Masses of hadrons containing heavy quarks and their QED corrections

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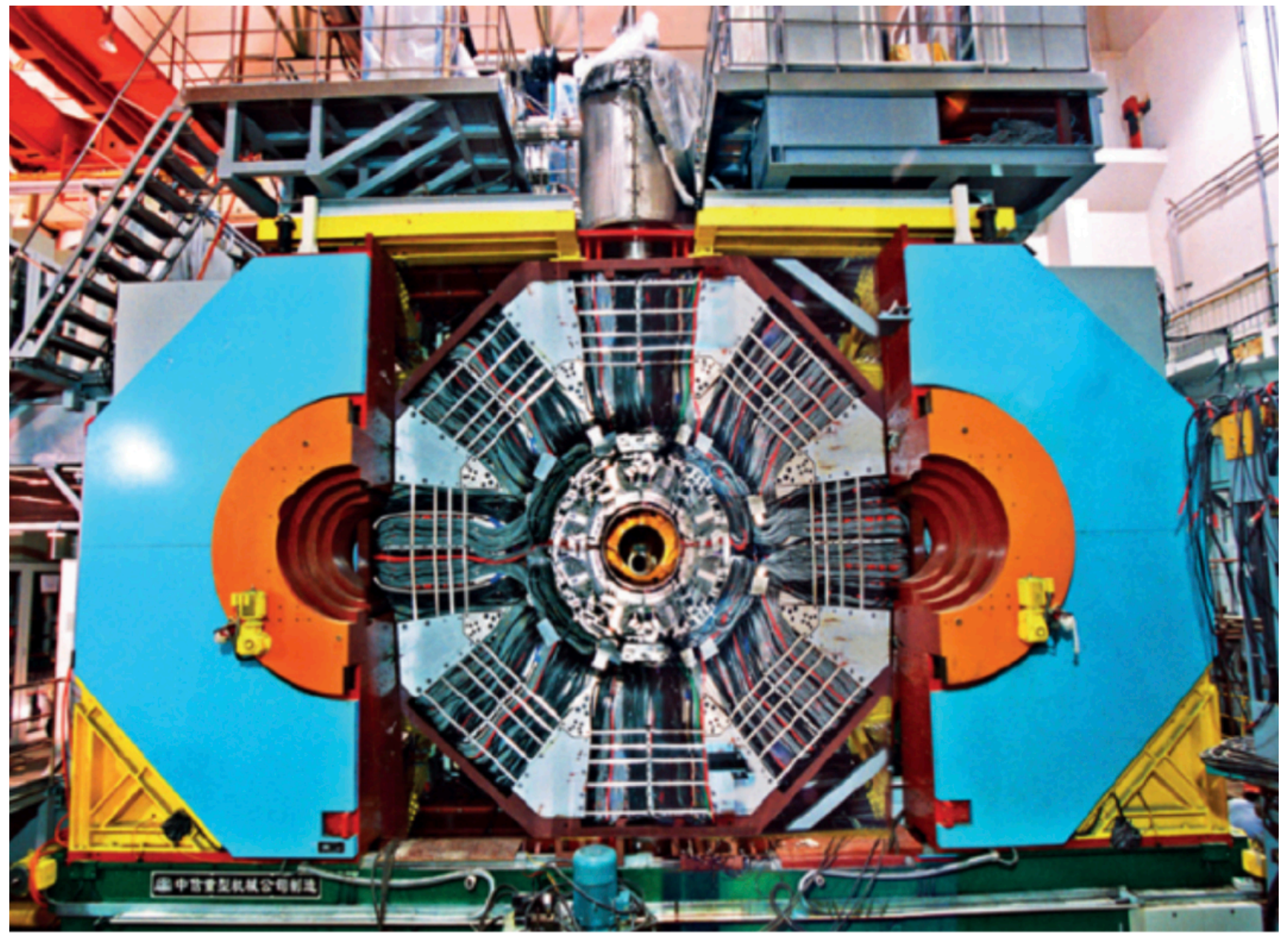
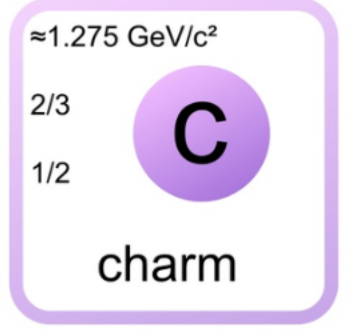
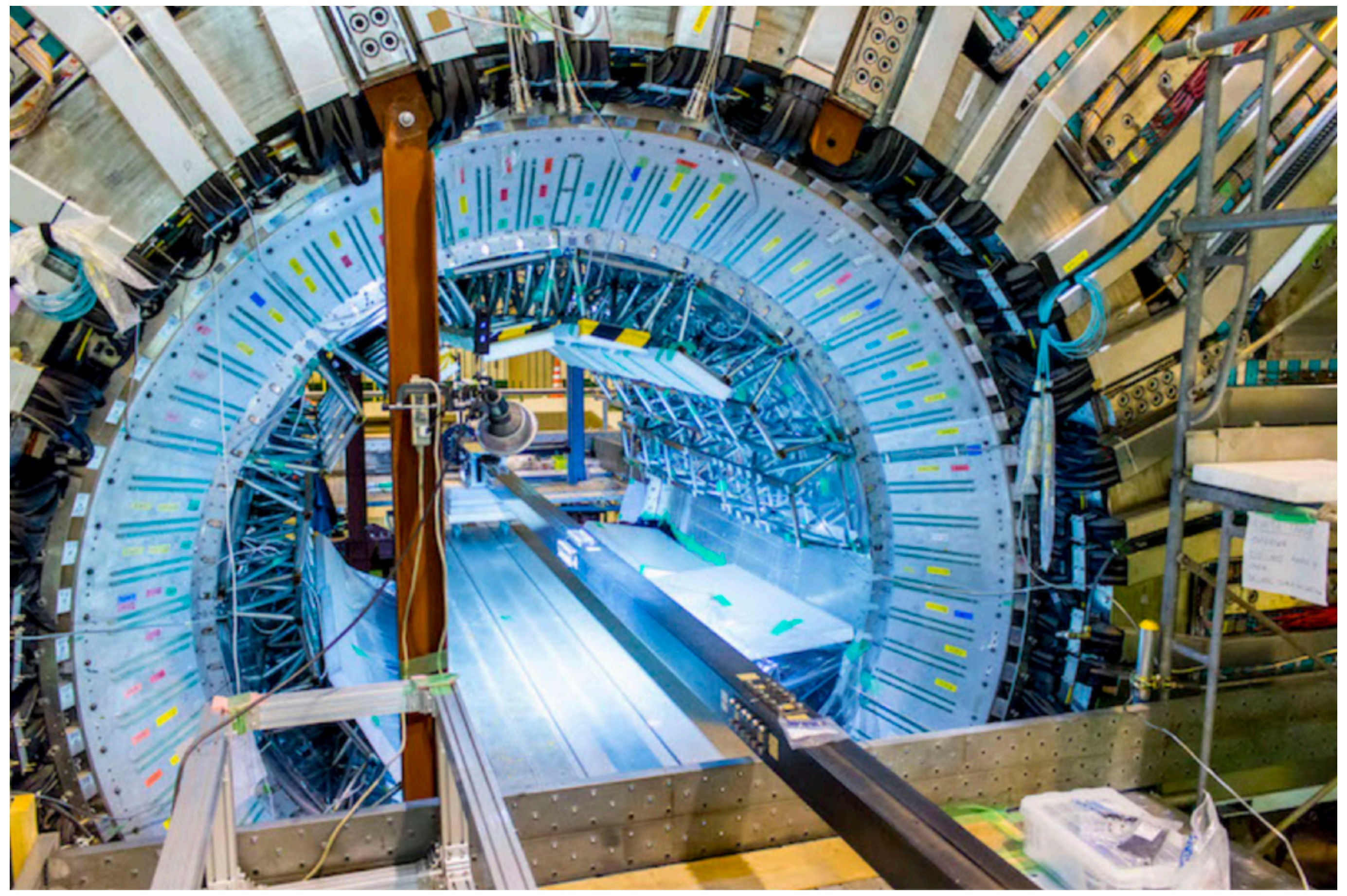
SM precision test

and their strong interaction inputs

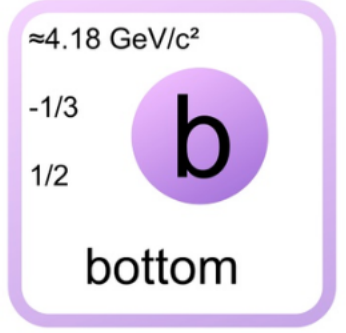


BESIII leads the international charm physics research

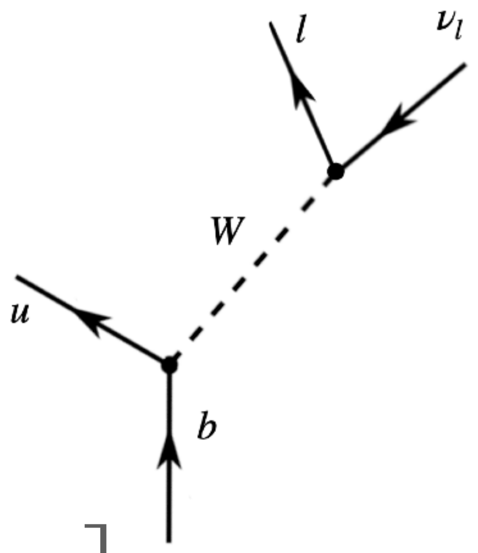




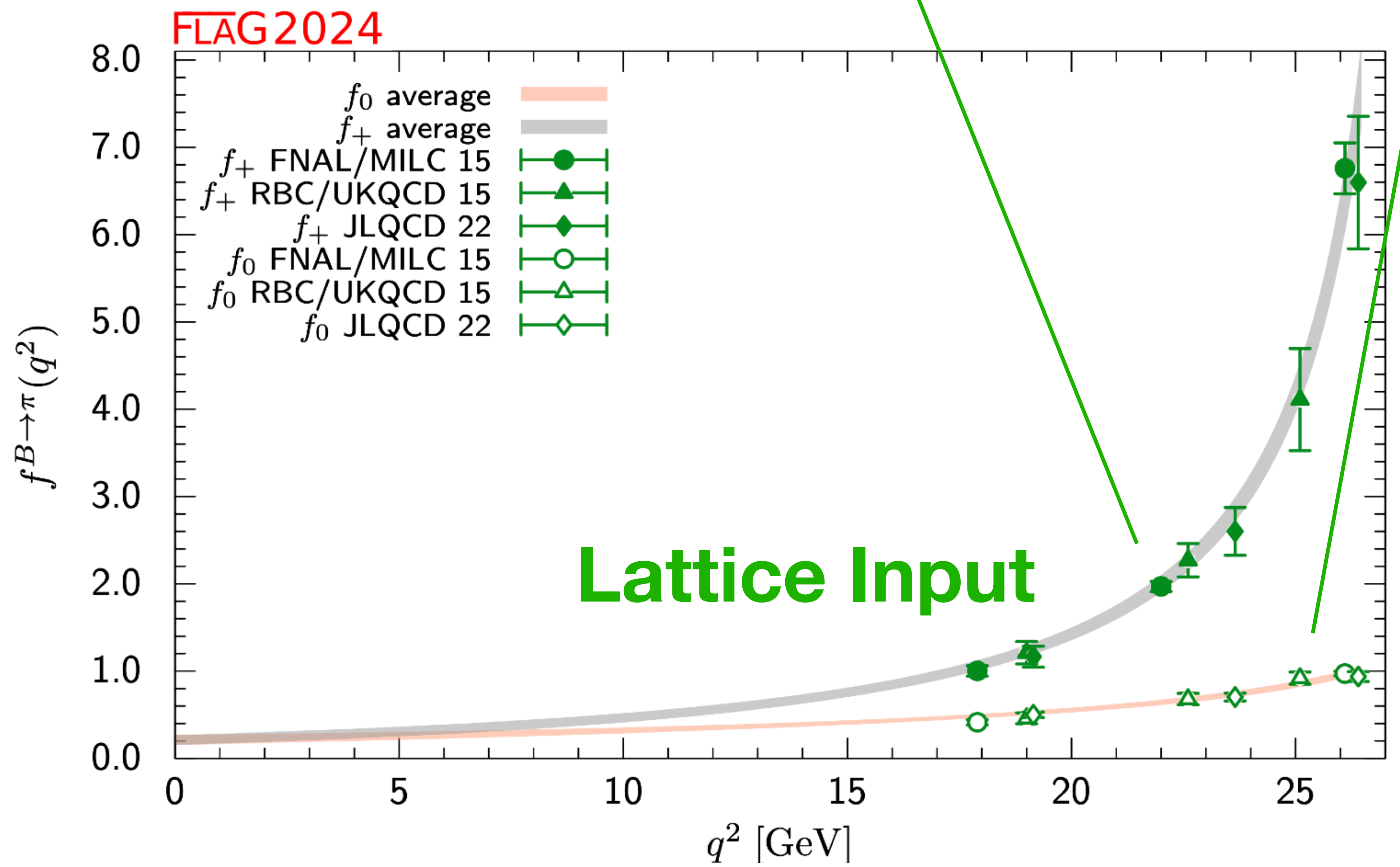
Belle2 completed upgrades and starts data collection.



Experimental results



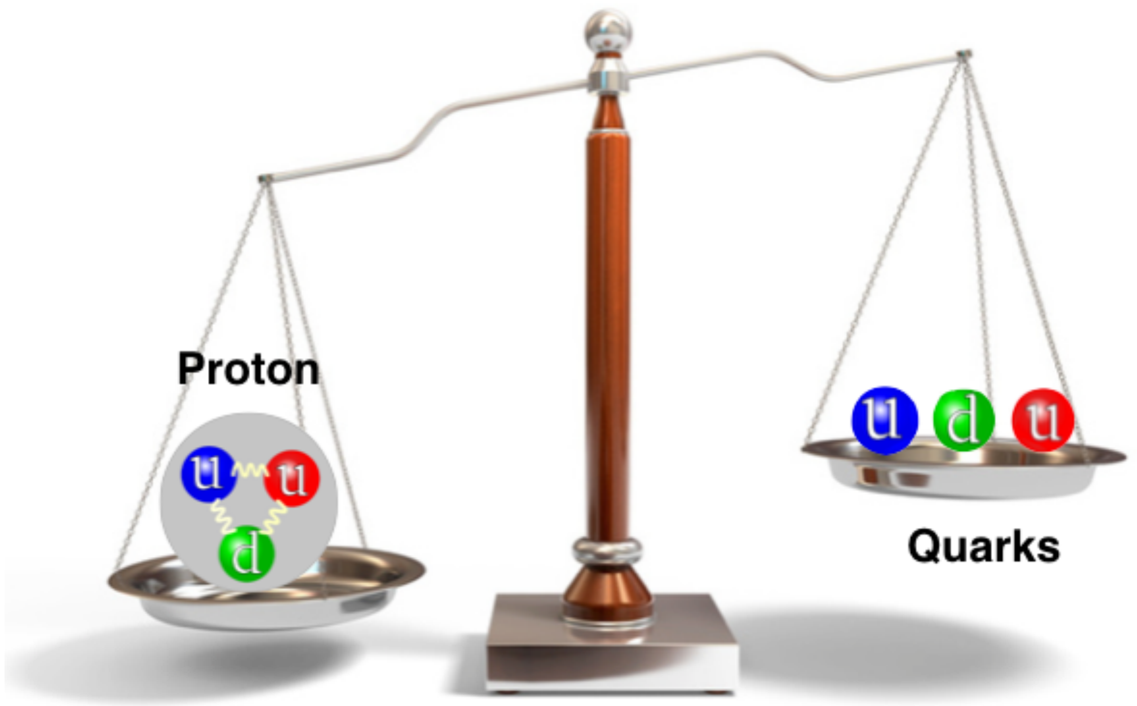
$$\frac{d\Gamma(B_{(s)}\to P\ell\nu)}{dq^2} = \frac{G_F^2|\eta_{EW}|^2|V_{ub}|^2}{24\pi^3}\frac{(q^2-m_\ell^2)^2\sqrt{E_P^2-m_P^2}}{q^4 m_{B_{(s)}}^2}\times\left[\left(1+\frac{m_\ell^2}{2q^2}\right)m_{B_{(s)}}^2(E_P^2-m_P^2)|f_+(q^2)|^2+\frac{3m_\ell^2}{8q^2}(m_{B_{(s)}}^2-m_P^2)^2|f_0(q^2)|^2\right]$$

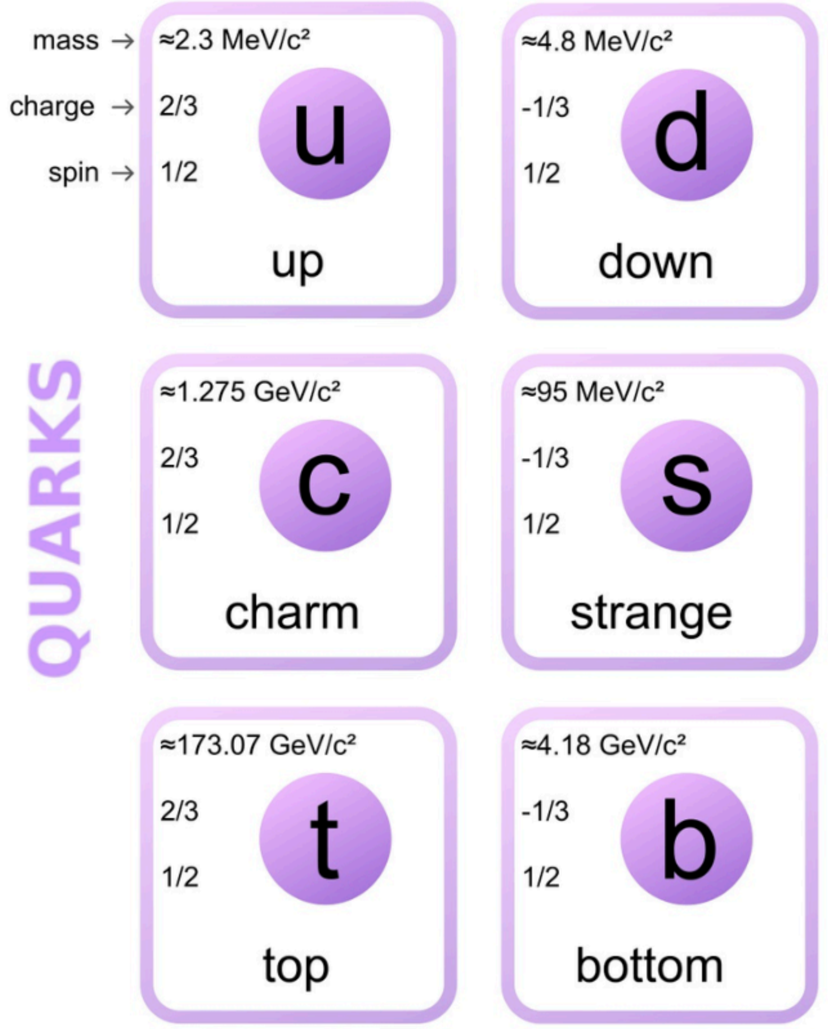


- Precision determination of CKM ME is the foundation of the standard model precision test;
- Extract all the CKM ME except V_{tb} requires inputs from the strong interaction predictions.

Baryon mass and spin

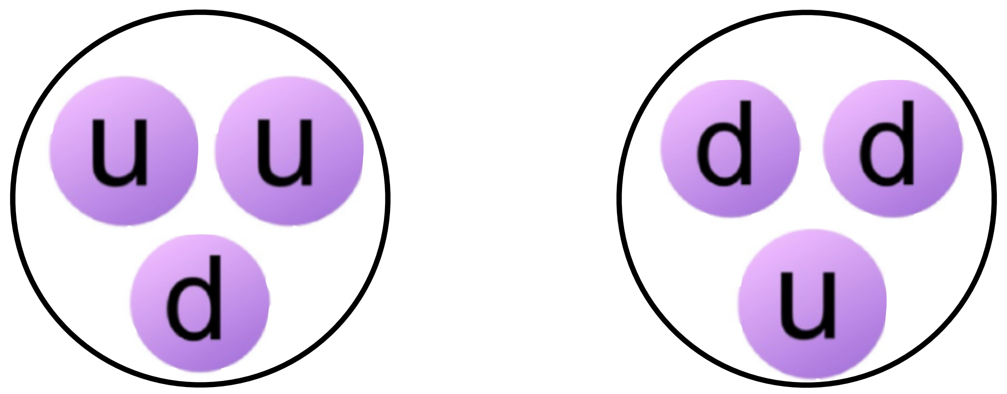
and their strong interaction origin



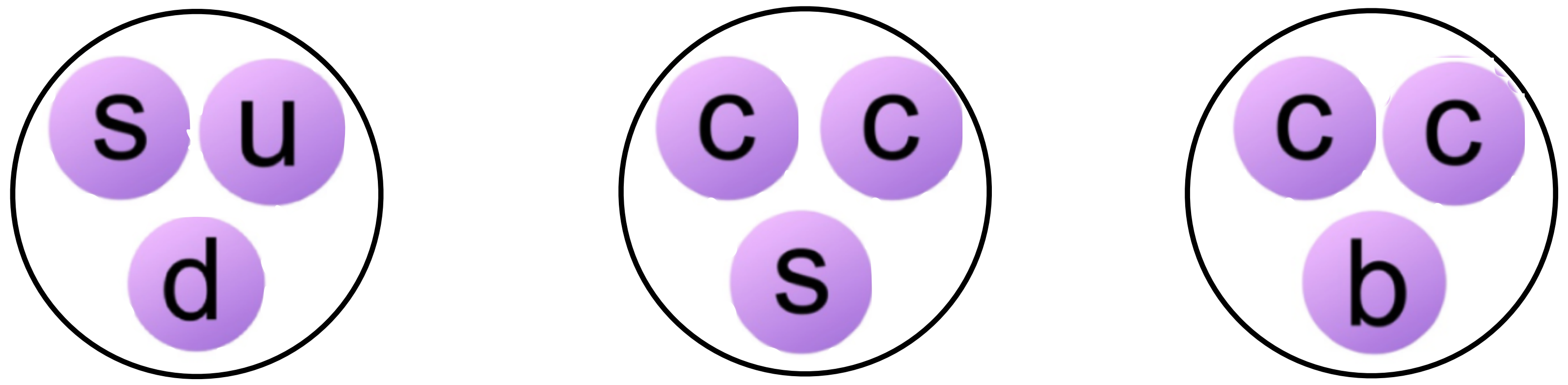


?

$m_p = 0.9383$ GeV $\quad m_n = 0.9396$ GeV



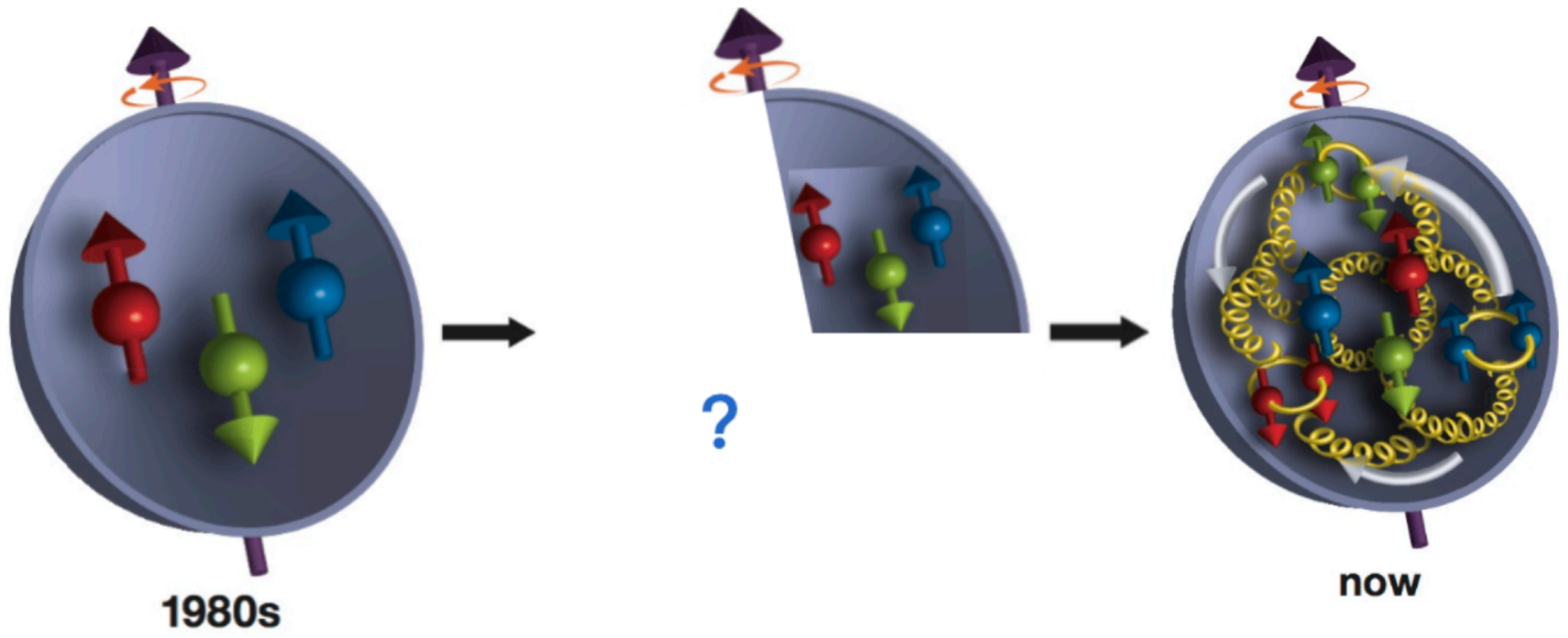
$m_\Lambda = 1.116$ GeV, $m_{\Omega_{cc}} = 3.519$ GeV, $m_{\Omega_{ccb}} = ?$



$$m_H = \langle \sum_q m_q \bar{q} q \rangle_H + \langle H_{en} \rangle_H + \frac{1}{4} \langle H_a \rangle_H$$

$S_\Lambda = ?$ $\quad S_{\Omega_{cc}} = ?$ $\quad S_{\Omega_{ccb}} = ?$

$$S_H = \frac{1}{2} \sum_q \Delta q + \Delta G + \sum_{i=q,g} L_i$$



?

Hadron masses

and their QED corrections

- The hadron mass and matrix elements in the real world require the full QCD+QED calculation. But since the lattice calculation can only reach an $\mathcal{O}(1\%)$ ($\mathcal{O}(0.1\%)$ in the heavy quark case) precision, one can expand the prediction in term of the polynomial of α and also $\delta_{\rm ISB} \equiv (m_d - m_u)/\Lambda_{\rm QCD}$:

$$\mathcal{M}^{\rm QCD+QED} = \mathcal{M}^{\rm isoQCD} + \alpha \mathcal{M}^{(0,1)} + \delta_{\rm ISB}\mathcal{M}^{(1,0)} + \mathcal{O}(\alpha^2, \alpha\delta_{\rm ISB}, \delta_{\rm ISB}^2)$$

- Naive power counting suggests that both ISB and QED corrections are 1%;

- There are kinds of known results for the ISB and QED corrections:

$$m_{\pi^+} - m_{\pi^0} = \sim 0_{\rm ISB} + 4.53_{\rm QED}(6)\ {\rm MeV}, \qquad m_{K^+} - m_{K^0} = -6.00_{\rm ISB}(15)\ {\rm MeV} + 2.07_{\rm QED}(15)\ {\rm MeV},$$

$$m_n - m_p = 2.52_{\rm ISB}(29)\ {\rm MeV} - 1.00_{\rm QED}(16)\ {\rm MeV},$$

$$m_{D^+} - m_{D^0} = 2.54_{\rm ISB}(13)\ {\rm MeV} + 2.14_{\rm QED}(13)\ {\rm MeV}, \qquad m_{B^+} - m_{B^0} = -1.88_{\rm ISB}(60)\ {\rm MeV} + 1.58_{\rm QED}(24)\ {\rm MeV}.$$

- The QED correction for m_{B^+} itself ?

BMWc, Science 347(2015)1452

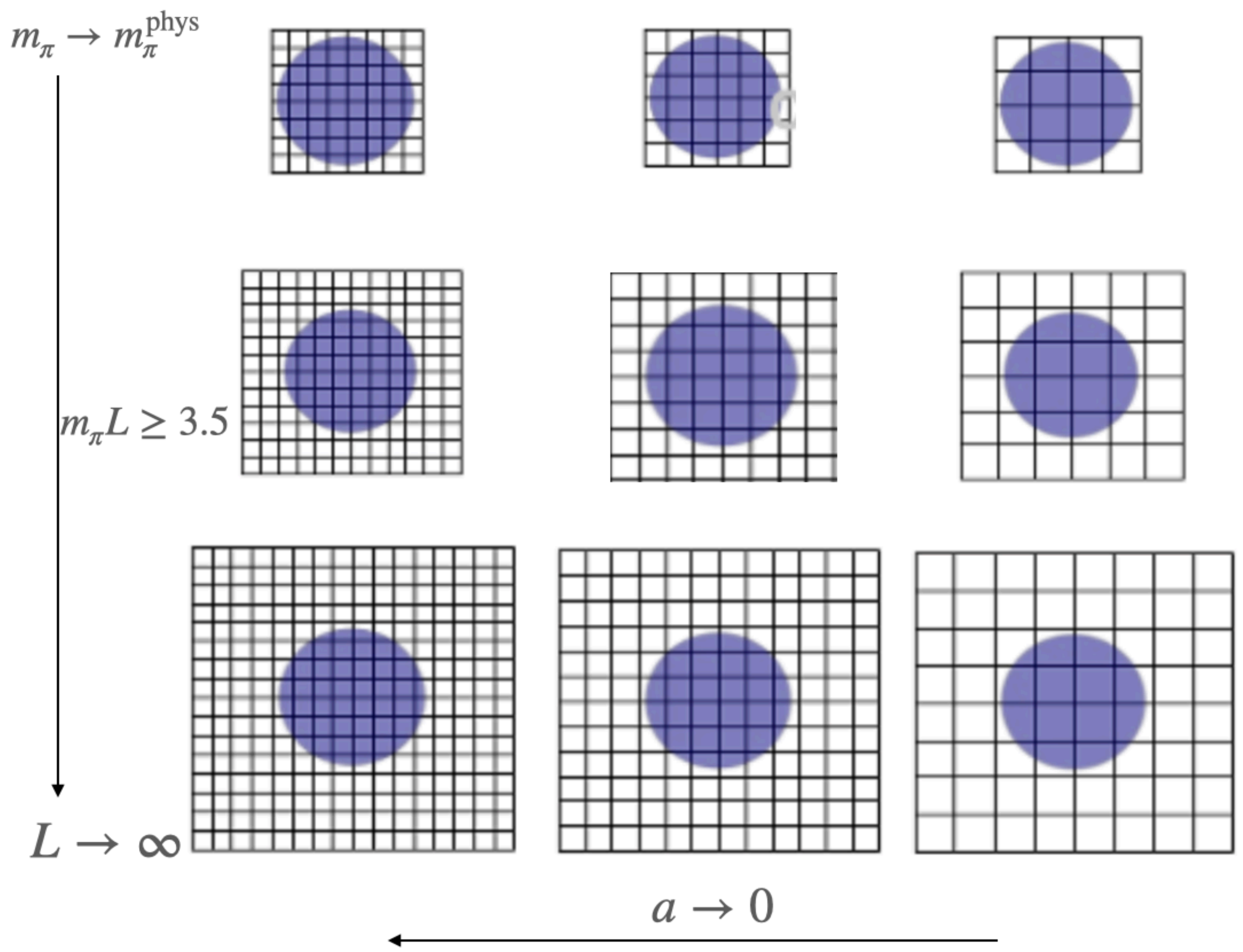
M. Rowe, R. Zwicky, JHEP(2023)089

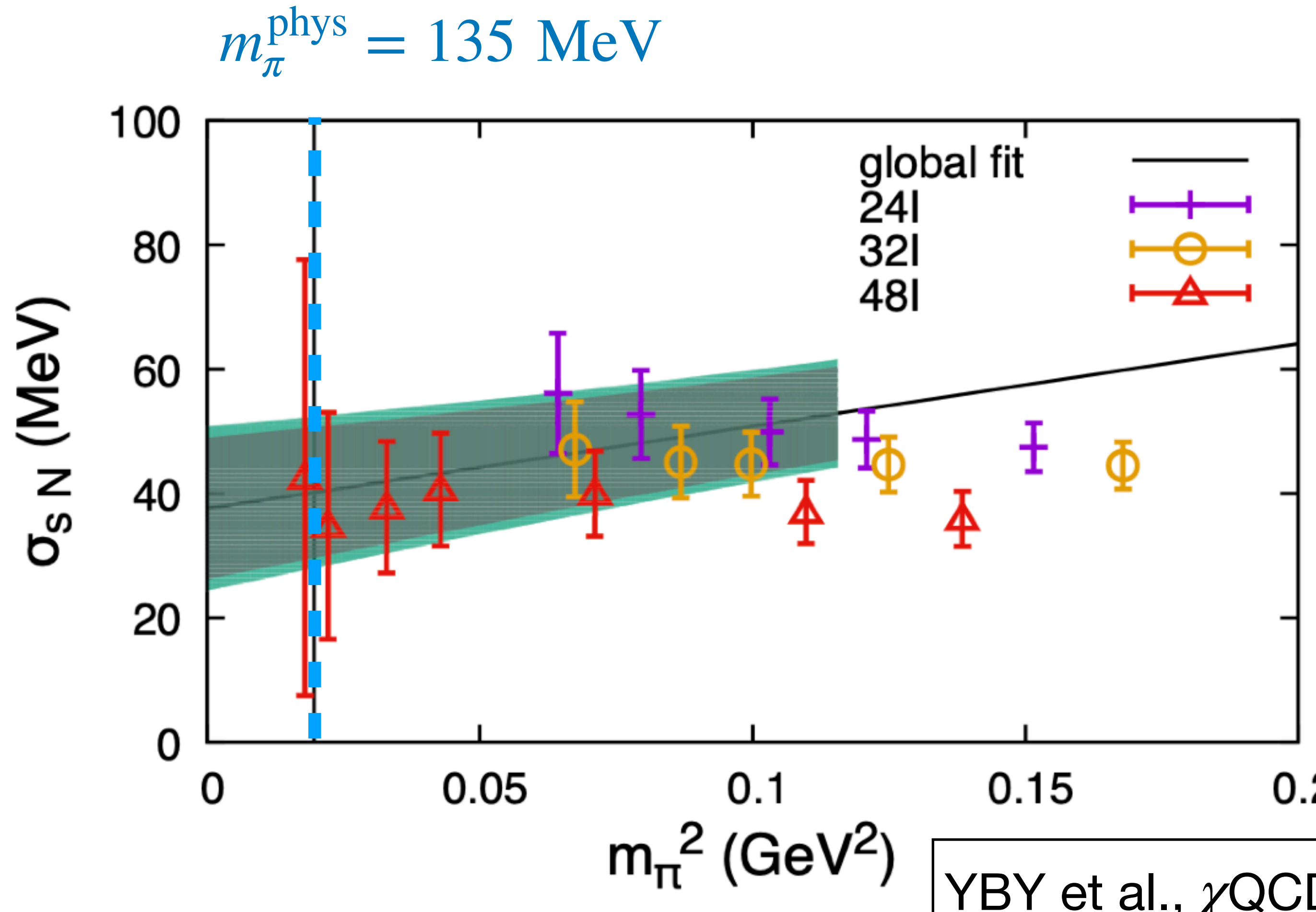
D. Giusti et al., PRD95(2017)114504

X. Feng et al., PRL128(2022)052003

Flavor physics

and their challenges on the lattice





YBY et al., χQCD, PRD94(2016)054503

To reach the same statistical precision for a physical light quark as we achieve for the strange quark:

- Computational cost would increase by a factor of **700**.

- Naive discretization error estimate $m_c^2 a^2 \leq 0.1$ requires $a \leq 0.06$ fm and then the computational cost would increase by a factor of **7**, compared to the $a \sim 0.1$ fm case.

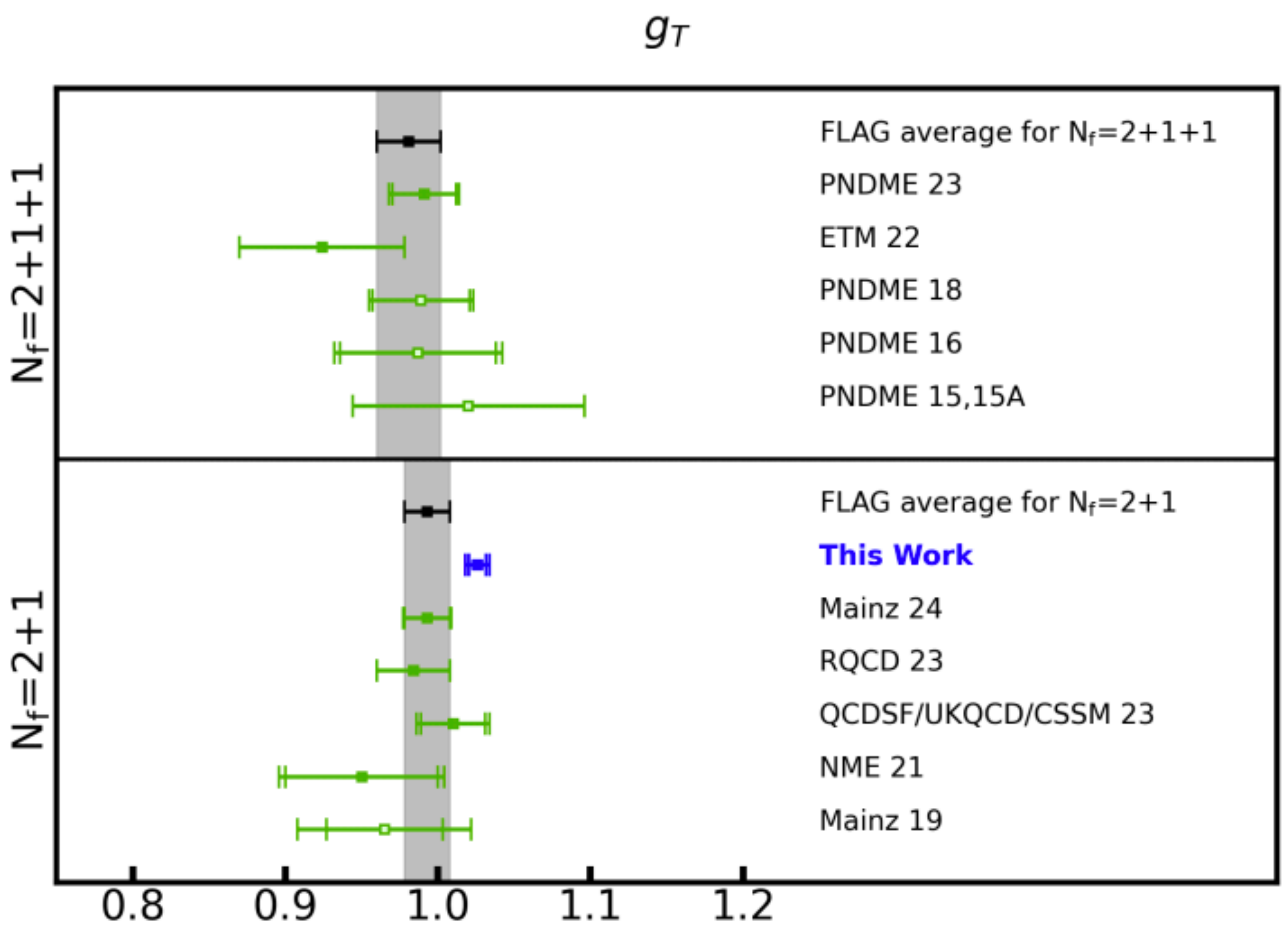
- Similar estimate $m_b^2 a^2 \leq 0.1$ requires $a \leq 0.013$ fm and then the factor is **4823.**

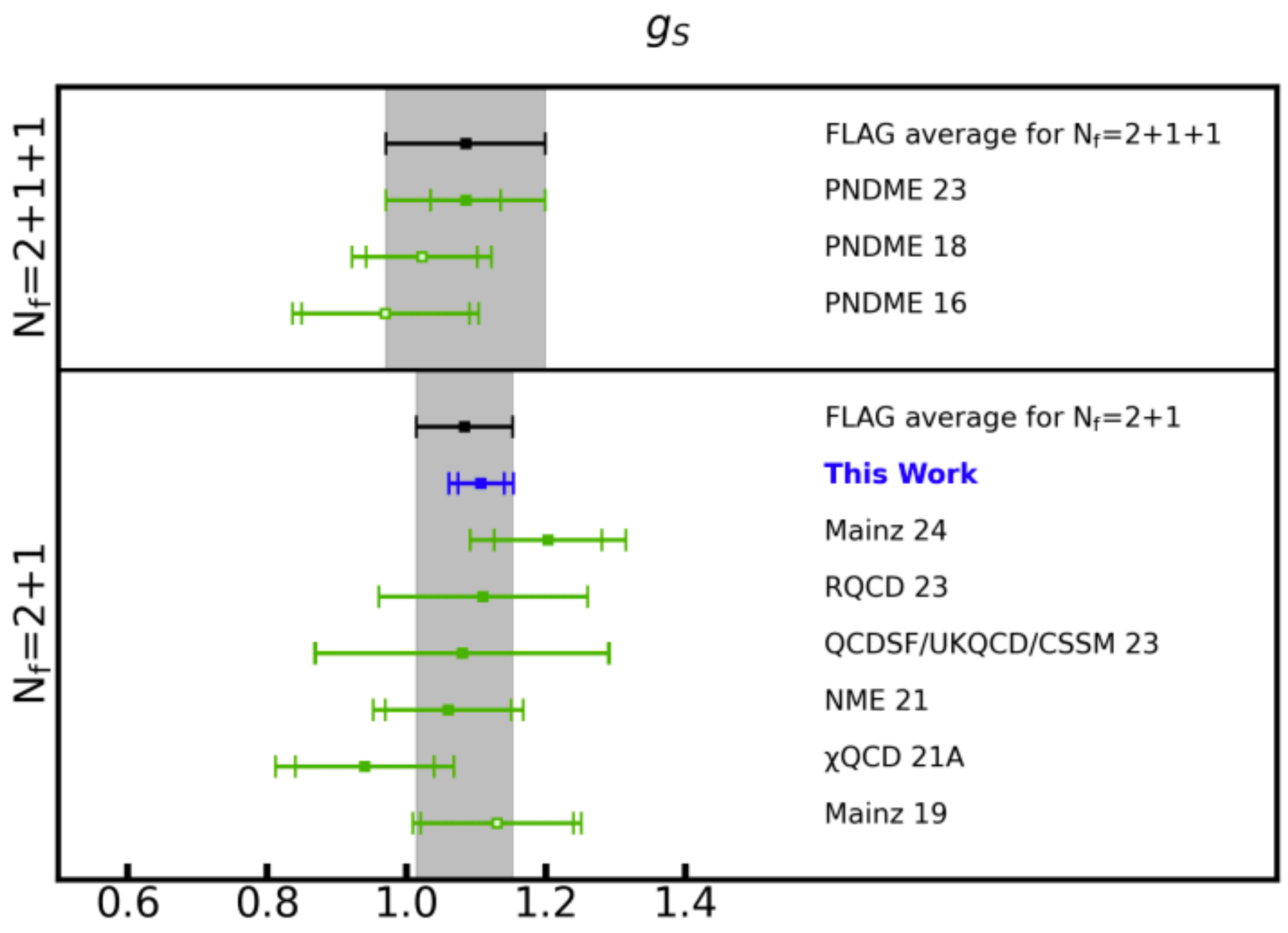
- The top quark information can only be accessed through the heavy quark extrapolations.

CLQCD ensembles

Predictions

Z.-H. Hu et al. [CLQCD], PRD109(2024)054507
H.Y. Du et al. [CLQCD], PRD111(2025)054504
M.C. Cai et al. [CLQCD], arXiv:2603.01846

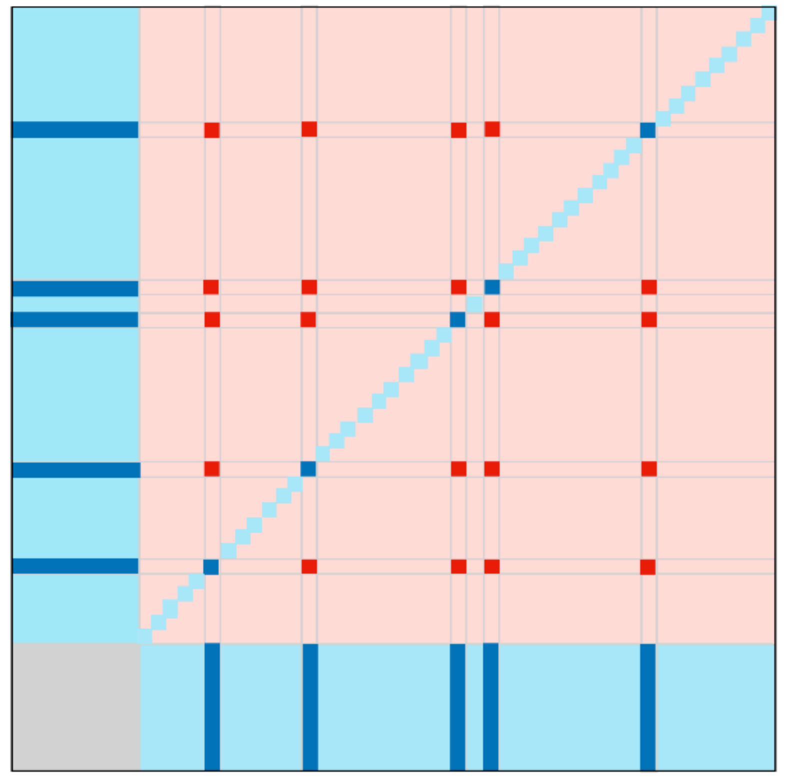




J.H. Wang et al. [CLQCD], arXiv:2511.02326

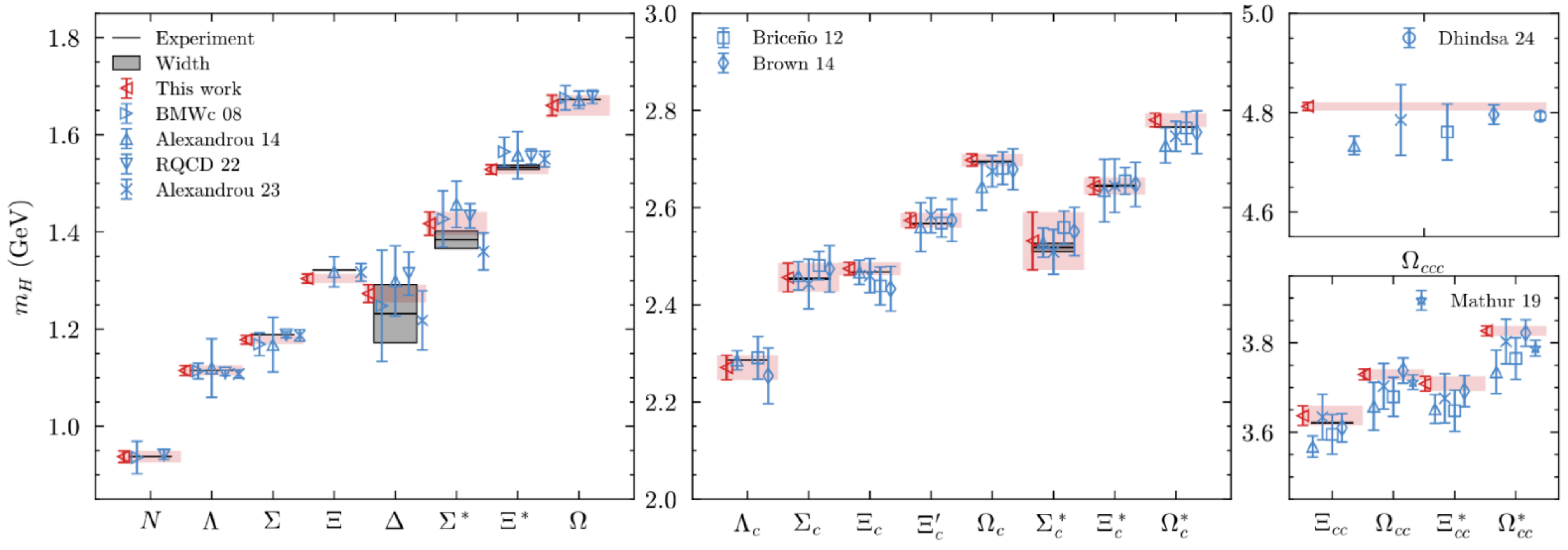
Outline

- **Light and strange quark**

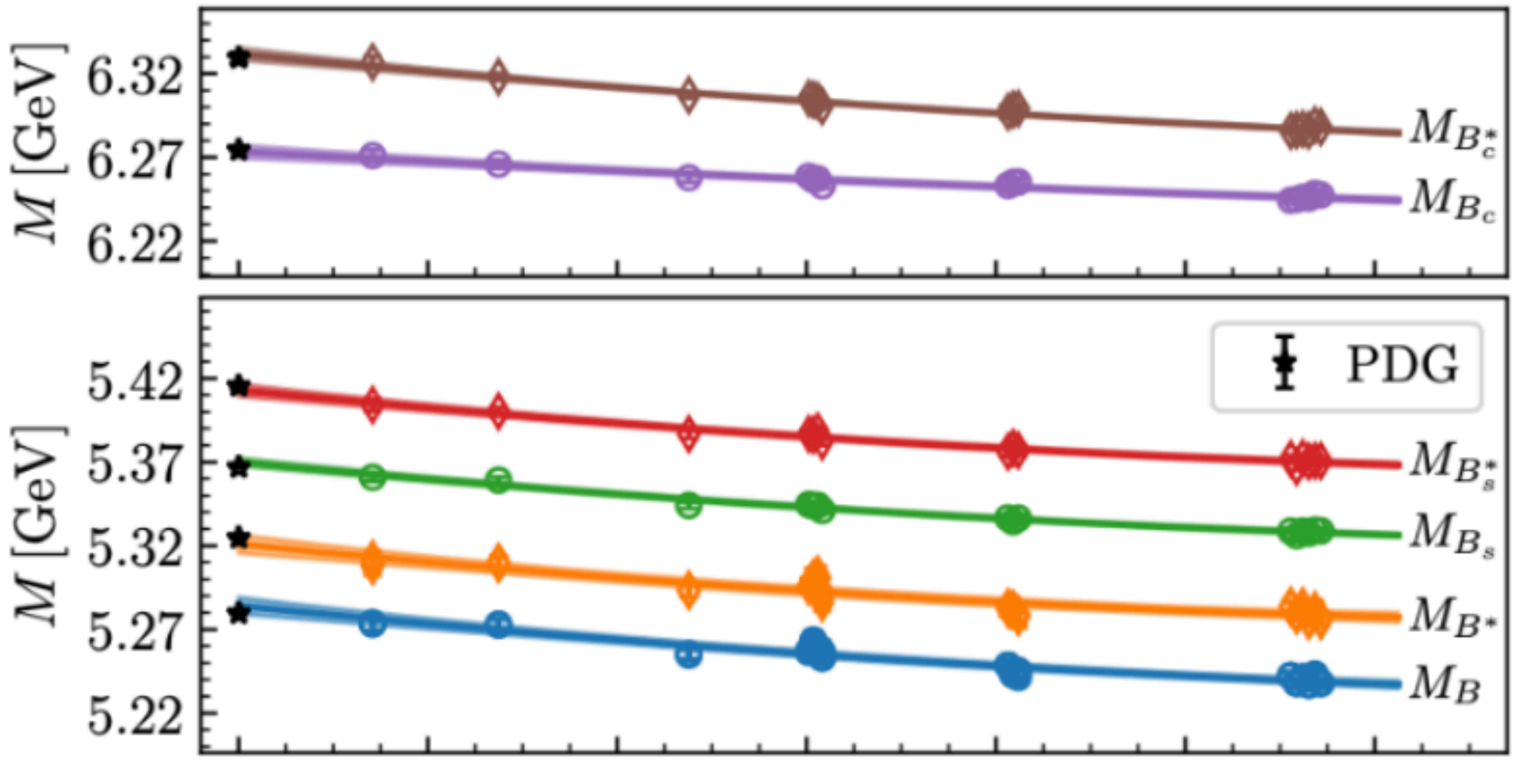


$$\Omega^{(2)}_{ij} = \begin{cases} 1 & \text{for } i,j \le N_e, \\ \Pi^{1}_{i=0}\frac{V-N_e-i}{N_{st}-i} & \text{for } i,j > N_e,\ i \neq j \\ \frac{V-N_e}{N_{st}} & \text{for the other cases,} \end{cases}$$

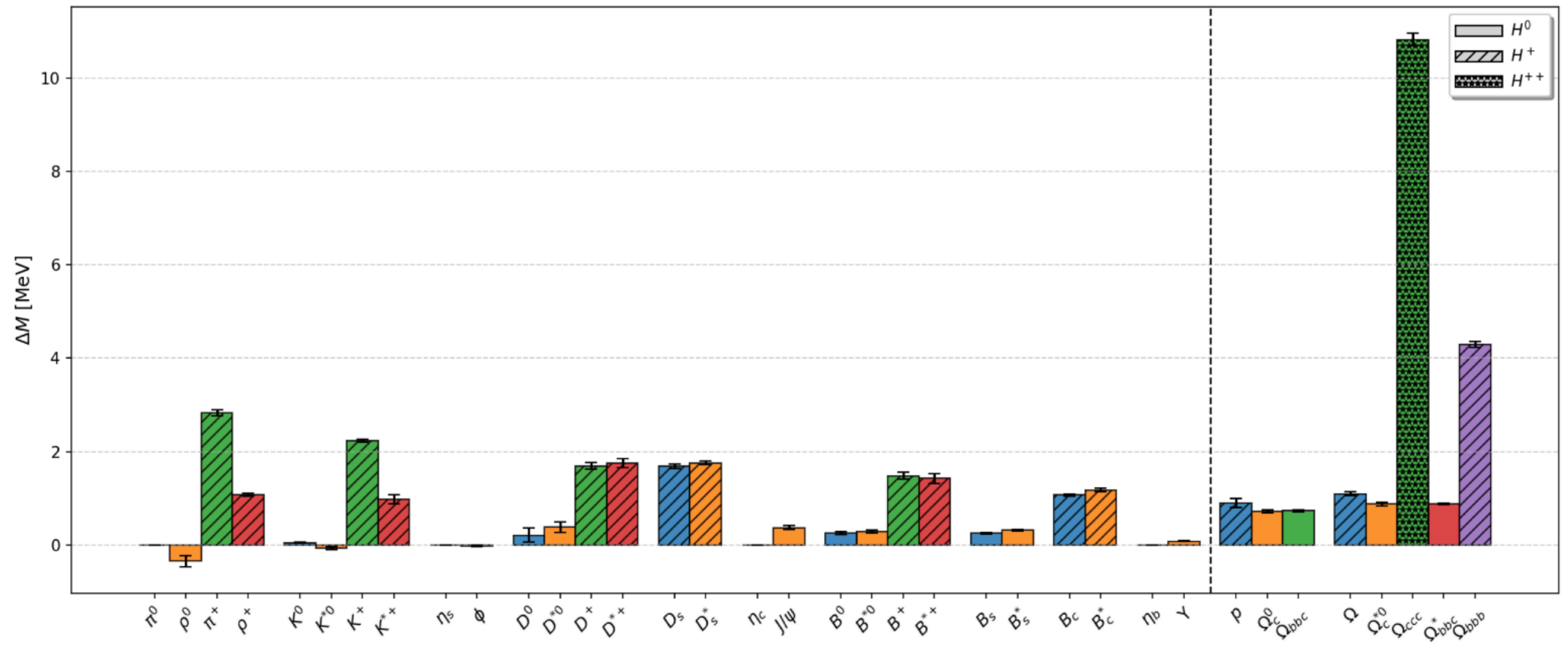
- Charm quark



- Bottom quark



- QED corrections

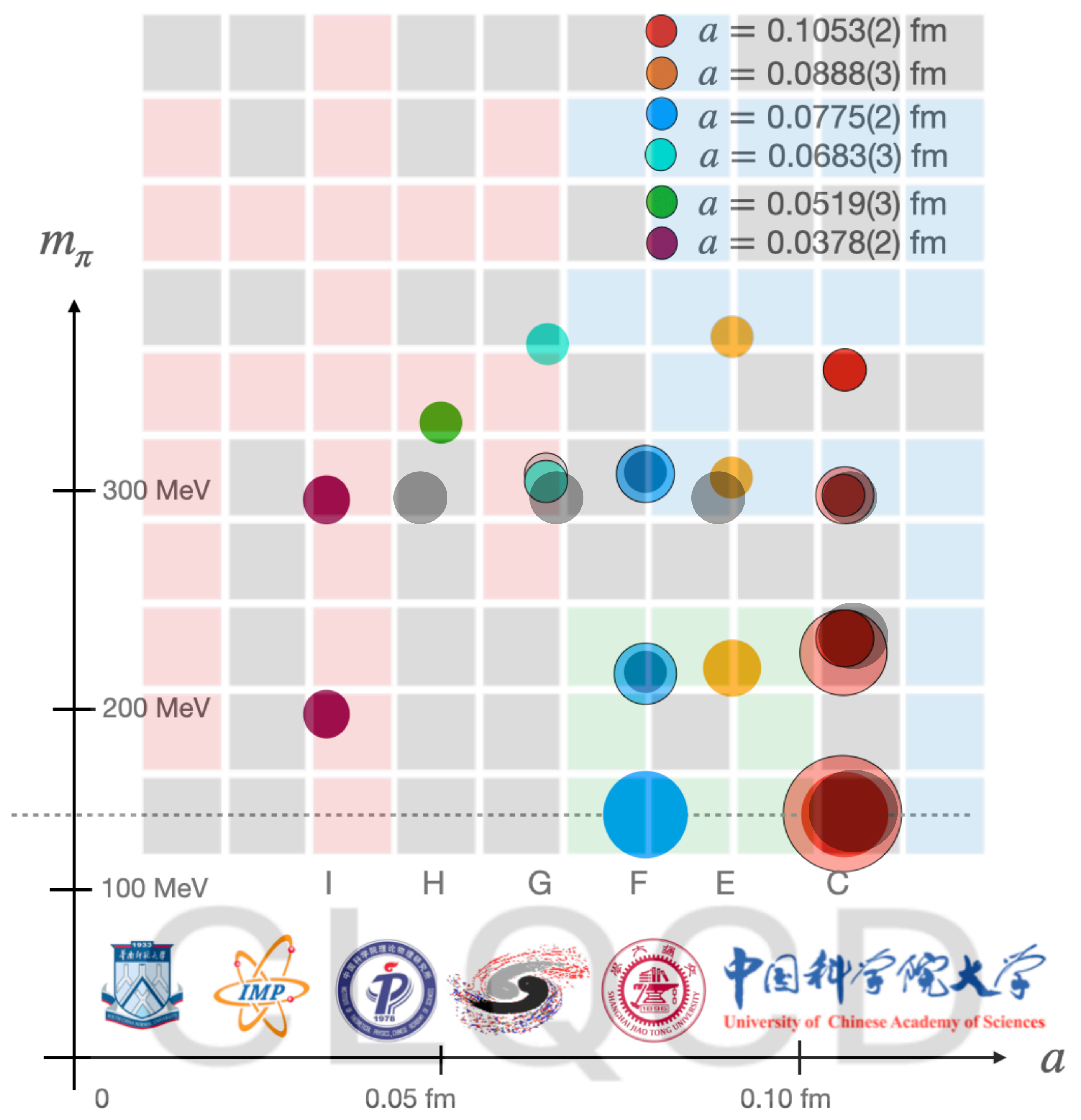


- Summary

Light and strange quarks

CLQCD ensembles

	Country/ Region	Smallest lattice spacing	No. of physical point ensembles	Largest spacial size	No. of fermion discretization
MILC	US	0.03 fm	5	5.8 fm	1
RBC	US	0.06 fm	3	5.5 fm	1
BMW	EN	0.05 fm	15	10 fm	2
CLS	EN	0.04 fm	2	5.5 fm	1
ETM	EN	0.05 fm	5	6.3 fm	1
PACS	JP	0.06 fm	3	10 fm	1
CLQCD	**CN**	**0.04 fm**	**4**	**6.7 fm**	**2**



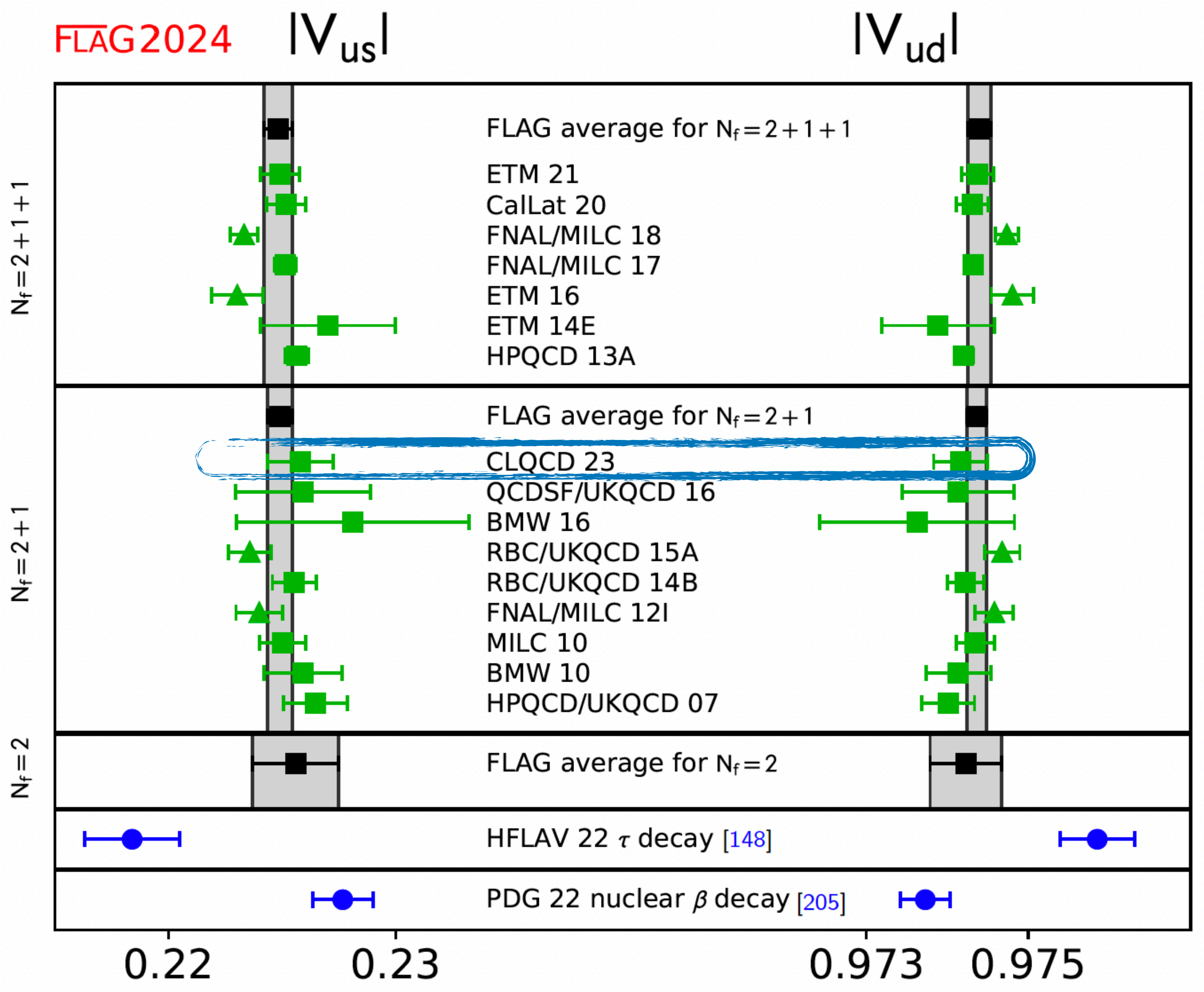
Fermion Discretization	No. of ensembles	Sea fermion flavors	Lattice spacing range (fm)	No. of ensembles at physical pion mass	Device for data generation	Device for data anslysis
Clover	20	2+1	0.038-0.105	3	CUDA GPU	Sugon DCU
HISQ	10	2+1+1	0.048-0.108	1	Sugon DCU	Sugon DCU

Light and strange quarks

CLQCD inputs for FLAG

Y. Aoki et al. *FLAG Review 2024* 2411.04268

Collaboration	Ref.	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	running	m_{ud}	m_s
CLQCD 23	[10]	A	★	★	★	★	*e*	3.60(11)(15)	98.8(2.9)(4.7)
ALPHA 19	[11]	A	○	★	★	★	*e*	3.54(12)(9)	95.7(2.5)(2.4)
Maezawa 16	[12]	A	■	★	★	★	*d*	–	92.0(1.7)
RBC/UKQCD 14B$^{\ominus}$	[13]	A	★	★	★	★	*d*	3.31(4)(4)	90.3(0.9)(1.0)
RBC/UKQCD 12$^{\ominus}$	[9]	A	★	○	★	★	*d*	3.37(9)(7)(1)(2)	92.3(1.9)(0.9)(0.4)(0.8)
PACS-CS 12*	[14]	A	★	■	■	★	*b*	3.12(24)(8)	83.60(0.58)(2.23)
Laiho 11	[15]	C	○	★	★	○	–	3.31(7)(20)(17)	94.2(1.4)(3.2)(4.7)
BMW 10A, 10B^{+}	[16, 17]	A	★	★	★	★	*c*	3.469(47)(48)	95.5(1.1)(1.5)
PACS-CS 10	[18]	A	★	■	■	★	*b*	2.78(27)	86.7(2.3)
MILC 10A	[19]	C	○	★	★	○	–	3.19(4)(5)(16)	–
HPQCD 10**	[20]	A	○	★	★	–	–	3.39(6)	92.2(1.3)
RBC/UKQCD 10A	[21]	A	○	○	★	★	*a*	3.59(13)(14)(8)	96.2(1.6)(0.2)(2.1)
Blum 10†	[22]	A	○	■	○	★	–	3.44(12)(22)	97.6(2.9)(5.5)
PACS-CS 09	[23]	A	★	■	■	★	*b*	2.97(28)(3)	92.75(58)(95)
HPQCD 09A$^{\oplus}$	[24]	A	○	★	★	–	–	3.40(7)	92.4(1.5)
MILC 09A	[25]	C	○	★	★	○	–	3.25 (1)(7)(16)(0)	89.0(0.2)(1.6)(4.5)(0.1)
MILC 09	[26]	A	○	★	★	○	–	3.2(0)(1)(2)(0)	88(0)(3)(4)(0)
PACS-CS 08	[27]	A	★	■	■	■	–	2.527(47)	72.72(78)
RBC/UKQCD 08	[28]	A	○	■	★	★	–	3.72(16)(33)(18)	107.3(4.4)(9.7)(4.9)
CP-PACS/ JLQCD 07	[29]	A	■	★	★	■	–	$3.55(19)(^{+56}_{-20})$	$90.1(4.3)(^{+16.7}_{-4.3})$
HPQCD 05	[30]	A	○	○	○	○	–	3.2(0)(2)(2)(0)‡	87(0)(4)(4)(0)‡
MILC 04, HPQCD/ MILC/UKQCD 04	[31, 32]	A	○	○	○	■	–	2.8(0)(1)(3)(0)	76(0)(3)(7)(0)



Flavor lattice average group, 2411.04268

- First domestic light quark masses and CKM matrix elements determination which enters the lattice QCD world averages;

Z.-H. Hu, B.-L. Hu, J.-H. Wang, et. al., CLQCD, PRD109(2024) 054507

- Reach the highest “green star” grade in the **chiral, continuum, infinite volume extrapolation and also renormalization.**

Light and strange quarks

- Most precise prediction on $g_{S,T}^{u-d}$ so far, among three new results appears 2025;

$$m_n^{\rm exp} - m_p^{\rm exp} = 1.293\ {\rm MeV}$$

$$m_n - m_p = m_u\left(\frac{\partial m_n}{\partial m_u} - \frac{\partial m_p}{\partial m_u}\right) + m_d\left(\frac{\partial m_n}{\partial m_d} - \frac{\partial m_p}{\partial m_d}\right) + \delta^{\rm QED} m_p^{\rm isoQCD} = (m_d - m_u) g_S^{u-d} + \delta^{\rm QED} m_p^{\rm isoQCD}$$

$$= (2.35(12)\ {\rm MeV})_{m_d - m_u} * 1.11(5)_{g_s} - 1.00(16)\ {\rm MeV}_{\rm QED}$$

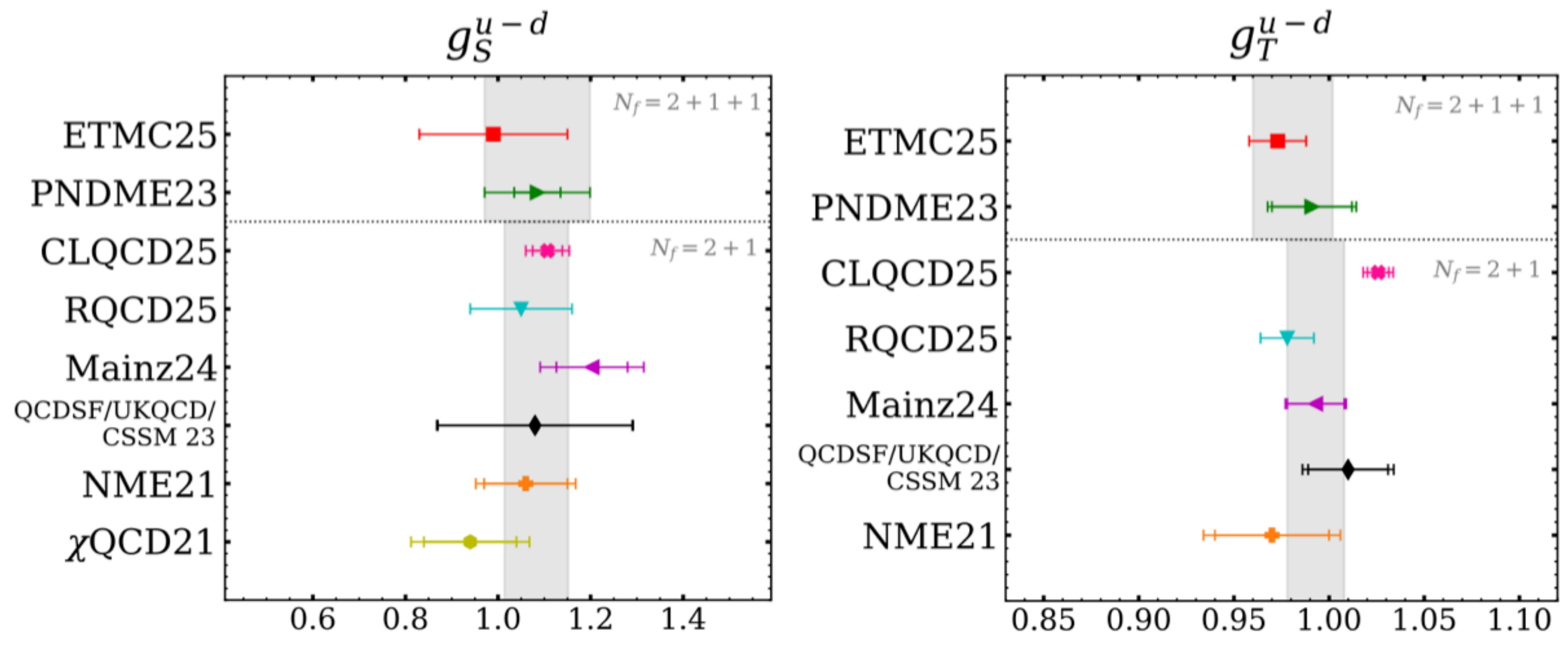
$$= 1.60[0.23]_{\rm tot}(11)_{g_s}(0.13)_{\rm ISB}(0.16)_{\rm QED}\ {\rm MeV}\,.$$

ISB : Flavor lattice average group, 2411.04268
QED: BMWc, Science 347(2015)1452

- Better control on all kinds of systematic uncertainties thanks to the large data set of CLQCD ensembles, and also high precision data using newly proposed "blending method".

✵ Nucleon isovector charges are well-studied by many collaborations: Recent results by

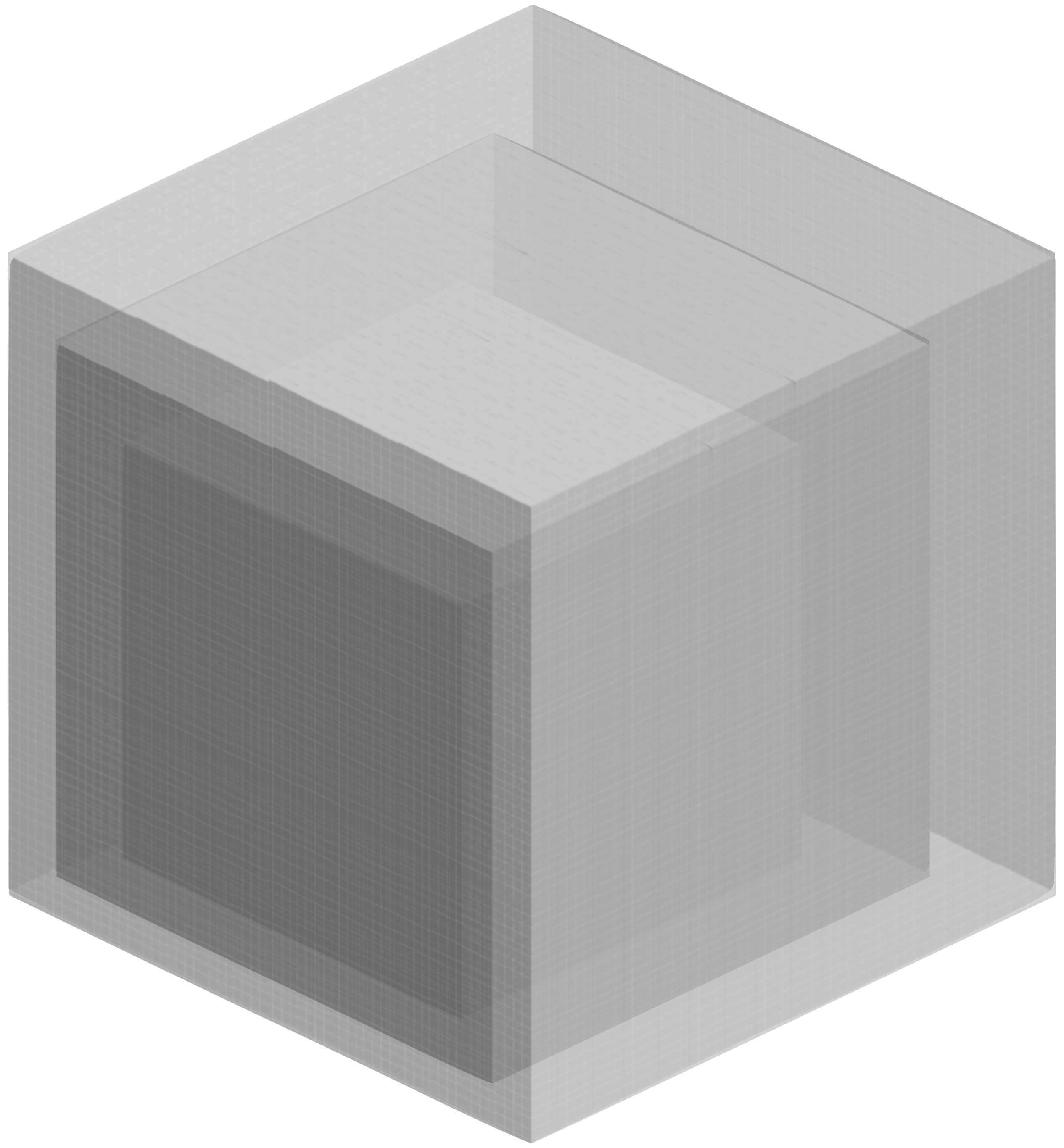
- RQCD: 47 CLS ensembles at 6 a~(0.038-0.098) fm, m_π~(480-130) MeV and multiple L
- ETMC: 4 ensembles at 4 a~(0.08-0.05) fm and m_π~140 MeV
- CLQCD: 16 Clover ensembles ar 4 a~(0.105-0.052) fm, m_π~(340-134) MeV and multiple L



"Hadron Structure in the Context of the EIC Program", Constantia Alexandrou, Lattice 2025 Plenary talk

Light and strange quarks

Fermion propagator on the lattice



Only the bosonic two-point function of the fermion, namely propagator, can be calculated on the lattice, but the all-to-all fermion propagator can be extremely costly:

- A modern $24^3 \times 64$ lattice will require ~15K$* and 820 TB to calculate and storage the all-to-all propagator for a given quark mass and configuration.

*Assuming A100 GPU can generate a 12-column propagator within 1min

- State-of-the-arts $48^3 \times 64$ lattice will requires **8x** and **64x** on the resource and storage, respectively.
- Realistic calculation will require $\mathcal{O}(100)$ configuration and repeat the calculation on multiple lattices.

	Cost efficency	Hadron state optimization	(N>2)-point functions
Tranditional	Low	Yes	Yes
Distillation	High	Yes	No
Sparsening	High	No	Yes

Light and strange quarks

- The “blending” method projects the identity matrix into exact low momentum modes plus stochastic samples of high momentum modes;

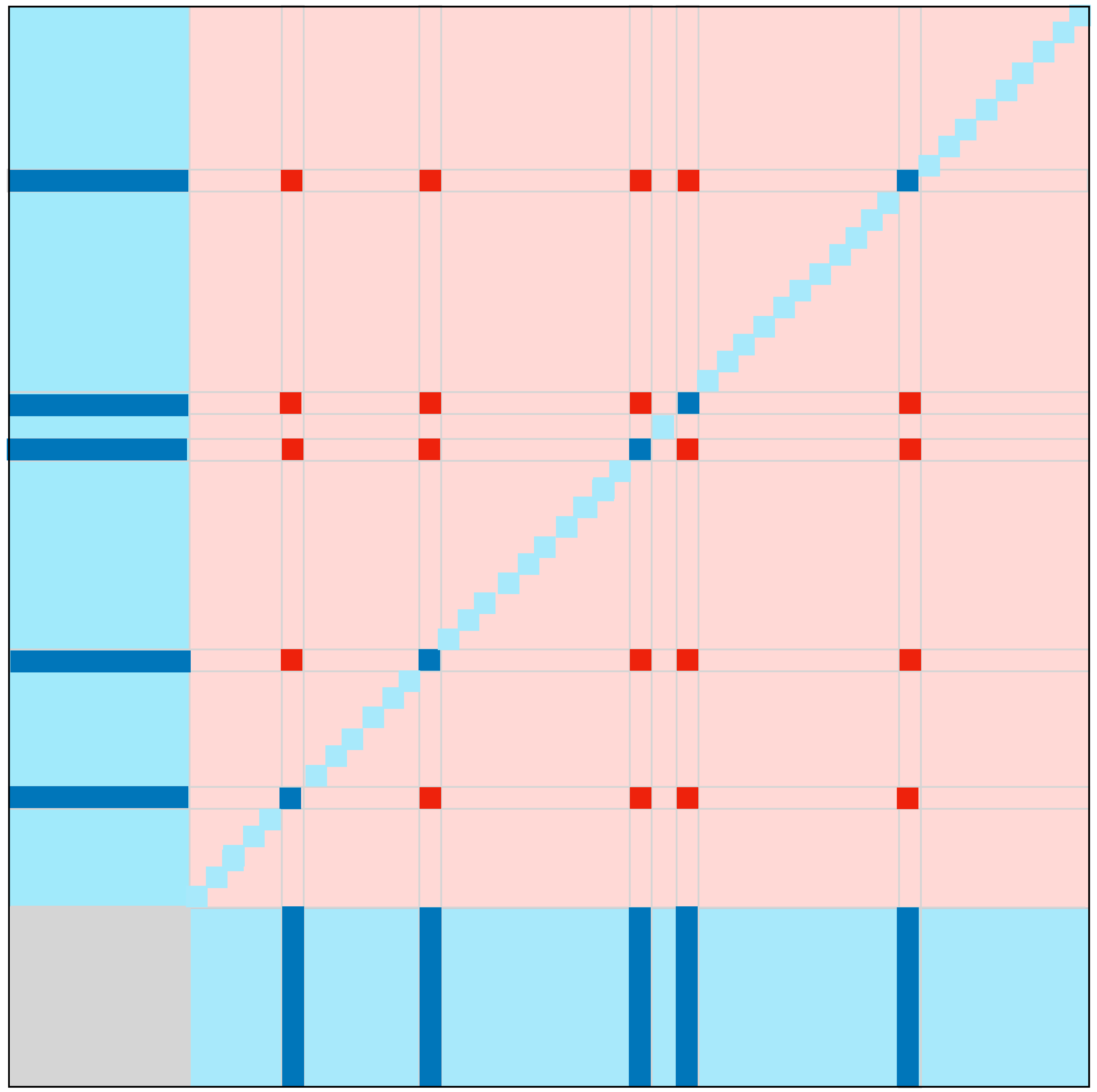
$$\hat{I} = \sum_{k=1}^{N_e} |\phi_k\rangle\langle\phi_k| + \sum_{k=N_e+1}^{N_e+N_{st}} \frac{N_c N_L^3 - N_e}{N_{st}} |\phi_k\rangle\langle\phi_k| + \mathcal{O}(\frac{1}{N_{st}^n})$$

- The the correlation function can be projected to the subspace of those modes:

$$\sum_{x,y,z,w} \langle \dots S(x,y)\mathcal{O}(y,z)S(z,w)\dots \rangle = \sum_{i,j,k,l} \langle \dots S_{ij}\mathcal{O}_{ik}S_{kl}\dots \rangle + \mathcal{O}(\frac{1}{N_{st}})$$

with the projected propagators

$$S_{ij} \equiv \int \mathrm{d}^3z\mathrm{d}^3w \langle \phi_i(z)|S(z,w)|\phi_j(w)\rangle$$



and projected and reweighed operators

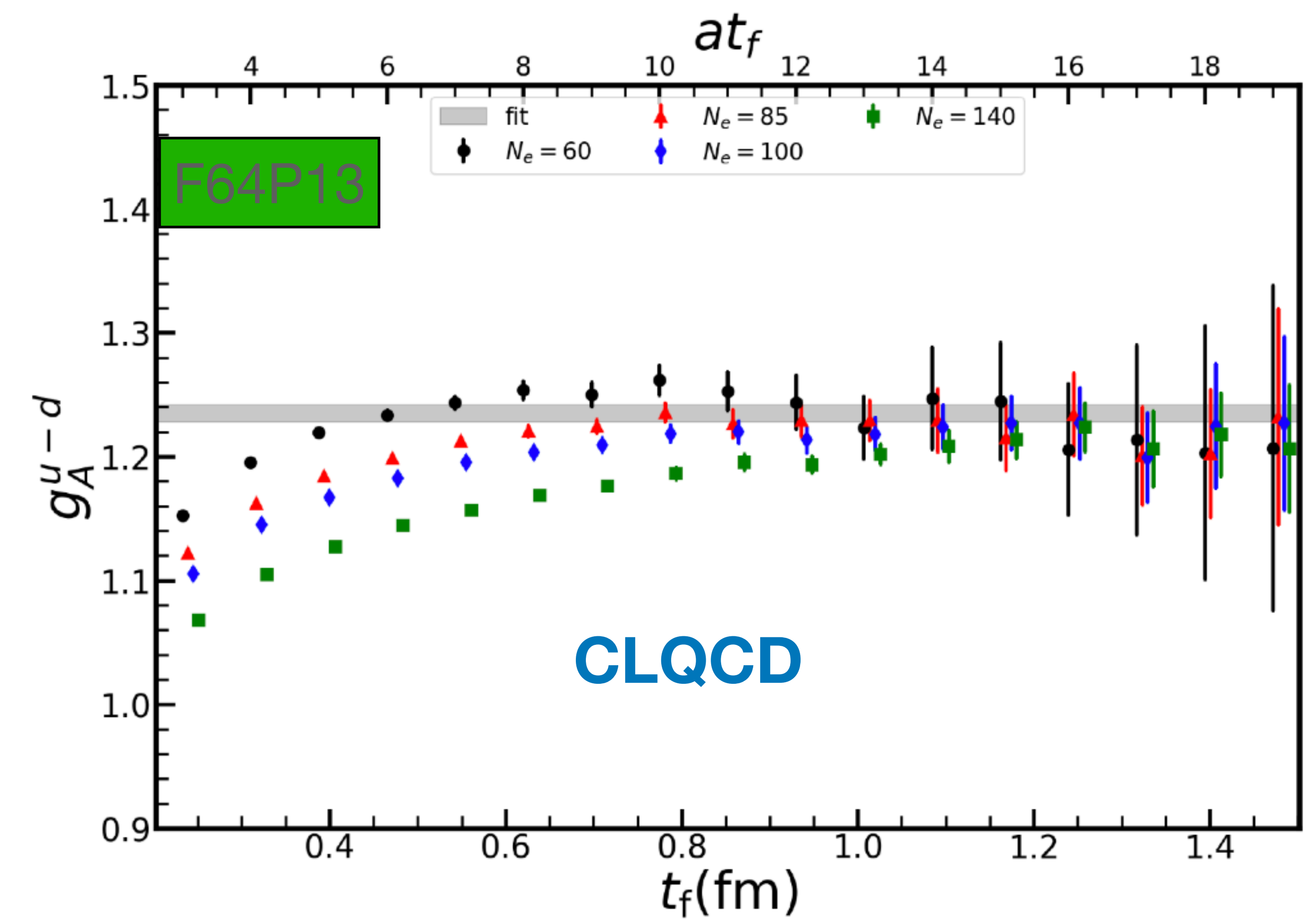
$$\mathcal{O}_{ij} \equiv \Omega_{ij}^{(2)} \int \mathrm{d}^3z\mathrm{d}^3w \langle \phi_i(z)|M_{\mathcal{O}}(z,w)|\phi_j(w)\rangle, \qquad M_{\mathcal{O}}(x,y) = \int \mathrm{d}^3z\mathrm{d}^3w \hat{I}_{x,z}\mathcal{O}(z,w)\hat{I}_{w,y} = \sum_{i=1}^{N_e+N_{st}} \sum_{j=1}^{N_e+N_{st}} |\phi_i(x)\rangle\mathcal{O}_{ij}\langle\phi_j(y)| + \mathcal{O}(\frac{1}{N_{st}}),$$

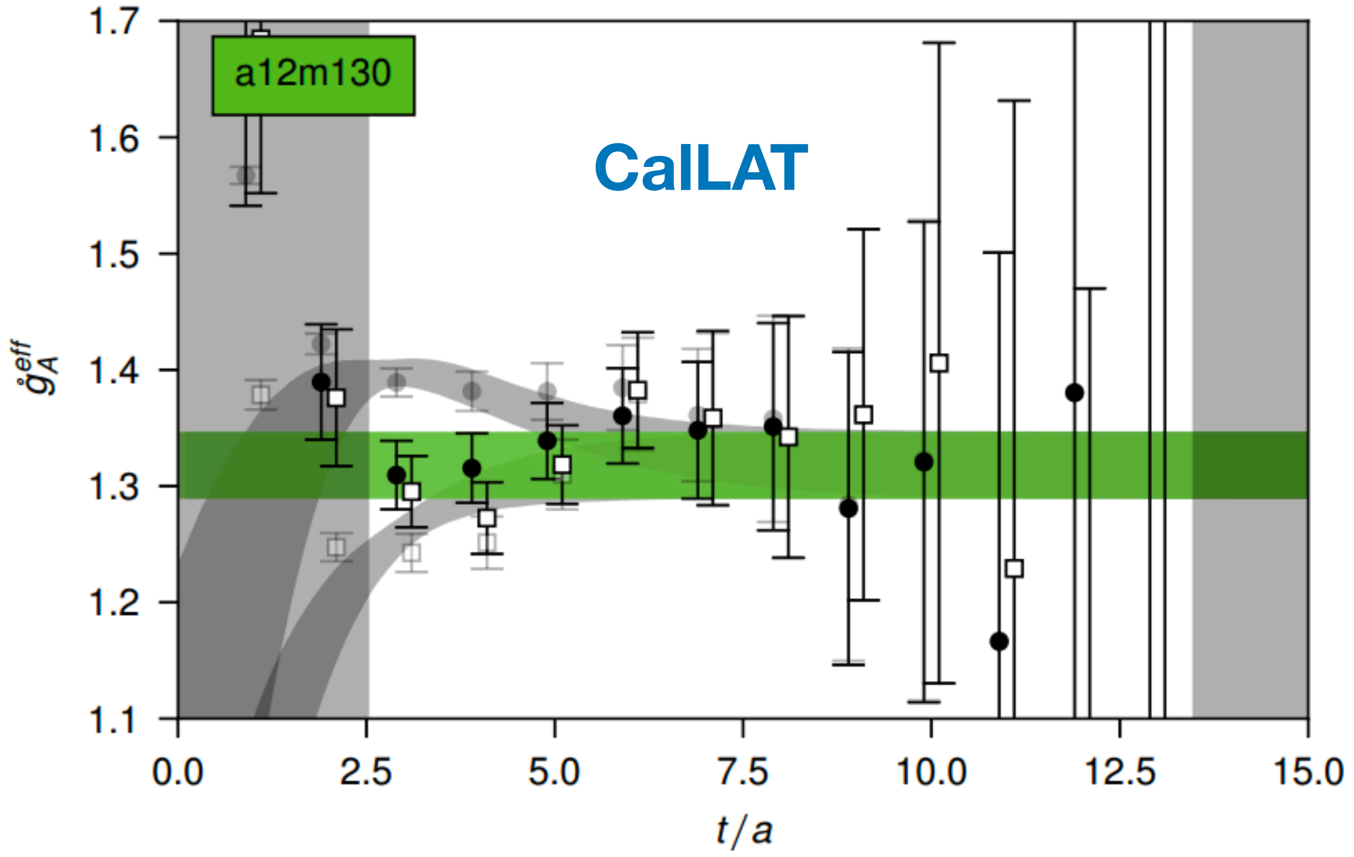
$$\Omega_{ij}^{(2)} = \begin{cases} 1 & \text{for } i,j \le N_e, \\ \Pi_{i=0}^{1} \frac{V-N_e-i}{N_{st}-i} & \text{for } i,j > N_e,\ i \ne j \\ \frac{V-N_e}{N_{st}} & \text{for the other cases,} \end{cases}$$

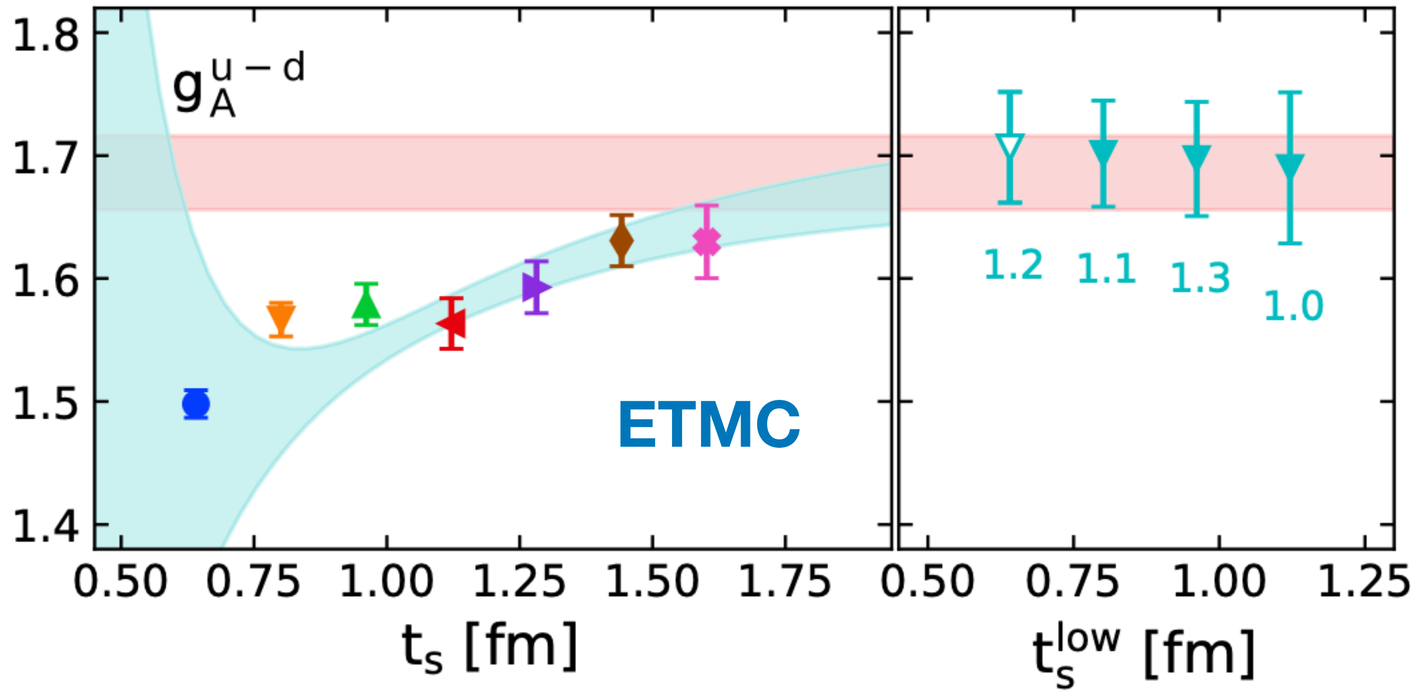
Z.C.Hu, J.H. Wang et. al., arXiv:2505.01719

Light and strange quarks

Cost comparison at physical pion mass







	Ensemble	L	T	a(fm)	m_π	n_{cfg}	g_A^{u-d}	Propagators	Propagators for 1% error
CalLAT	a12m130	48	64	0.121	131	1000	1.29(03)	0.03M	0.15M
CLQCD	F64P13	64	128	0.078	134	**40**	**1.24(01)**	**0.34M**	**0.11M**
ETMC	cB211.072.64	64	128	0.080	139	750	1.29(02)	1.71M	5.5M
RQCD	D452	64	128	0.076	156	1000	1.19(25)	0.01M	5.2M
PNDME	a09m130	64	96	0.090	138	1290	1.32(03)	1.69M	11.2M

C.C.Chang et. al., [CalLAT], Nature 558(2018)91

Z.C.Hu, et. al.,[CLQCD] in preparation

C.Alexandrou et. al., [ETMC], PRD102(2020)054517

G.S.Bali et. al., [RQCD], PRD108(2023)094512

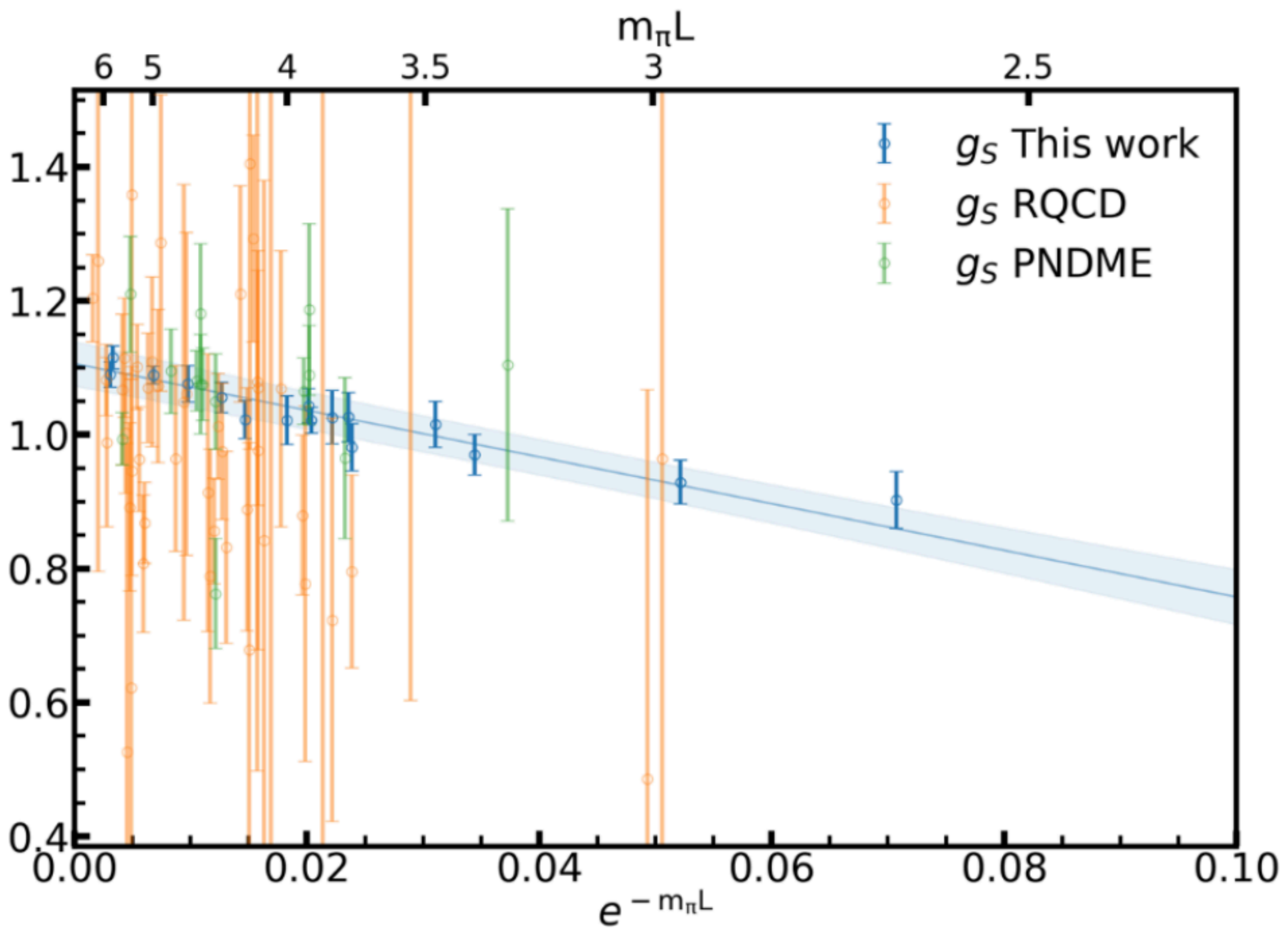
Y.C.Jang et. al., [PNDME], PRD109(2023)014503

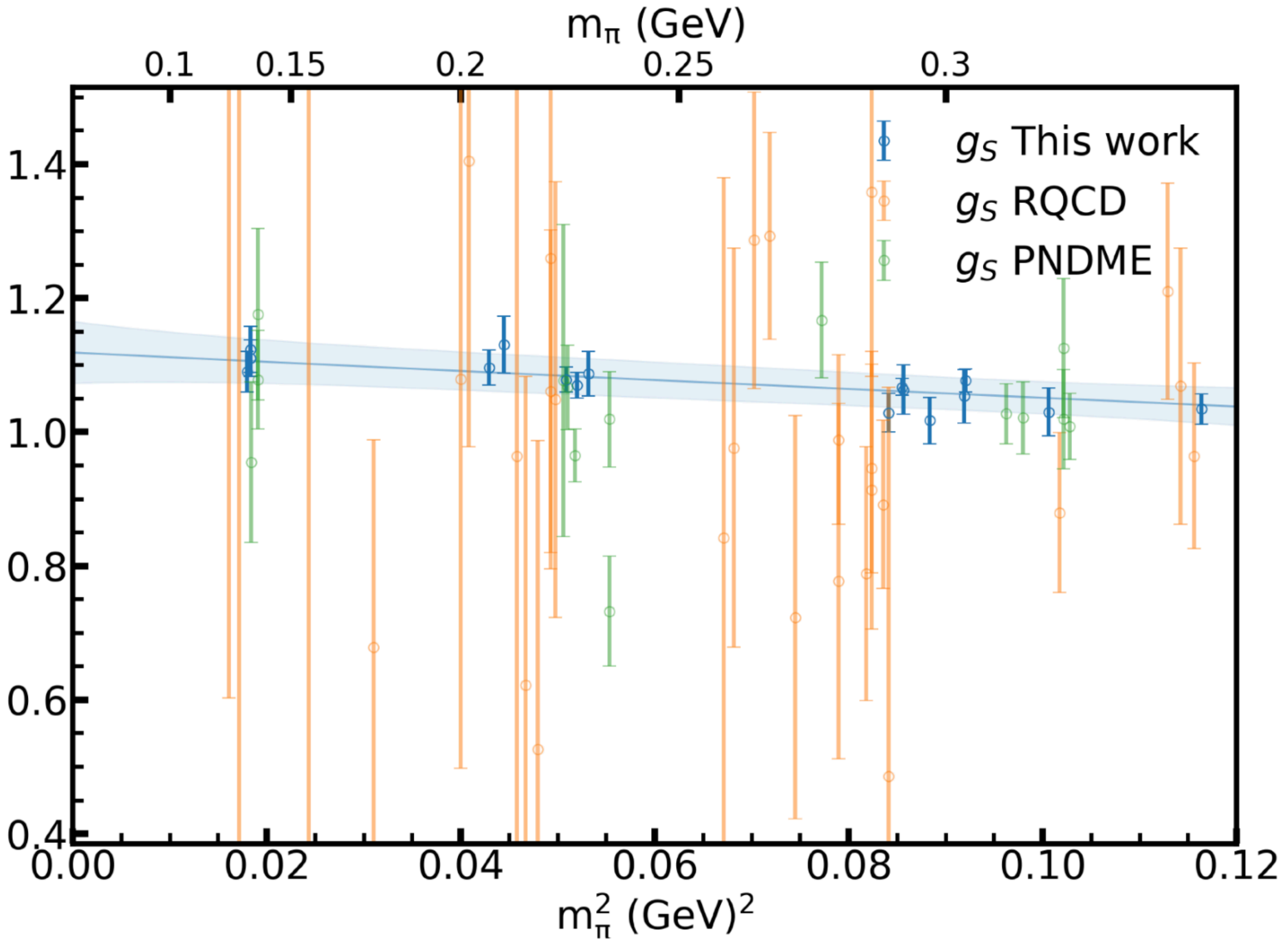
- Advantage becomes much more significant at the physical pion mass, except the CalLAT results which is only available at much larger lattice spacing;

- And also provide much more information on different source-current-sink separations and nucleon interpolation fields, which allow us to have much better control on the excited state contaminations.

Light and strange quarks

Control on systematic uncertainties





J.H. Wang et al. [CLQCD], arXiv:2511.02326

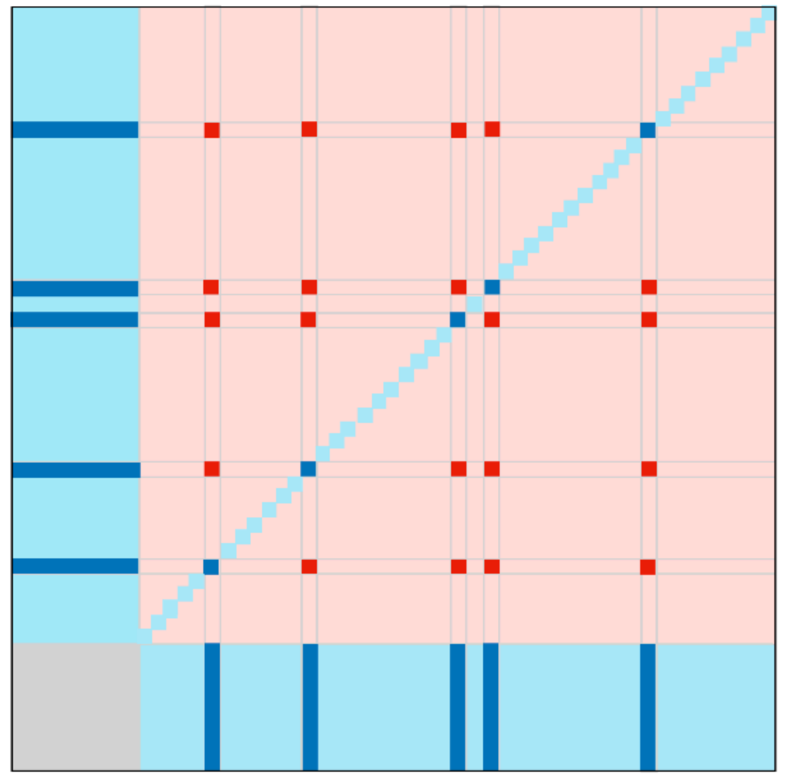
Ensemble	L	T	a(fm)	
C48P14/C64P14	48/64	96	0.105	135
C24P23/C32P23/C48P23	24/32/48	64/96	0.105	227
C24P29/C32P29/C48P29	24/32/48	64/72	0.105	290
F32P21/F48P21	32/48	96	0.077	210
F32P30/F48P30	32/48	64/96	0.077	300

- Best parameter combinations to constraint the finite volume effects at different lattice spacing and light quark masses;

- Much better signal at physical pion mass compared to all the previous results.

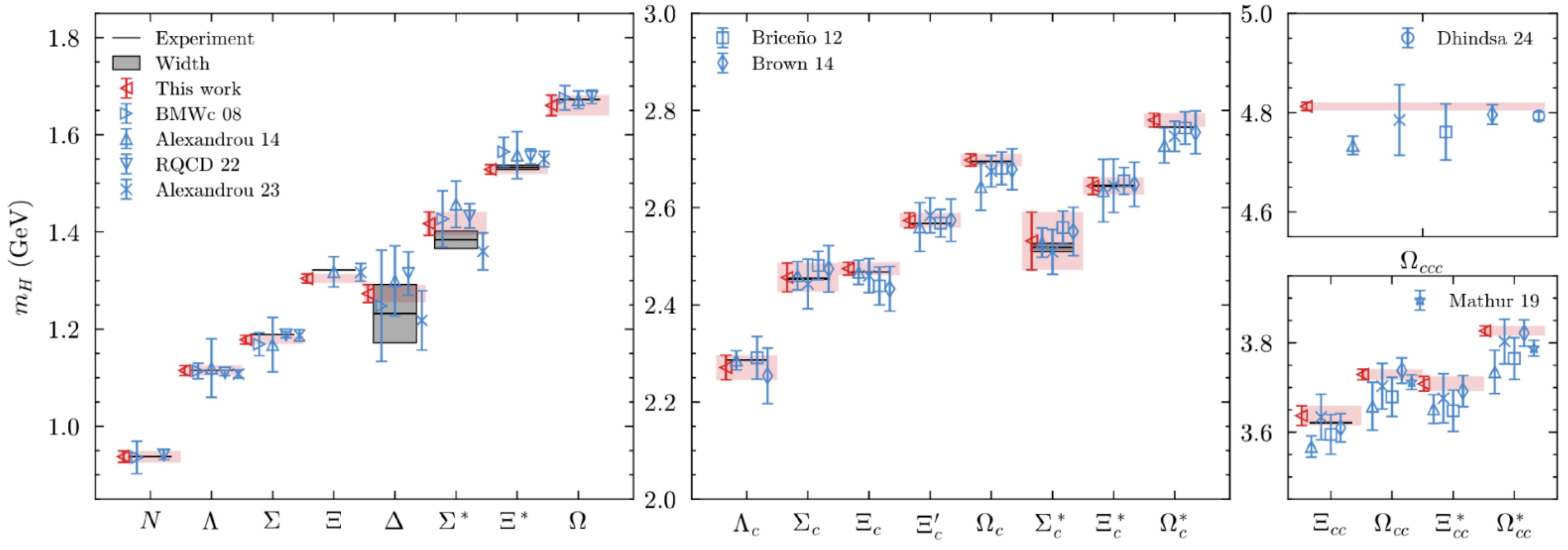
Outline

- Light and strange quark

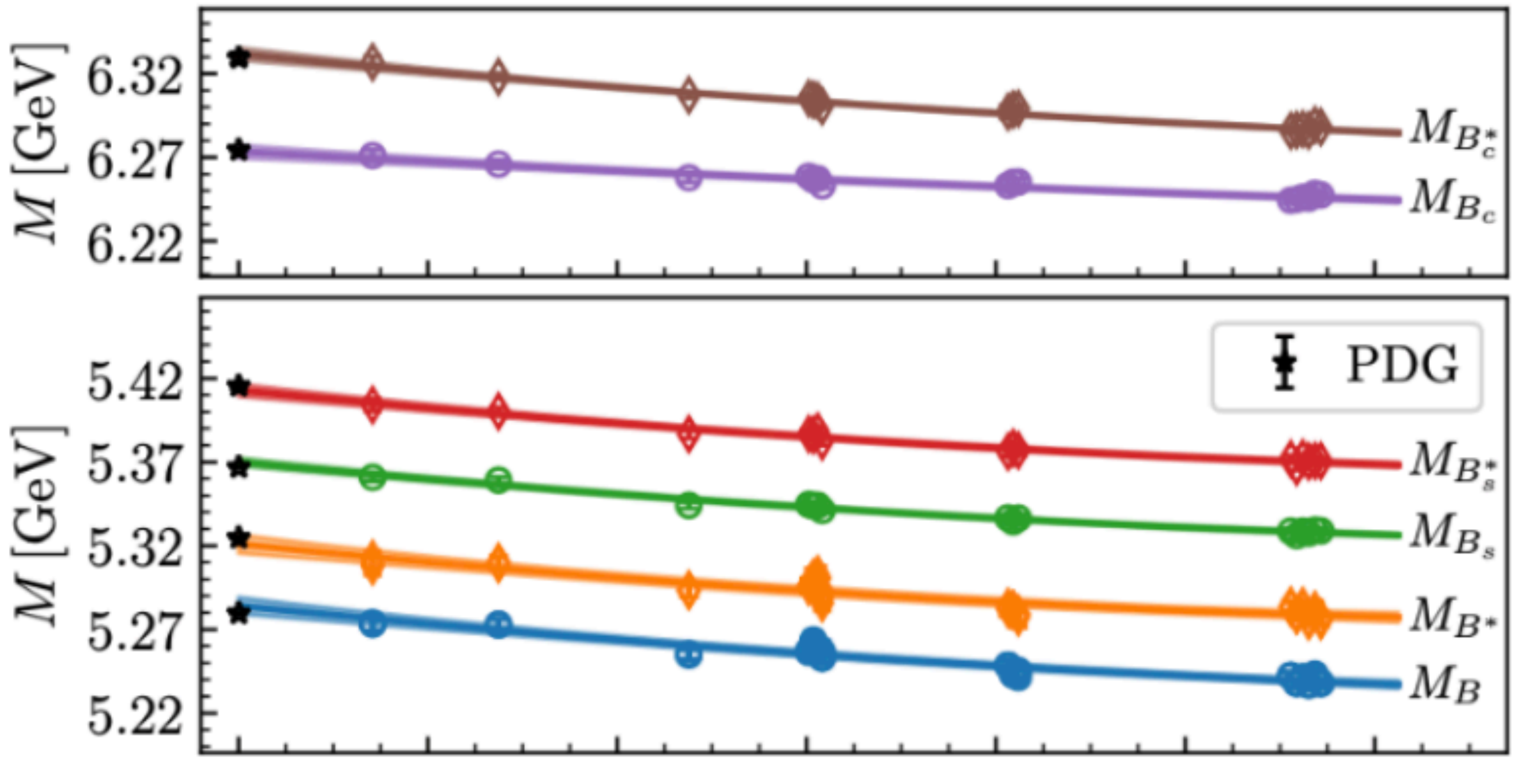


$$\Omega_{ij}^{(2)} = \begin{cases} 1 & \text{for } i,j \le N_e, \\ \Pi_{i=0}^{1} \frac{V-N_e-i}{N_{st}-i} & \text{for } i,j > N_e,\ i \ne j \\ \frac{V-N_e}{N_{st}} & \text{for the other cases,} \end{cases}$$

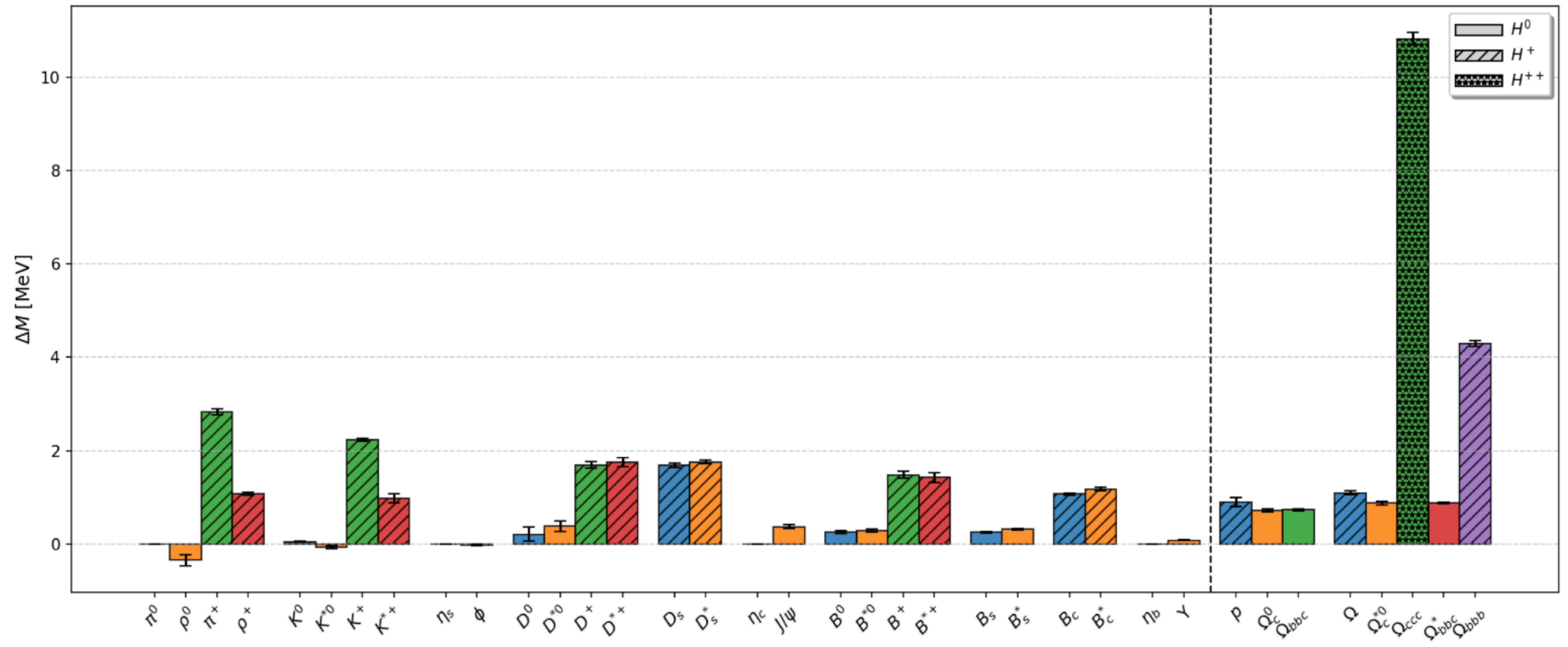
- **Charm quark**



- Bottom quark



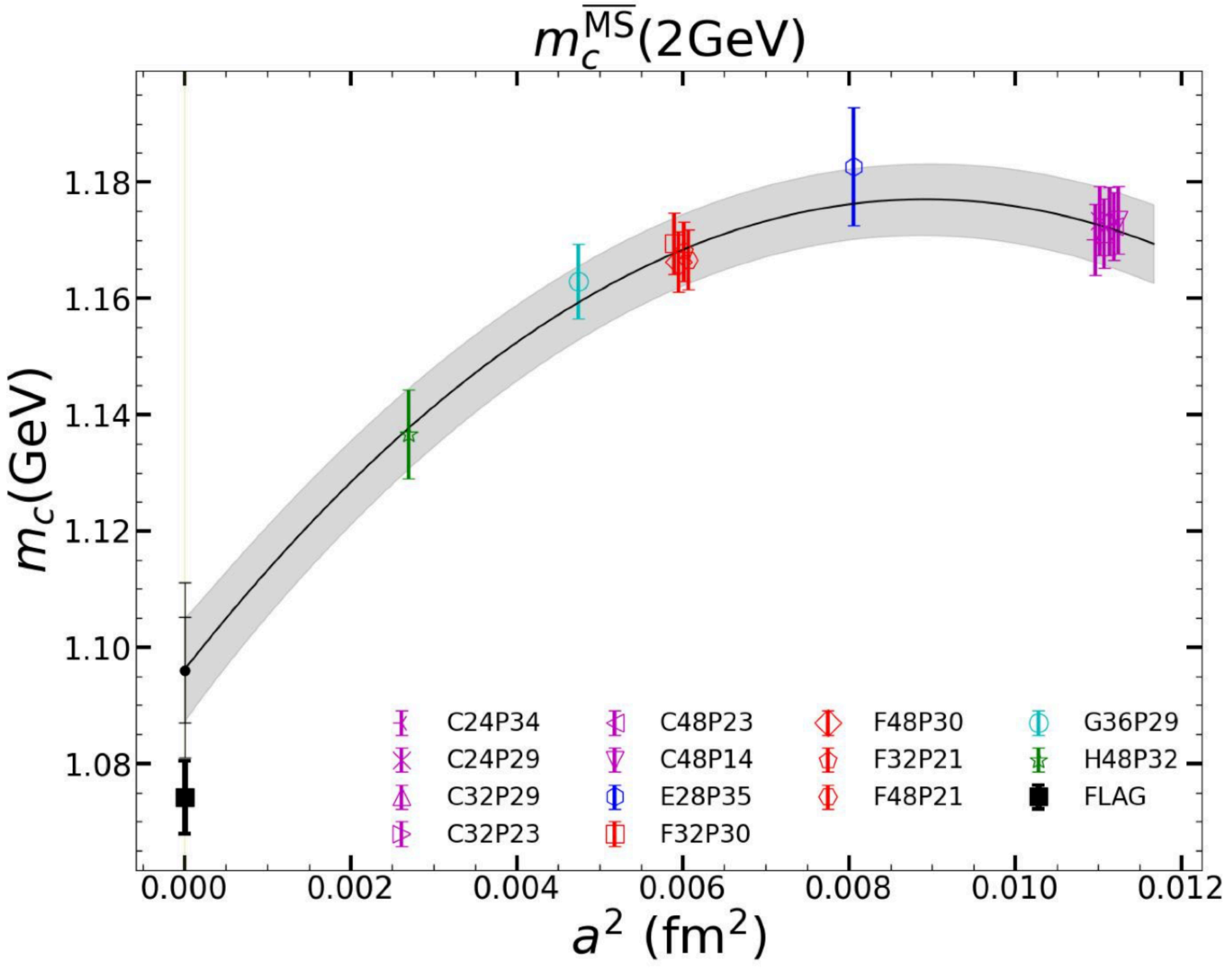
- QED corrections

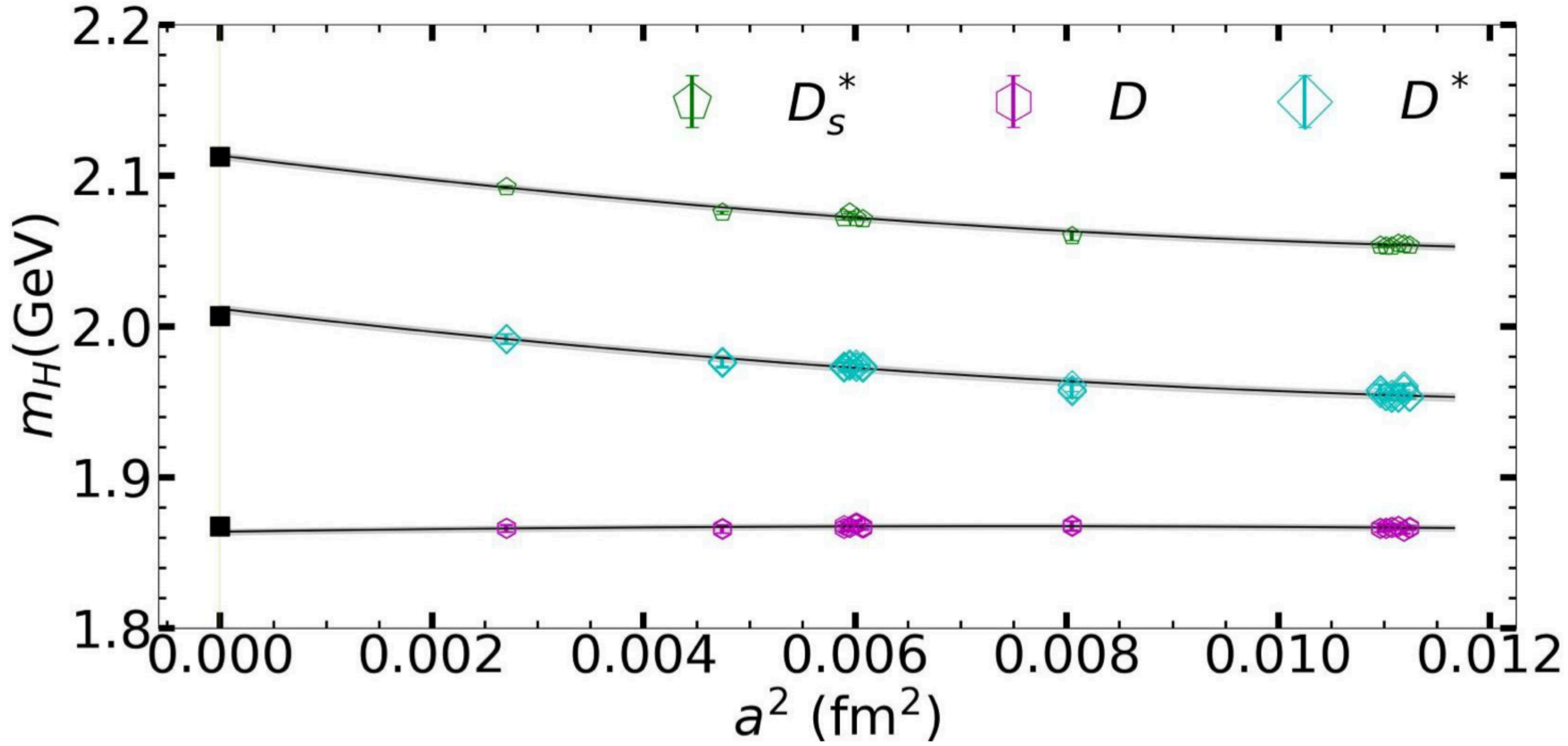


- Summary

Charm quark

Quark mass v.s
Hadron masses





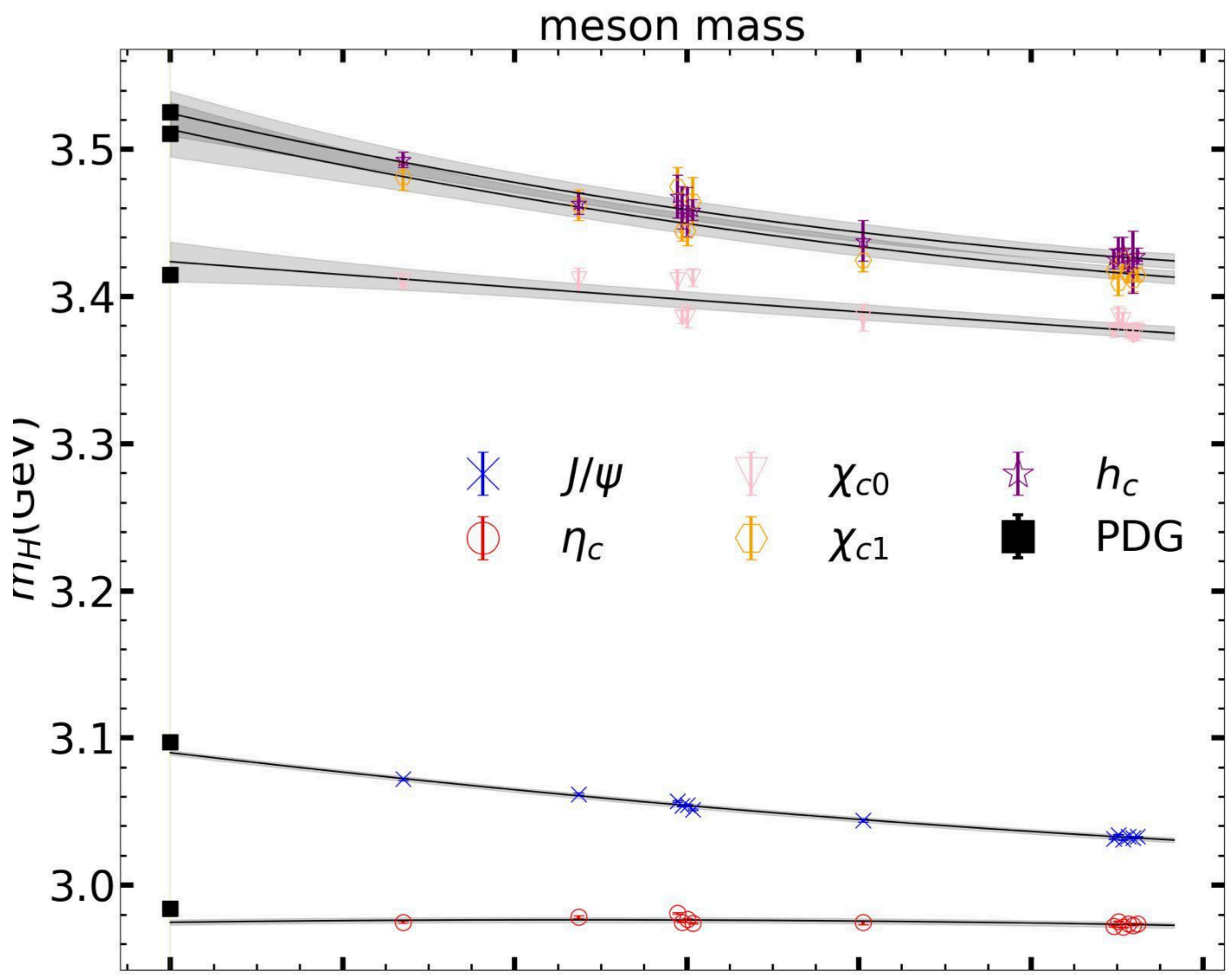
Hai-Yang Du, B.L. Hu, et. al., CLQCD, PRD111(2025) 054504

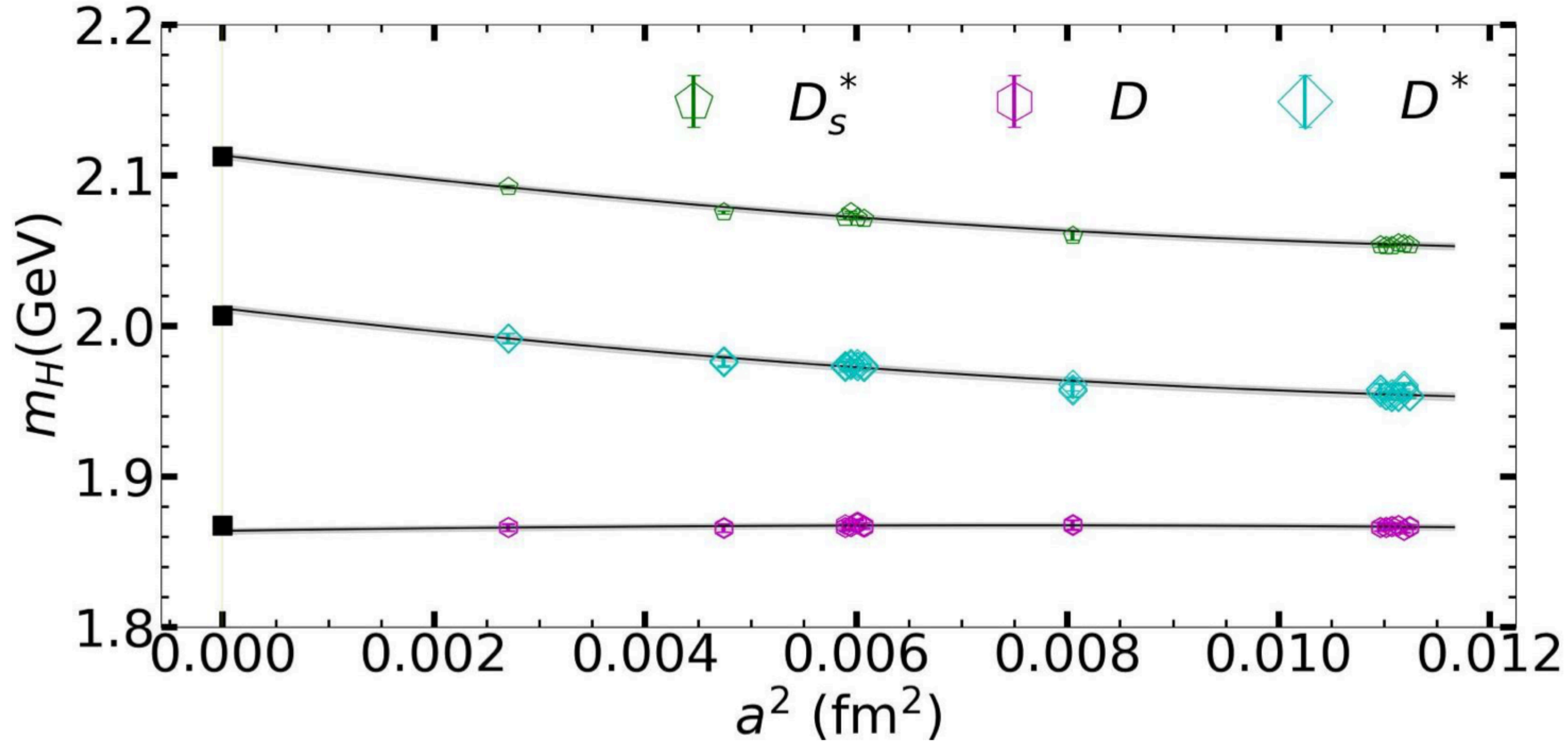
- Using $m_{D_s}^{\rm QCD} = m_{D_s}^{\rm phys} - \Delta^{\rm QED} m_{D_s} = 1.9667(15)$ GeV as input to determine the quark mass on each sample at each lattice spacing;
- Most of the $\mathcal{O}(m_c^2 a^2)$ errors in the hadron mass are absorbed into the definition of $m_c(a)$;
- The systematic uncertainty of the scale setting (~0.5%) is also majorly cancelled for the masses of other hadron containing charm quark, e.g., we predict $m_D = 1.864(2)(2)$ GeV.

Charm quark

- $m_D^{\pm} - m_D^0 = 2.9(2)_{\rm QCD} + 2.4(5)_{\rm QED} = 5.3(2)(5)$ MeV agrees with the PDG value 4.8(1) MeV well;

- $m_{J/\psi} - m_{\eta_c} = 116(3)$ MeV agree with previous HPQCD pure QCD prediction 119(1) MeV.

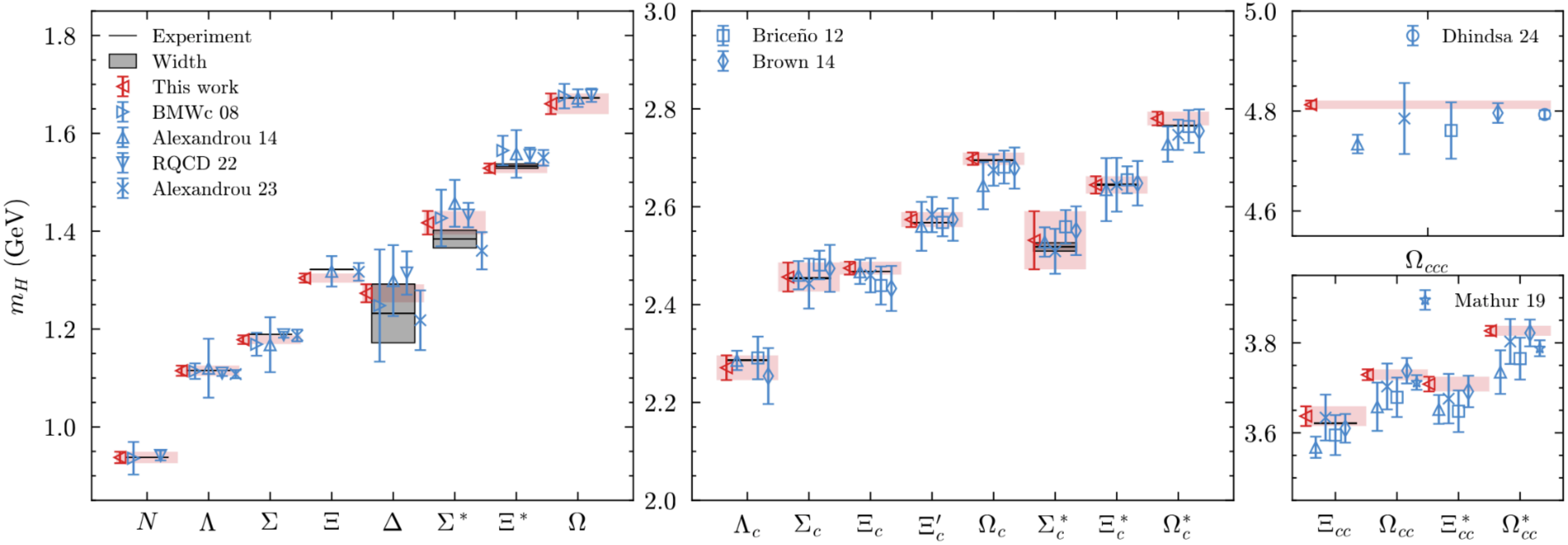




- P-wave charmonium masses also agree with PDG well, as $m_{1P} - m_{1S} = 461(19)$ MeV.

Hai-Yang Du, B.L. Hu, et. al., CLQCD, PRD111(2025) 054504

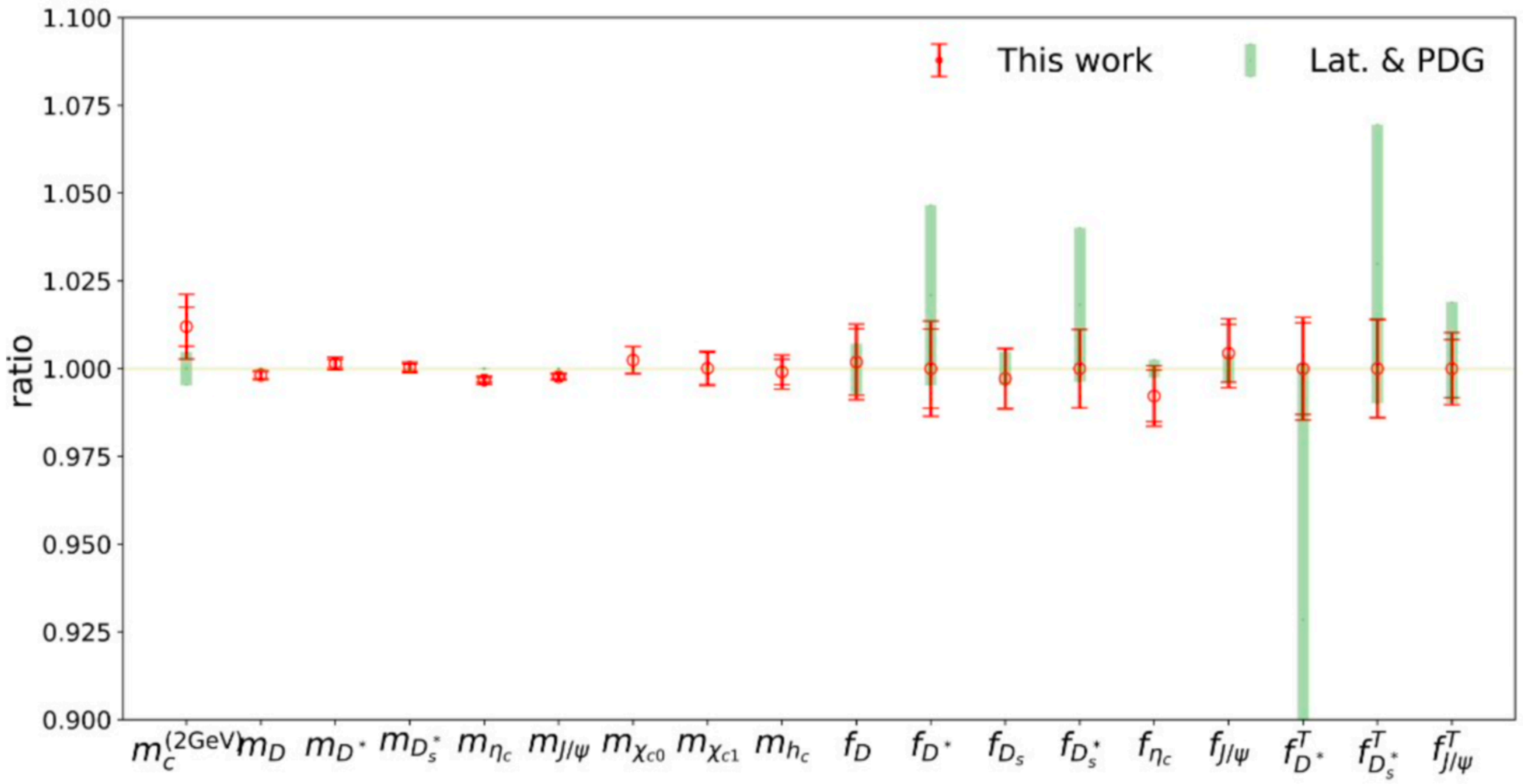
Charm quark



B.-L. Hu, et. al., CLQCD, arXiv: 2411.18402

- Predict the ground state $\frac{1}{2}^+$ and $\frac{3}{2}^+$ masses at 1% level, while QED effects are missing;
- Agree with the previous results well with better precision in most of the cases, and more systematic.

Charm quark



Hai-Yang Du, B.L. Hu, et. al., CLQCD, PRD111(2025) 054504

$f_{D^+} = 0.2113(33)_{\rm lat}$ MeV

$f_{D^+}|V_{cd}| = 45.8(1.1)_{\rm exp}$ MeV $\longrightarrow |V_{cd}| = 0.2168(33)_{\rm lat}(52)_{\rm exp}$

$f_{D_s^+}|V_{cs}| = 243.5(2.7)_{\rm exp}$ MeV $\longrightarrow |V_{cs}| = 0.975(13)_{\rm lat}(11)_{\rm exp}$

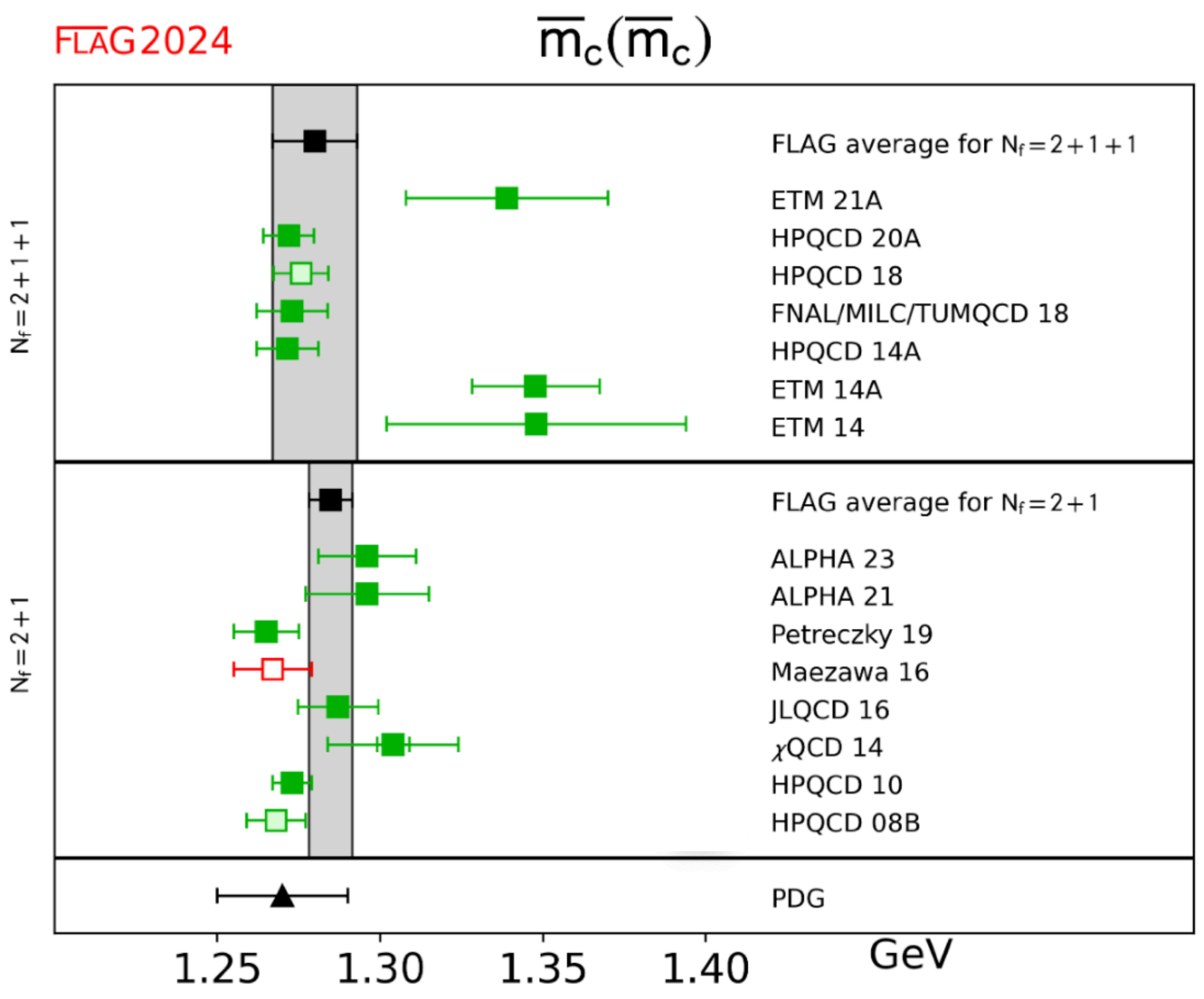
$f_{D_s^+} = 0.2498(33)_{\rm lat}$ MeV

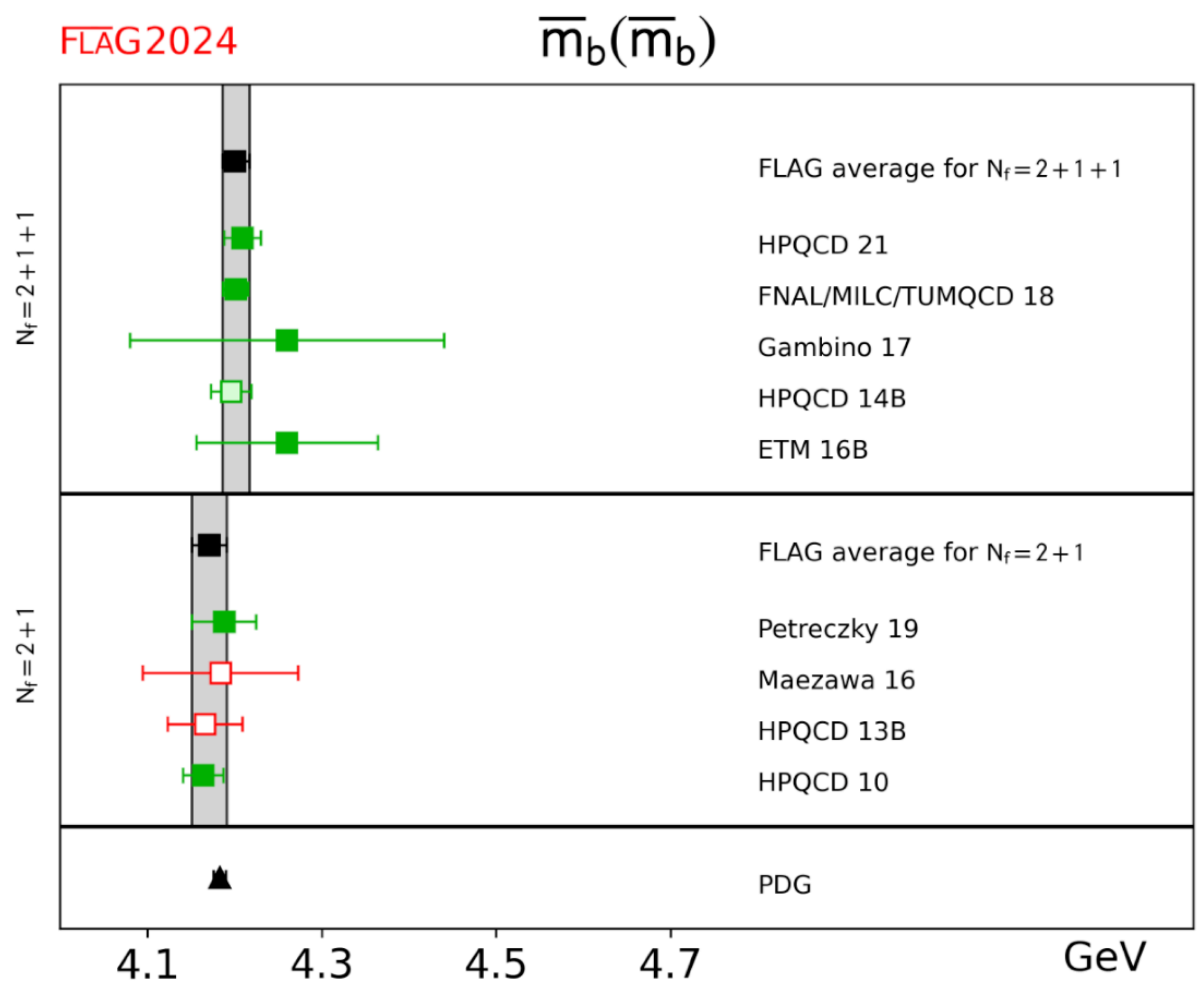
- Verified the unitarity of CKM matrix elements involving the charm quark:

$$|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 0.999(25)(22).$$

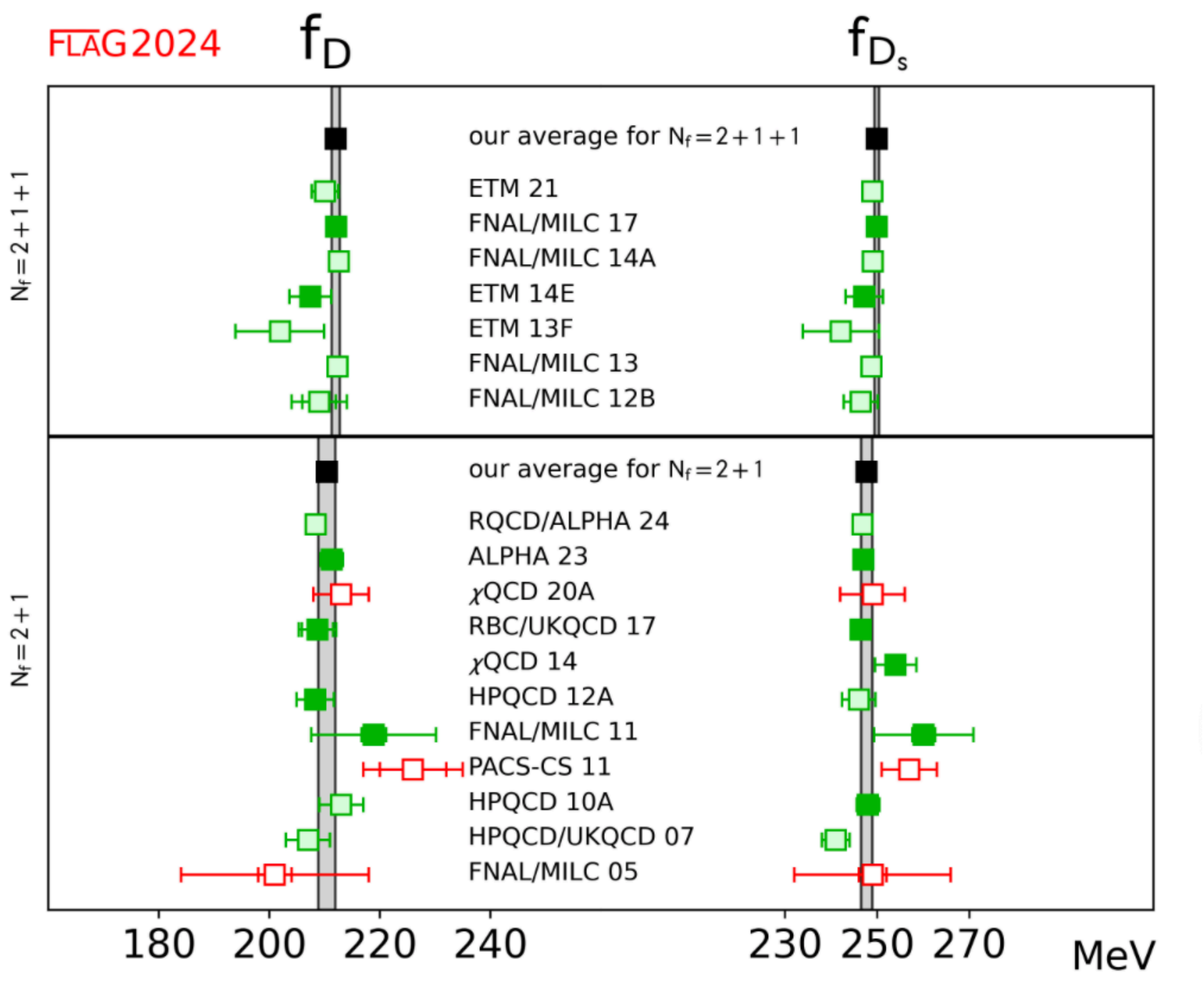
- Also provide the most precise f_{D^*} and $f_{D_s^*}$ so far.

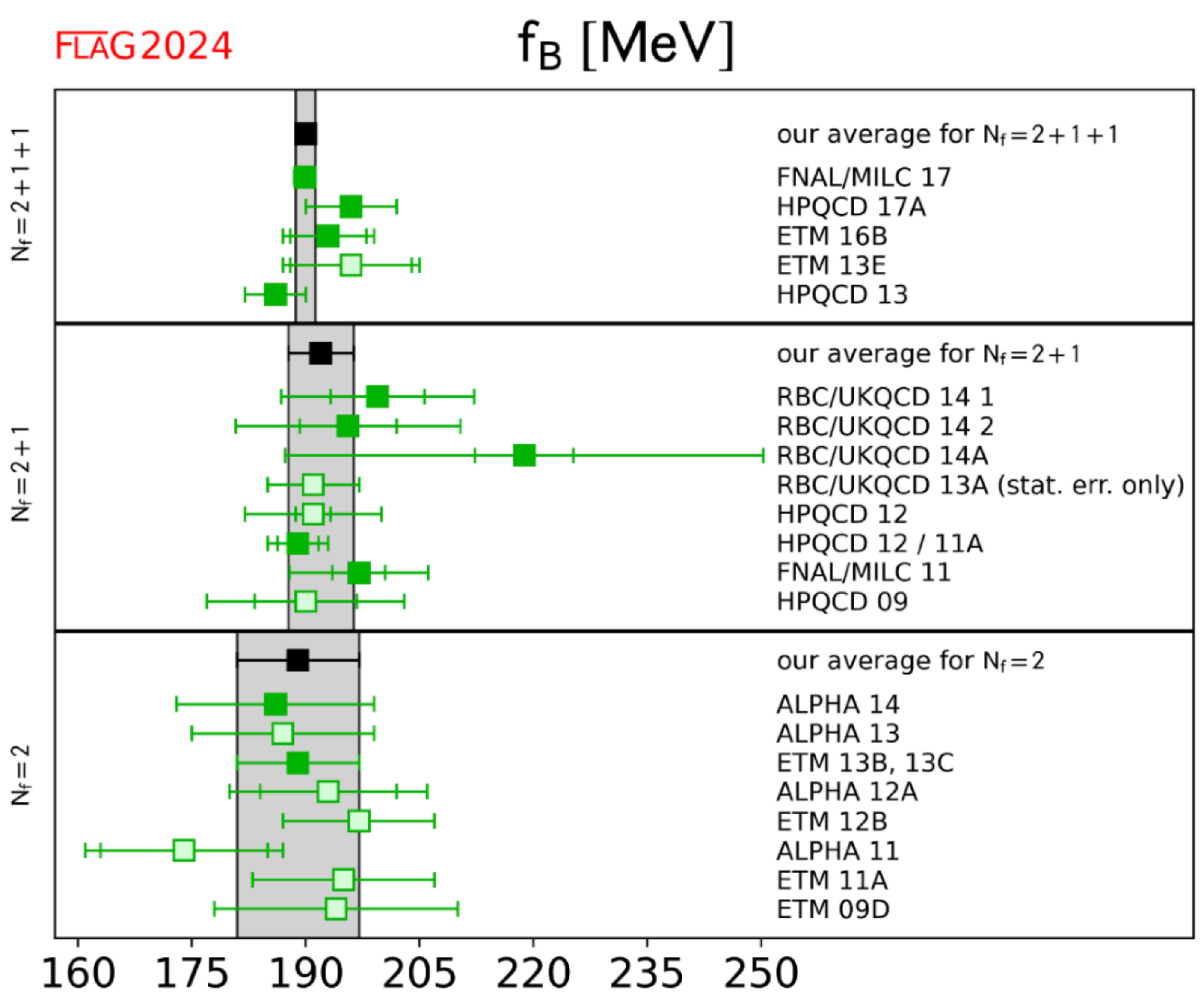
Charm sea effects

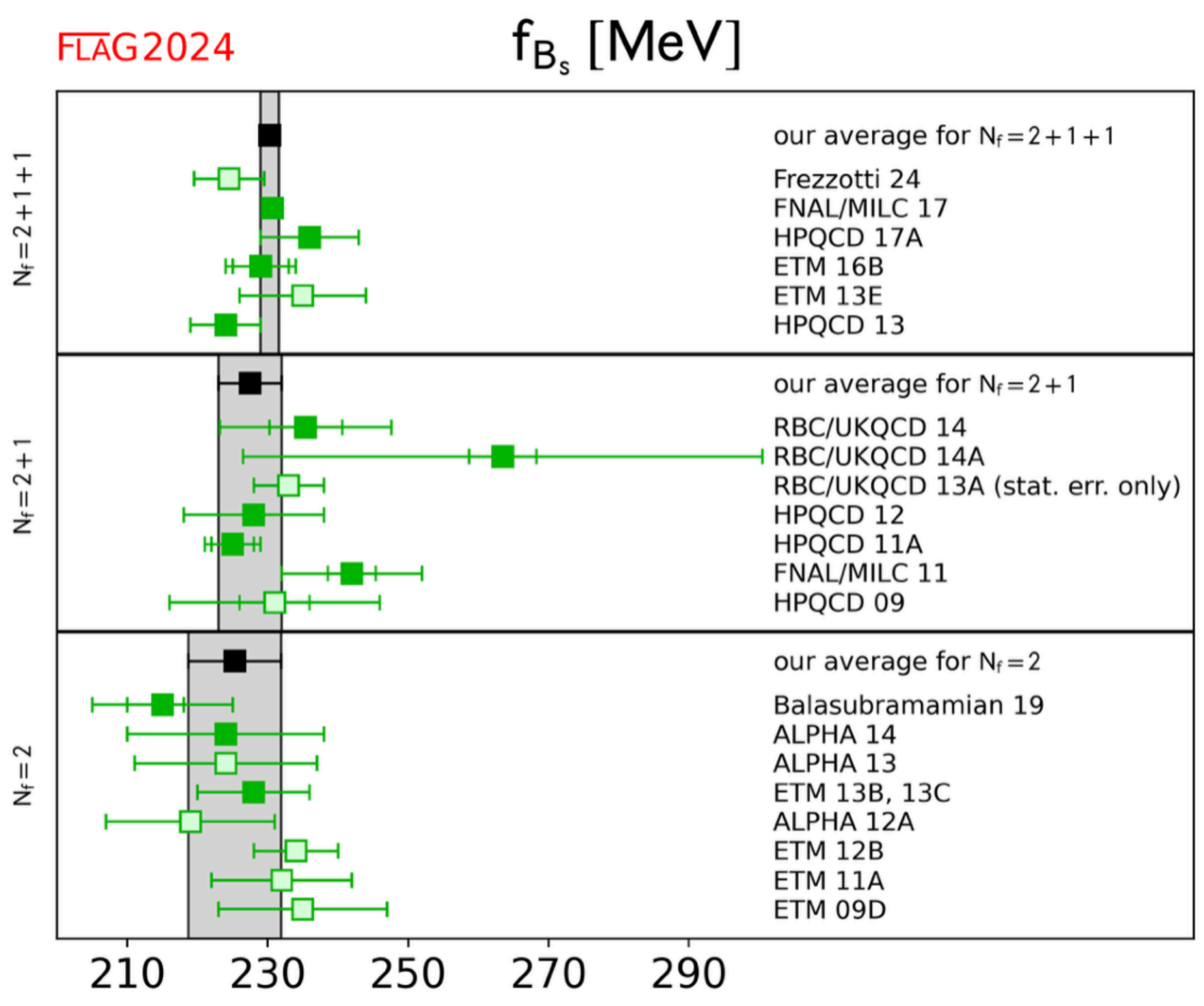




- Impact from the charm sea is unlikely to be large, but shall be a systematic uncertainty to be controlled.
- But adding the charm sea to the existed CLQCD ensemble with clover fermion is highly non-trivial.







Fermion actions

- The propagator of the naive fermion has $1/m$ IR poles at $pa = (0/\pi, 0/\pi, 0/\pi, 0/\pi)$, which is different from the continuum theory.

- Staggered fermion redefined $\psi^{\rm st}(x) = \gamma_4^{x_4}\gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\psi(x)$:

$$\frac{1+\gamma_5}{2}\psi^{\rm st}(x) = \psi^{\rm st}((\sum_\mu x_\mu)\,\%\,2 \equiv 0)$$

$$\frac{1-\gamma_5}{2}\psi^{\rm st}(x) = \psi^{\rm st}((\sum_\mu x_\mu)\,\%\,2 \equiv 1)$$

- $\bar\psi(x)\psi(x) = \bar\psi^{\rm st}(x)\psi^{\rm st}(x)$; $\bar\psi(x)\gamma_5\psi(x) = (-1)^{\sum_\mu x_\mu}\bar\psi^{\rm st}(x)\psi^{\rm st}(x)$
- $\bar\psi(x)\gamma_\mu\psi(x \pm \hat\mu a) = \bar\psi^{\rm st}(x)\gamma_\mu^{\rm st}\psi^{\rm st}(x \pm \hat\mu a)$, with $\{\gamma_1^{\rm st}, \gamma_2^{\rm st}, \gamma_3^{\rm st}, \gamma_4^{\rm st}\} = \{(-1)^{x_4}, (-1)^{x_1+x_4}, (-1)^{x_1+x_2+x_4}, 1\}$;
- 16 IR poles → 4 IR poles.
- Mixing between IR poles can be suppressed with kinds of the improvement, likes the so-call highly-improved staggered quark (HISQ).

Parameter tuning

Staggered v.s. Clover

	C24P34	C24P29	C32P29	C32P23
$\tilde{L}^3 \times \tilde{T}$	$24^3 \times 64$	$24^3 \times 72$	$32^3 \times 64$	$32^3 \times 64$
$\hat{\beta}$			6.20	
a (fm)			0.10530(18)	
$\tilde{m}_l^b$	-0.2770	-0.2770	-0.2770	-0.2790
$\tilde{m}_s^b$	-0.2310	-0.2400	-0.2400	-0.2400
m_l^R (MeV)	22.90(19)	16.94(12)	17.35(11)	10.55(11)
m_s^R (MeV)	111.41(16)	87.46(10)	88.16(10)	84.48(07)

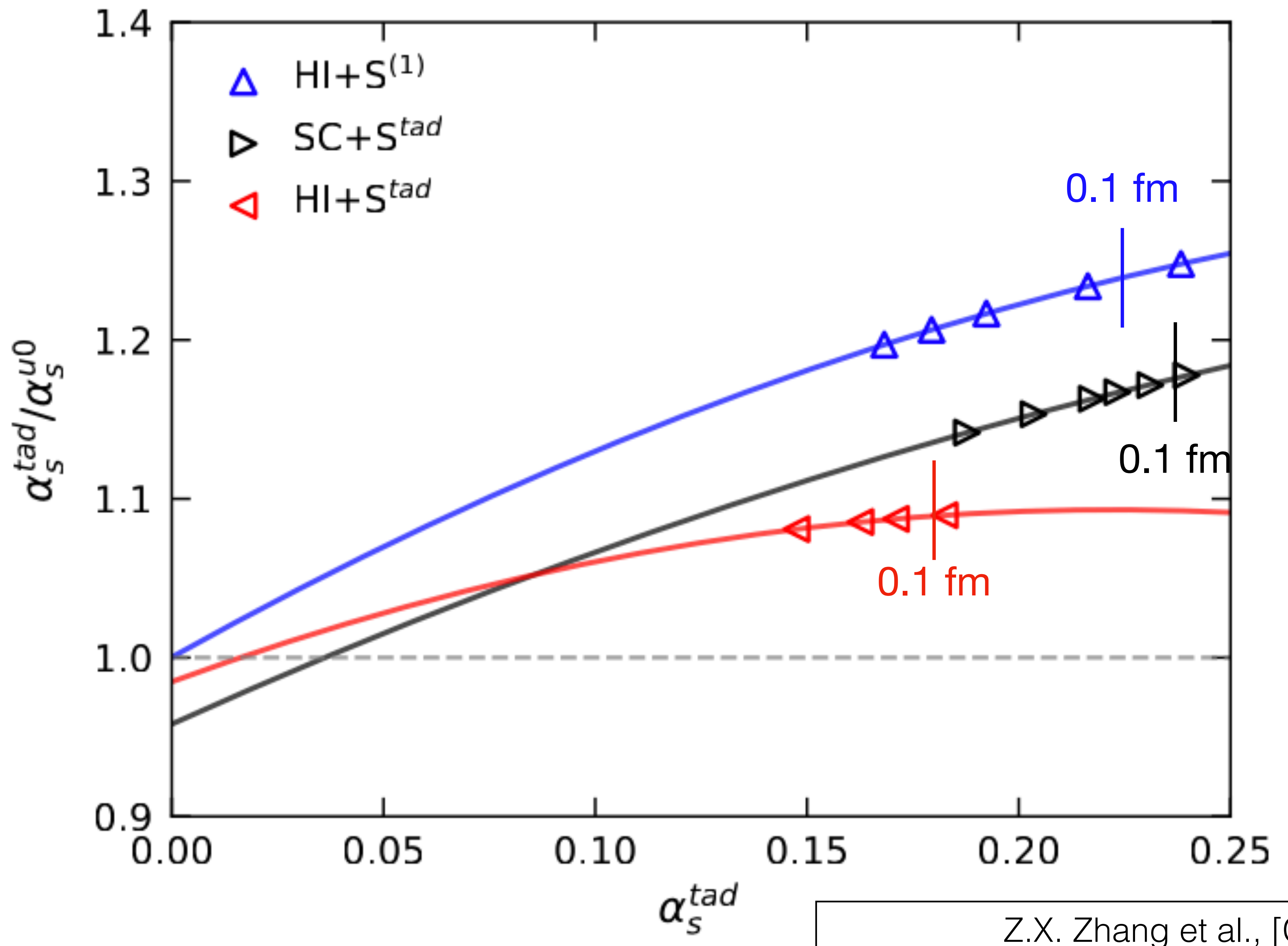
Ensemble	$\hat{\beta}$	$\tilde{L}^3 \times \tilde{T}$	u_0	$\tilde{m}_l^{\mathrm{b}}$	$\tilde{m}_s^{\mathrm{b}}$	$\tilde{m}_c^{\mathrm{b}}$
c24P31s	7.29	$24^3 \times 48$	0.879440(3)	0.00944	0.055	0.5555
c24P31	7.29	$24^3 \times 48$	0.879452(3)	0.00944	0.04721	0.5555
c32P31	7.29	$32^3 \times 48$	0.879451(2)	0.00944	0.04721	0.5555
c24P22	7.29	$24^3 \times 48$	0.879469(3)	0.00472	0.04721	0.5555
c32P22	7.29	$32^3 \times 48$	0.879468(2)	0.00472	0.04721	0.5555
c48P13	7.29	$48^3 \times 48$	0.879472(1)	0.00174	0.04721	0.5555
e32P31	7.54	$32^3 \times 64$	0.886360(2)	0.007434	0.03715	0.4371
g32P31	7.75	$32^3 \times 64$	0.891434(1)	0.00579	0.02895	0.34
g48P31	7.75	$48^3 \times 64$	0.891432(1)	0.00579	0.02895	0.34
h48P31	8.20	$48^3 \times 96$	0.900600(1)	0.003526	0.01763	0.207

Z.X. Zhang et al., [CLQCD], arXiv:2512.19265, accepted by PRD

- The critical quark mass of the HISQ fermion is exactly zero.
- Thus we can require $m_s/m_l = \{5,10,27\}$ and tune m_s with $m_{\eta_s} = 690$ MeV to obtain the light quark masses which correspond to $m_\pi = \{310,220,135\}$ MeV;
- And obtain the physical charm quark mass by requiring $m_c/m_s \sim 12$.
- Thus we just need to tune $\hat{\beta} \propto 10/g^2$, $\tilde{m}_s^b \equiv m_s^b a$ for $m_{\eta_s} = 690$ MeV and given lattice spacing a, plus the self consistent u_0.
- For the clover ensembles, we need to tune $\hat{\beta} \propto 10/g^2$, $\tilde{m}^b_{l,s,c}$ separately, plus the self consistent u_0 and v_0.

Bare coupling

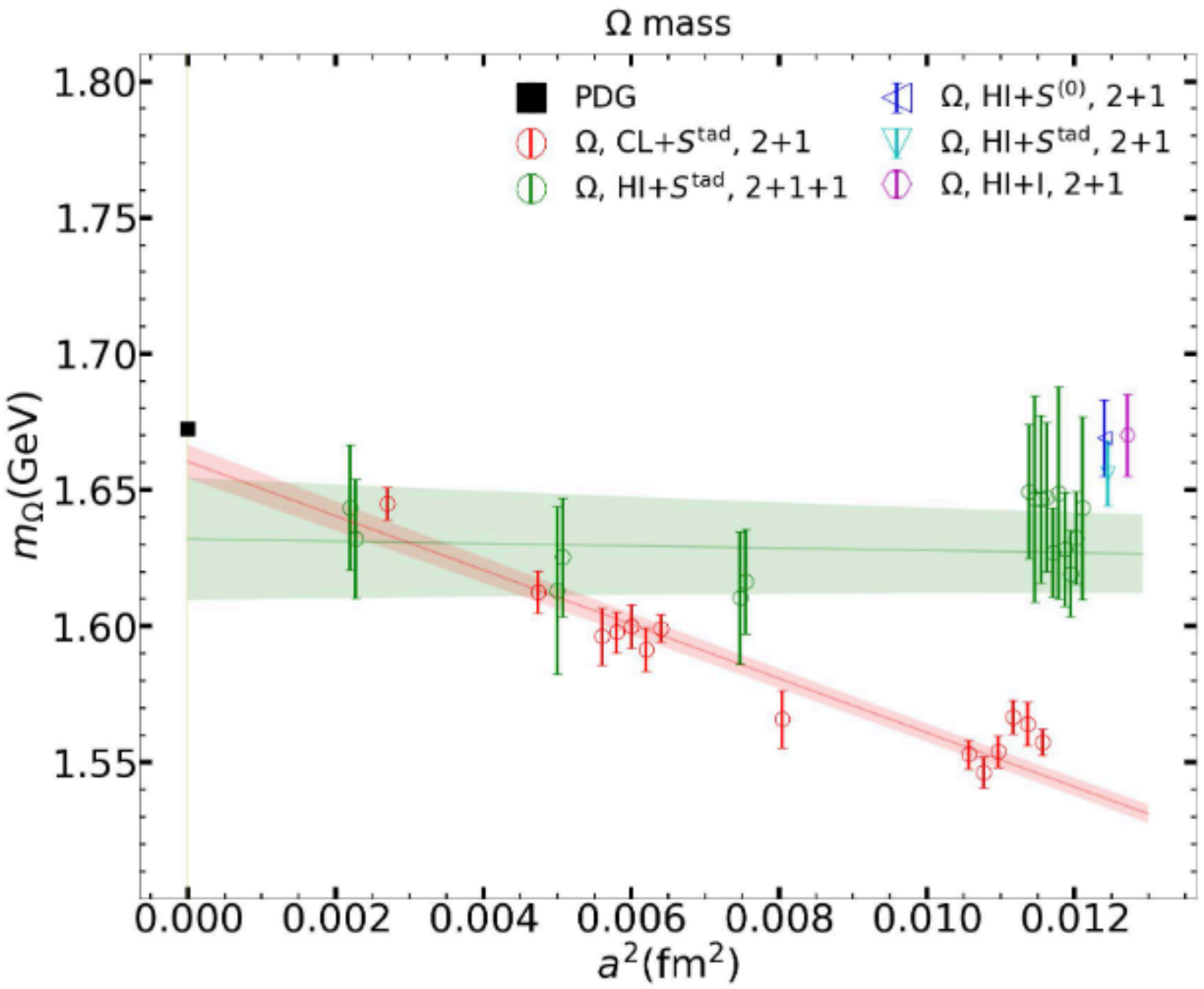
Action	N_f	$\hat{\beta}$	a (fm)	u_0	$v_0^{(\mathrm{HYP})}$	$\tilde{m}_l^{b(\mathrm{HYP})}$
HI+S$^{\mathrm{tad}}$	2+1+1	7.29	0.1084(04)	0.87944	0.9862	-0.0529
		7.54	0.0867(03)	0.88636	0.9878	-0.0438
		7.75	0.0710(03)	0.89143	0.9889	-0.0397
		8.20	0.0473(03)	0.90060	0.9905	-0.0329
HI+S$^{(0)}$	2+1	6.74	0.1114(06)	0.87296	0.9852	-0.0578
HI+S$^{\mathrm{tad}}$	2+1	7.21	0.1116(06)	0.87727	0.9858	-0.0558
HI+I	2+1	8.32	0.1128(07)	0.88668	0.9870	-0.0485
SC+S$^{\mathrm{tad}}$	2+1	6.20	0.1052(06)	0.85545	0.9830	-0.0328
		6.41	0.0775(05)	0.86346	0.9851	-0.0208
		6.72	0.0520(03)	0.87338	0.9871	-0.0135
HI+S$^{(1)}$	2+1+1	6.00	0.1222(03)	0.86373	0.9836	-0.0708
		6.30	0.0879(02)	0.87417	0.9863	-0.0514
		6.72	0.0566(01)	0.88578	0.9887	-0.0398
		7.00	0.0426(01)	0.89218	0.9897	-0.0365

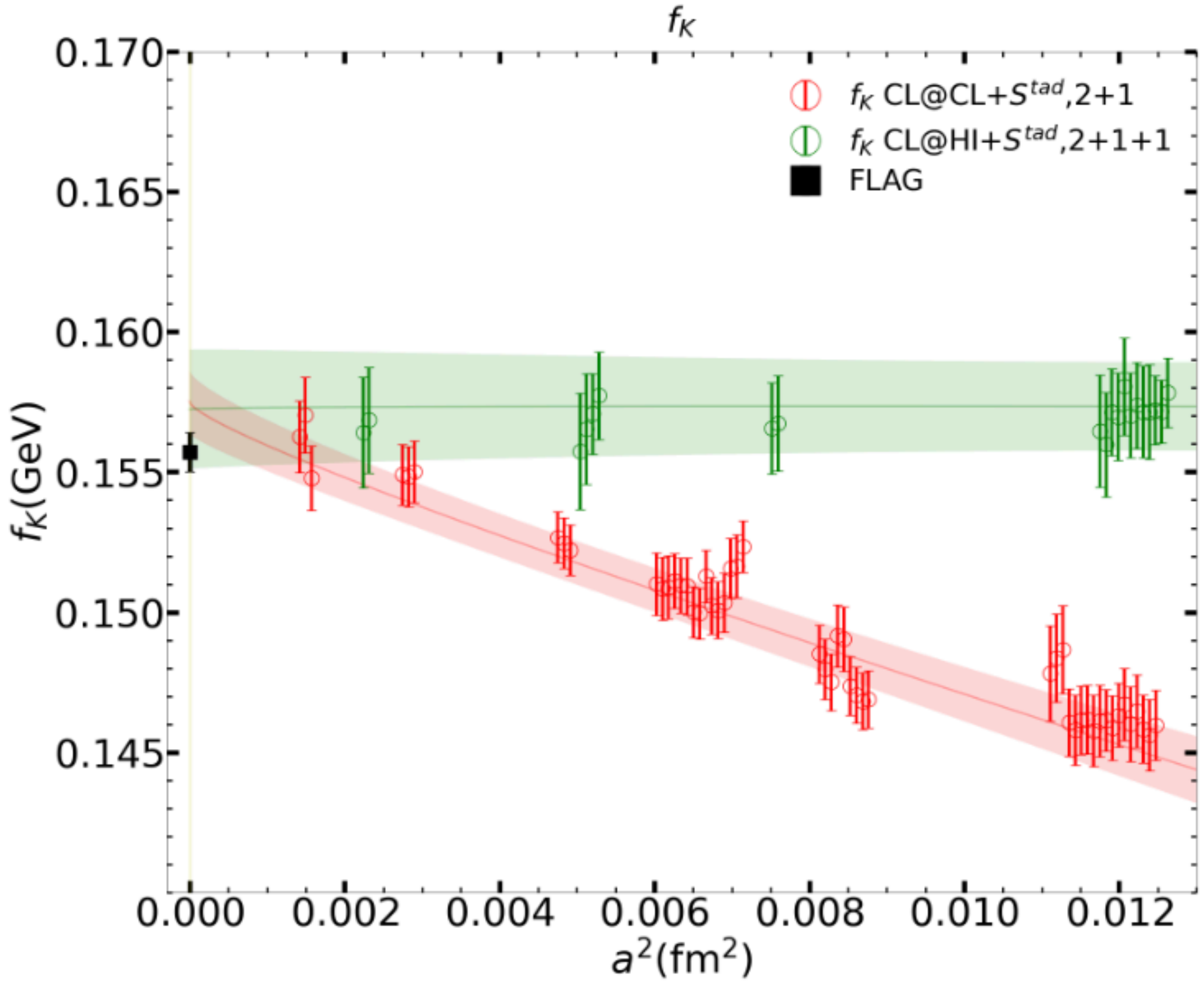


Z.X. Zhang et al., [CLQCD], arXiv:2512.19265, accepted by PRD

- The bare coupling $\hat{\beta} = (1 - 8c_1)6/g^2$ can be very different with various fermion and gauge discretizations;
- Tadpole improved effective coupling from tadpole improvement $\alpha_s^{\mathrm{tad}} \equiv \dfrac{6(1 - 8c_1 u_0^2)}{4\pi\hat{\beta}u_0^4}$ is somehow closer but still different.
- Effective coupling from 1-loop approximation of tadpole factor, $\alpha_s^{u_0} \equiv -\dfrac{4}{3.06839}\log u_0$ for the Symanzik gauge action, should be consistent with α_s^{tad} in the weak coupling (continuum) limit, but different at finite lattice spacing.

Charm quark



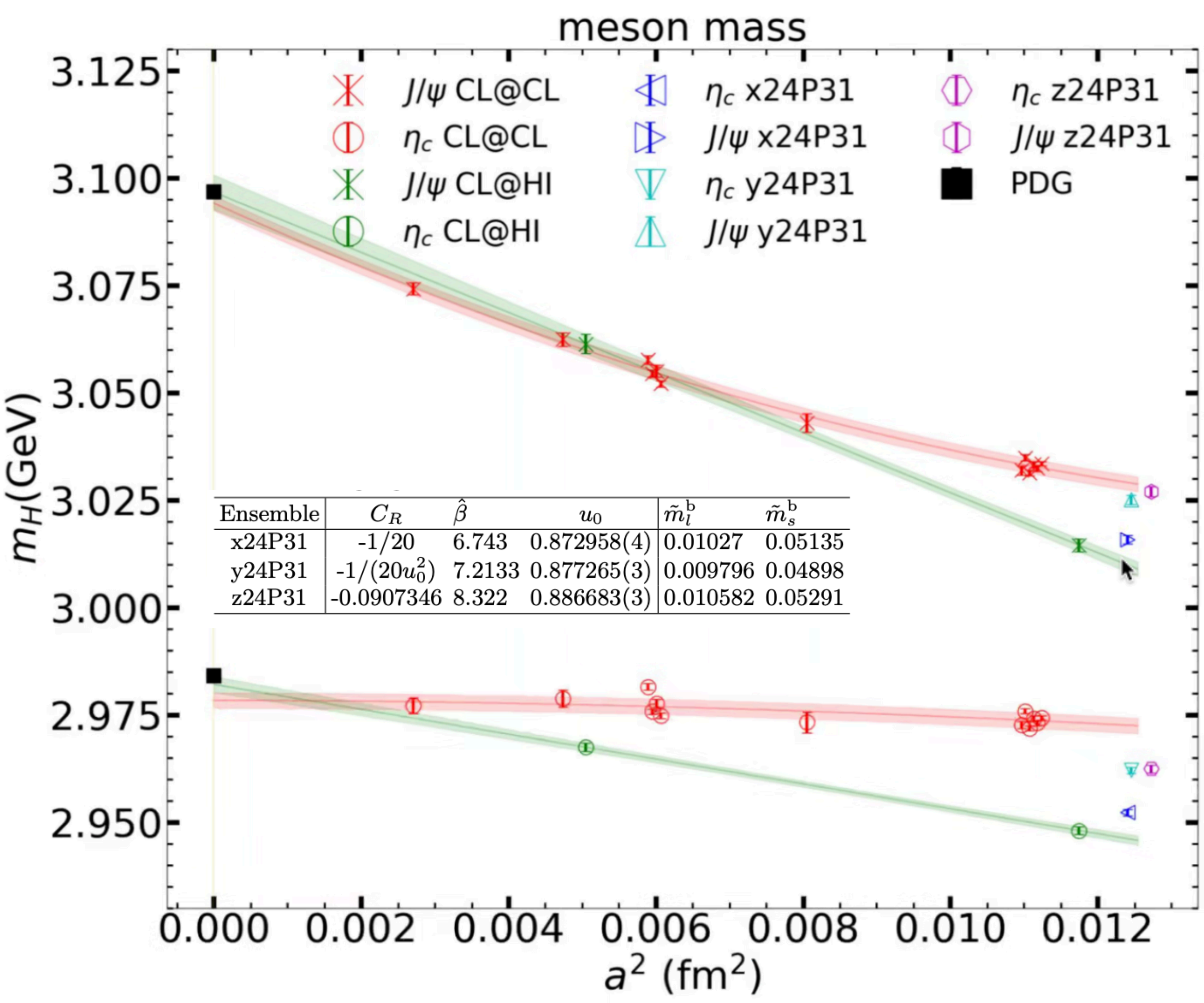


- Discretization errors are also smaller for m_Ω and f_K with similar continuum limit which suggest negligible charm sea effects.

T.W Lin et al. [CLQCD], arXiv:2603.04230

Systematic uncertainties

Charm sea effects



Ensemble	C_R	$\hat{\beta}$	u_0	$\tilde{m}_l^b$	$\tilde{m}_s^b$
x24P31	-1/20	6.743	0.872958(4)	0.01027	0.05135
y24P31	$-1/(20u_0^2)$	7.2133	0.877265(3)	0.009796	0.04898
z24P31	-0.0907346	8.322	0.886683(3)	0.010582	0.05291

CL@CL: 2+1 flavor Clover ensemble

CL@HI: 2+1+1 flavor HISQ ensemble

x/y/z24P31: 2+1 flavor HISQ ensemble with different gauge actions

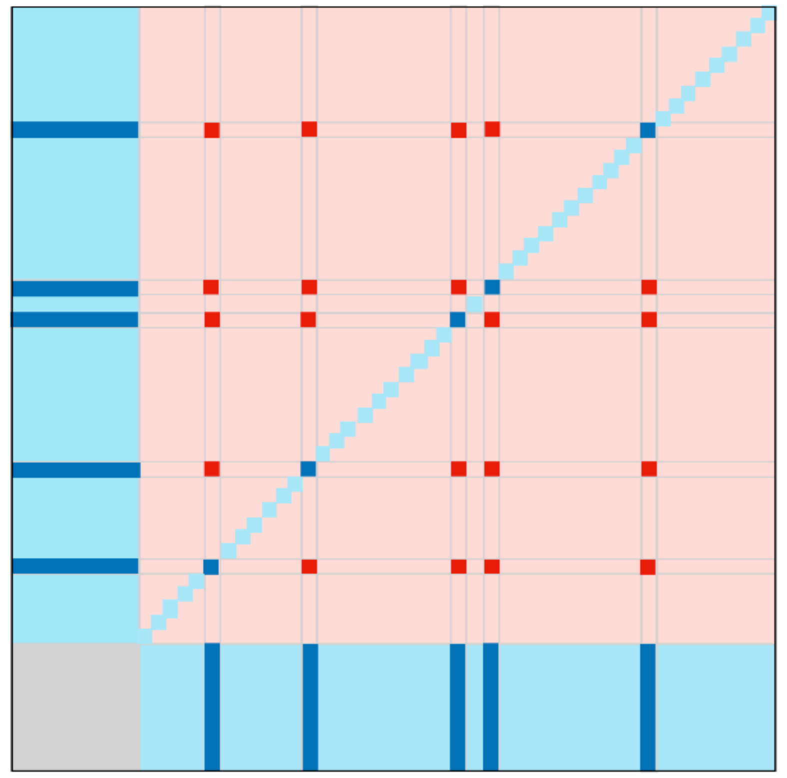
For the S-wave charmonium with the charm quark mass determined using the physical pure QCD m_{D_s}, compared with the values from 2+1 flavor clover ensembles, those from the 2+1+1 flavor HISQ ensembles using the valence clover fermion have the following features:

- The discretization error becomes larger for m_{η_c};
- But comparable for $m_{J/\psi}$ at small lattice spacing with smaller a^4 correction.
- And then the hyperfine splitting has smaller discretization error.
- The impact from charm sea and gauge actions are can be highly non-trivial.

T.W Lin et al. [CLQCD], arXiv:2603.04230

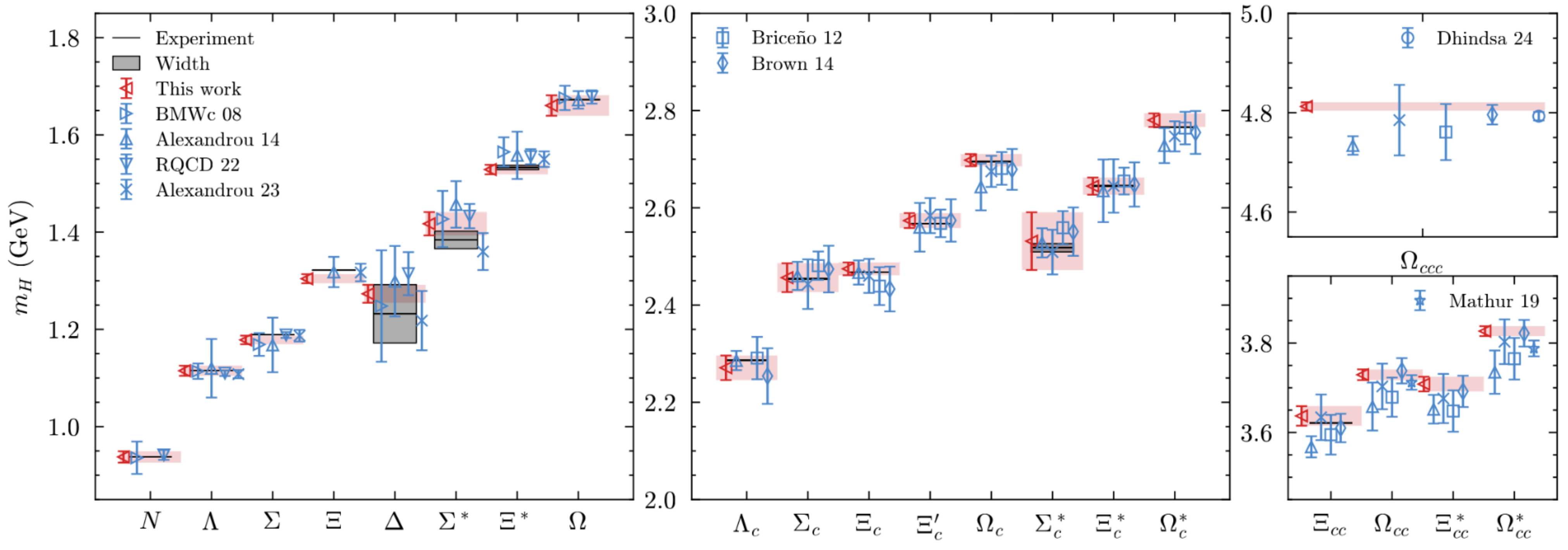
Outline

- Light and strange quark

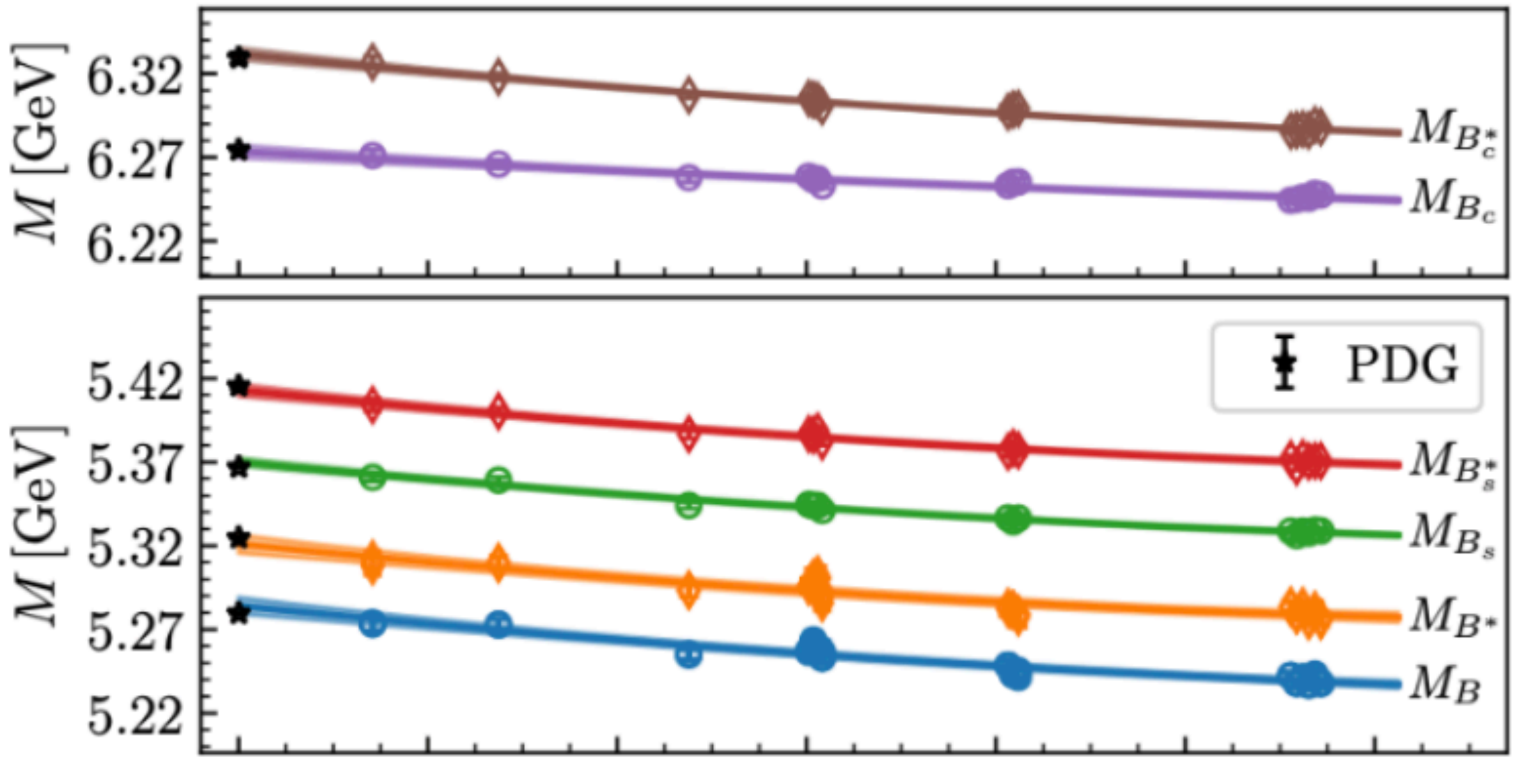


$$\Omega^{(2)}_{ij} = \begin{cases} 1 & \text{for } i, j \le N_e, \\ \Pi^{1}_{i=0} \frac{V - N_e - i}{N_{st} - i} & \text{for } i, j > N_e,\ i \neq j \\ \frac{V - N_e}{N_{st}} & \text{for the other cases,} \end{cases}$$

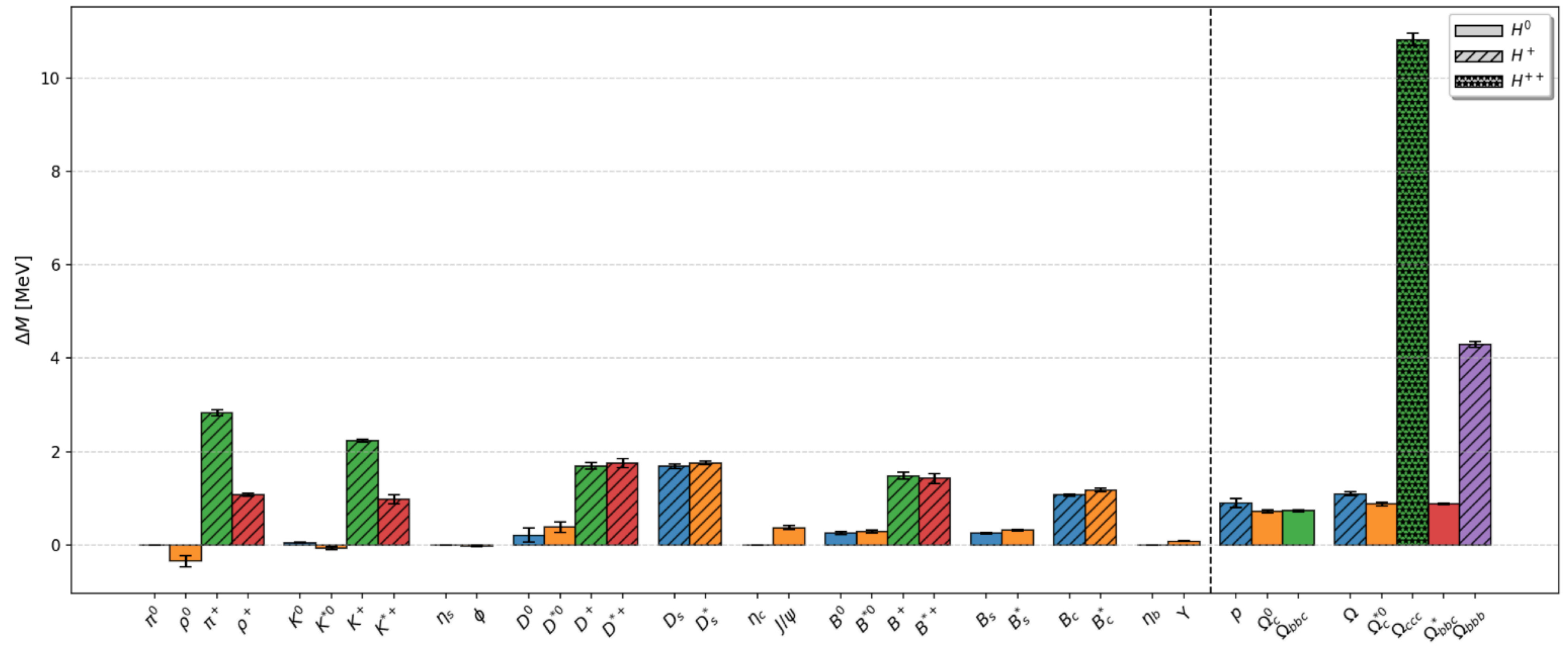
- Charm quark



- **Bottom quark**



- QED corrections



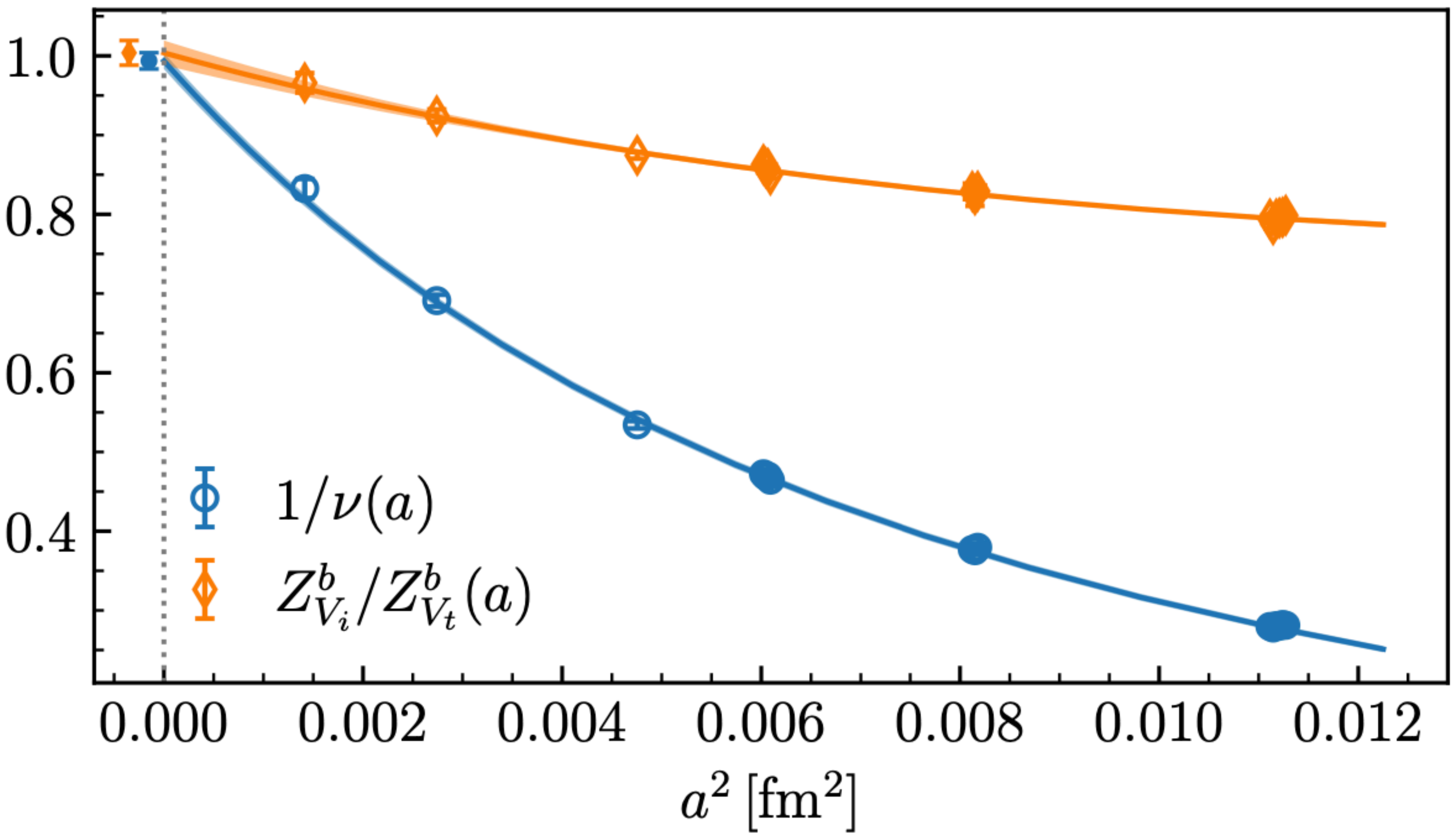
- Summary

Bottom quark

Restoration of isotropy in the continuum limit

$$S_Q = a^4 \sum_x \bar{Q}\Big[m_Q + \gamma_4\nabla_4 - \frac{a}{2}\nabla_4^2 + \nu \sum_{i=1}^{3}\left(\gamma_i\nabla_i - \frac{a}{2}\nabla_i^2\right) - c_E(\nu,u_0)\frac{a}{2}\sum_{i=1}^{3}\sigma_{i4}F_{i4} - c_B(\nu,u_0)\frac{a}{2}\sum_{i>j=1}^{3}\sigma_{ij}F_{ij}\Big]Q,$$

$c_E(\nu,u_0) = (1+\nu)/(2u_0^3),\ c_B(\nu,u_0) = \nu/(u_0^3)$ L.M. Liu et al. PRD81(2010)094505



M.C. Cai et al. [CLQCD], arXiv:2603.01846

- Tune ν to ensure $E_H^2 = m_H^2 + p_H^2$;
- Practical calculation suggests

$$\nu = \frac{\sinh(c_\nu m_H a)}{c_\nu m_H a}(1.006(10) + 0.258\ \mathrm{GeV}^{-2}(m_\pi^2 - m_{\pi,\mathrm{phys}}^2))$$

  with $c_\nu = 0.621(3)$;
- Comparable with the free quark case $c_\nu = 0.5$;
- $\nu \to 1$ with either $m_H \to 0$ or $a \to 0$;
- The anisotropy of the matrix elements also approaches 1 in the continuum limit:

$$\frac{Z_{V_z}}{Z_{V_t}} \equiv \frac{\langle H|V_t|H\rangle}{\langle H|V_z|H\rangle}\frac{p_z}{E} = 0.99(2) - 0.015(2)m_H^2a^2.$$

Bottom quark

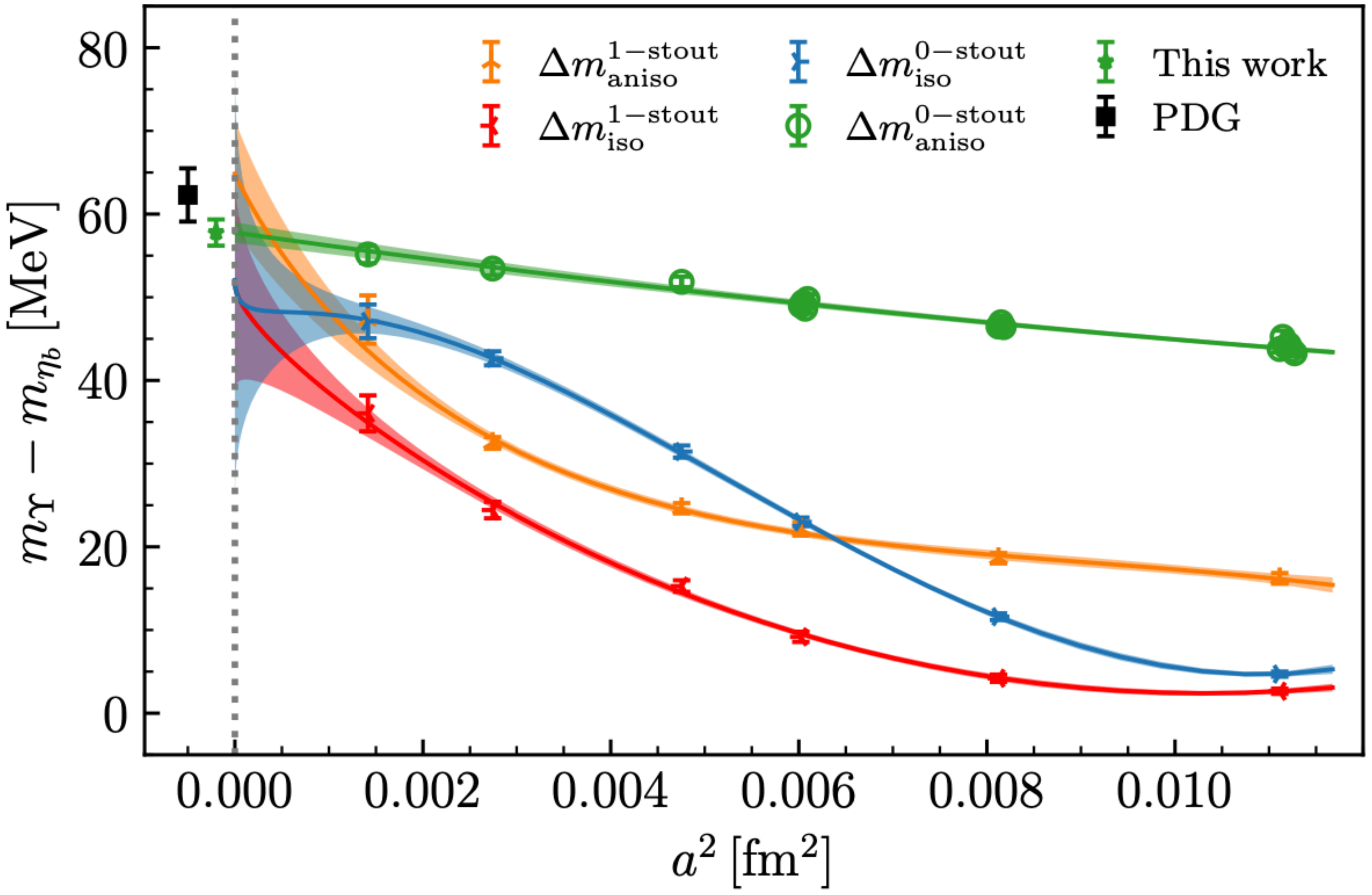
$\Delta^b_{\rm HFS} \equiv M_\Upsilon - M_{\eta_b}$

$$S_Q = a^4 \sum_x \bar{Q}\Big[m_Q + \gamma_4\nabla_4 - \frac{a}{2}\nabla_4^2 + \nu \sum_{i=1}^{3}\Big(\gamma_i\nabla_i - \frac{a}{2}\nabla_i^2\Big) - c_E(\nu,u_0)\frac{a}{2}\sum_{i=1}^{3}\sigma_{i4}F_{i4} - c_B(\nu,u_0)\frac{a}{2}\sum_{i>j=1}^{3}\sigma_{ij}F_{ij}\Big]Q,$$

$\Delta^b_{\rm HFS} \equiv M_\Upsilon - M_{\eta_b}$

$c_E(\nu,u_0) = (1+\nu)/(2u_0^3),\ c_B(\nu,u_0) = \nu/(u_0^3)$

L.M. Liu et al. PRD81(2010)094505



Isotropic action with smeared gauge field, as **used by lighter flavors**

$\Delta^b_{\rm HFS}(a = 0.105\ {\rm fm}, \nu = 1, u_0 \sim 0.95) \sim 3\ {\rm MeV}$

Change gluon discretization → Isotropic action with **original** gauge field

$\Delta^b_{\rm HFS}(a = 0.105\ {\rm fm}, \nu = 1, u_0 \sim 0.85) \sim 9\ {\rm MeV}$

Introduce anisotropy → **Anisotropic** action with smeared gauge field

$\Delta^b_{\rm HFS}(a = 0.105\ {\rm fm}, \nu = 3.6, u_0 \sim 0.95) \sim 15\ {\rm MeV}$

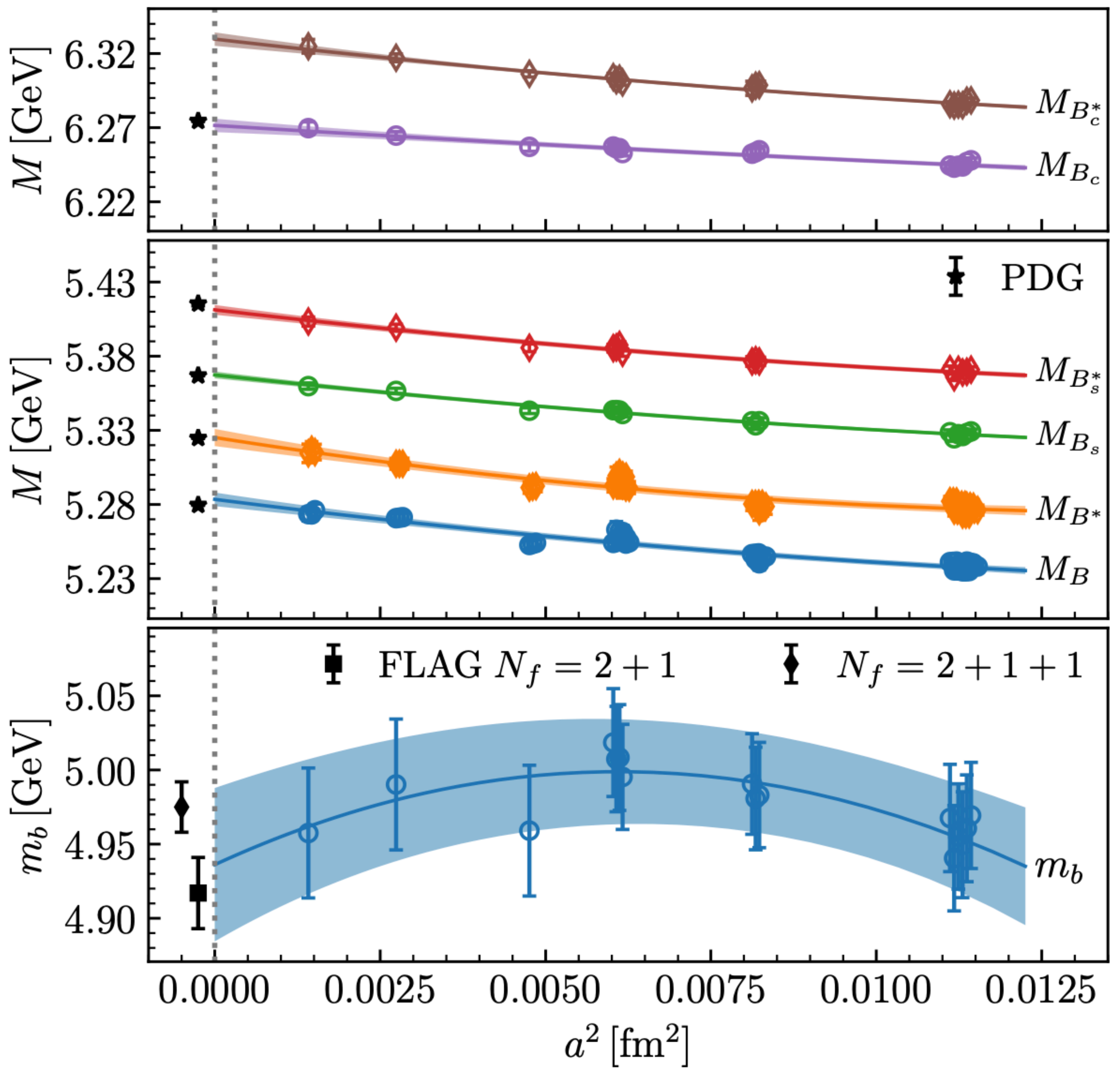
Introduce anisotropy / Change gluon discretization → **Anisotropic** action with **original** gauge field

$\Delta^b_{\rm HFS}(a = 0.105\ {\rm fm}, \nu = 3.6, u_0 \sim 0.85) \sim 45\ {\rm MeV}$

- Different discretized fermion action share the same continuum limit but vey different discretization errors!

M.C. Cai et al. [CLQCD], arXiv:2603.01846

Bottom quark

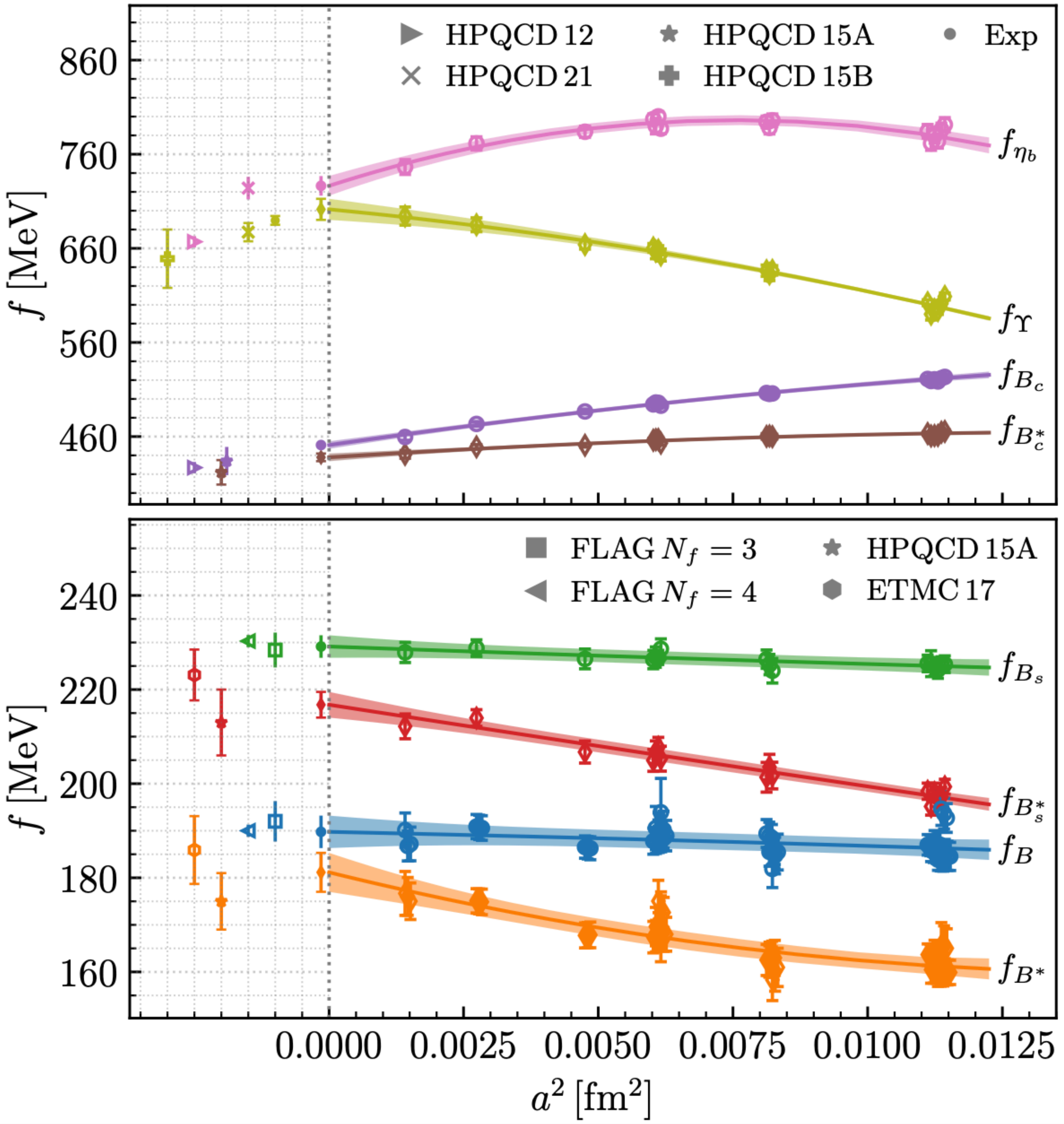


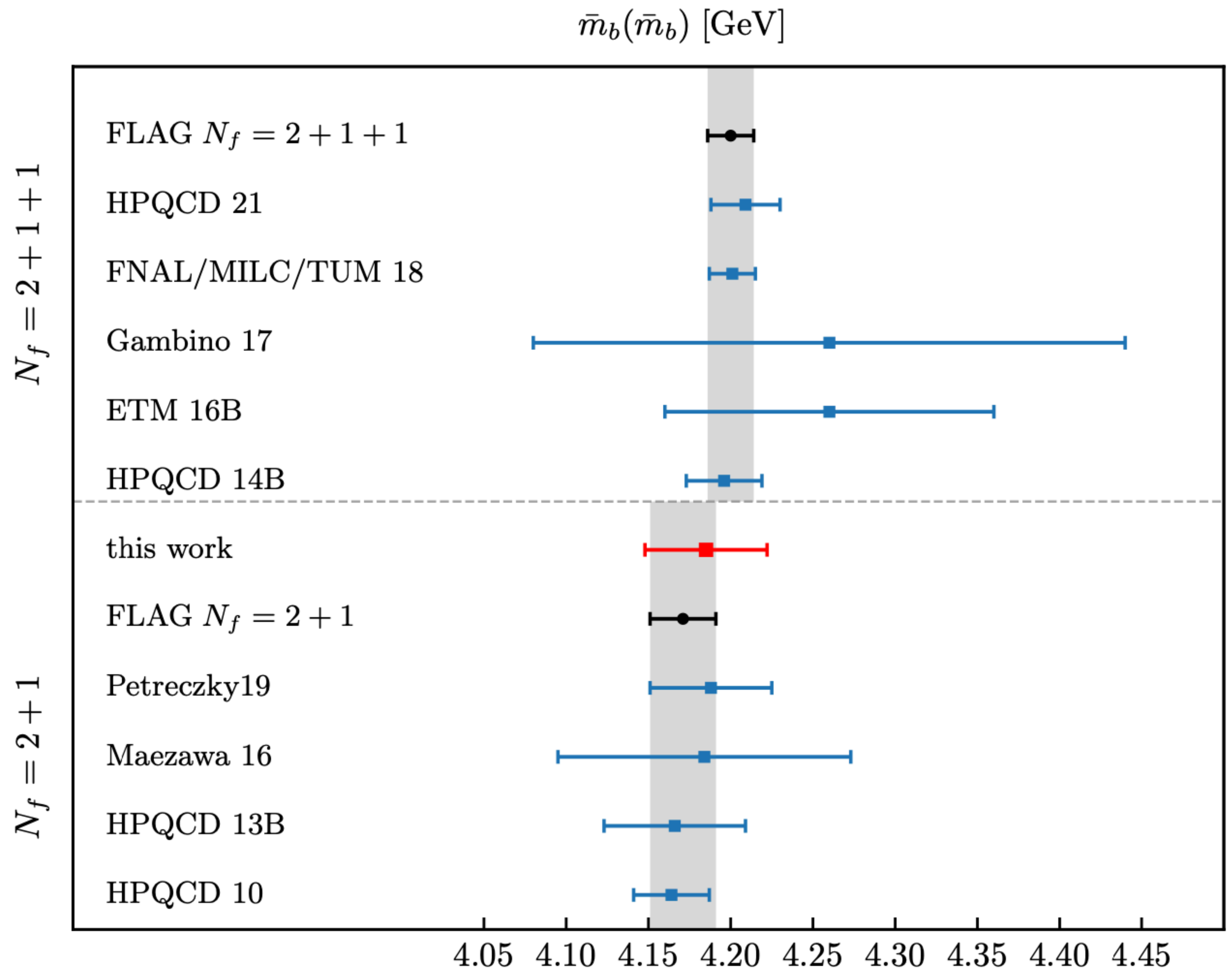
	m_B	m_{B^*}	m_{B_s}	$m_{B_s^*}$	m_{B_c}	$m_{B_c^*}$	m_{η_b}
M [GeV]	5.2835(46)	5.3252(58)	5.3673(23)	5.4112(33)	6.2715(45)	6.3297(45)	9.4026(12)
PDG	5.2796(02)	5.3248(02)	5.3669(01)	5.4154(14)	6.2745(03)	6.3295(30)	9.3987(20)
χ^2/dof	0.93	0.82	0.65	0.53	0.73	0.60	0.78
Q	0.60	0.81	0.84	0.93	0.76	0.88	0.71

- Discretization errors of both the bottom quark and hadron masses are at 1% level at the coarsest lattice spacing;
- Combining the ISB and QED corrections gives
  $m_{B^0}^{\rm phys} = 5283.5(4.6) + 1.1(0.1)_{\rm ISB} + 0.2(0.1)_{\rm QED} = 5284.8(4.6)$ GeV
  $m_{B^+}^{\rm phys} = 5283.5(4.6) - 1.1(0.1)_{\rm ISB} + 1.7(0.2)_{\rm QED} = 5284.1(4.6)$ GeV
  and $m_{B_s}^{\rm phys} = 5367.3(2.3) + 0.2(0.2)_{\rm QED} = 5367.5(2.3)$ GeV.
- Using the $M_{B_s}^{\rm phys}$ to determine the bottom quark mass can results 0.5(2.5) MeV shift on all the predictions (1(5) MeV for $M_{\eta_b/\Upsilon}$).

M.C. Cai et al. [CLQCD], arXiv:2603.01846

Bottom quark





- Discretization errors of the decay constants of the bottomed hadron are highly suppressed compared to naive $\mathcal{O}(m_b^2 a^2)$ power counting ($m_b^2 a^2 \sim 7$ at $a \sim 0.105$ fm) and also previous HPQCD result;
- Verify the relation $f_{B^*}/f_B = f_{B_s^*}/f_{B_s} = f_{B_c^*}/f_{B_c} = f_\Upsilon/f_{\eta_b}$ with so far the highest precision.
- Most precise prediction of $m_b(m_b)$ except those using the staggered fermion.

M.C. Cai et al. [CLQCD], arXiv:2603.01846

Bottom quark

Summary of our standard model parameter determinations

$w_0 = 0.1736(9)$ fm

$m_{\pi^0} = 134.98$ MeV $\rightarrow$ $(m_u + m_d)^{\overline{\rm MS}(2\ {\rm GeV})}/2 = 3.60(19)$ MeV $\leftarrow$ $m_{\pi^\pm}(m_u, m_d)$

$m_{K^0}^{\rm QCD} = 497.44(02)$ MeV

$m_{K^\pm}^{\rm QCD} = 491.44(15)$ MeV

$\rightarrow$

$m_u^{\overline{\rm MS}(2\ {\rm GeV})} = 2.45(30)$ MeV

$m_d^{\overline{\rm MS}(2\ {\rm GeV})} = 4.74(14)$ MeV

$m_s^{\overline{\rm MS}(2\ {\rm GeV})} = 98.8(5.5)$ MeV

$\leftarrow$ $m_K(m_u^{val}, m_d^{val}, m_l^{sea}, m_s^{sea})$

$m_{D_s}^{\rm QCD} = 1966.7(1.5)$ MeV $\rightarrow$ $m_c^{\overline{\rm MS}(m_c)} = 1293(13)(11)$ MeV $\leftarrow$ $m_{D_s}(m_c^{val}, m_s^{val}, m_l^{sea}, m_s^{sea})$

$m_\Upsilon = 9460.4(1)$ MeV $\rightarrow$ $m_b^{\overline{\rm MS}(m_b)} = 4.16(4)(2)$ GeV $\leftarrow$ $m_\Upsilon(m_b^{val}, m_l^{sea}, m_s^{sea})$

$\frac{f_K}{f_\pi} = 1.1907(76)_{\rm lat}$

$\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi} = 0.27683(29)_{\rm exp}(20)_{\rm th}$

$1 = |V_{ud}|^2 + |V_{us}^2| + |V_{ub}|^2$

$|V_{ud}| = 0.9740(03)_{\rm lat}(01)_{\rm ph}$

$|V_{us}| = 0.2265(14)_{\rm lat}(03)_{\rm ph}$

$|V_{ub}| = 4.06(13)_{\rm lat}(65)_{\rm exp} \times 10^{-3}$

$f_{B^+}|V_{ub}| = 0.77(12)_{\rm exp}$ MeV

$f_{B^+} = 0.190(3)_{\rm lat}$ MeV

$f_{D^+} = 0.2113(33)_{\rm lat}$ MeV

$f_{D^+}|V_{cd}| = 45.8(1.1)_{\rm exp}$ MeV

$|V_{cd}| = 0.2168(33)_{\rm lat}(52)_{\rm exp}$

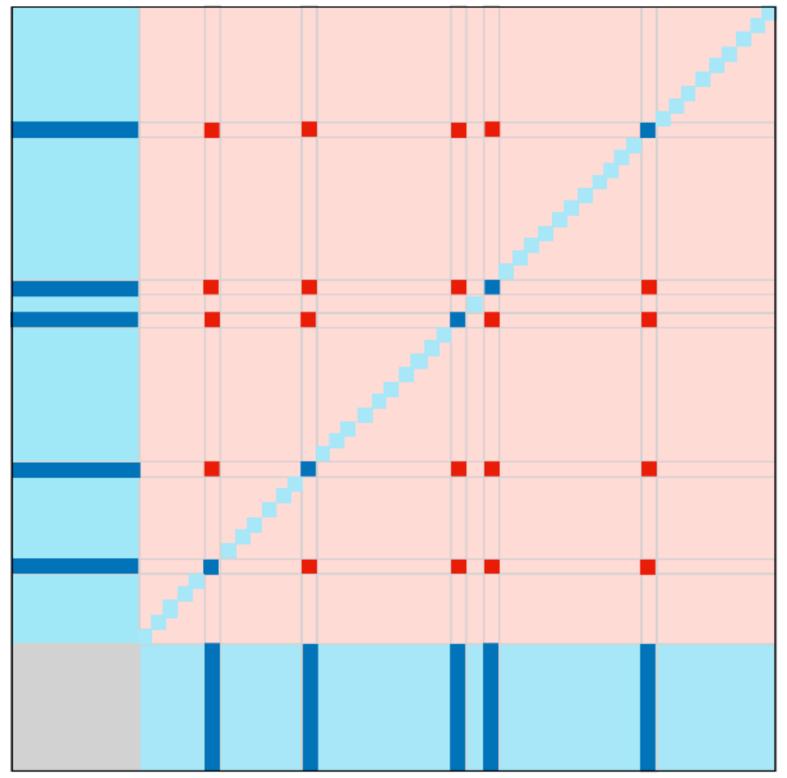
$|V_{cs}| = 0.975(13)_{\rm lat}(11)_{\rm exp}$

$f_{D_s^+}|V_{cs}| = 243.5(2.7)_{\rm exp}$ MeV

$f_{D_s^+} = 0.2498(33)_{\rm lat}$ MeV

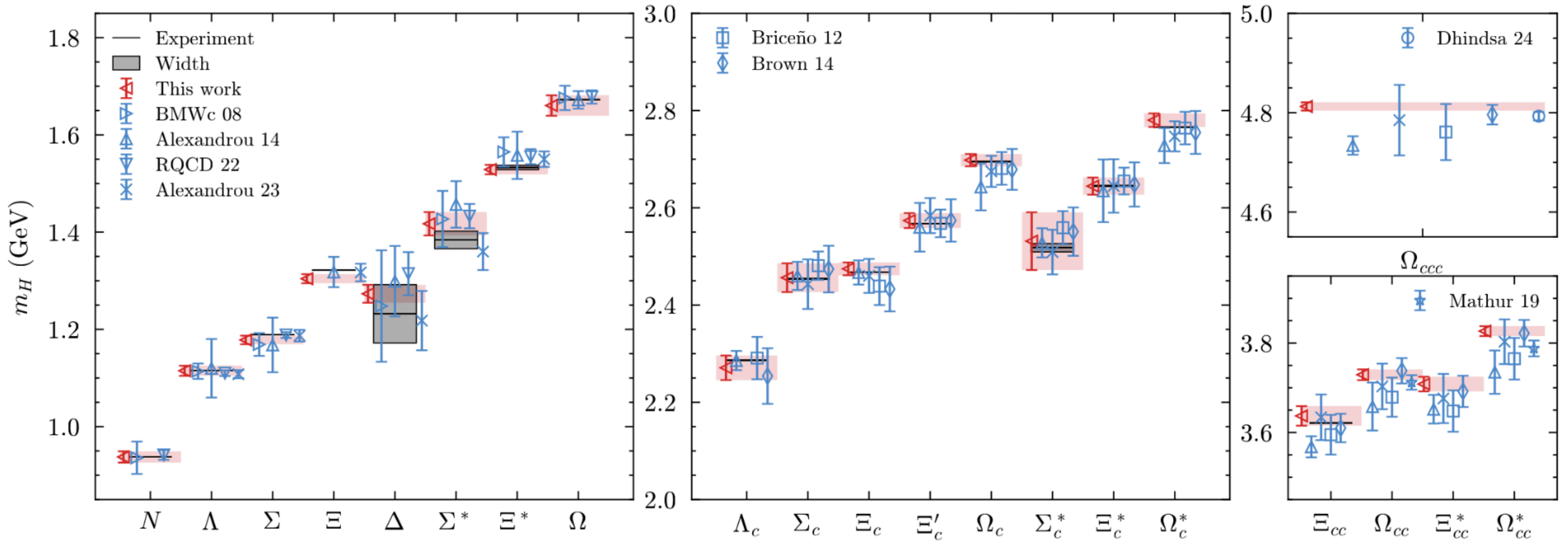
Outline

- Light and strange quark

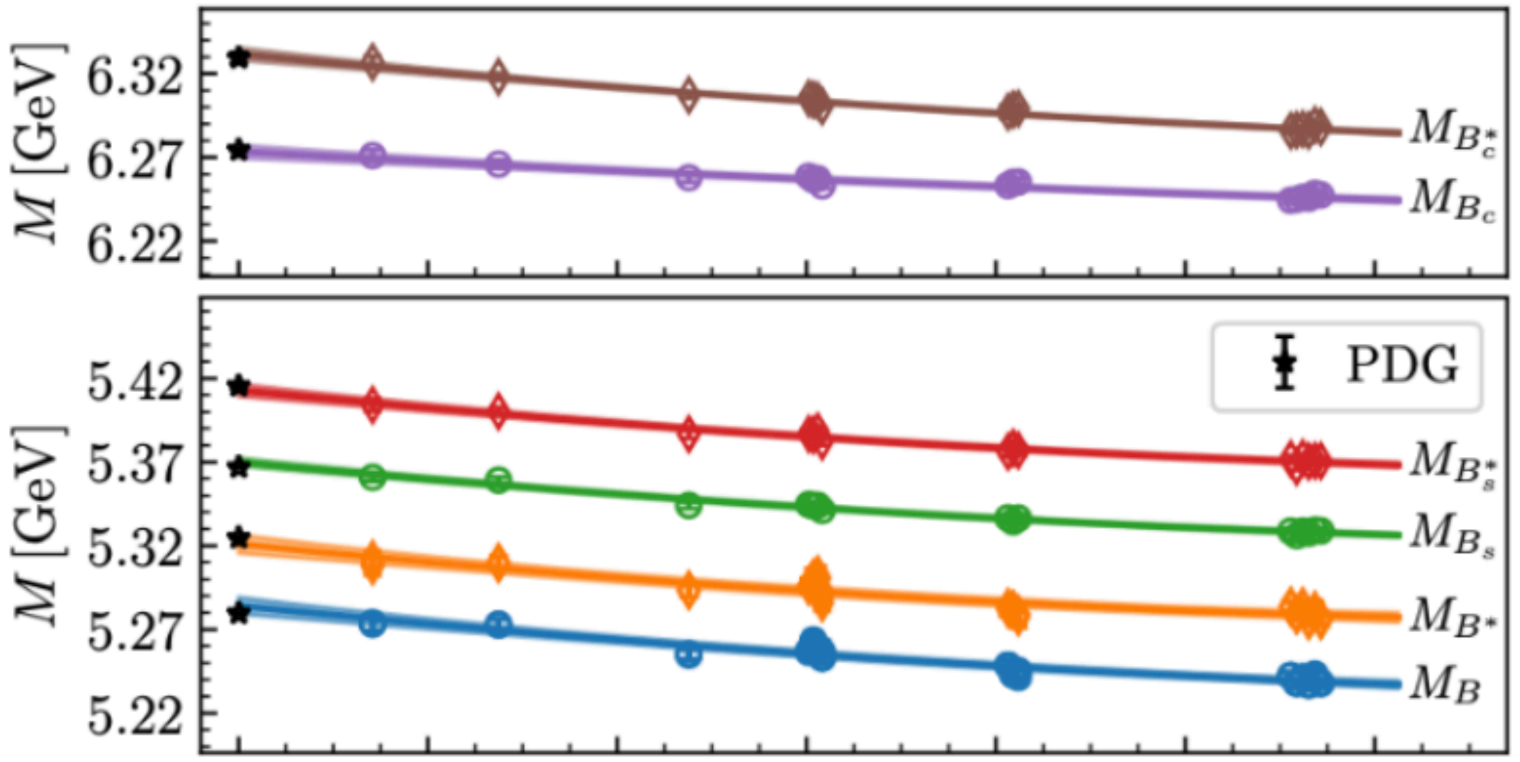


$$\Omega^{(2)}_{ij} = \begin{cases} 1 & \text{for } i,j \le N_e, \\ \Pi^{1}_{i=0}\frac{V-N_e-i}{N_{st}-i} & \text{for } i,j > N_e,\ i \neq j \\ \frac{V-N_e}{N_{st}} & \text{for the other cases,} \end{cases}$$

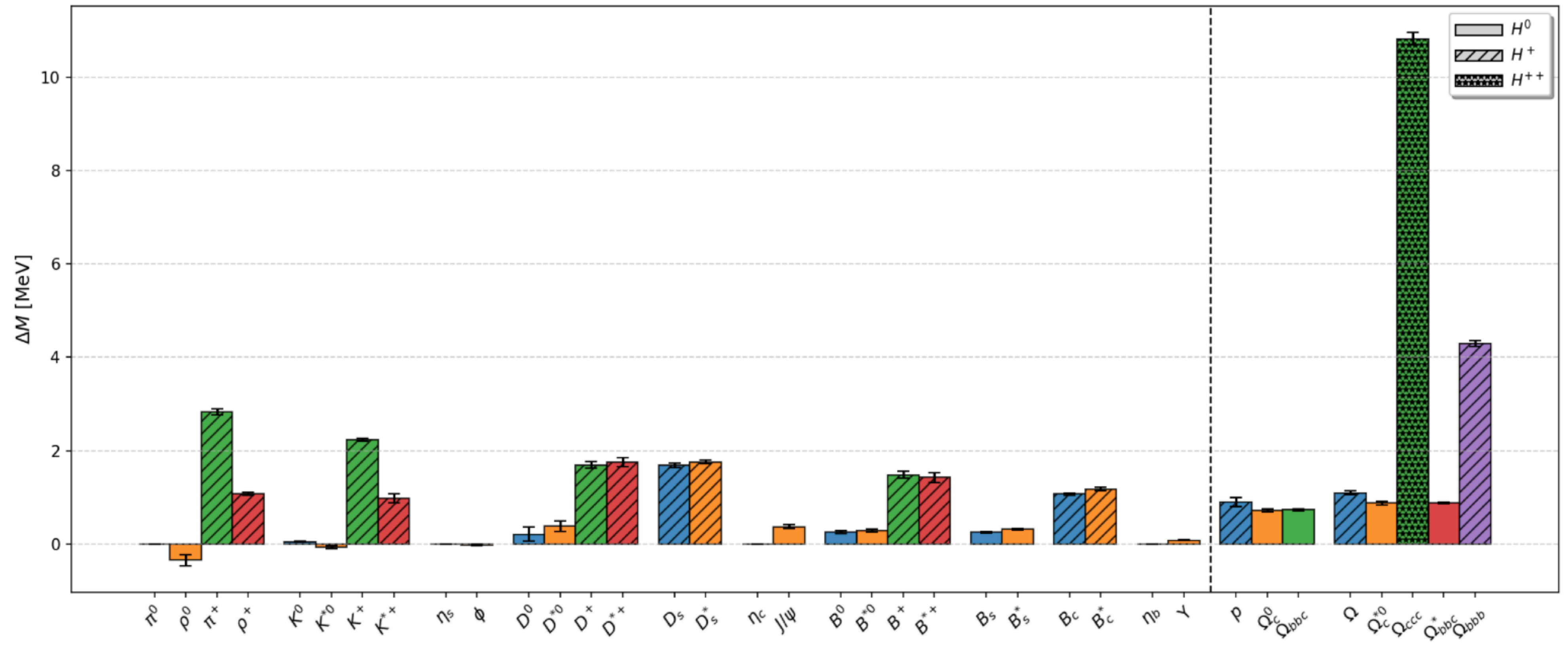
- Charm quark



- Bottom quark



- **QED corrections**

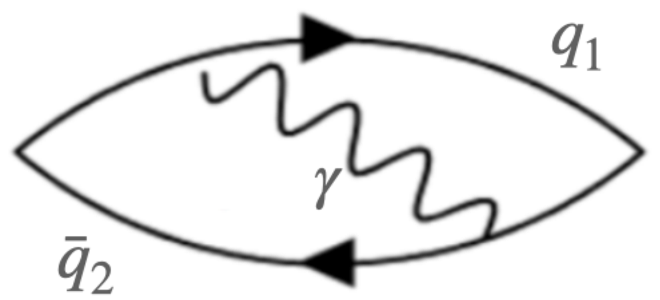


- Summary

QED corrections

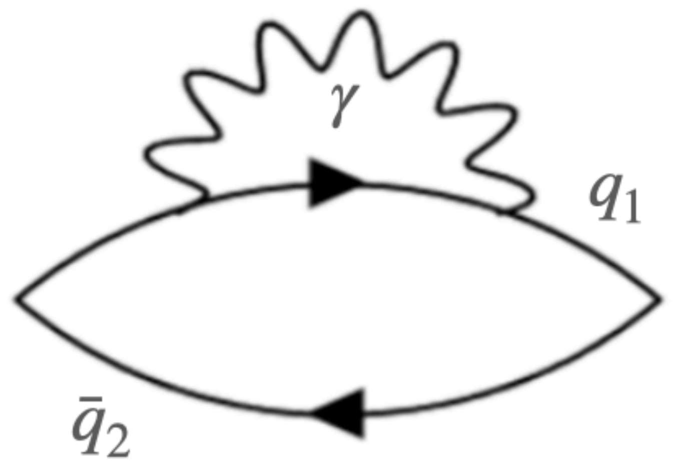
- The QCD+QED calculation can be done under the quenched QED approximation using QED_L for the valence fermion:

$$U_\mu^{\mathrm{QCD+QED}} = U_\mu^{\mathrm{QCD}} e^{-\mathrm{i}ee_q A_\mu},\ A_\mu(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-\mathrm{i}p\cdot x} A_\mu(p),\ P_{A_\mu(p)}|_{\vec{p}\neq 0} \propto e^{-\frac{1}{2N_V}\hat{p}^2 A_\mu^2(p)}$$



QED UV finite (at 1-loop level)

$$-e_{q_1}e_{q_2}\delta_{\mathrm{QED}}^{\mathrm{int}} m_\pi(m_q + \delta_{\mathrm{QED}} m_q) \simeq -e_{q_1}e_{q_2}\delta_{\mathrm{QED}}^{\mathrm{int}} m_\pi(m_q)$$



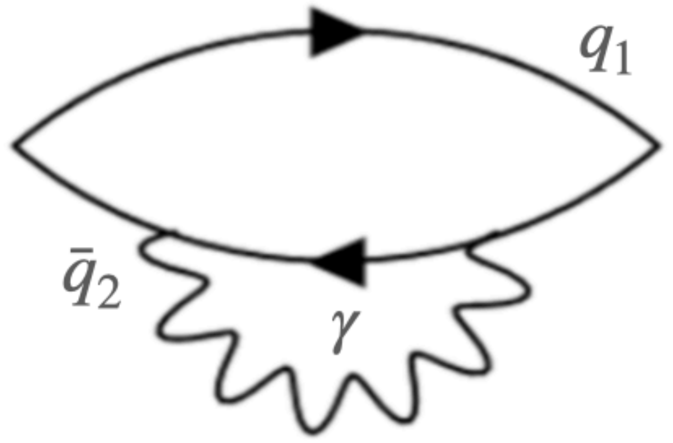
$$e_{q_1}^2 \delta_{\mathrm{QED}}^{\mathrm{self}} m_H$$

QED UV finite

$$\delta_{\mathrm{QED}}^{\mathrm{self}} m_\pi(m_q + \delta_{\mathrm{QED}} m_q)$$

$$\simeq \delta_{\mathrm{QED}}^{\mathrm{self}} m_\pi(m_q) + \langle \bar{q}q\rangle_\pi \delta_{\mathrm{QED}} m_q$$

QED UV divergent   QED UV divergent



$$e_{q_2}^2 \delta_{\mathrm{QED}}^{\mathrm{self}} m_H$$

- QED correction of the hadron masses suffer from additional UV divergence and require further renormalization in the quark mass;

- The PQU scheme requires

$$\delta_{\mathrm{QED}} m_q = e_q^2 \frac{\delta_{\mathrm{QED}}^{\mathrm{int}} m_{\eta_q} - 2\delta_{\mathrm{QED}}^{\mathrm{self}} m_{\eta_q}}{2\langle \bar{q}q\rangle_{\eta_q}}$$

  to cancel the UV divergence of $\delta_{\mathrm{QED}}^{\mathrm{self}} m_{\eta_q}$, and ensure $\delta_{\mathrm{QED}} m_{\eta_q} = 0$.

- And then the final correction to the hadron mass can be significantly smaller than 1% with proper quark mass renormalizations.

Y.Y. Liu et al. [CLQCD], in preparation

QED corrections

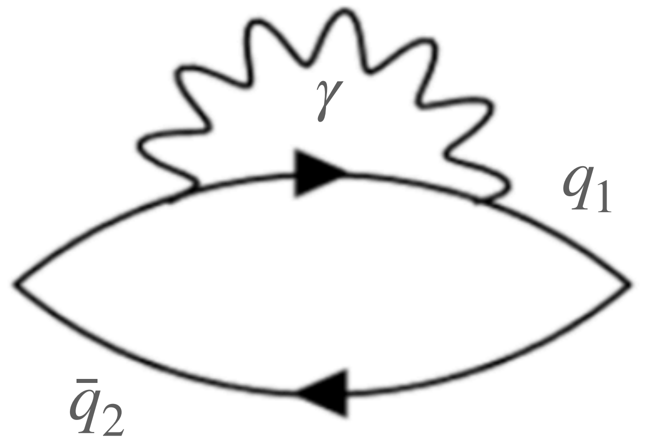
Pion case

QED UV finite (at 1-loop level)

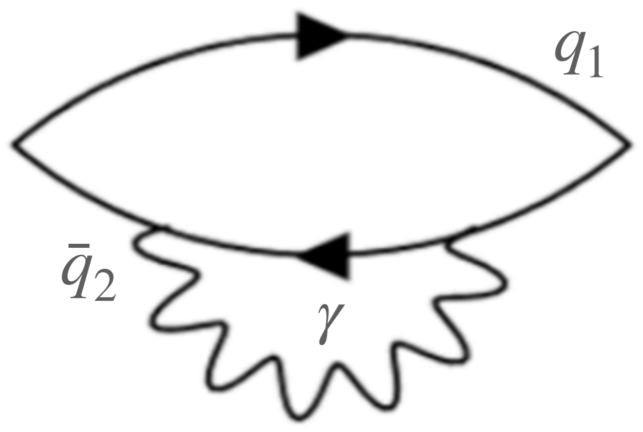
$$\delta^{\rm int}_{\rm QED} m_\pi (m_q + \delta_{\rm QED} m_q) \simeq \delta^{\rm int}_{\rm QED} m_\pi (m_q)$$

QED UV finite — QED UV divergent — QED UV divergent

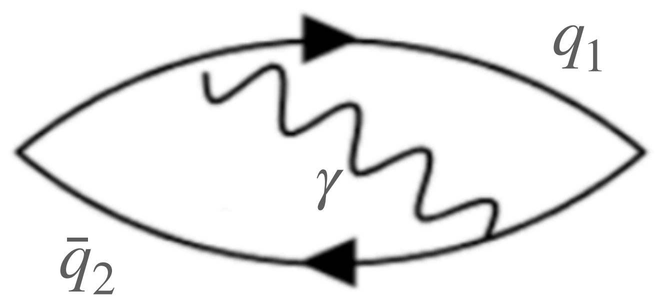
$$\delta^{\rm self}_{\rm QED} m_\pi (m_q + \delta_{\rm QED} m_q) \simeq \delta^{\rm self}_{\rm QED} m_\pi (m_q) + \langle \bar{q} q \rangle_\pi \delta_{\rm QED} m_q$$



$$e^2_{q_1} \delta^{\rm self}_{\rm QED} m_H$$



$$e^2_{q_2} \delta^{\rm self}_{\rm QED} m_H$$



$$-e_{q_1} e_{q_2} \delta^{\rm int}_{\rm QED} m_H$$

$$\delta^{\rm self}_{\rm QED} m_\pi = 4.6 \text{ MeV} \qquad \delta^{\rm int}_{\rm QED} m_\pi = 9.2 \text{ MeV}$$

$$\delta_{\rm QED} m_{\pi^+} = (e_u^2 + e_d^2) \delta^{\rm self}_{\rm QED} m_\pi + e_u e_{\bar{d}} \delta^{\rm int}_{\rm QED} m_\pi$$
$$= \frac{5}{9} 4.6 \text{ MeV} + \frac{2}{9} 9.2 \text{ MeV} = 4.6 \text{ MeV}$$

Self energy
+
Repulsive interaction

$$\delta_{\rm QED} m_{\pi^0} = (e_u^2 + e_d^2) \delta^{\rm self}_{\rm QED} m_\pi + \frac{1}{2} (e_u e_{\bar{u}} + e_d e_{\bar{d}}) \delta^{\rm int}_{\rm QED} m_\pi$$
$$= \frac{5}{9} 4.6 \text{ MeV} - \frac{1}{2} \frac{5}{9} 9.2 \text{ MeV} = 0$$

Self energy
+
Attractive interaction

Y.Y. Liu et al. [CLQCD], in preparation

QED corrections

Self energy v.s. Interaction

$$\delta_{\text{QED}} m_H = \sum_q e_q^2 \delta_{\text{QED}}^{\text{self,q}} m_H + \sum_q \sum_{q' \neq q} e_q e_{q'} \delta_{\text{QED}}^{\text{int,qq}'} m_H$$

$$\delta_{\text{QED}} m_{B_0} = e_b^2 \delta_{\text{QED}}^{\text{self,b}} m_{B_0} + e_d^2 \delta_{\text{QED}}^{\text{self,d}} m_{B_0} + e_b e_d \delta_{\text{QED}}^{\text{int,bd}} m_{B_0}$$
$$= \frac{1}{9} 5\ \text{MeV} + \frac{1}{9} 0.6\ \text{MeV} - \frac{1}{9} 4\ \text{MeV}$$
$$= 0.2\ \text{MeV}$$

Self energy
+
Attractive interaction

$$\delta_{\text{QED}} m_{B_+} = e_b^2 \delta_{\text{QED}}^{\text{self,b}} m_{B_+} + e_u^2 \delta_{\text{QED}}^{\text{self,u}} m_{B_+} + e_b e_d \delta_{\text{QED}}^{\text{int,bd}} m_{B_+}$$
$$= \frac{1}{9} 5\ \text{MeV} + \frac{4}{9} 0.6\ \text{MeV} + \frac{2}{9} 4\ \text{MeV}$$
$$= 1.6\ \text{MeV}$$

Self energy
+
Repulsive interaction

$$\delta_{\text{QED}} m_{\Omega^*_{bbc}} = 2 e_b^2 \delta_{\text{QED}}^{\text{self,b}} m_{\Omega^*_{bbc}} + e_c^2 \delta_{\text{QED}}^{\text{self,c}} m_{\Omega^*_{bbc}} + e_b^2 \delta_{\text{QED}}^{\text{int,bb}} m_{\Omega^*_{bbc}} + 2 e_b e_c \delta_{\text{QED}}^{\text{int,bc}} m_{\Omega^*_{bbc}}$$
$$= 2 \frac{1}{9} 5\ \text{MeV} + \frac{4}{9} 3.5\ \text{MeV} + \frac{1}{9} 4\ \text{MeV} - 2 \frac{2}{9} 5\ \text{MeV}$$
$$= 0.8\ \text{MeV}$$

Self energy
+
Repulsive interaction
+
Attractive interaction

$$\delta_{\text{QED}} m_{\Omega_{ccc}} = 3 e_c^2 \delta_{\text{QED}}^{\text{self,b}} m_{\Omega_{ccc}} + 3 c_c^2 \delta_{\text{QED}}^{\text{int,cc}} m_{\Omega_{ccc}}$$
$$= 3 \frac{4}{9} 3.5\ \text{MeV} + 3 \frac{4}{9} 5\ \text{MeV}$$
$$= 11\ \text{MeV}$$

Self energy
+
Repulsive interaction

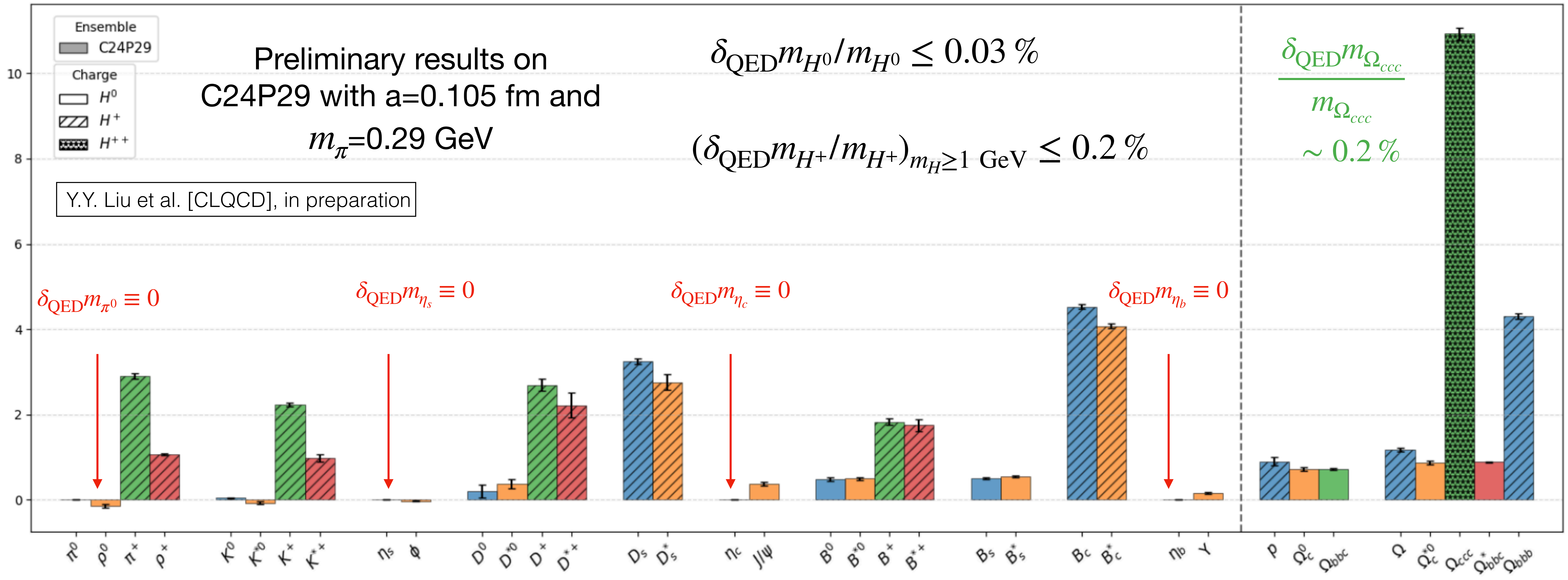
Y.Y. Liu et al. [CLQCD], in preparation

QED corrections

$$\delta_{\text{QED}} m_q = e_q^2 \frac{\delta_{\text{QED}}^{\text{int},\bar{q}q} m_{\eta_q} - \delta_{\text{QED}}^{\text{self},q} m_{\eta_q} - \delta_{\text{QED}}^{\text{self},\bar{q}} m_{\eta_q}}{2\langle \bar{q}q \rangle_{\eta_q}},$$

$$\delta_{\text{QED}} m_H = \sum_q e_q^2 (\delta_{\text{QED}}^{\text{self},q} m_H + \delta_{\text{QED}} m_q \langle \bar{q}q \rangle_H) + \sum_q \sum_{q' \neq q} e_q e_{q'} \delta_{\text{QED}}^{\text{int},qq'} m_H$$

Total correction, [MeV]



Summary

- Blending method provide new possibilities to improve the statistical precision for physical light quark mass;
- Charm and bottom physics can now be predicted with suppressed discretization error with kinds of the improvements;
- QED correction of the hadron masses can also be predicted precisely.