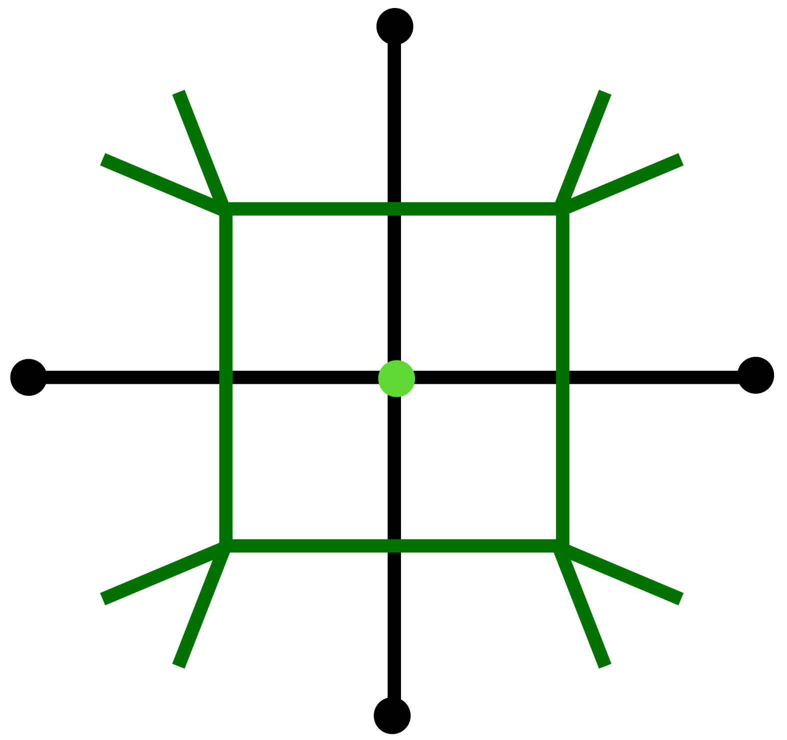
On the other hand, let us consider the following *conformal integrals* in flat space.

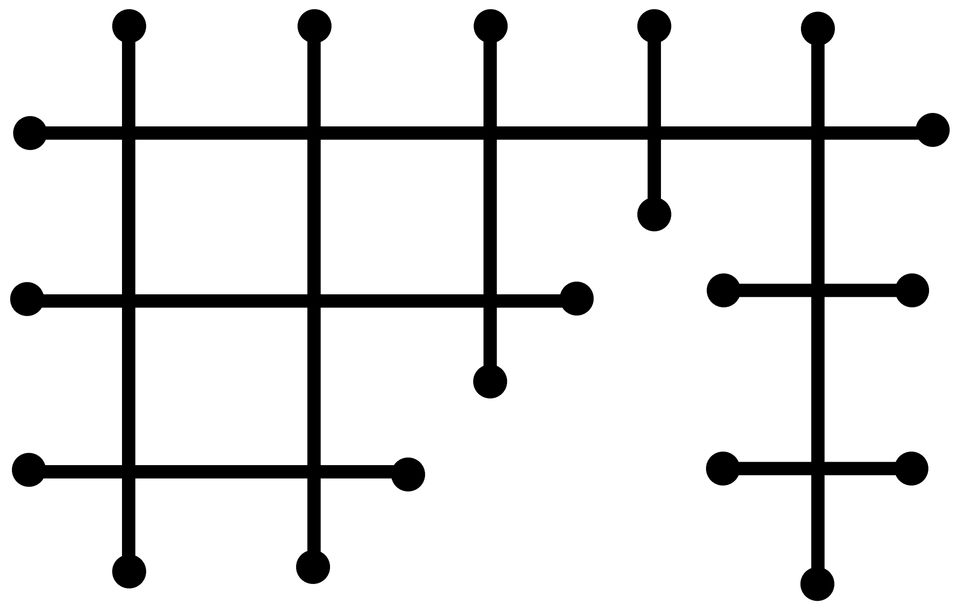
D-dimensional conformal integral



$$I_n = \int \frac{d^D x_0}{\prod_{j=1}^{n} (x_{j0}^2 + m_j^2)^{\Delta_i}}, \quad \sum_i \Delta_i = D$$

- A generalization of the box diagram



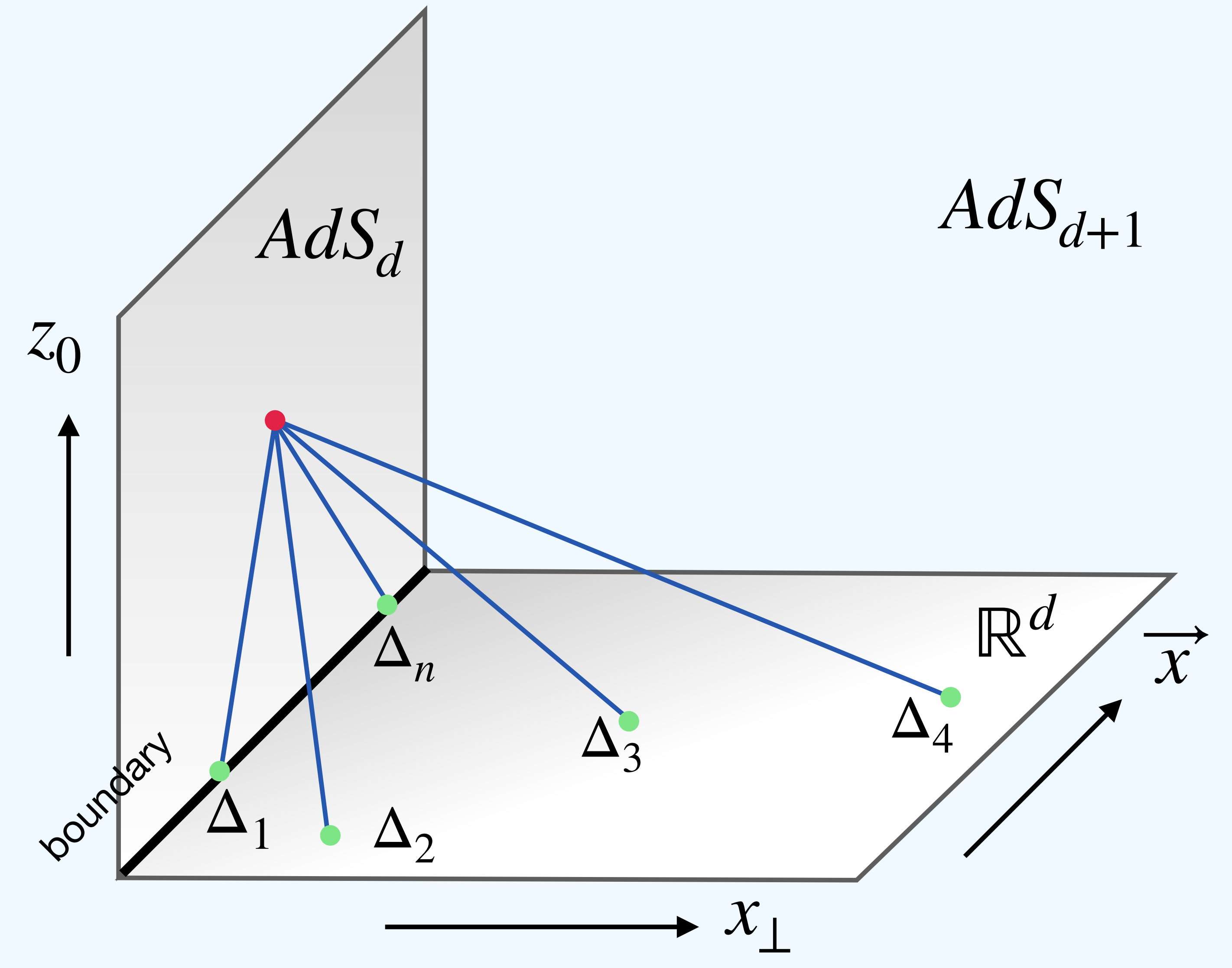


which is a special case of the so-called *fishnet diagrams*.

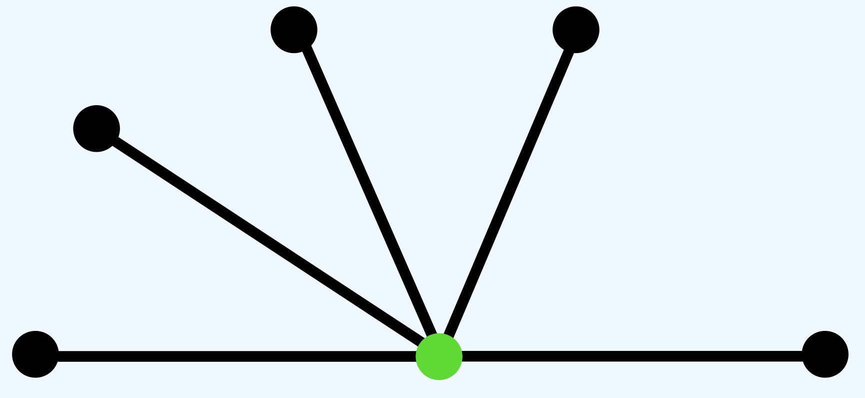
- Fishnet diagrams appear in an integrable deformation of N=4 SYM [Gurdogan, Kazakov '15]. This motivated the study of integrability properties of these integrals.

- *Yangian invariance* was first proven for the massless case for n=4,6, then streamlined and proven in the massive case for all n [Loebbert, Miczajka, Muller, Munkler].

1^{st} observation



D-dimensional conformal integral



$$I_n = \int \frac{d^D x_0}{\prod_{j=1}^{n} (x_{j0}^2 + m_j^2)^{\Delta_i}}, \quad \sum_i \Delta_i = D$$

- These two objects *are* essentially *the same*.
- The conformal integrals are *Yangian invariant* [Loebbert, Miczajka, Muller, Munkler]. Hence these Witten diagrams also have *Yangian symmetry*.

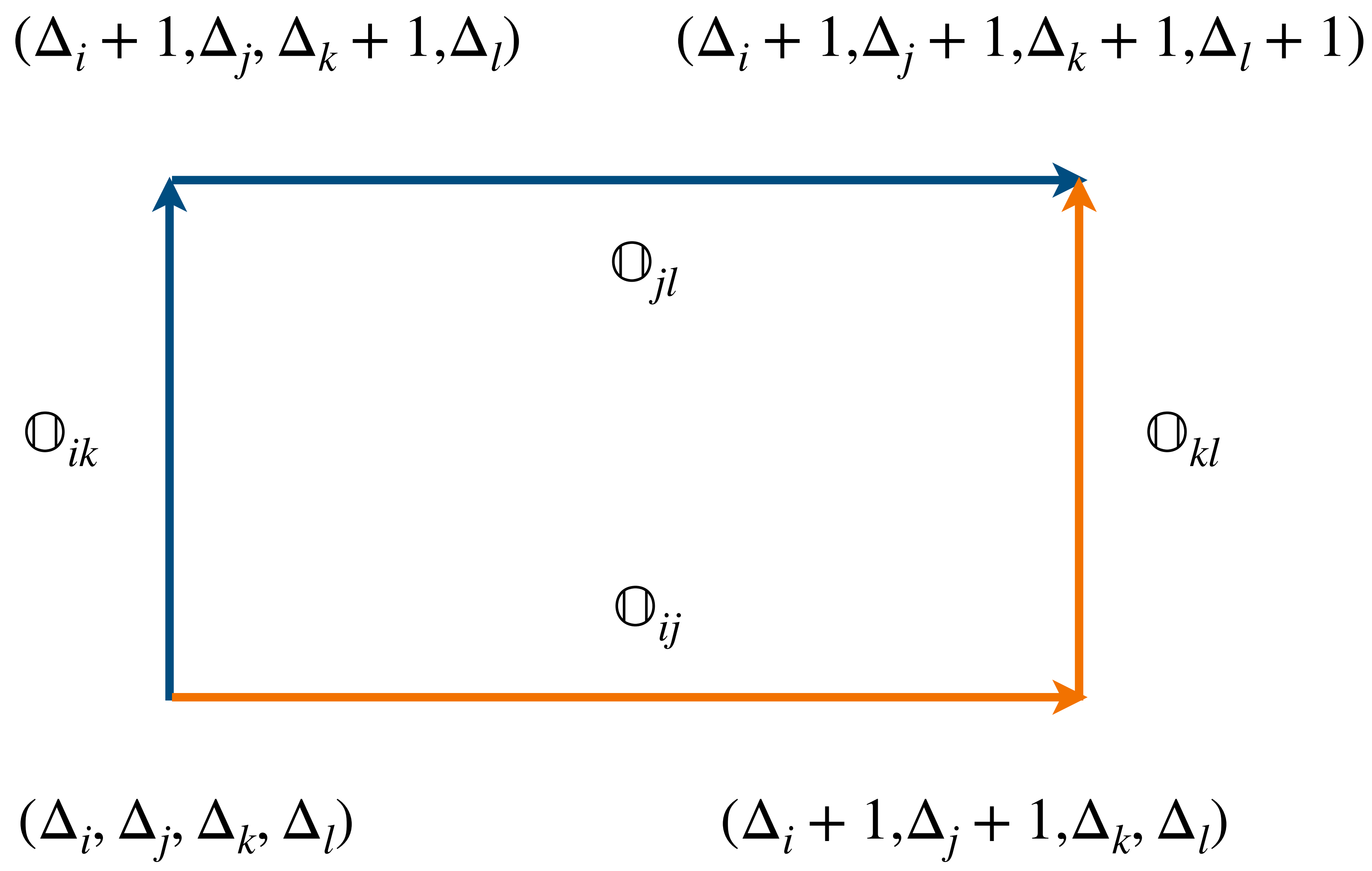
2^{nd} observation

- The Witten diagrams satisfy a web of *differential recursion relations*, generalizing those of D-functions. These relations shift the external conformal dimensions.

- Different ways of changing dimensions must agree. This leads to *consistency conditions*.

- We will show that the *Yangian invariance* condition is *equivalent* to whole set of *consistency conditions*.

- The latter can written down in a totally explicit way. They have a simpler form and are valid for any number of points.



In the rest of the talk:

- Prove the equivalence between Witten diagrams and conformal integrals.

- Recursion relations, consistency conditions. Sketch the proof of the equivalence of consistency conditions and Yangian invariance.

- Possible implications.

$W_n \leftrightarrow I_n$

In fact, the BCFT contact diagrams were already systematically studied for totally different reasons [Rastelli, Zhou '17]. So we can just borrow results from there.

Using Schwinger parameterization and integrating out the AdS coordinates, we get

$$W = C_n \int_0^\infty \prod_{i=1}^n dt_i t_i^{\Delta_i - 1} e^{-\sum_{i<j} t_i t_j P_{ij} - (\sum_{i=1}^n t_i m_i)^2}$$

Recall

$$W_n = \int \frac{dz_0 d^{d-1}\vec{z}}{z_0^d} \prod_{i=1}^n G_{B\partial}^{\Delta_i}(z, \vec{x}_i, m_i)$$

$$G_{B\partial}^{\Delta_i}(z, \vec{x}_i, m_i) = \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x}_i)^2 + m_i^2} \right)^{\Delta_i}$$

where we defined $P_{ij} = x_{ij}^2 + (m_i - m_j)^2$.

Note the integral is essentially *independent of d*! More precisely, all dependence is in the factor C,

$$C_n = \pi^{\frac{d-1}{2}} \Gamma[\frac{\sum_{i=1}^n \Delta_i - d + 1}{2}] \prod_{i=1}^n \Gamma^{-1}[\Delta_i]$$

We define the *d-independent correlator* $\widetilde{W} = C_n^{-1} W$.

$W_n \leftrightarrow I_n$

Now let's take a step back and integrate out only the *radial* coordinate of AdS. We find instead

$$\widetilde{W} = \frac{\pi^{\frac{1-d}{2}}}{2} \int d^{d-1}\vec{z} \int_0^\infty \prod_{i=1}^{n} dt_i t_i^{\Delta_i - 1} \Big(\sum_{i=1}^{n} t_i \Big)^{\frac{d-1-\sum_{i=1}^{n}\Delta_i}{2}} e^{-\sum_{i=1}^{n} t_i((\vec{z}-\vec{x}_i)^2+m_i^2)}$$

We can use the d-independence to set $d = D + 1$ where we recall $D = \sum_i \Delta_i$. Then we get

$$\widetilde{W} = \frac{\pi^{-\frac{D}{2}}}{2} \int d^{D}\vec{z} \int_0^\infty \prod_{i=1}^{n} dt_i t_i^{\Delta_i - 1} e^{-\sum_{i=1}^{n} t_i((\vec{z}-\vec{x}_i)^2+m_i^2)}$$

which is noting but I_n after using Schwinger parameterization!

Recall

$$I_n = \int \frac{d^D x_0}{\prod_{j=1}^{n} (x_{j0}^2 + m_j^2)^{\Delta_i}} ,$$

Recursion relations

Let us return to the representation

$$W = C_n \int_0^\infty \prod_{i=1}^n dt_i t_i^{\Delta_i - 1} e^{-\sum_{i<j} t_i t_j P_{ij} - (\sum_{i=1}^n t_i m_i)^2}$$

Taking derivatives w.r.t. P_{ij} and m_i gives the following *differential recursion relations*

$$\mathbb{O}_{ij} W = \frac{2\Delta_i \Delta_j}{d - 1 - \sum_i \Delta_i} W \Big|_{\Delta_{i,j} \to \Delta_{i,j}+1} \qquad \mathbb{D}_i W = \frac{4m_i^2 \Delta_i (\Delta_i + 1)}{d - 1 - \sum_i \Delta_i} W \Big|_{\Delta_i \to \Delta_i + 2}$$

where $\mathbb{D}_i = m_i N_i - 2 \sum_{j \neq i} m_i m_j \mathbb{O}_{ij}$ and $\mathbb{O}_{ij}$, N_i are derivative w.r.t. P_{ij} and m_i

$$\mathbb{O}_{ij} = \frac{\partial}{\partial P_{ij}} \Big|_{P,m}, \qquad N_i = \frac{\partial}{\partial m_i} \Big|_{P,m}$$

Recursion relations

For these relations to be mutually compatible, we need to have the following highly nontrivial consistency conditions

$$\mathbb{O}_{ij} W = \frac{2\Delta_i \Delta_j}{d - 1 - \sum_i \Delta_i} W \Big|_{\Delta_{i,j} \to \Delta_{i,j}+1}$$

$$\mathbb{D}_i W = \frac{4m_i^2 \Delta_i(\Delta_i + 1)}{d - 1 - \sum_i \Delta_i} W \Big|_{\Delta_i \to \Delta_i+2}$$

$$(\mathbb{O}_{ij}\mathbb{O}_{kl} - \mathbb{O}_{ik}\mathbb{O}_{jl})W = 0\ , \quad i, l \neq j, k\ ,$$

$$\mathbb{D}_i \mathbb{O}_{kl} W = 2m_i^2 \mathbb{O}_{ik}\mathbb{O}_{il} W\ , \quad i, j, k \text{ all different}\ ,$$

$$\mathbb{D}_j \mathbb{D}_k W = 4m_j^2 m_k^2 \mathbb{O}_{jk}\mathbb{O}_{jk} W\ , \qquad j \neq k\ .$$

A natural question: do they have a *symmetry origin*? The answer is yes!

Yangian invariance

The D-dimensional integral has an $SO(D,2)$ conformal symmetry, generated by

$$\mathrm{P}_j^{\hat{\mu}} = -i\partial_{x_j}^{\hat{\mu}}\,, \qquad \mathrm{L}_j^{\hat{\mu}\hat{\nu}} = ix_j^{\hat{\mu}}\partial_{x_j}^{\hat{\nu}} - ix_j^{\hat{\nu}}\partial_{x_j}^{\hat{\mu}}\,, \qquad \mathrm{D}_j = -i(x_{j,\mu}\partial_{x_j}^{\mu} + m_j\partial_{m_j} + \Delta_j)\,,$$

$$\mathrm{K}_j^{\hat{\mu}} = -2ix_j^{\hat{\mu}}(x_{j,\nu}\partial_{x_j}^{\nu} + m_j\partial_{m_j} + \Delta_j) + i(x_j^2 + m_j^2)\partial_{x_j}^{\hat{\mu}}\,.$$

The Yangian is an infinite dimensional extension generated by the above *level-zero* generators and the following *level-one* generators [Drinfeld]

$$\widehat{\mathrm{J}}^{\,a} = \frac{1}{2} f^{a}{}_{bc} \sum_{j<k}^{n} \mathrm{J}_j^c \mathrm{J}_k^b + \sum_{j=1}^{n} s_j \mathrm{J}_j^a\,.$$

Higher levels are generated by commutators. Because I_n is annihilated by both level-zero and level-one, the invariance under higher-level generators is guaranteed.

Yangian invariance

Moreover, $\widehat{\mathrm{J}}^{\,a}$ transform in the *adj* rep of level-zero. So invariance under the whole Yangian boils down to just that of one generator, which we can choose to be $\widehat{\mathrm{P}}^{\mu}$.

We also have *permutation* symmetry. As a result, annihilation by $\widehat{\mathrm{P}}^{\mu}$ is equivalent to annihilation by the following *two-site* operator $\widehat{\mathrm{P}}^{\mu}_{jk}$ for any j, k [Loebbert, Miczajka, Muller, Munkler]

$$\widehat{\mathrm{P}}^{\mu}_{jk} = \frac{i}{2}\Big(\mathrm{P}^{\mu}_{j}\mathrm{D}_{k} + \mathrm{P}_{j,\nu}\mathrm{L}^{\mu\nu}_{k} - i\Delta_{k}\mathrm{P}^{\mu}_{j} - (j \leftrightarrow k)\Big)\ , \qquad \widehat{\mathrm{P}}^{\mu} = \sum_{k>j=1}^{n} \widehat{\mathrm{P}}^{\mu}_{jk}\ .$$

Furthermore, it was observed that I_n is invariant under an extra level-one operator $\widehat{\mathrm{P}}^{\mu}_{jk,\text{extra}}$

$$\widehat{\mathrm{P}}^{\mu}_{jk,\text{extra}} = \frac{i}{2}\big(\mathrm{P}_{j,D+1}\mathrm{L}^{\mu,D+1}_{k} - (j \leftrightarrow k)\big)$$

The *full Yangian invariance condition* is:

$$\widehat{\mathrm{P}}^{\mu}_{jk,}I_n = 0\ , \qquad \widehat{\mathrm{P}}^{\mu}_{jk,\text{extra}}I_n = 0\ .$$

Yangian invariance

We claim that the invariance condition is equivalent to the consistency conditions. We will only sketch the proof for the simpler *massless* case with $m_i = 0$. The proof for the *massive* case is *similar* but technically more complicated.

In the massless case, the operator $\mathbb{D}_i$ vanishes. Therefore, there is only one class of consistency conditions which remains. Also the extra level-one operator $\widehat{\mathrm{P}}^{\mu}_{jk,\text{extra}}$ vanishes automatically. Consequently, we need to prove

$$(\mathbb{O}_{ij}\mathbb{O}_{kl} - \mathbb{O}_{ik}\mathbb{O}_{jl})W = 0 \qquad \Leftrightarrow \qquad \widehat{\mathrm{P}}^{\mu}_{jk,}W = 0$$

Yangian invariance

To show this, we first write $\widehat{\mathrm{P}}^{\mu}_{jk}$ explicitly as

$$\widehat{\mathrm{P}}^{\mu}_{jk} = \frac{i}{2}\Big(X^{\nu\mu\rho}\partial_{x_j,\rho}\partial_{x_k,\nu} + (2\Delta_j + m_j\partial_{m_j})\partial^{\mu}_{x_k} - (2\Delta_k + m_k\partial_{m_k})\partial^{\mu}_{x_j}\Big)$$

where $X^{\nu\mu\rho} = x^{\nu}_{jk}\eta^{\mu\rho} + x^{\rho}_{jk}\eta^{\mu\nu} - x^{\mu}_{jk}\eta^{\nu\rho}$. Since the Witten diagrams are functions of P_{ij}, the derivatives can be written in terms of those of P_{ij}

$$\partial^{\mu}_{x_j} = 2\sum_{i\neq j} x^{\mu}_{ji}\mathbb{O}_{ij}\ ,$$

$$\partial^{\rho}_{x_j}\partial^{\nu}_{x_k} = 4\sum_{i\neq k}\sum_{l\neq j} x^{\rho}_{jl}x^{\nu}_{ki}\mathbb{O}_{jl}\mathbb{O}_{ki} - 2\eta^{\rho\nu}\mathbb{O}_{jk}$$

Yangian invariance

By using

$$X^{\nu\mu\rho}x^{\rho}_{jl}x^{\nu}_{ki} = \frac{1}{2}\left(\mathrm{T}^{\mu}_{jk}P_{jk}P_{li} - \mathrm{T}^{\mu}_{ji}P_{ji}P_{kl} - \mathrm{T}^{\mu}_{jl}P_{jl}P_{ki} + \mathrm{T}^{\mu}_{ki}P_{ki}P_{jl} + \mathrm{T}^{\mu}_{kl}P_{kl}P_{ij} - \mathrm{T}^{\mu}_{il}P_{il}P_{jk}\right)$$

with $\mathrm{T}^{\mu}_{ab} = \dfrac{x^{\mu}_{ab}}{P_{ab}}$, we can write the action of $\widehat{\mathrm{P}}^{\mu}_{jk}$ in the form of

$$-2i\widehat{\mathrm{P}}^{\mu}_{jk}W = \sum_{a<b} \mathrm{T}^{\mu}_{ab}E_{ab} \ .$$

The coefficients E_{ab} have the same dimensions as W and the structures T^{μ}_{ab} are *independent*.

Yangian invariance then requires

$$E_{ab} = 0 \ .$$

Yangian invariance

These coefficient functions are

$$E_{il} = -2P_{il}P_{jk}(\mathbb{O}_{jl}\mathbb{O}_{ik} - \mathbb{O}_{ji}\mathbb{O}_{kl})W$$

$$i, l \neq j, k$$

$$E_{ki} = 2\Big\{\sum_{l\neq j,k} P_{ki}P_{jl}\mathbb{O}_{jl}\mathbb{O}_{ki} + 2P_{ki}P_{jk}\mathbb{O}_{jk}\mathbb{O}_{ki} + \sum_{l\neq j,k} P_{ki}P_{jl}\mathbb{O}_{ji}\mathbb{O}_{kl} + 2\Delta_j P_{ki}\mathbb{O}_{ki}\Big\}W$$

$$E_{jl} = -E_{ki}\Big|_{j\leftrightarrow k, i\leftrightarrow l} \qquad E_{jk} = 2\Big\{\sum_{i,l\neq j,k} P_{jk}P_{il}\mathbb{O}_{jl}\mathbb{O}_{ki} - 2P_{jk}^2\mathbb{O}_{jk}\mathbb{O}_{jk} - (2 - D + 2\Delta_j + 2\Delta_k)P_{jk}\mathbb{O}_{jk}\Big\}W\,.$$

The condition $E_{il} = 0$ gives the consistency condition $(\mathbb{O}_{ij}\mathbb{O}_{kl} - \mathbb{O}_{ik}\mathbb{O}_{jl})W = 0$. The other conditions can be shown to vanish after using this condition and conformal invariance.

The *massive case* is similar: $E_{il} = 0$ gives $(\mathbb{O}_{ij}\mathbb{O}_{kl} - \mathbb{O}_{ik}\mathbb{O}_{jl})W = 0$, and $E_{ki} = 0$ gives $\mathbb{D}_i\mathbb{O}_{kl}W = 2m_i^2\mathbb{O}_{ik}\mathbb{O}_{il}W$. $E_{jk} = 0$ yields no new relations. The remaining condition $\mathbb{D}_j\mathbb{D}_kW = 4m_j^2m_k^2\mathbb{O}_{jk}\mathbb{O}_{jk}W$ follows from $\widehat{\mathrm{P}}^{\mu}_{jk,\mathrm{extra}}W = 0$.

Conclusions and outlook

- A new connection between integrability and holography.

- Contact Witten diagrams are Yangian invariant conformal integrals in flat space.

- An array of differential recursion relations. Their consistency conditions are the same as the Yangian invariance condition.

Many things to explore!

- From the consistency conditions

  - In some simple cases, Yangian invariance was shown to completely determine the conformal integrals [Loebbert, Miczajka, Muller, Munkler].

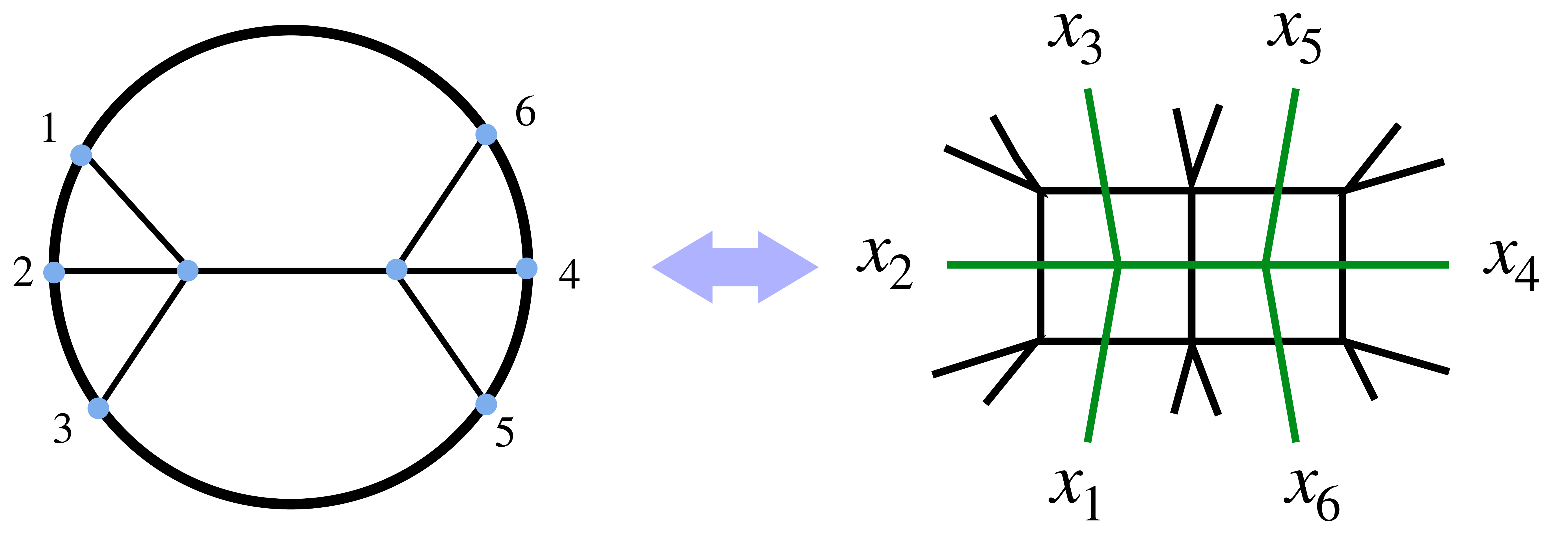
  - In terms of consistency conditions, this condition now takes a simpler form the redundancies have been removed. We can use them to explicitly compute more complicated integrals.

  - What's the structure of these constraints as a system of differential operators?

  - Does that lead to new insight about integrability?

- Beyond contact diagrams: integrability properties of exchange diagrams.

Certain exchange diagrams and conformal integrals are known to be the same [Paulos, Spradlin, Volovich ’12, Ma, XZ ‘22]. The Yangian invariance of these two-loop integrals are also proven [Loebbert, Miczajka, Muller, Munkler]. E.g.,



$\Delta = 1$ for all legs

However, the general story is not yet clear.

- Supersymmetry and super-Yangian invariance.

  - Here we have restricted ourselves to the bosonic case. We can also consider the Yangian of $PSU(2,2|4)$ and study the Yangian invariant correlators.

  - Presumably, these will be some “super D-functions”.

  - It would be very interesting to see if the super Yangian invariance condition can give rise to an alternative derivation of the general results of holographic correlators on $AdS_5 \times S^5$ [Rastelli, Zhou ’16].

Thank you!