Categorical symmetry induced superconformal field theories

Qing-Rui Wang

Yau Mathematical Sciences Center, Tsinghua University

2022.8.26

第三届全国场论与弦论学术研讨会

Collaborators: Jin Chen, Ji-Yao Chen, Babak Haghighat, Wei Li, Junshen Wang

Motivations

Motivations

- Generalizing group (invertible) symmetries to fusion category (non-invertible) symmetries

Motivations

- Generalizing group (invertible) symmetries to fusion category (non-invertible) symmetries
- Understanding bulk/boundary (TQFT/CFT) correspondence using categorical symmetries

Motivations

- Generalizing group (invertible) symmetries to fusion category (non-invertible) symmetries
- Understanding bulk/boundary (TQFT/CFT) correspondence using categorical symmetries
- Finding new constraints on the IR theories with a given UV theory

Outline

- Introduction to fusion category and categorical symmetry
- Categorical symmetry in 3D TQFT
  $\mathbb{Z}_2$ gauge theory $\rightarrow$ Dijkgraaf-Witten G gauge theory $\rightarrow$ Turaev-Viro-Levin-Wen model
- CFT on the 2D boundary
  categorical symmetry in Ising and tricritical Ising CFT
- Categorical symmetry in 3D Fermionic TQFT
  superfusion category, fermionic Turaev-Viro-Levin-Wen model
- SCFT on the 2D boundary
  tricritical Ising minimal model, parafermion model

From group to representations

From group to representations

- Group

  $g \cdot h = gh, \quad 1, \quad (g \cdot h) \cdot k = g \cdot (h \cdot k), \quad g \cdot g^{-1} = 1$

From group to representations

- Group

  $$g \cdot h = gh, \quad 1, \quad (g \cdot h) \cdot k = g \cdot (h \cdot k), \quad g \cdot g^{-1} = 1$$

- Group representation

  $$\rho_1 \otimes \rho_2 \sim \oplus_i \, \rho_i, \quad 1, \quad (\rho_1 \otimes \rho_2) \otimes \rho_3 \sim \rho_1 \otimes (\rho_2 \otimes \rho_3), \quad \rho \otimes \bar{\rho} \sim 1 \oplus \ldots$$

From group to representations

- Group

$$g \cdot h = gh, \quad 1, \quad (g \cdot h) \cdot k = g \cdot (h \cdot k), \quad g \cdot g^{-1} = 1$$

- Group representation

$$\rho_1 \otimes \rho_2 \sim \oplus_i \rho_i, \quad 1, \quad (\rho_1 \otimes \rho_2) \otimes \rho_3 \sim \rho_1 \otimes (\rho_2 \otimes \rho_3), \quad \rho \otimes \bar{\rho} \sim 1 \oplus \ldots$$

- Tannaka duality: we can reconstruct G from Rep(G)

  (including reps, CG or 3-j symbols , 6-j symbols)

From group to representations

- Group
  $$g \cdot h = gh, \quad 1, \quad (g \cdot h) \cdot k = g \cdot (h \cdot k), \quad g \cdot g^{-1} = 1$$
- Group representation
  $$\rho_1 \otimes \rho_2 \sim \oplus_i \rho_i, \quad 1, \quad (\rho_1 \otimes \rho_2) \otimes \rho_3 \sim \rho_1 \otimes (\rho_2 \otimes \rho_3), \quad \rho \otimes \bar{\rho} \sim 1 \oplus \ldots$$
- Tannaka duality: we can reconstruct G from Rep(G)
  (including reps, CG or 3-j symbols , 6-j symbols)
- Slogan for category: Morphisms (maps) are more important than objects (sets)

From group to representations

- Group

  $g \cdot h = gh, \quad 1, \quad (g \cdot h) \cdot k = g \cdot (h \cdot k), \quad g \cdot g^{-1} = 1$

- Group representation

  $\rho_1 \otimes \rho_2 \sim \oplus_i \rho_i, \quad 1, \quad (\rho_1 \otimes \rho_2) \otimes \rho_3 \sim \rho_1 \otimes (\rho_2 \otimes \rho_3), \quad \rho \otimes \bar{\rho} \sim 1 \oplus \ldots$

- Tannaka duality: we can reconstruct G from Rep(G)

  (including reps, CG or 3-j symbols , 6-j symbols)

- Slogan for category: Morphisms (maps) are more important than objects (sets)
- 人的本质是一切社会关系的总和

From group to representations

- Group

  $g \cdot h = gh, \quad 1, \quad (g \cdot h) \cdot k = g \cdot (h \cdot k), \quad g \cdot g^{-1} = 1$

- Group representation

  $\rho_1 \otimes \rho_2 \sim \oplus_i \rho_i, \quad 1, \quad (\rho_1 \otimes \rho_2) \otimes \rho_3 \sim \rho_1 \otimes (\rho_2 \otimes \rho_3), \quad \rho \otimes \bar{\rho} \sim 1 \oplus \dots$

- Tannaka duality: we can reconstruct G from Rep(G)
  (including reps, CG or 3-j symbols , 6-j symbols)
- Slogan for category: Morphisms (maps) are more important than objects (sets)
- 人的本质是一切社会关系的总和
- Particles are defined by its interactions with others

Fusion category

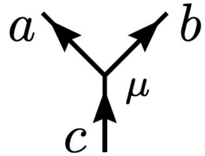
Fusion category

- simple objects {a,b,c,…}

G-irreps

Fusion category

- simple objects {a,b,c,…}
- morphism



$= |a, b; c, \mu\rangle \in V_c^{ab}$

G-irreps

CG coefficient

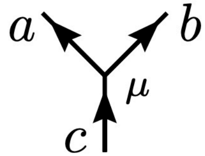
$a \otimes b = \bigoplus_c N_c^{a,b} c$

Fusion category

- simple objects {a,b,c,…}

G-irreps

- morphism

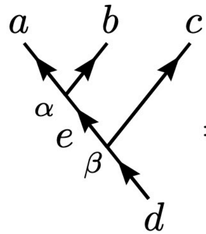


$= |a,b;c,\mu\rangle \in V_c^{ab}$

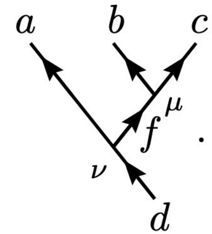
CG coefficient

$a \otimes b = \bigoplus_c N_c^{a,b} c$

- associator



$= \sum_{f,\mu,\nu} \left[F_d^{abc}\right]_{(e,\alpha,\beta)(f,\mu,\nu)}$



.

6-j symbol

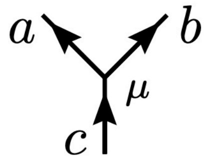
$(a \otimes b) \otimes c \to a \otimes (b \otimes c)$

Fusion category

- simple objects {a,b,c,…}

G-irreps

- morphism

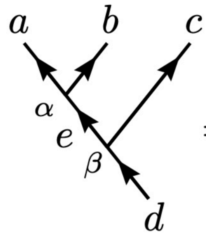


$= |a, b; c, \mu\rangle \in V_c^{ab}$

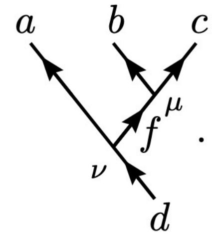
CG coefficient

$a \otimes b = \oplus_c N_c^{a,b} c$

- associator



$= \sum_{f,\mu,\nu} \left[F_d^{abc}\right]_{(e,\alpha,\beta)(f,\mu,\nu)}$

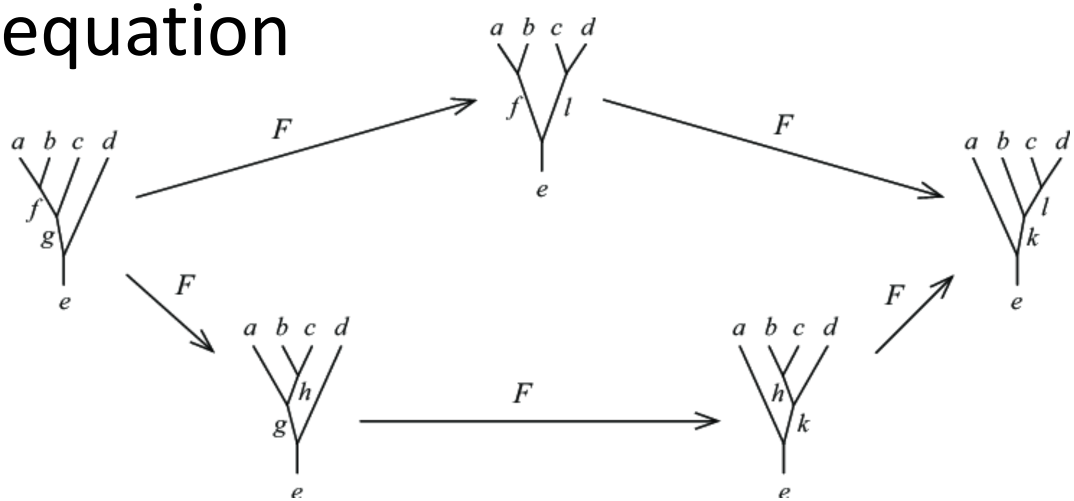


.

6-j symbol

$(a \otimes b) \otimes c \to a \otimes (b \otimes c)$

- pentagon equation



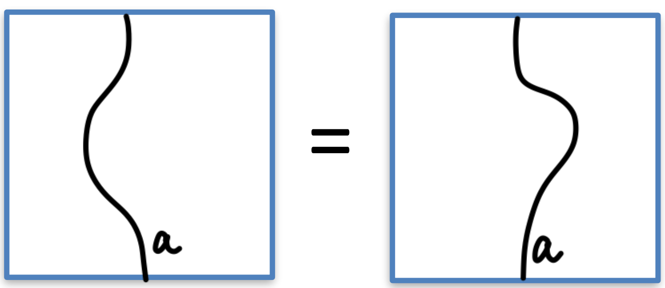
Topological defect line and categorical symmetry in 2D

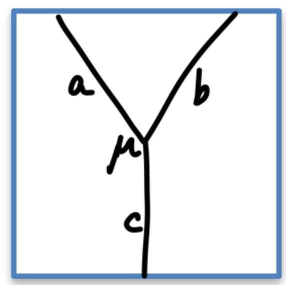
Topological defect line and categorical symmetry in 2D

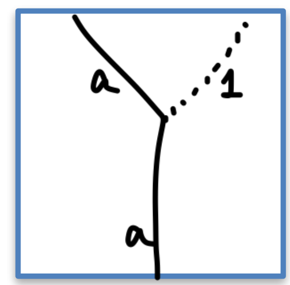
- {Topological defect lines} is a fusion category for 2D QFT (not only in CFT)

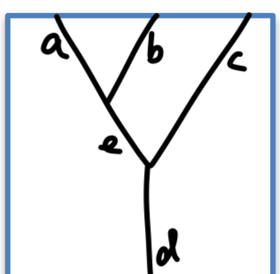
Topological defect line and categorical symmetry in 2D

- {Topological defect lines} is a fusion category for 2D QFT (not only in CFT)

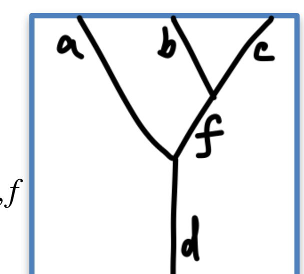






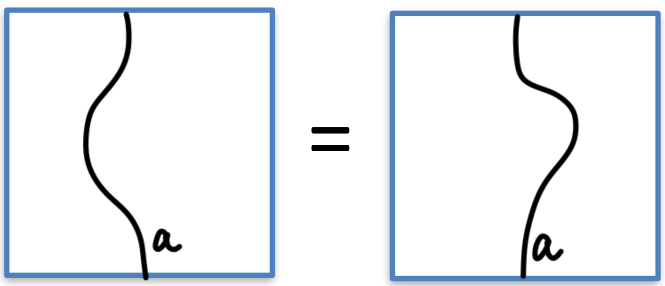


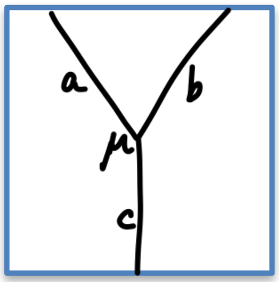
$$= \sum \left(F_d^{a,b,c}\right)_{e,f}$$

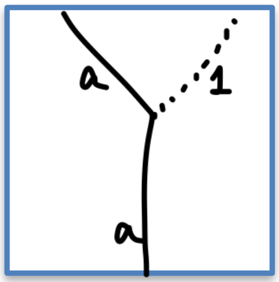


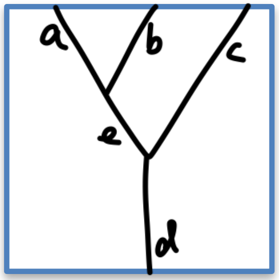
Topological defect line and categorical symmetry in 2D

- {Topological defect lines} is a fusion category for 2D QFT (not only in CFT)
- Group-like symmetry if the defect is invertible: $a \otimes \bar{a} = 1$

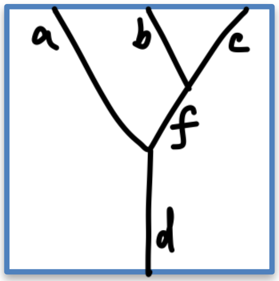






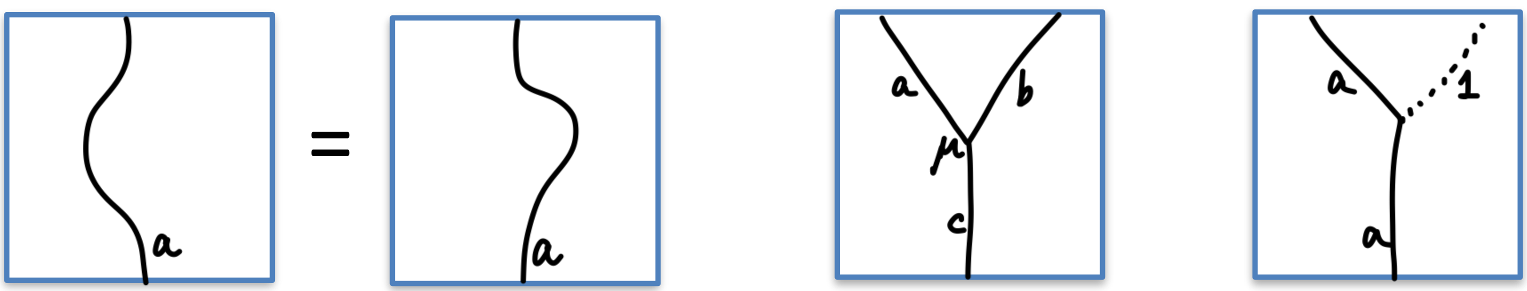


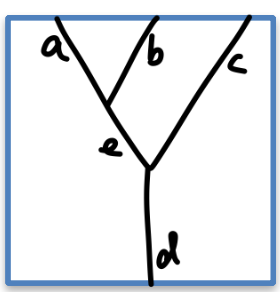
$$= \sum \left(F_d^{a,b,c}\right)_{e,f}$$



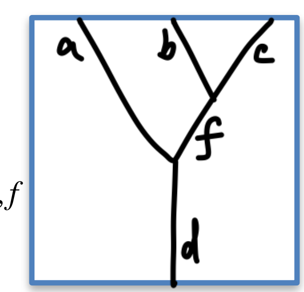
Topological defect line and categorical symmetry in 2D

- {Topological defect lines} is a fusion category for 2D QFT (not only in CFT)
- Group-like symmetry if the defect is invertible: $a \otimes \bar{a} = 1$
- Can be generalized to higher dimensions





$$= \sum \left(F^{a,b,c}_{d}\right)_{e,f}$$



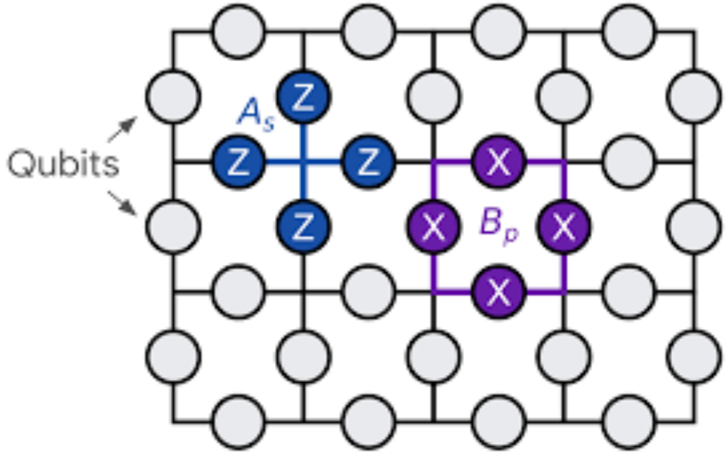
Outline

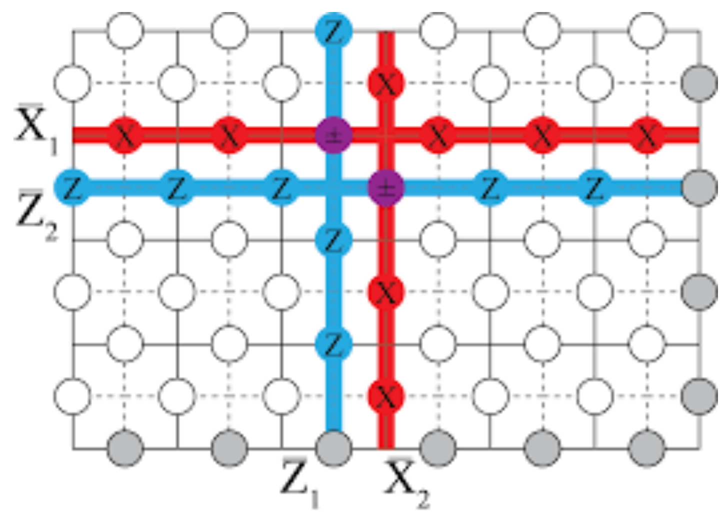
- Introduction to fusion category and categorical symmetry
- Categorical symmetry in 3D TQFT
  $\mathbb{Z}_2$ gauge theory $\rightarrow$ Dijkgraaf-Witten G gauge theory $\rightarrow$ Turaev-Viro-Levin-Wen model
- CFT on the 2D boundary
  categorical symmetry in Ising and tricritical Ising CFT
- Categorical symmetry in 3D Fermionic TQFT
  superfusion category, fermionic Turaev-Viro-Levin-Wen model
- SCFT on the 2D boundary
  tricritical Ising minimal model, parafermion model

3D toric code

- Toric code model (Kitaev 1997) = lattice $\mathbb{Z}_2$ gauge theory

$$H = -\sum_s A_s - \sum_p B_p$$





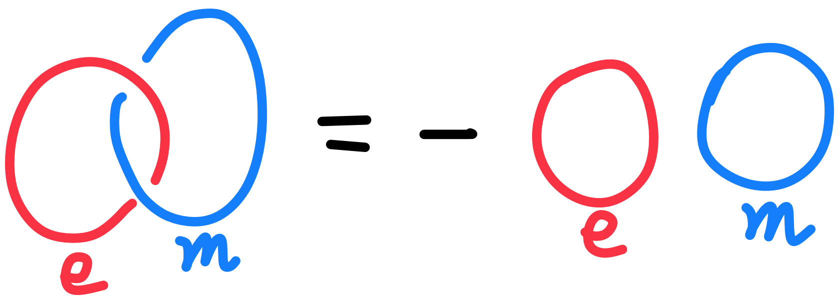
Wilson/’t Hooft loops as 1-form symmetries

Wilson/’t Hooft loops as 1-form symmetries

- Excitations of toric code: {1, e, m, f=em}
- Wilson/’t Hooft lines as worldlines of Electric/magnetic charges

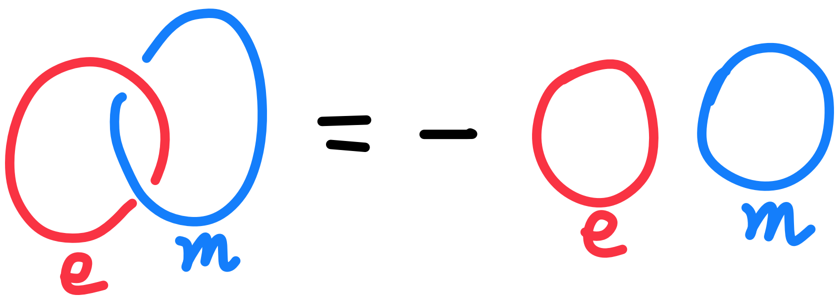
Wilson/'t Hooft loops as 1-form symmetries

- Excitations of toric code: {1, e, m, f=em}
- Wilson/'t Hooft lines as worldlines of Electric/ magnetic charges



Wilson/'t Hooft loops as 1-form symmetries

- Excitations of toric code: {1, e, m, f=em}
- Wilson/'t Hooft lines as worldlines of Electric/magnetic charges
- Wilson/'t Hooft loops generate invertible anomalous $(\mathbb{Z}_2)^2$ 1-form symmetry (codim-2 submanifold)



Categorical symmetry in 3D Dijkgraaf-Witten G gauge theory

Categorical symmetry in 3D Dijkgraaf-Witten G gauge theory

- 3D gauge theory with gauge group G, 3-cocycle $\omega_3 \in H^3(BG, U(1))$

Categorical symmetry in 3D Dijkgraaf-Witten G gauge theory

- 3D gauge theory with gauge group G, 3-cocycle $\omega_3 \in H^3(BG, U(1))$
- Excitations:
  flux ('t Hooft line): conjugacy class of G
  charge (Wilson line): irrep of G
  bound state: twisted irrep of centralizer of flux

Categorical symmetry in 3D Dijkgraaf-Witten G gauge theory

- 3D gauge theory with gauge group G, 3-cocycle $\omega_3 \in H^3(BG, U(1))$
- Excitations:
  flux ('t Hooft line): conjugacy class of G
  charge (Wilson line): irrep of G
  bound state: twisted irrep of centralizer of flux
- Excitations described by representation of twisted quantum double $D^{\omega_3}(G)$

Categorical symmetry in 3D Dijkgraaf-Witten G gauge theory

- 3D gauge theory with gauge group G, 3-cocycle $\omega_3 \in H^3(BG, U(1))$
- Excitations:
  flux ('t Hooft line): conjugacy class of G
  charge (Wilson line): irrep of G
  bound state: twisted irrep of centralizer of flux
- Excitations described by representation of twisted quantum double $D^{\omega_3}(G)$
- Wilson/'t Hooft loops as non-invertible categorical symmetries if G is non-Abelian

3D Turaev-Viro-Levin-Wen model

3D Turaev-Viro-Levin-Wen model

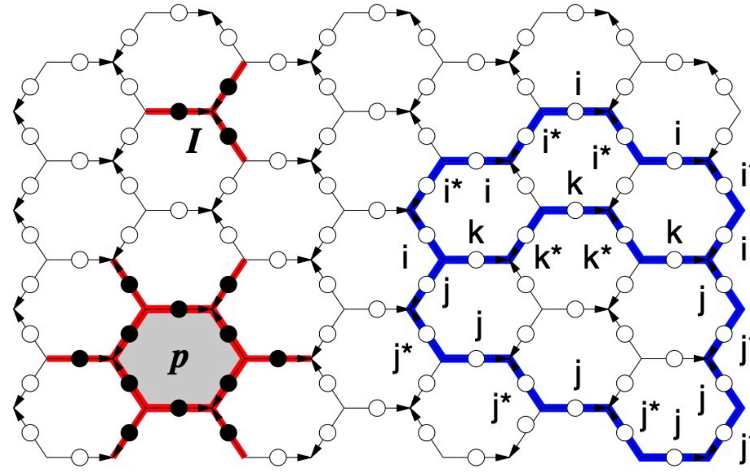
- 3D manifold invariants
  (Jones, Witten, Reshetikhin-Turaev, …)

3D Turaev-Viro-Levin-Wen model

- 3D manifold invariants
  (Jones, Witten, Reshetikhin-Turaev, …)
- Generalization of gauge theory

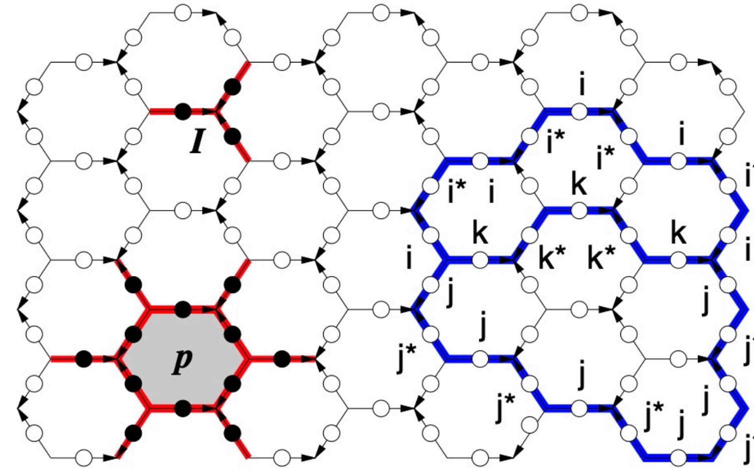
3D Turaev-Viro-Levin-Wen model

- 3D manifold invariants
  (Jones, Witten, Reshetikhin-Turaev, …)
- Generalization of gauge theory
- Hilbert space

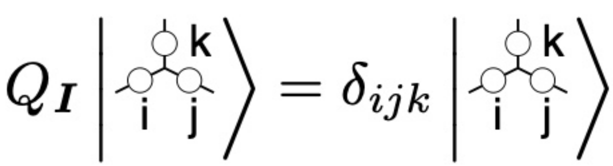


3D Turaev-Viro-Levin-Wen model

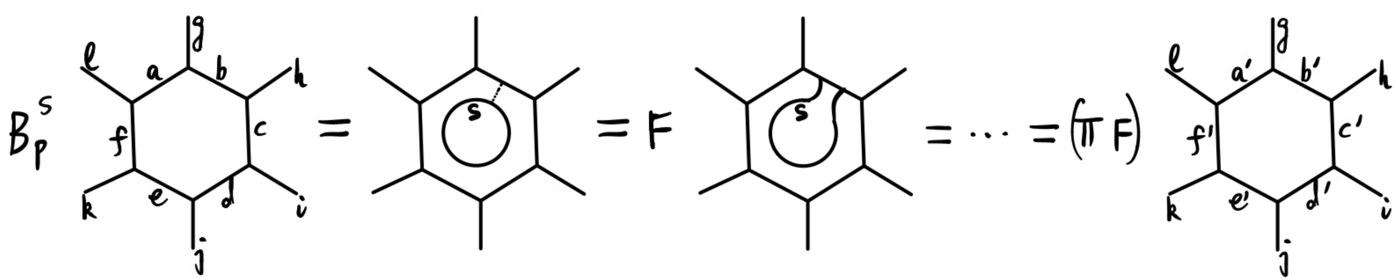
- 3D manifold invariants
  (Jones, Witten, Reshetikhin-Turaev, …)
- Generalization of gauge theory
- Hilbert space
- Hamiltonian



$$H = -\sum_{I} Q_{I} - \sum_{p} B_{p}$$



$$B_{p} = \sum_{s=0}^{N} a_s B_{p}^{s}$$



Categorical symmetry in Turaev-Viro-Levin-Wen model

Categorical symmetry in Turaev-Viro-Levin-Wen model

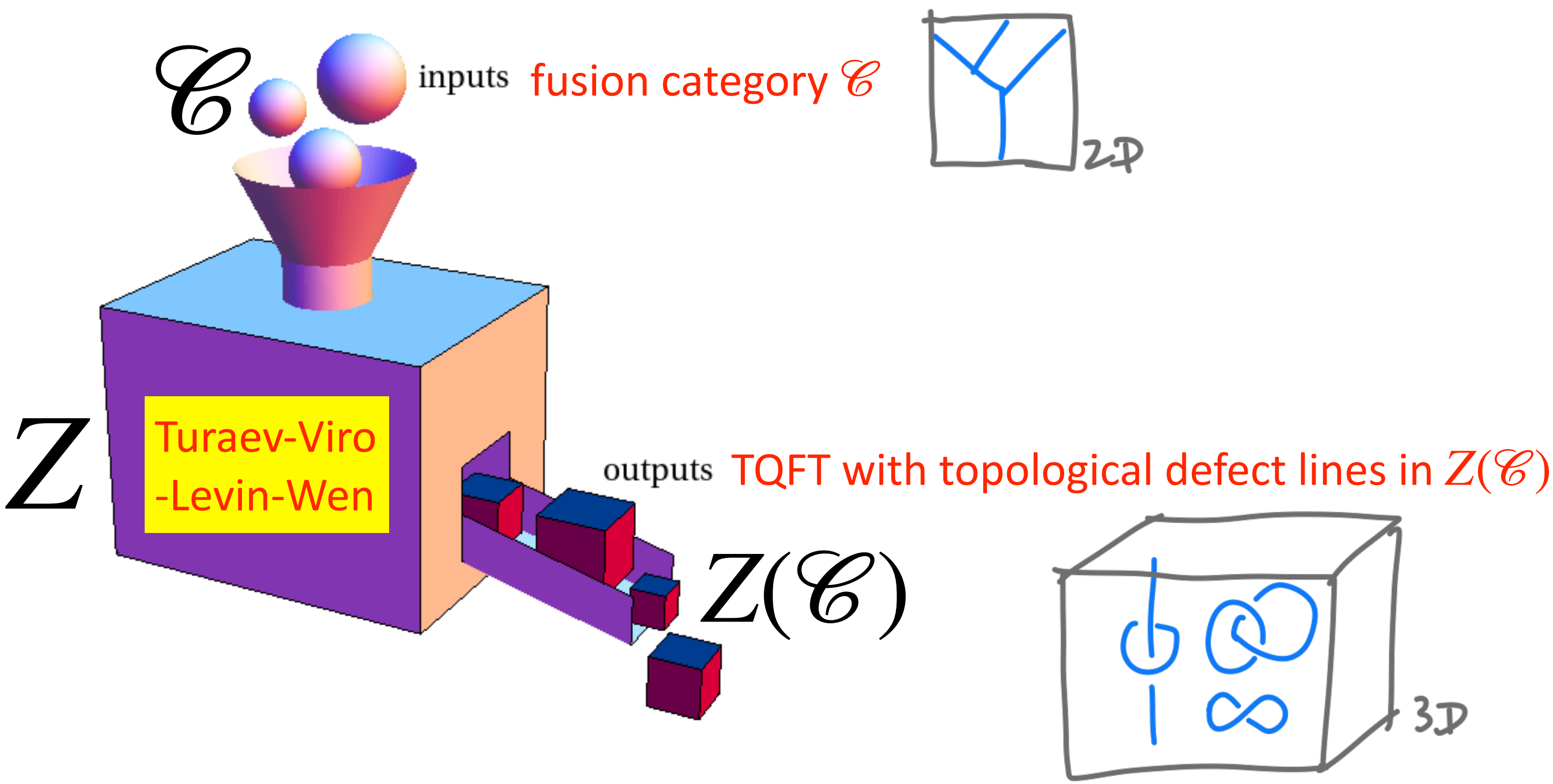
- Input fusion category $\mathscr{C}$

Categorical symmetry in Turaev-Viro-Levin-Wen model

- Input fusion category $\mathscr{C}$
- Output TQFT excitations in Drinfeld center $Z(\mathscr{C})$: topological defect lines as categorical symmetry

Categorical symmetry in Turaev-Viro-Levin-Wen model

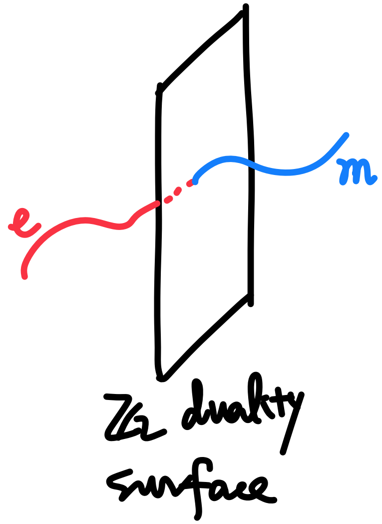
- Input fusion category $\mathscr{C}$
- Output TQFT excitations in Drinfeld center $Z(\mathscr{C})$: topological defect lines as categorical symmetry



Gauging e-m duality: 3D Z(Ising) TQFT

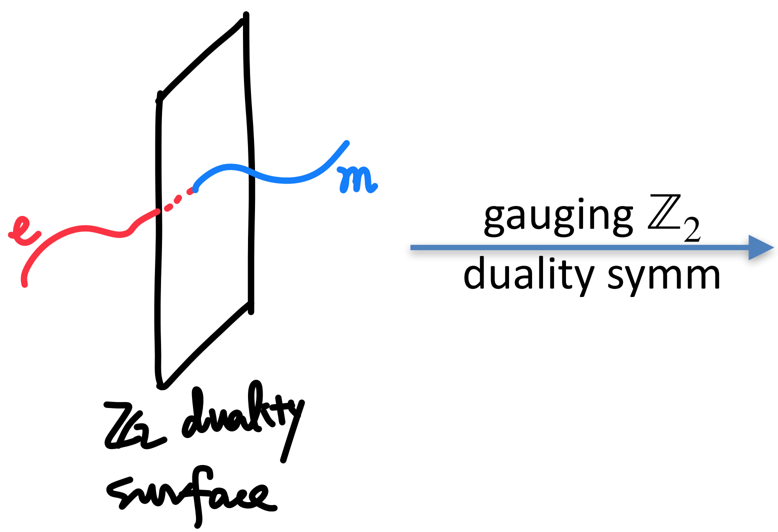
Gauging e-m duality: 3D Z(Ising) TQFT

- 3D $\mathbb{Z}_2$ gauge theory, e-m duality topological defect surface (0-form $\mathbb{Z}_2$ symmetry)



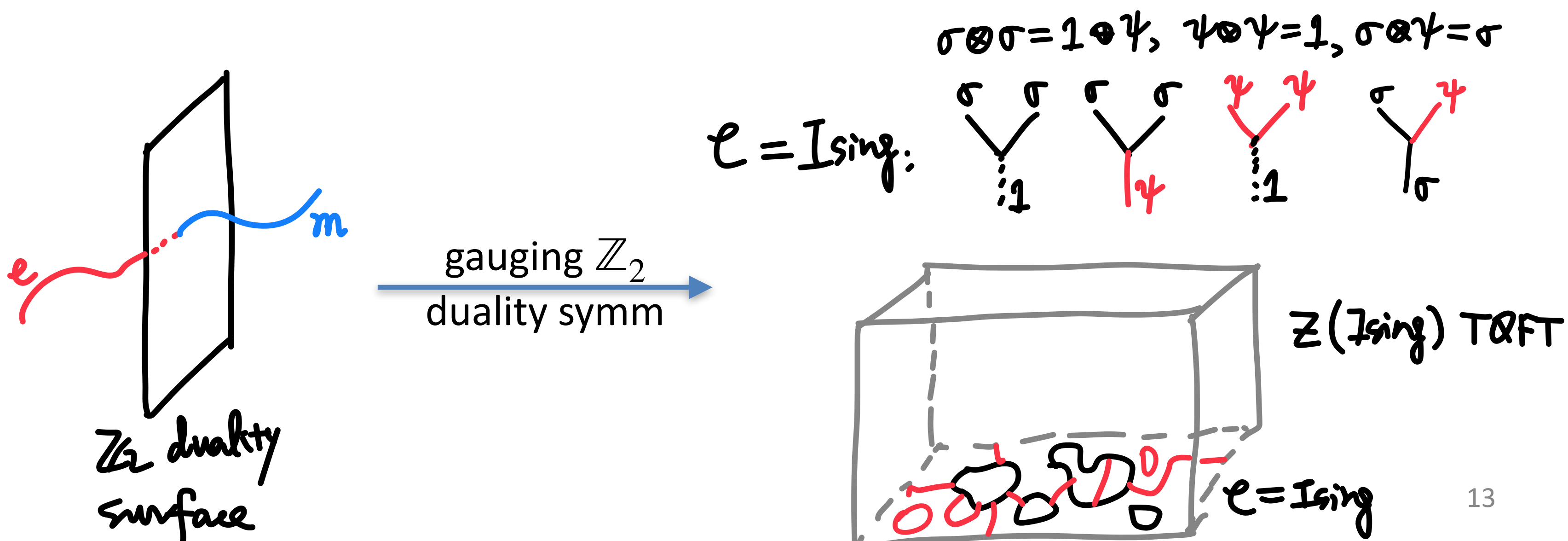
Gauging e-m duality: 3D Z(Ising) TQFT

- 3D $\mathbb{Z}_2$ gauge theory, e-m duality topological defect surface (0-form $\mathbb{Z}_2$ symmetry)
- Gauging e-m duality $\mathbb{Z}_2$ symmetry → Turaev-Viro-Levin-Wen model with input category $\mathscr{C}$=Ising → 3D TQFT $Z(\mathscr{C}$=Ising) = Ising $\boxtimes$ $\overline{\text{Ising}}$



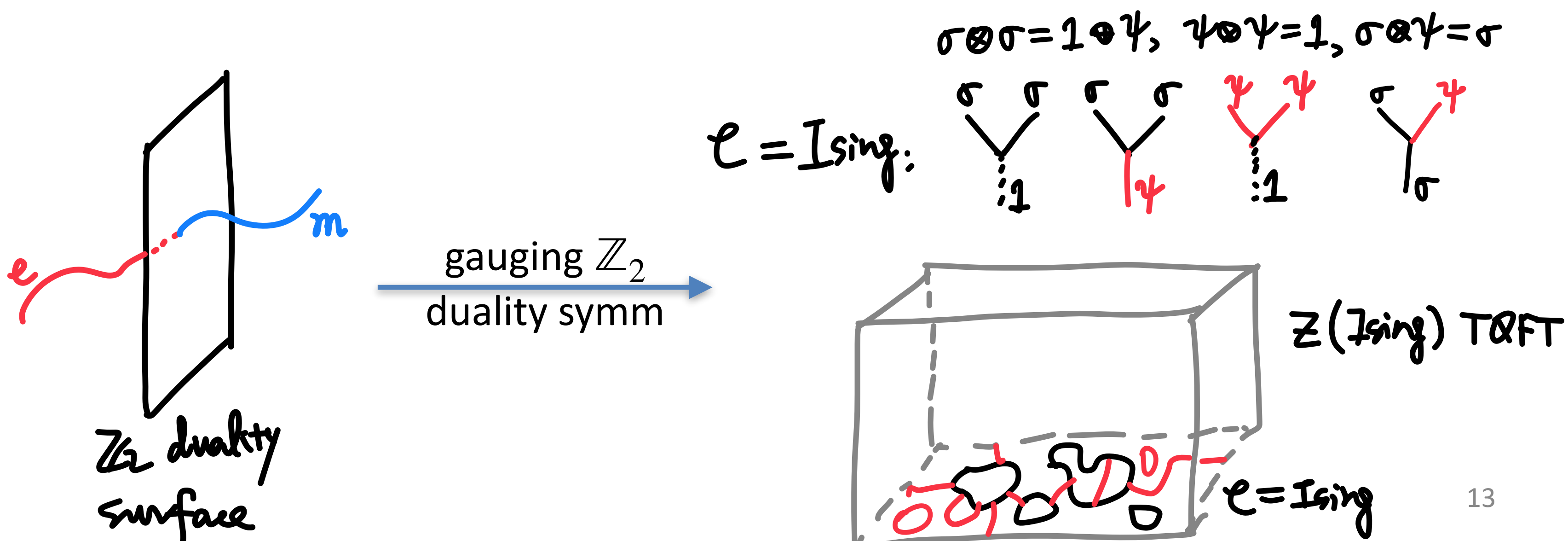
Gauging e-m duality: 3D Z(Ising) TQFT

- 3D $\mathbb{Z}_2$ gauge theory, e-m duality topological defect surface (0-form $\mathbb{Z}_2$ symmetry)
- Gauging e-m duality $\mathbb{Z}_2$ symmetry $\rightarrow$ Turaev-Viro-Levin-Wen model with input category $\mathscr{C}$=Ising $\rightarrow$ 3D TQFT $Z(\mathscr{C}\text{=Ising})$ = Ising $\boxtimes$ $\overline{\text{Ising}}$



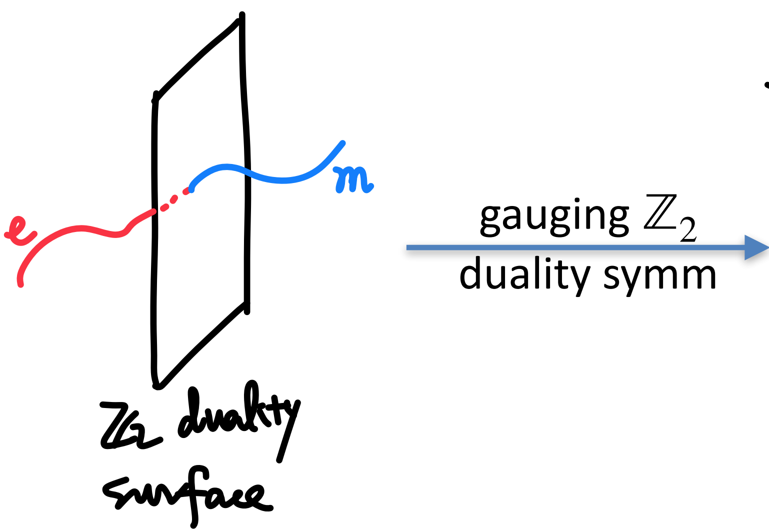
Gauging e-m duality: 3D Z(Ising) TQFT

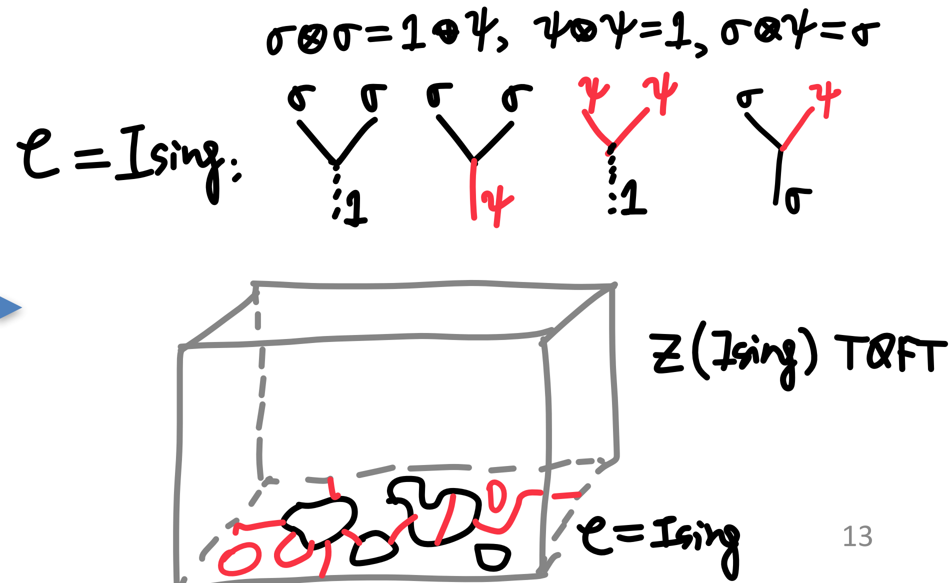
- 3D $\mathbb{Z}_2$ gauge theory, e-m duality topological defect surface (0-form $\mathbb{Z}_2$ symmetry)
- Gauging e-m duality $\mathbb{Z}_2$ symmetry → Turaev-Viro-Levin-Wen model with input category $\mathscr{C}$=Ising → 3D TQFT $Z(\mathscr{C}\text{=Ising})$ = Ising $\boxtimes$ $\overline{\text{Ising}}$
- Gauge theory to non-gauge theory



Gauging e-m duality: 3D Z(Ising) TQFT

- 3D $\mathbb{Z}_2$ gauge theory, e-m duality topological defect surface (0-form $\mathbb{Z}_2$ symmetry)
- Gauging e-m duality $\mathbb{Z}_2$ symmetry → Turaev-Viro-Levin-Wen model with input category $\mathscr{C}$=Ising → 3D TQFT $Z(\mathscr{C}$=Ising) = Ising $\boxtimes$ $\overline{\text{Ising}}$
- Gauge theory to non-gauge theory
- {gauge theories} is NOT closed under gauging





Outline

- Introduction to fusion category and categorical symmetry
- Categorical symmetry in 3D TQFT
  $\mathbb{Z}_2$ gauge theory $\rightarrow$ Dijkgraaf-Witten G gauge theory $\rightarrow$ Turaev-Viro-Levin-Wen model
- CFT on the 2D boundary
  categorical symmetry in Ising and tricritical Ising CFT
- Categorical symmetry in 3D Fermionic TQFT
  superfusion category, fermionic Turaev-Viro-Levin-Wen model
- SCFT on the 2D boundary
  tricritical Ising minimal model, parafermion model

2D boundary theory

2D boundary theory

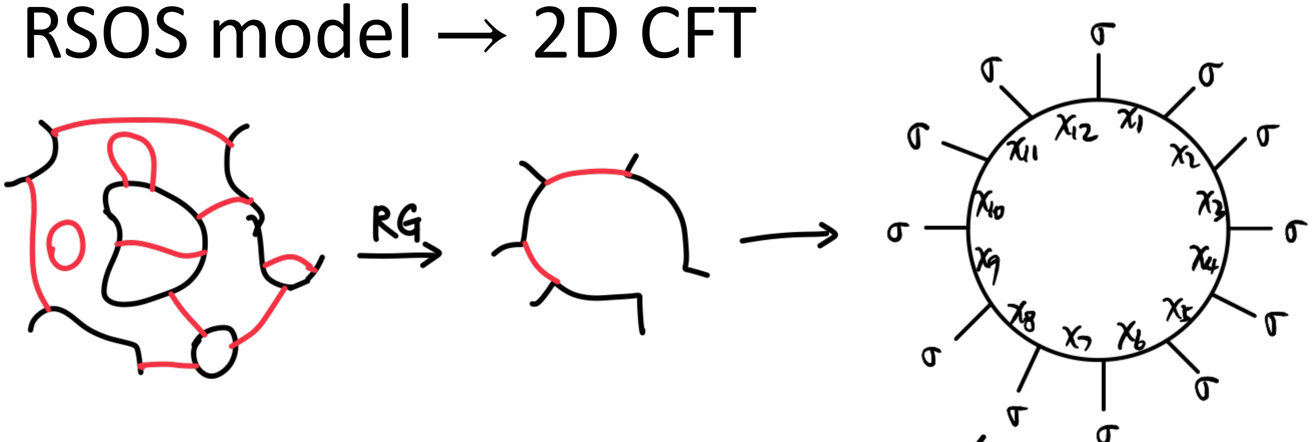
- Bulk: 3D Z(Ising) TQFT, boundary: which 2D CFT?

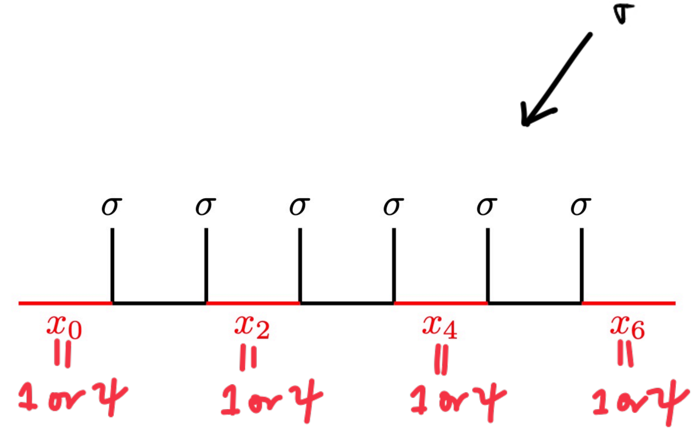
2D boundary theory

- Bulk: 3D Z(Ising) TQFT, boundary: which 2D CFT?
- 3D bulk RG → 1+1D boundary anyon chain → RSOS model → 2D CFT

2D boundary theory

- Bulk: 3D Z(Ising) TQFT, boundary: which 2D CFT?
- 3D bulk RG → 1+1D boundary anyon chain → RSOS model → 2D CFT



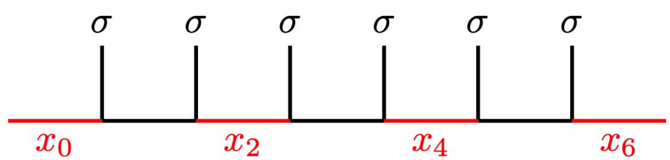


anyon chain -> spin chain

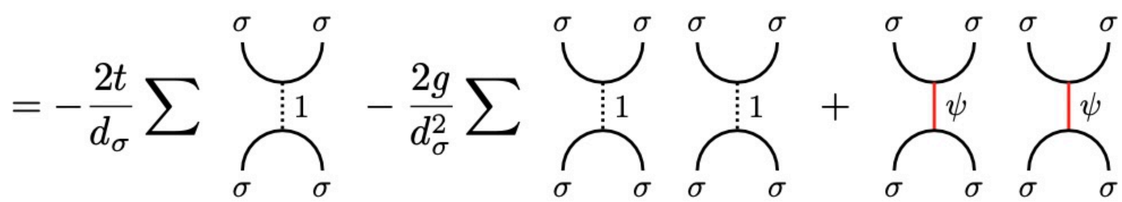
2D boundary Hamiltonian and phase diagram

2D boundary Hamiltonian and phase diagram

- Hamiltonian from Temperley-Lieb algebra

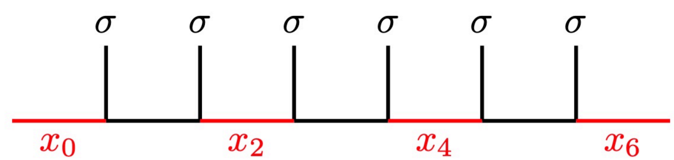


$$H^b_{\text{Ising}} = -2t\sum_i P_i^{(1)} - 2g\sum_i \left(P_i^{(1)} \otimes P_{i+2}^{(1)} + P_i^{(\psi)} \otimes P_{i+2}^{(\psi)}\right)$$

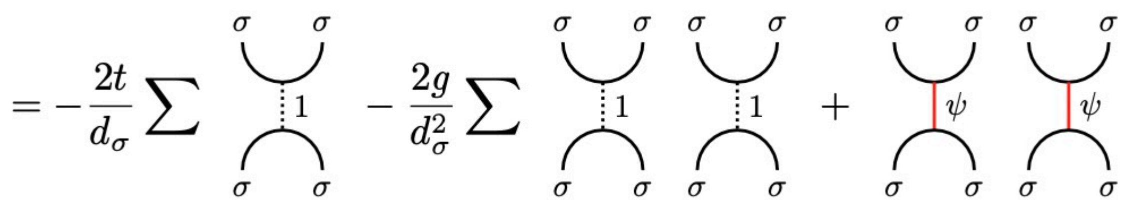


2D boundary Hamiltonian and phase diagram

- Hamiltonian from Temperley-Lieb algebra



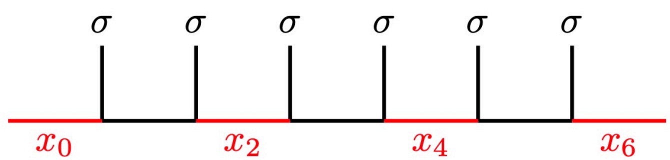
$$H^b_{\text{Ising}} = -2t\sum_i P_i^{(1)} - 2g\sum_i \left(P_i^{(1)} \otimes P_{i+2}^{(1)} + P_i^{(\psi)} \otimes P_{i+2}^{(\psi)}\right)$$



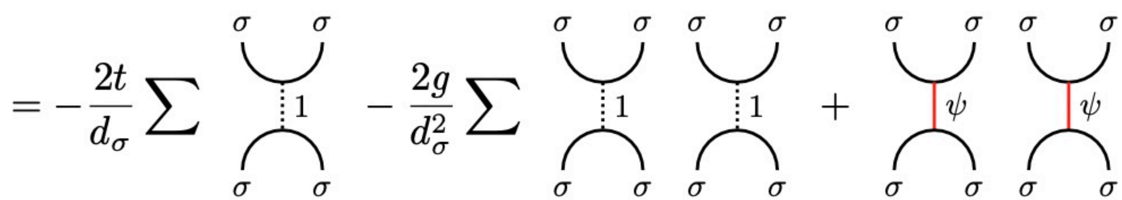
$$H^b_{\text{Ising}} = -t\sum_i X_i - t\sum_{\langle ij\rangle} Z_i Z_j - g\sum_{\langle ij\rangle} X_i X_j - g\sum_{\langle ijk\rangle} Z_i Z_k,$$

2D boundary Hamiltonian and phase diagram

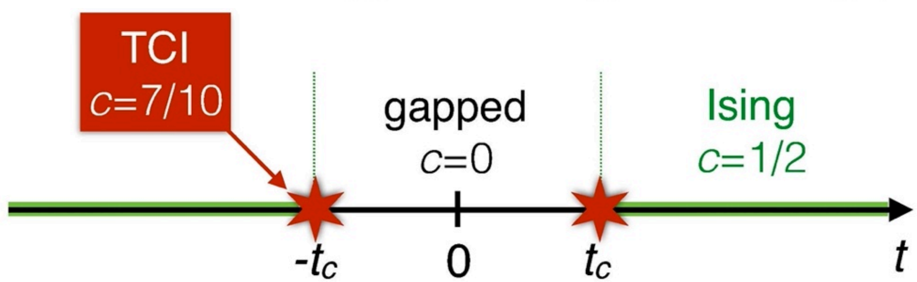
- Hamiltonian from Temperley-Lieb algebra
- Phase diagram (Rahmani-Zhu-Franz-Affleck 2015)



$$H^b_{\text{Ising}} = -2t \sum_i P_i^{(1)} - 2g \sum_i \left(P_i^{(1)} \otimes P_{i+2}^{(1)} + P_i^{(\psi)} \otimes P_{i+2}^{(\psi)} \right)$$



$$H^b_{\text{Ising}} = -t \sum_i X_i - t \sum_{\langle ij \rangle} Z_i Z_j - g \sum_{\langle ij \rangle} X_i X_j - g \sum_{\langle ijk \rangle} Z_i Z_k,$$



Categorical symmetry of CFT

Categorical symmetry of CFT

- Q: Why 2D Ising and tricritical Ising CFTs can appear on the boundary of 3D Z(Ising) TQFT?

Categorical symmetry of CFT

- Q: Why 2D Ising and tricritical Ising CFTs can appear on the boundary of 3D Z(Ising) TQFT?
- Ising CFT primaries = $\{1, \epsilon, \sigma\}$ = Ising

Categorical symmetry of CFT

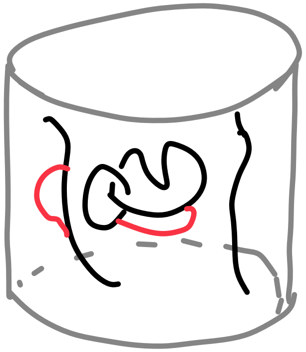
- Q: Why 2D Ising and tricritical Ising CFTs can appear on the boundary of 3D Z(Ising) TQFT?
- Ising CFT primaries = $\{1, \epsilon, \sigma\}$ = Ising
- Tricritical Ising CFT primaries = $\{1, \epsilon, \epsilon', \epsilon'', \sigma, \sigma'\}$ = Ising $\boxtimes$ Fibonacci = $\{1, \epsilon'', \sigma'\} \boxtimes \{1, \epsilon'\}$, with $\sigma' \times \sigma' = 1 + \epsilon''$ and $\epsilon' \times \epsilon' = 1 + \epsilon'$

Categorical symmetry of CFT

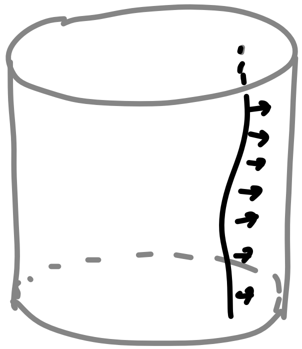
- Q: Why 2D Ising and tricritical Ising CFTs can appear on the boundary of 3D Z(Ising) TQFT?
- Ising CFT primaries = $\{1, \epsilon, \sigma\}$ = Ising
- Tricritical Ising CFT primaries = $\{1, \epsilon, \epsilon', \epsilon'', \sigma, \sigma'\}$ = Ising $\boxtimes$ Fibonacci = $\{1, \epsilon'', \sigma'\} \boxtimes \{1, \epsilon'\}$, with $\sigma' \times \sigma' = 1 + \epsilon''$ and $\epsilon' \times \epsilon' = 1 + \epsilon'$
- Claim: 2D CFT on the boundary should have Ising categorical symmetry

Categorical symmetry of CFT

- Q: Why 2D Ising and tricritical Ising CFTs can appear on the boundary of 3D Z(Ising) TQFT?
- Ising CFT primaries = $\{1, \epsilon, \sigma\}$ = Ising
- Tricritical Ising CFT primaries = $\{1, \epsilon, \epsilon', \epsilon'', \sigma, \sigma'\}$ = Ising $\boxtimes$ Fibonacci = $\{1, \epsilon'', \sigma'\} \boxtimes \{1, \epsilon'\}$, with $\sigma' \times \sigma' = 1 + \epsilon''$ and $\epsilon' \times \epsilon' = 1 + \epsilon'$
- Claim: 2D CFT on the boundary should have Ising categorical symmetry



3D TQFT



TDL acting on 2D

Outline

- Introduction to fusion category and categorical symmetry
- Categorical symmetry in 3D TQFT
  $\mathbb{Z}_2$ gauge theory $\rightarrow$ Dijkgraaf-Witten G gauge theory $\rightarrow$ Turaev-Viro-Levin-Wen model
- CFT on the 2D boundary
  categorical symmetry in Ising and tricritical Ising CFT
- Categorical symmetry in 3D Fermionic TQFT
  superfusion category, fermionic Turaev-Viro-Levin-Wen model
- SCFT on the 2D boundary
  tricritical Ising minimal model, parafermion model

Superfusion category

Superfusion category

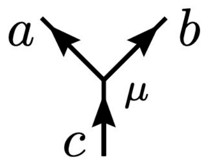
- Q: How to add fermion to fusion category

Superfusion category

- Q: How to add fermion to fusion category
- simple objects: m-type if $\text{End}(a) = \mathbb{C}$, q-type if $\text{End}(a) = \mathbb{C}^{1|1} = \mathbb{C}l_1$

Superfusion category

- Q: How to add fermion to fusion category
- simple objects: m-type if $\mathrm{End}(a) = \mathbb{C}$, q-type if $\mathrm{End}(a) = \mathbb{C}^{1|1} = \mathbb{C}l_1$
- morphism is super vector space: can be bosonic or fermionic

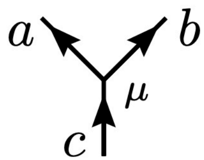


$= |a, b; c, \mu\rangle \in V_c^{ab}$

μ is $\mathbb{Z}_2$-graded

Superfusion category

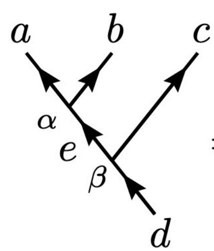
- Q: How to add fermion to fusion category
- simple objects: m-type if End$(a) = \mathbb{C}$, q-type if End$(a) = \mathbb{C}^{1|1} = \mathbb{C}l_1$
- morphism is super vector space: can be bosonic or fermionic



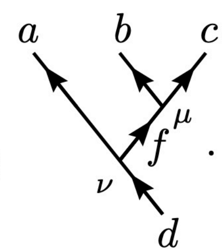
$= |a, b; c, \mu\rangle \in V_c^{ab}$

μ is $\mathbb{Z}_2$-graded

- associator



$= \sum_{f,\mu,\nu} \left[F_d^{abc}\right]_{(e,\alpha,\beta)(f,\mu,\nu)}$



.

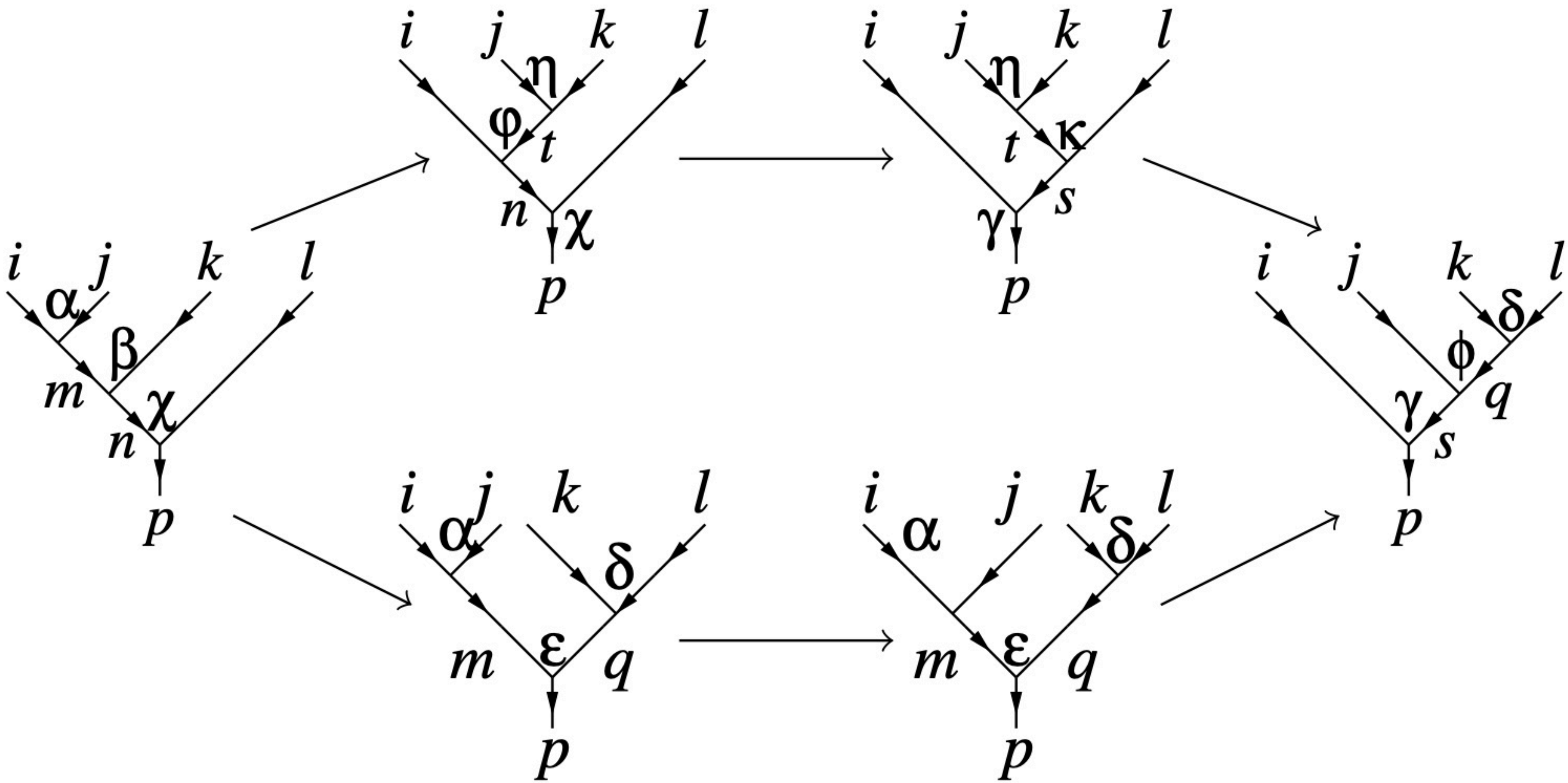
$$F = F_b c_\alpha^\dagger c_\beta^\dagger c_\nu c_\mu$$

Superfusion category

- superpentagon equation

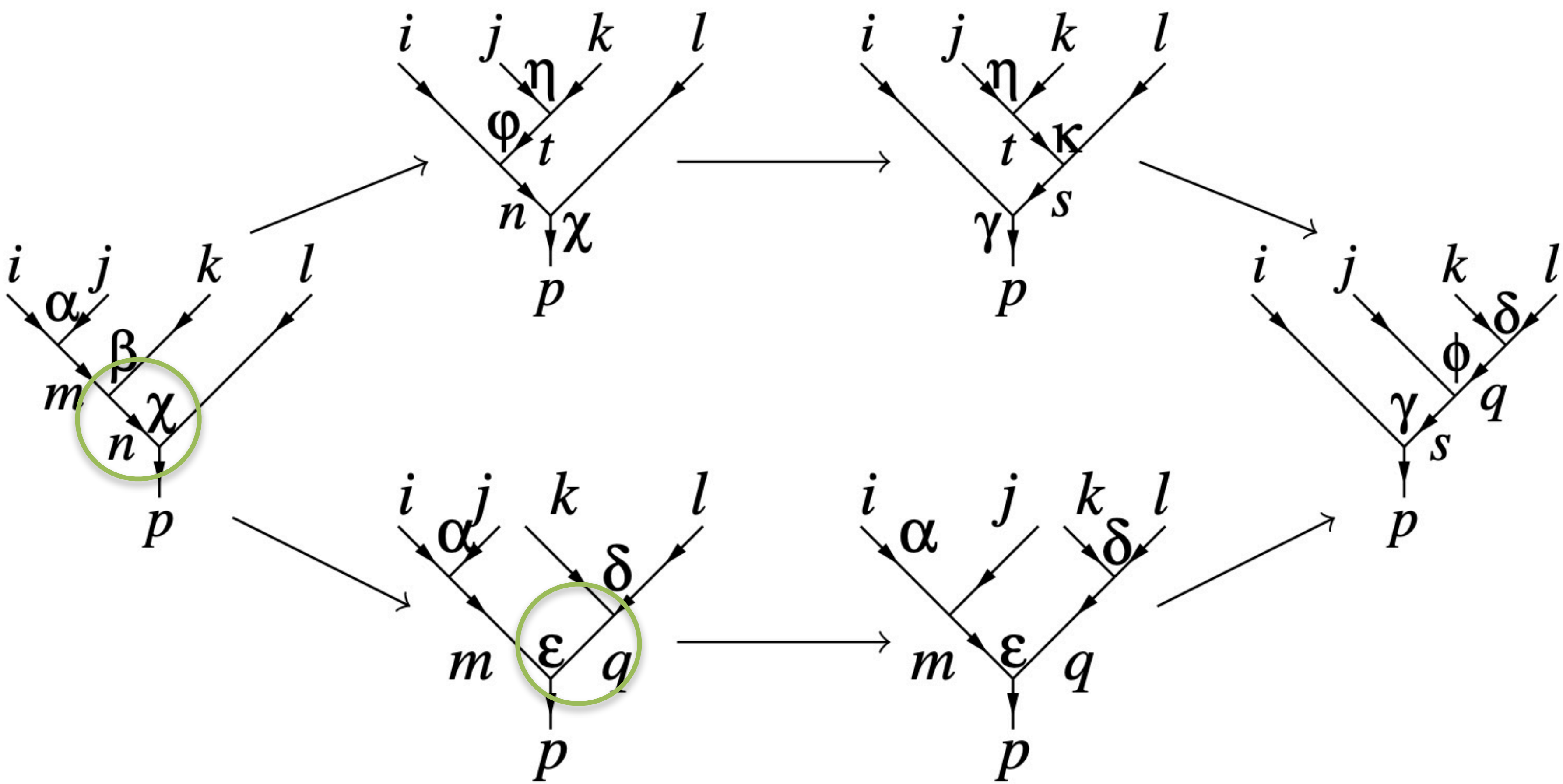
Superfusion category

- superpentagon equation



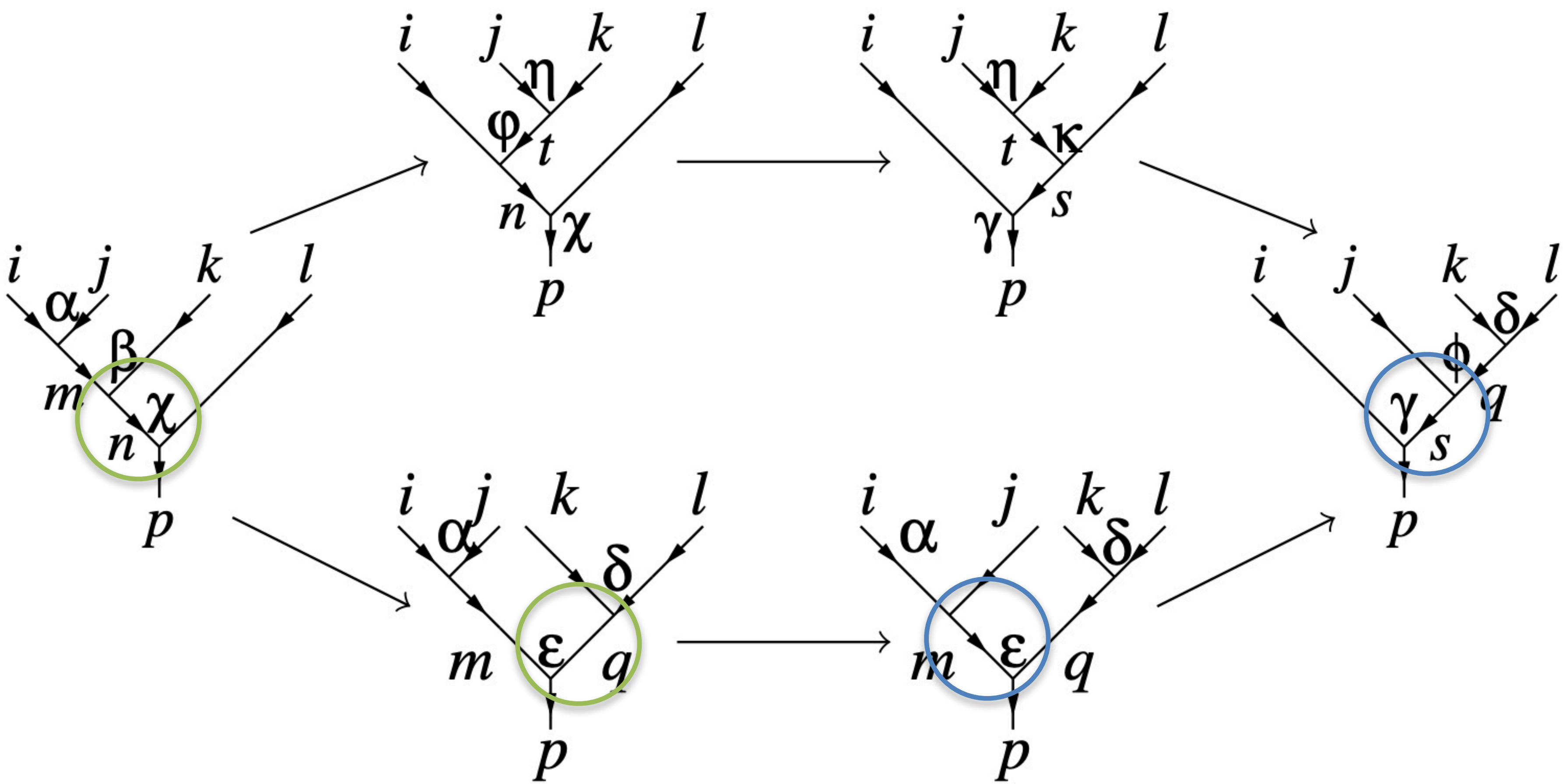
Superfusion category

- superpentagon equation



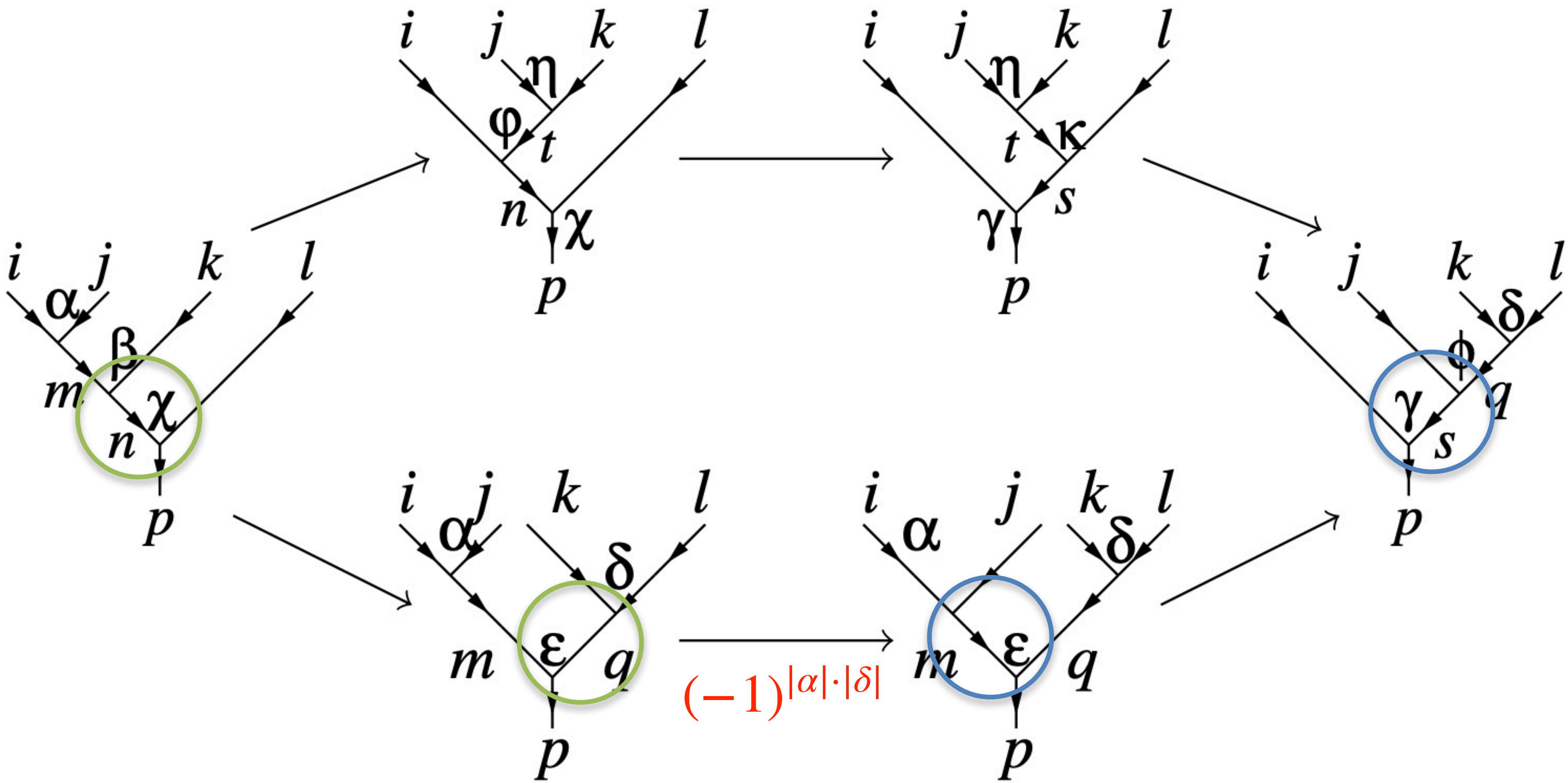
Superfusion category

- superpentagon equation



Superfusion category

- superpentagon equation



Fermionic Turaev-Viro-Levin-Wen model

Fermionic Turaev-Viro-Levin-Wen model

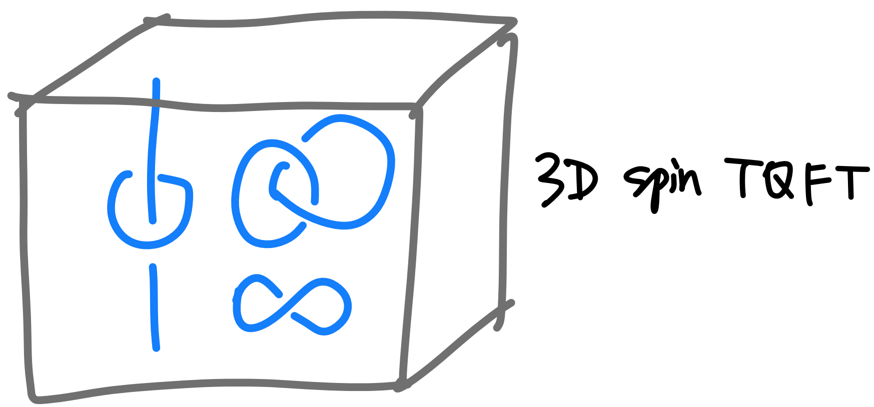
- A construction of 3D spin-TQFT

Fermionic Turaev-Viro-Levin-Wen model

- A construction of 3D spin-TQFT
- Input: superfusion category $\mathscr{C}_f$
- Output: Drinfeld center $Z(\mathscr{C}_f)$

Fermionic Turaev-Viro-Levin-Wen model

- A construction of 3D spin-TQFT
- Input: superfusion category $\mathscr{C}_f$
- Output: Drinfeld center $Z(\mathscr{C}_f)$
- Excitation worldline as fermionic categorical symmetry



Outline

- Introduction to fusion category and categorical symmetry
- Categorical symmetry in 3D TQFT
  $\mathbb{Z}_2$ gauge theory $\rightarrow$ Dijkgraaf-Witten G gauge theory $\rightarrow$ Turaev-Viro-Levin-Wen model
- CFT on the 2D boundary
  categorical symmetry in Ising and tricritical Ising CFT
- Categorical symmetry in 3D Fermionic TQFT
  superfusion category, fermionic Turaev-Viro-Levin-Wen model
- SCFT on the 2D boundary
  tricritical Ising minimal model, parafermion model

Fermionic CFT on the 2D boundary

Fermionic CFT on the 2D boundary

- Q: which fermionic CFT on the 2D boundary?

Fermionic CFT on the 2D boundary

- Q: which fermionic CFT on the 2D boundary?
- One simple approach: fermionization of both 3D bulk (fermion condensation) and 2D boundary (Jordan-Wigner transformation, Hsieh-Nakayama-Tachikawa 2020)

Fermionic CFT on the 2D boundary

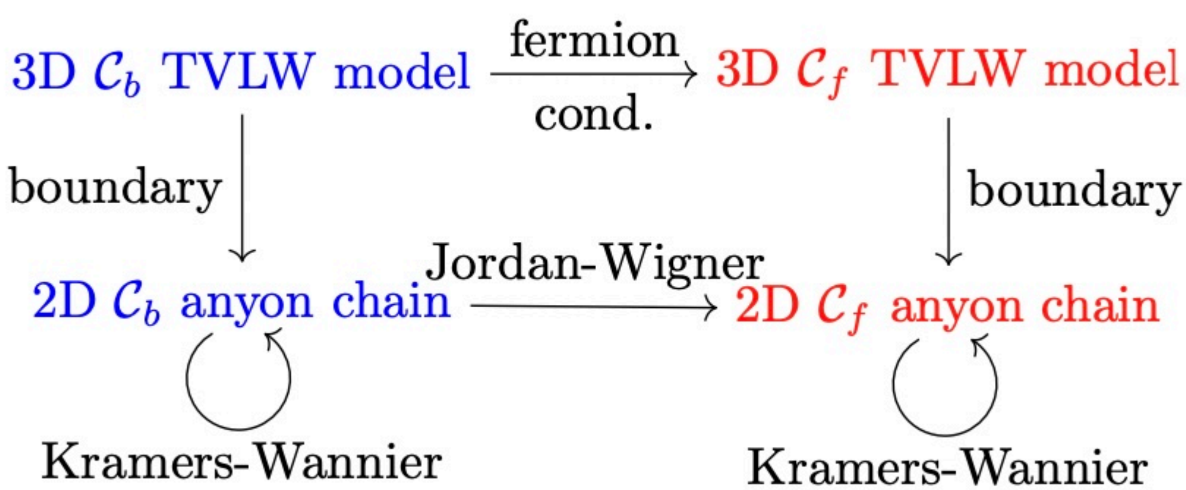
- Q: which fermionic CFT on the 2D boundary?
- One simple approach: fermionization of both 3D bulk (fermion condensation) and 2D boundary (Jordan-Wigner transformation, Hsieh-Nakayama-Tachikawa 2020)
- $\mathscr{C}_b = \mathscr{C}$ is a fusion category with fermion f

Fermionic CFT on the 2D boundary

- Q: which fermionic CFT on the 2D boundary?
- One simple approach: fermionization of both 3D bulk (fermion condensation) and 2D boundary (Jordan-Wigner transformation, Hsieh-Nakayama-Tachikawa 2020)
- $\mathscr{C}_b = \mathscr{C}$ is a fusion category with fermion f
- $\mathscr{C}_f = \mathscr{C}/f$ is a superfusion category

Fermionic CFT on the 2D boundary

- Q: which fermionic CFT on the 2D boundary?
- One simple approach: fermionization of both 3D bulk (fermion condensation) and 2D boundary (Jordan-Wigner transformation, Hsieh-Nakayama-Tachikawa 2020)
- $\mathscr{C}_b = \mathscr{C}$ is a fusion category with fermion f
- $\mathscr{C}_f = \mathscr{C}/f$ is a superfusion category



Example: Ising/f

Example: Ising/f

- $\mathscr{C}$=Ising, $\mathscr{C}_f$=Ising/f

Example: Ising/f

- $\mathscr{C}$=Ising, $\mathscr{C}_f$=Ising/f

$$H^b_{\text{Ising}} = -t\sum_i X_i - t\sum_{\langle ij\rangle} Z_iZ_j - g\sum_{\langle ij\rangle} X_iX_j - g\sum_{\langle ijk\rangle} Z_iZ_k$$

$$H^f_{\text{Ising}} = it\sum_{\langle ij\rangle}\gamma_i\gamma_j + g\sum_{\langle ijkl\rangle}\gamma_i\gamma_j\gamma_k\gamma_l$$

Example: Ising/f

- $\mathscr{C}$=Ising, $\mathscr{C}_f$=Ising/f
- Related by Jordan-Wigner transformation (fermionization) for all parameters

$$H^b_{\text{Ising}} = -t\sum_i X_i - t\sum_{\langle ij\rangle} Z_i Z_j - g\sum_{\langle ij\rangle} X_i X_j - g\sum_{\langle ijk\rangle} Z_i Z_k$$

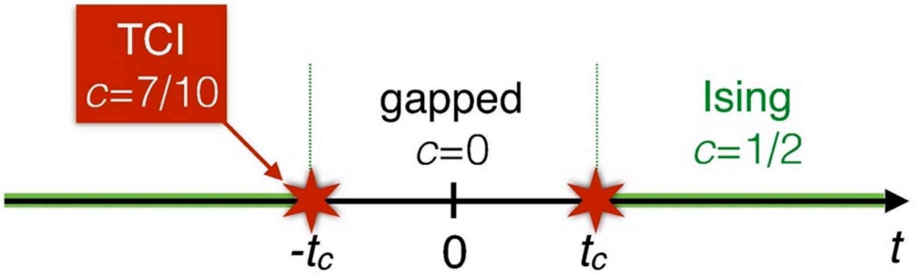
$$H^f_{\text{Ising}} = it\sum_{\langle ij\rangle} \gamma_i\gamma_j + g\sum_{\langle ijkl\rangle} \gamma_i\gamma_j\gamma_k\gamma_l$$

Example: Ising/f

- $\mathscr{C}$=Ising, $\mathscr{C}_f$=Ising/f
- Related by Jordan-Wigner transformation (fermionization) for all parameters

$$H^b_{\text{Ising}} = -t\sum_i X_i - t\sum_{\langle ij\rangle} Z_iZ_j - g\sum_{\langle ij\rangle} X_iX_j - g\sum_{\langle ijk\rangle} Z_iZ_k$$

$$H^f_{\text{Ising}} = it\sum_{\langle ij\rangle}\gamma_i\gamma_j + g\sum_{\langle ijkl\rangle}\gamma_i\gamma_j\gamma_k\gamma_l$$



Another example: Self-dual N-states Potts model

Another example: Self-dual N-states Potts model

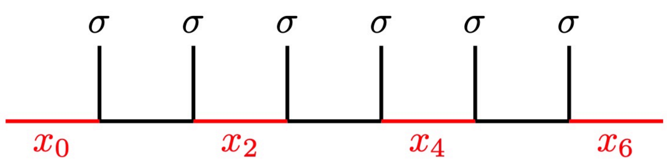
- Gauging e-m duality symmetry in 3D $\mathbb{Z}_N$ gauge theory

Another example: Self-dual N-states Potts model

- Gauging e-m duality symmetry in 3D $\mathbb{Z}_N$ gauge theory
- 3D TVLW TQFT from Tambara-Yamagami category = $\{1,2,...,N,\sigma\}$ with $\sigma \times \sigma = 1 + \ldots + N$

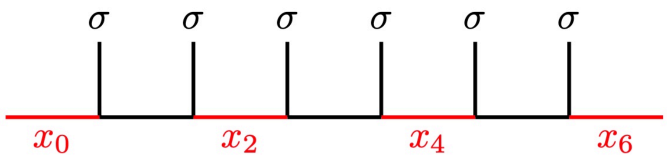
Another example: Self-dual N-states Potts model

- Gauging e-m duality symmetry in 3D $\mathbb{Z}_N$ gauge theory
- 3D TVLW TQFT from Tambara-Yamagami category = $\{1,2,\ldots,N,\sigma\}$ with $\sigma\times\sigma = 1+\ldots+N$
- 2D boundary anyon chain



Another example: Self-dual N-states Potts model

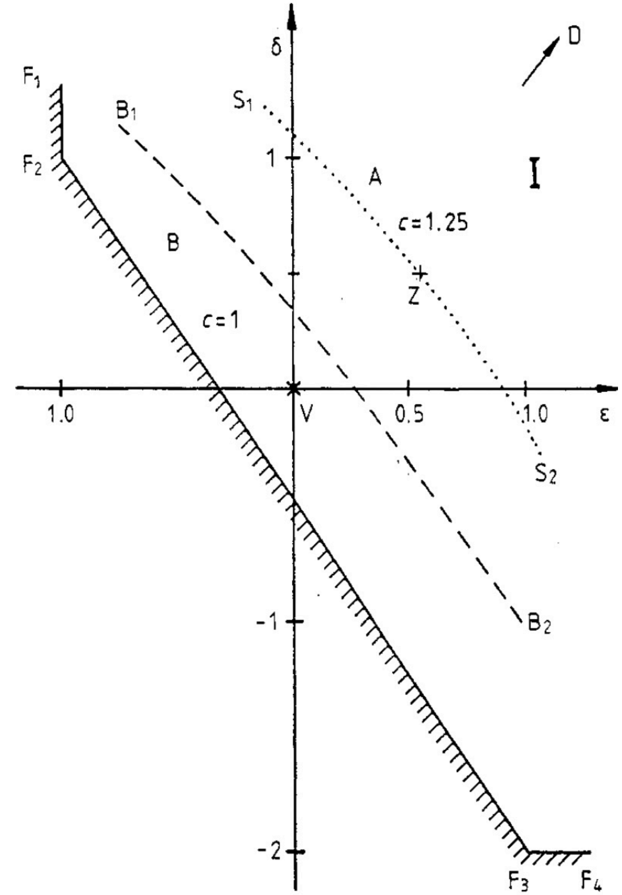
- Gauging e-m duality symmetry in 3D $\mathbb{Z}_N$ gauge theory
- 3D TVLW TQFT from Tambara-Yamagami category = $\{1,2,\ldots,N,\sigma\}$ with $\sigma\times\sigma = 1+\ldots+N$
- 2D boundary anyon chain



- One natural Hamiltonian: self-dual N-state Potts model

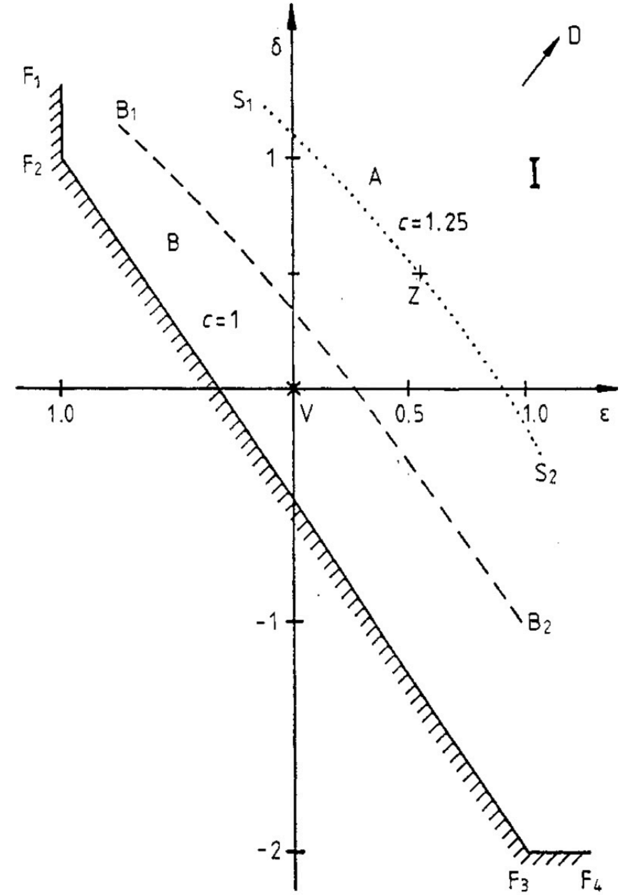
$$H^b_{\mathrm{TY}} = \sum_i \sum_{k=0}^{N-1} J_k (Z_i^k \otimes Z_{i+1}^{-k} + X_i^k)$$

6-states Potts model



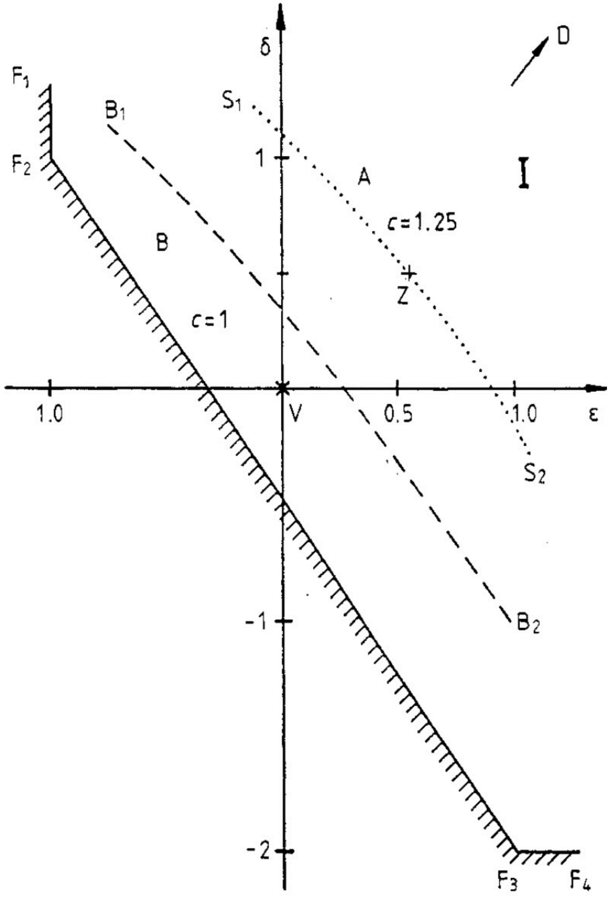
6-states Potts model

- Phase diagram [Schutz 1989]



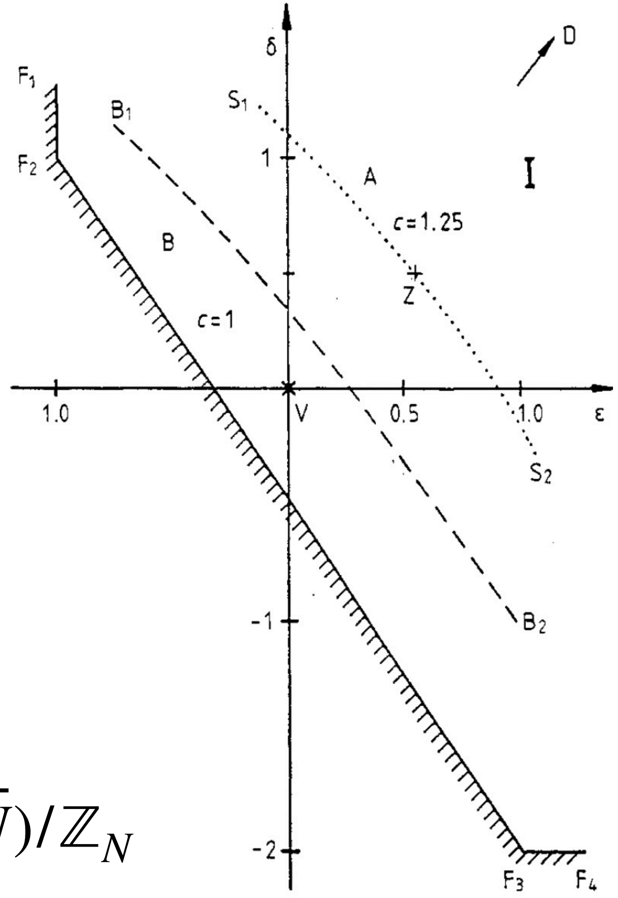
6-states Potts model

- Phase diagram [Schutz 1989]
- All phase diagram is gapless (protected by self-duality / categorical symmetry) [Levin 2019]

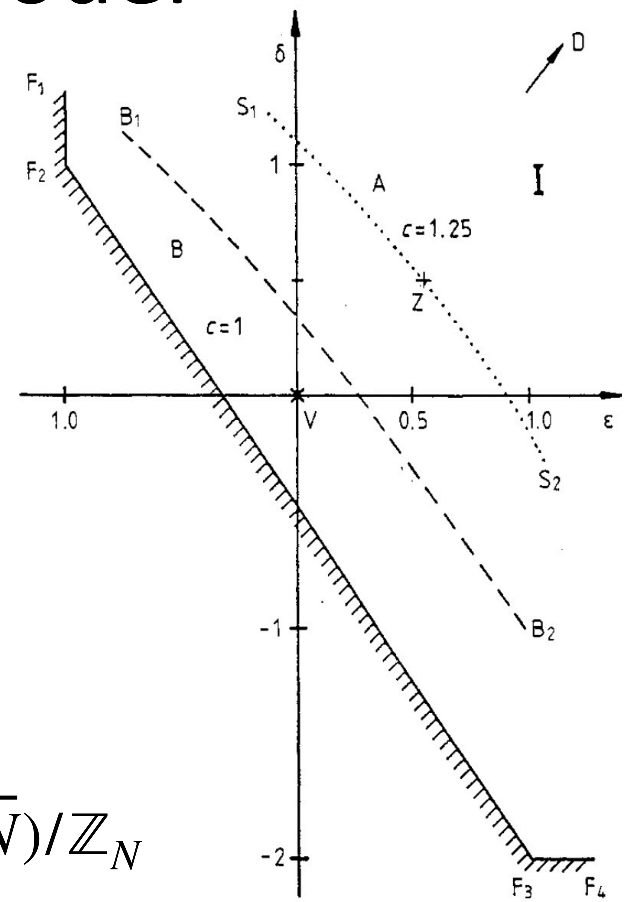


6-states Potts model

- Phase diagram [Schutz 1989]
- All phase diagram is gapless (protected by self-duality / categorical symmetry) [Levin 2019]
- Some special point:
  $\mathbb{Z}_6$ Potts 1st order transition
  c=5/4, $\mathbb{Z}_6$ parafermion
  c=1, $U(1)_{12}$, $Z(R=\sqrt{2N}) = Z(R=\sqrt{2N})/\mathbb{Z}_N$



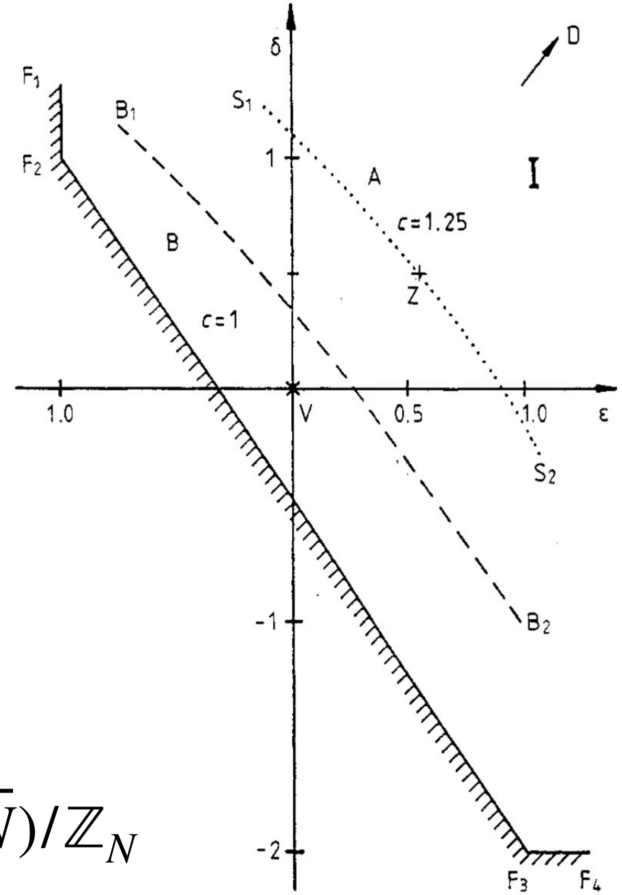
6-states Potts model



- Phase diagram [Schutz 1989]
- All phase diagram is gapless (protected by self-duality / categorical symmetry) [Levin 2019]
- Some special point:
  $\mathbb{Z}_6$ Potts 1st order transition
  c=5/4, $\mathbb{Z}_6$ parafermion
  c=1, $U(1)_{12}$, $Z(R=\sqrt{2N}) = Z(R=\sqrt{2N})/\mathbb{Z}_N$
- All appeared CFT have Tambara-Yamagami category topological defect line

6-states Potts model

- Phase diagram [Schutz 1989]
- All phase diagram is gapless (protected by self-duality / categorical symmetry) [Levin 2019]
- Some special point:
  $\mathbb{Z}_6$ Potts 1st order transition
  c=5/4, $\mathbb{Z}_6$ parafermion
  c=1, $U(1)_{12}$, $Z(R = \sqrt{2N}) = Z(R = \sqrt{2N})/\mathbb{Z}_N$



- All appeared CFT have Tambara-Yamagami category topological defect line
- Supersymmetry after fermionization

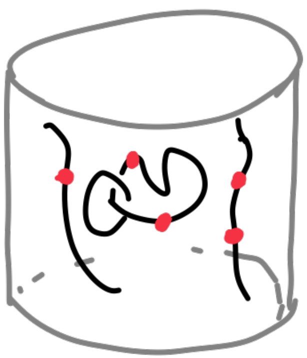
Categorical-symmetry induced supersymmetry

Categorical-symmetry induced supersymmetry

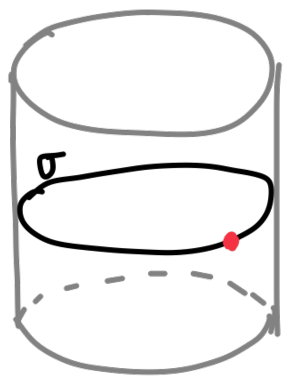
- Fermionic categorical symmetry $\rightarrow$ all energy levels are bosonic/fermionic degenerate

Categorical-symmetry induced supersymmetry

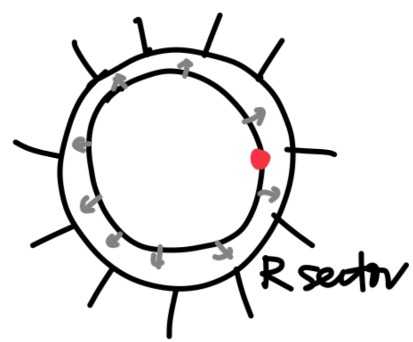
- Fermionic categorical symmetry → all energy levels are bosonic/fermionic degenerate



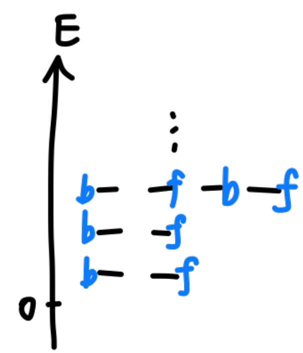
3D spin TQFT



fermionic TDL acting on 2D boundary



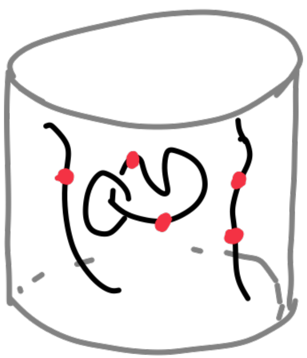
anyon chain
$[\mathscr{L}, H] = 0$
$\{\mathscr{L}, (-1)^F\} = 0$



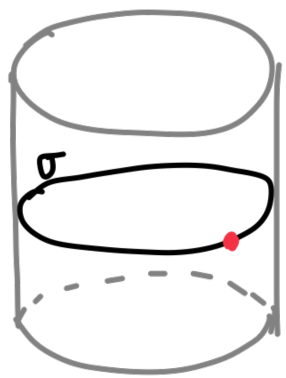
degenerate levels

Categorical-symmetry induced supersymmetry

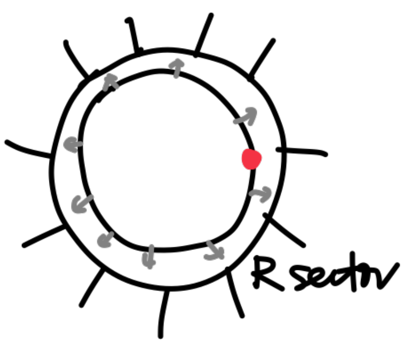
- Fermionic categorical symmetry → all energy levels are bosonic/fermionic degenerate
- Claim: There exists enhanced fermionic chiral algebra if $N = 2$ (mod 4)



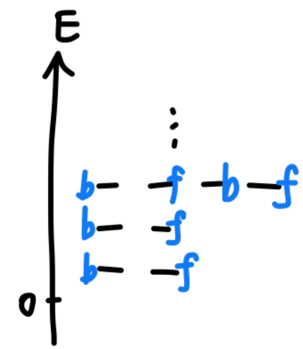
3D spin TQFT



fermionic TDL acting on 2D boundary



anyon chain
$[\mathscr{L}, H] = 0$
$\{\mathscr{L}, (-1)^F\} = 0$



degenerate levels

Summary

- Universal way to construct CFT on 2D boundary of 3D TQFT of TVLW model
- Categorical symmetry as IR CFT constraints on the phase diagram
- Categorical-symmetry induced supersymmetry in fermionic models

Summary

- Universal way to construct CFT on 2D boundary of 3D TQFT of TVLW model
- Categorical symmetry as IR CFT constraints on the phase diagram
- Categorical-symmetry induced supersymmetry in fermionic models

Thank you!