Computational AG and Integrable models

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Based on the works

Y. Jiang and Y. Zhang, *JHEP* 1803 (2018) 084, arXiv: 1710.04693

J. Jacobsen, ***Y. Jiang***, Y. Zhang, *JHEP* 03 (2019) 152, arXiv: 1812.00447

Z. Bajnok, J. Jacobsen, ***Y. Jiang***, R. Nepomechie, Y. Zhang,
JHEP 06 (2020) 169, arXiv: 2002.09019

Y. Jiang, R. Wen, Y. Zhang, 2109.10568

J. Boehm, J. Jacobsen, ***Y. Jiang***, Y. Zhang, *to appear*

Part I.

Introduction to basic ideas

Equation

$$q(x) = x^5 - 5x^4 + 7x^3 + 5x^2 - 21x + 7$$

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Questions

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- Compute the **sum** of the function **over all solutions** ?

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Computational algebraic Geometry

Although the questions we ask are somewhat trivial to solve for a single variable. They become highly non-trivial in the multi-variable cases and are among the main themes of modern computational algebraic geometry.

Numerical Method

Solution

1. By fundamental theorem of algebra, there are 5 solutions
2. Solve the equation numerically (up to 25 digits)

$$x_1 = -1.428817701781382219822436$$

$$x_2 = 0.3819660112501051517954132$$

$$x_3 = 2.618033988749894848204587$$

$$x_4 = 1.714408850890691109911218 - 1.399984900087945731206127i$$

$$x_5 = 1.714408850890691109911218 + 1.399984900087945731206127i$$

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$$F(x_1) = 39.5573572063554668510040$$

$$F(x_2) = 0.853322962757606348443172$$

$$F(x_3) = -674.760282669717313308150$$

$$F(x_4) = 299.037255462756332508564 - 107.837305569845322316012i$$

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$$F(x_1) + \cdots + F(x_5) = -36.2{\color{red}750915}750915{\color{red}75091}58 \approx -\frac{99031}{2730}$$

Rational number !

Analytical Method

- **Linear space** spanned by

$$e_1 = x^4, \quad e_2 = x^3, \quad e_3 = x^2, \quad e_4 = x, \quad e_5 = 1$$

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- Divide $F(x)$ by $q(x)$, find the **remainder**

$$F(x) = a(x)q(x) + r(x)$$

$$r(x) = -\frac{144}{5}x^4 + \frac{81}{2}x^3 + \frac{491}{15}x^2 - \frac{23311}{195}x + \frac{842}{21}$$

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- Construct a matrix of the remainder in the linear space

$r(x)e_i = a_i(x)q(x) + r_i(x)$	$r_i(x) = M_{ij}e_j$

This matrix is called the **companion matrix** of the function

Analytical Method

$$M_F = \begin{pmatrix} -\frac{1910212}{1365} & \frac{801854}{195} & -\frac{24539}{13} & -\frac{4688677}{390} & \frac{303429}{65} \\ -\frac{43347}{65} & \frac{203171}{105} & -\frac{8341}{15} & -5222 & \frac{11893}{6} \\ -\frac{1699}{6} & \frac{292093}{390} & -\frac{9913}{210} & -\frac{19719}{10} & \frac{1449}{2} \\ -\frac{207}{2} & \frac{703}{3} & \frac{4769}{195} & -\frac{59294}{105} & \frac{1008}{5} \\ -\frac{144}{5} & \frac{81}{2} & \frac{491}{15} & -\frac{23311}{195} & \frac{842}{21} \end{pmatrix}$$

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Remarks

1. The result is exact, no need to solve equations
2. It is clear that the final result should be a rational number.

Notions of algebraic geometry

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Polynomial ring $\mathbb{C}[x]$

All polynomials in x
with complex coefficients

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A finite dimensional linear space $\mathrm{Q}_q = \mathbb{C}[x]/\mathrm{I}_q$

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Standard basis

All monomials that cannot be divided by $\mathrm{LT}[q(x)]$

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Key results from AG

- Polynomial $F(x) \mapsto M_F$ is mapped to a **numerical matrix**
- **Dimension** of Q_q = number of solutions of $q(x) = 0$

Baby problem

Real problem

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- Equation $q(x) = 0$

Real problem

- Bethe ansatz equations

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- Function of one variable

  $F(x)$

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- Function of rapidities

  $F(u_1, u_2, \cdots, u_N)$

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  (Highly non-trivial !!)

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- Calculate the sum

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Real problem

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- Number of solutions of Bethe ansatz equations

  (Highly non-trivial !!)

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$$\sum_{\text{sol BAE}} F(u_1, \cdots, u_N)$$

Algebraic Geometry

Polynomial ring

$\mathbb{C}[u_1, \cdots, u_N]$

All polynomials in

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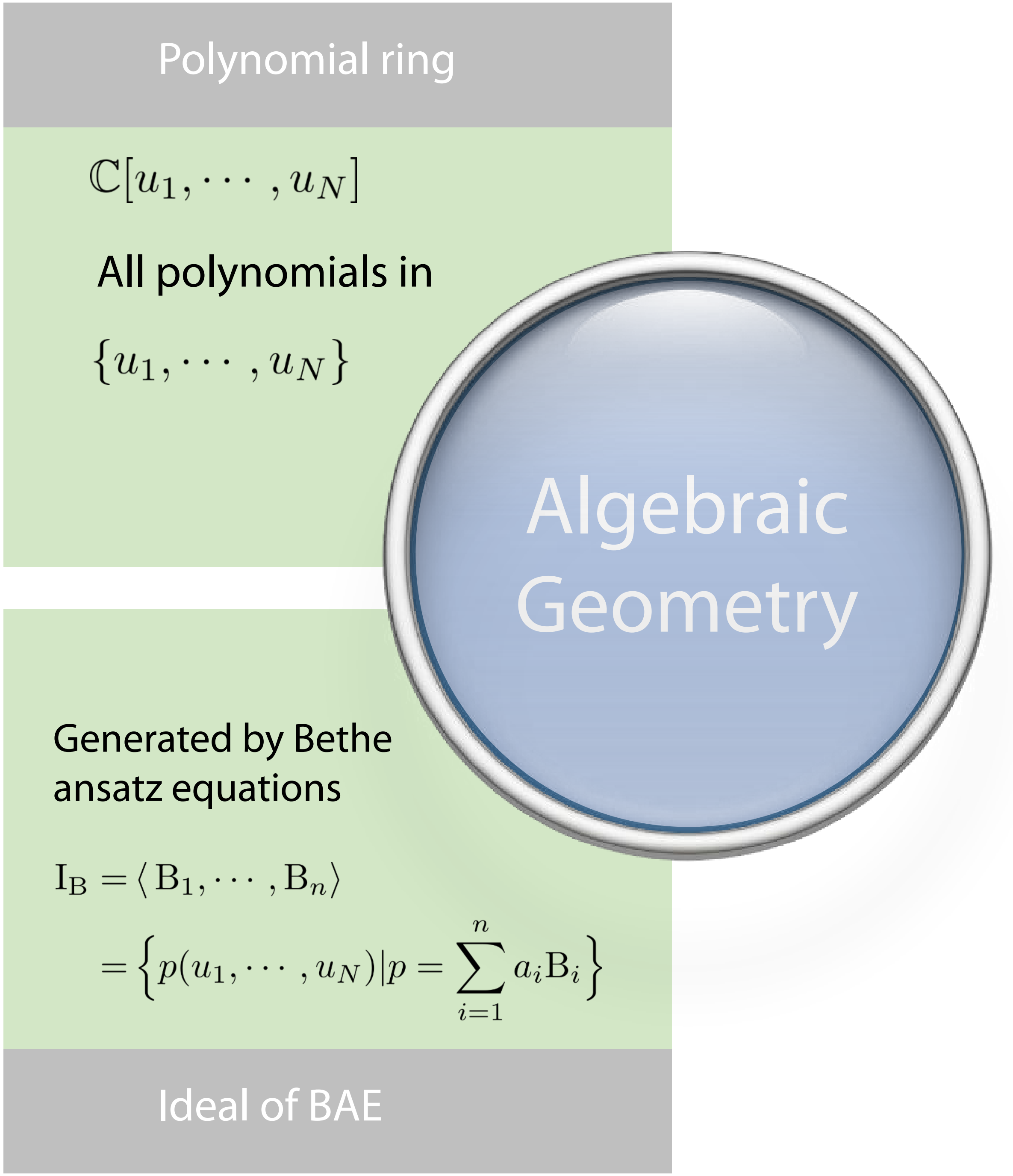
Algebraic Geometry



Polynomial ring
$\mathbb{C}[u_1, \cdots, u_N]$
All polynomials in
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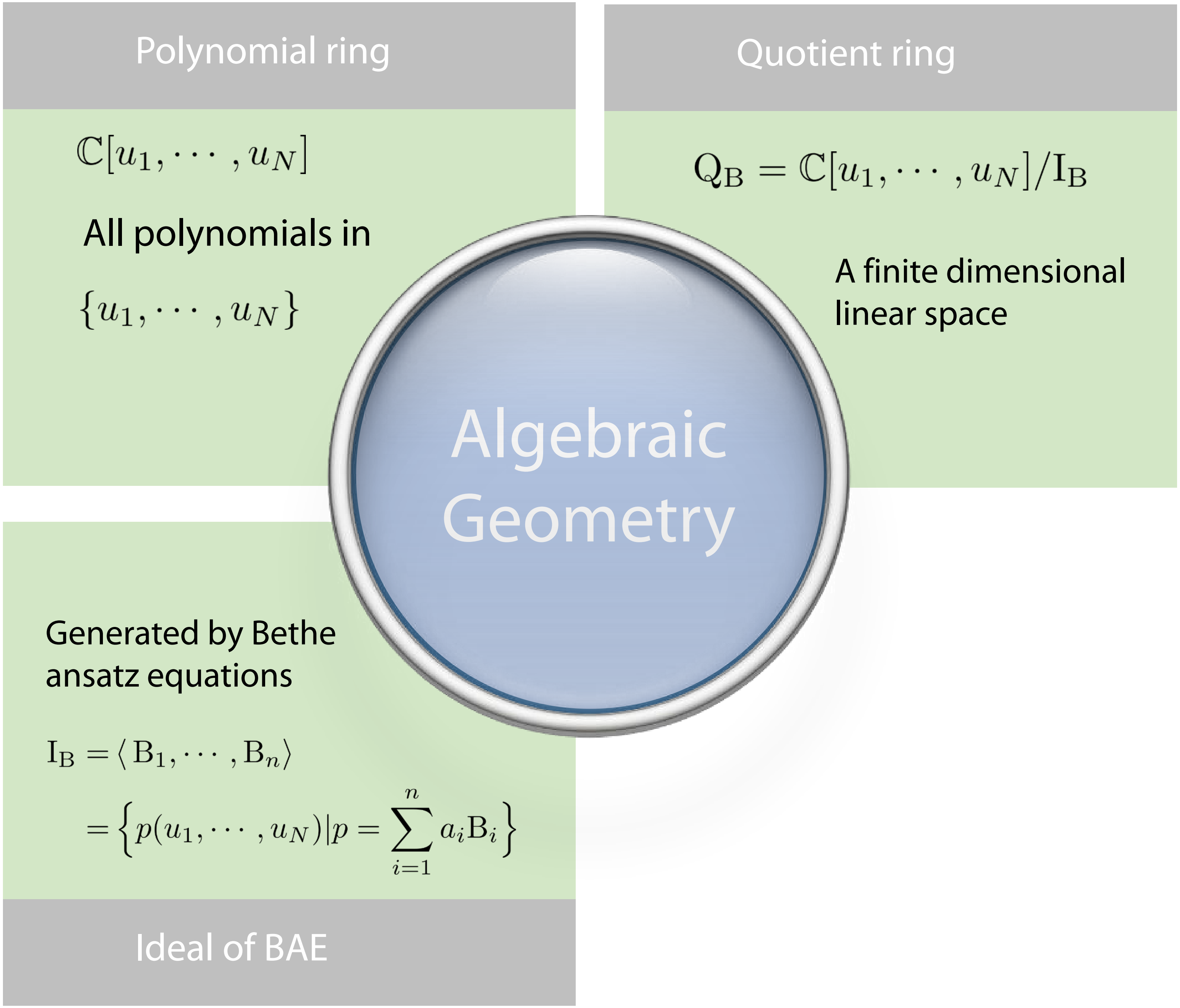
Algebraic Geometry
Generated by Bethe ansatz equations
$\mathrm{I_B} = \langle \mathrm{B}_1, \cdots, \mathrm{B}_n \rangle$
$= \left\{ p(u_1, \cdots, u_N) | p = \sum_{i=1}^{n} a_i \mathrm{B}_i \right\}$
Ideal of BAE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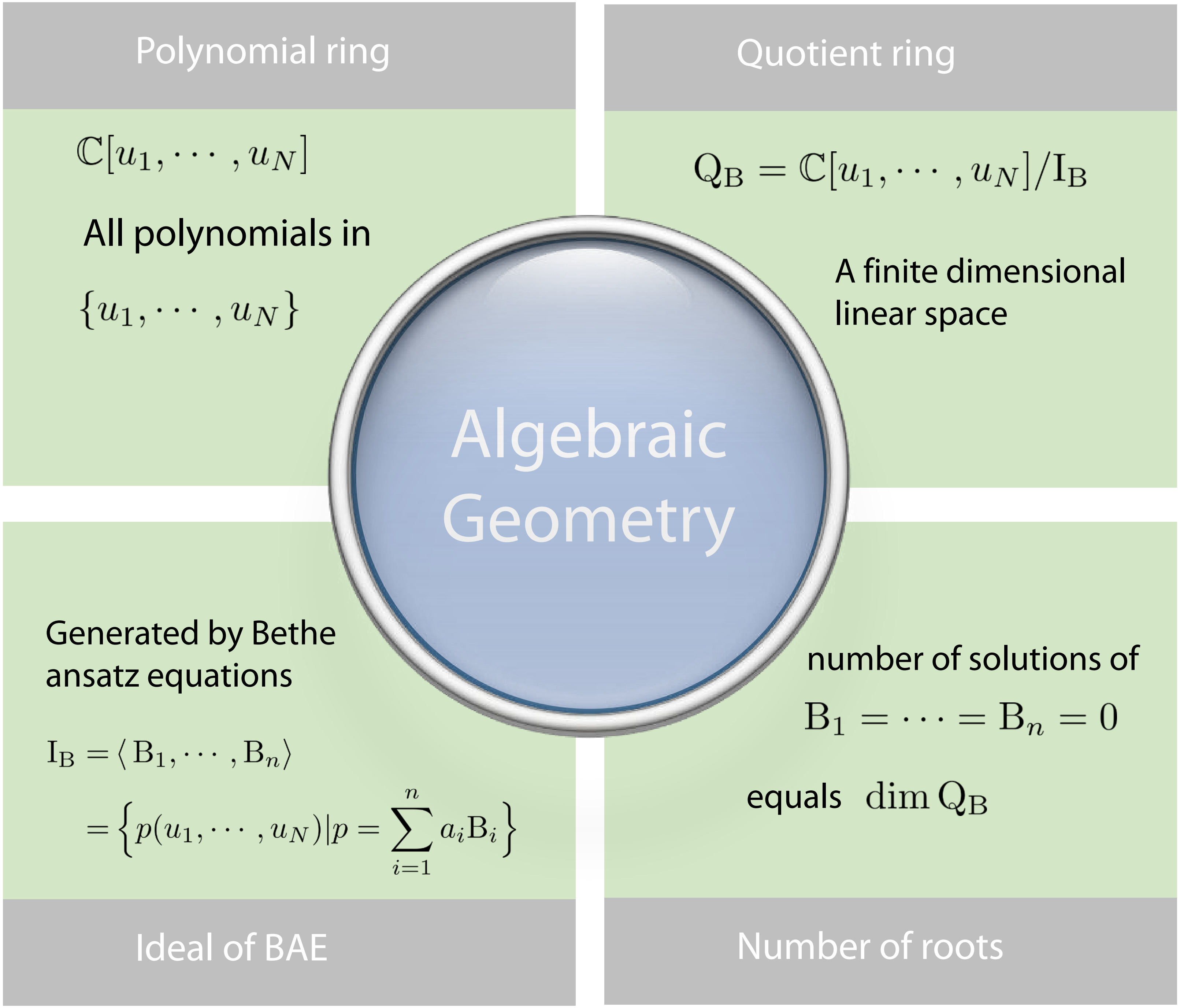
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Ideal of BAE
number of solutions of
$\mathrm{B}_1=\cdots=\mathrm{B}_n=0$
equals $\dim \mathrm{Q_B}$
Number of roots

Differences

More (variables) is different !

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Single variable

BAE $= q(x) = x^3 - 2x^2 + 7 = 0$

"Remainder" of polynomials "divided" by BAE is well-defined

All remainders in the linear space $\mathrm{Span}_{\mathbb{C}}(x^2, x, 1)$

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Multi variable

$$f_1 = y^2 - 1 \qquad f_2 = xy - 1 \qquad F(x,y) = x^2y + xy^2 + y^2$$

We see that $F(x,y) = (x+1)\, f_1 + x\, f_2 + (2x+1)$

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The remainder is **not unique** !

Groebner Basis

Ideals can be generated by different basis

$$\mathrm{I_B} = \langle\, \mathrm{B}_1, \cdots, \mathrm{B}_n \rangle = \langle\, \mathrm{G}_1, \cdots, \mathrm{G}_s \rangle$$

The Groebner basis : remainders are
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For very simple cases, we can compute it by hand using known algorithms

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It is important to have brilliant students !

—— **W. Groebner**

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All monomials that cannot be divided by $\mathrm{LT}[\mathrm{G}_i]$,
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Bruno Buchberger

Simple Example

$f_1 = y^2 - 1 \quad f_2 = xy - 1$

$$\langle f_1, f_2 \rangle = \langle \mathrm{G}_1, \mathrm{G}_2 \rangle$$

$\mathrm{G}_1 = y^2 - 1 \quad \mathrm{G}_2 = x - y$

Choose the order, $x \succ y$
we have

$$\mathrm{LT}[\mathrm{G}_1] = y^2$$

$$\mathrm{LT}[\mathrm{G}_2] = x$$

The basis of $\mathbb{C}[x, y]/\langle f_1, f_2 \rangle$ is given by $\{y, 1\}$

Indeed, easy to see we have 2 solutions

Properties

$$M_{P_1 \pm P_2} = M_{P_1} \pm M_{P_2}$$

$$M_{P_1 \cdot P_2} = M_{P_1} \cdot M_{P_2}$$

$$M_{P_1 / P_2} = M_{P_1} \cdot M_{P_2}^{-1}$$

Important result

$$\sum_{\text{sol}} P(\mathbf{s}) = \text{Tr}\, M_P$$

Companion Matrix

For any e_j , find $P(\mathbf{s}) e_j = \sum_{k=1}^{S} a_k \, G_k + r_j(\mathbf{s})$

Expand in terms of basis $r_j(\mathbf{s}) = \sum_{k=1}^{S} M_{jk} \, e_k$

The matrix $(M_P)_{ij} = M_{ij}$ is called the companion matrix of $P(s_1, \cdots, s_K)$

Example

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$$F_1 = x^4y^2 + 3xy + 1 \qquad F_2 = y^3 + y^2 - 2$$

$$P(x,y) = \frac{x^3}{3} + \frac{y^3}{7} + 4xy(x+y) + 2x + 1$$

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Numerical approach

- The equations $F_1 = F_2 = 0$ has 12 solutions, solve numerically
- Plug each solution to $P(x,y)$, each term is irrational
- Take the sum $\mathrm{P} = \sum_{12\ \mathrm{sol}} P(x,y) \approx \frac{104}{7}$ Rational number !

Analytical approach

Analytical approach

1

Groebner basis of the system $\langle F_1, F_2 \rangle = \langle \mathrm{G}_1, \mathrm{G}_2 \rangle$

$$\mathrm{G}_1 = 3xy^2 + 3xy + y + 2x^4 + 1$$

$$\mathrm{G}_2 = y^3 + y^2 - 2$$

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2

Standard basis of **quotient ring** : all monomials that cannot be divided by x^4 and y^3, 12 terms in total

$$\{e_1 = x^3y^2, \, , e_2 = x^3y \cdots , e_{11} = y, e_{12} = 1\}$$

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$$\{e_1 = x^3y^2, , e_2 = x^3y \cdots , e_{11} = y, e_{12} = 1\}$$

3

Compute the **companion matrix**

$$P(x,y)e_1 = a_1\, G_1 + a_2\, G_2 + R_1(x,y)$$

$$R_1(x,y) = \sum_{j=1}^{12} (M_P)_{1j}\; e_j$$

$$(M_P)_{1j} = (\frac{8}{7}, -\frac{9}{7}, \cdots, 0, -2)$$

The full matrix takes the following form

$$M_P = \frac{1}{42}\begin{pmatrix} 48 & -54 & 12 & -504 & 0 & -14 & 504 & -420 & -1008 & -168 & 0 & -84 \\ 6 & 54 & -54 & -7 & -511 & 0 & -504 & 0 & -420 & -42 & -210 & 0 \\ -27 & -21 & 54 & 0 & -7 & -511 & -210 & -714 & 0 & 0 & -42 & -210 \\ 252 & 336 & -336 & 48 & -54 & 12 & -504 & 0 & -14 & 0 & -168 & 0 \\ -168 & 84 & 336 & 6 & 54 & -54 & -7 & -511 & 0 & 0 & 0 & -168 \\ 168 & 0 & 84 & -27 & -21 & 54 & 0 & -7 & -511 & -84 & -84 & 0 \\ -168 & 0 & 336 & 252 & 336 & -336 & 48 & -54 & 12 & 0 & 0 & -14 \\ 168 & 0 & 0 & -168 & 84 & 336 & 6 & 54 & -54 & -7 & -7 & 0 \\ 0 & 168 & 0 & 168 & 0 & 84 & -27 & -21 & 54 & 0 & -7 & -7 \\ 14 & 0 & 0 & -168 & 0 & 336 & 252 & 336 & -336 & 48 & -12 & 12 \\ 0 & 14 & 0 & 168 & 0 & 0 & -168 & 84 & 336 & 6 & 54 & -12 \\ 0 & 0 & 14 & 0 & 168 & 0 & 168 & 0 & 84 & -6 & 0 & 54 \end{pmatrix}$$

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252 & 336 & -336 & 48 & -54 & 12 & -504 & 0 & -14 & 0 & -168 & 0 \\
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\end{pmatrix}$$

$$\operatorname{Tr} M_P = \frac{104}{7}$$

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$$
M_P = \frac{1}{42}\begin{pmatrix}
48 & -54 & 12 & -504 & 0 & -14 & 504 & -420 & -1008 & -168 & 0 & -84 \\
6 & 54 & -54 & -7 & -511 & 0 & -504 & 0 & -420 & -42 & -210 & 0 \\
-27 & -21 & 54 & 0 & -7 & -511 & -210 & -714 & 0 & 0 & -42 & -210 \\
252 & 336 & -336 & 48 & -54 & 12 & -504 & 0 & -14 & 0 & -168 & 0 \\
-168 & 84 & 336 & 6 & 54 & -54 & -7 & -511 & 0 & 0 & 0 & -168 \\
168 & 0 & 84 & -27 & -21 & 54 & 0 & -7 & -511 & -84 & -84 & 0 \\
-168 & 0 & 336 & 252 & 336 & -336 & 48 & -54 & 12 & 0 & 0 & -14 \\
168 & 0 & 0 & -168 & 84 & 336 & 6 & 54 & -54 & -7 & -7 & 0 \\
0 & 168 & 0 & 168 & 0 & 84 & -27 & -21 & 54 & 0 & -7 & -7 \\
14 & 0 & 0 & -168 & 0 & 336 & 252 & 336 & -336 & 48 & -12 & 12 \\
0 & 14 & 0 & 168 & 0 & 0 & -168 & 84 & 336 & 6 & 54 & -12 \\
0 & 0 & 14 & 0 & 168 & 0 & 168 & 0 & 84 & -6 & 0 & 54
\end{pmatrix}
$$

$$
\mathrm{Tr}\, M_P = \frac{104}{7}
$$

Comments

- No need to solve any equations
- The final result is rational number

Part II.

Applications