Direct (Symbol) Integration

Mostly based on [2304.01776]

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Concentrate all your fire
on the nearest starship!

MPL and the Symbol

$$G(a_1, a_2, \cdots, a_n; z) := \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \cdots, a_n; t),$$

$$G(\overbrace{0, \cdots, 0}^{n}; z) := \frac{1}{n!} \log^n z, \quad G(; z) := 1.$$

For our purposes, the following property of the symbol suffices:

$$\mathrm{d}F = H\,\mathrm{d}w \quad \Longrightarrow \quad \mathcal{S}[\,F\,] = \mathcal{S}[\,H\,] \otimes w.$$

With $a_0 \equiv z$ and $a_{n+1} \equiv 0$,

$$\mathrm{d}G(\vec{a}; z) = \sum_{k=1}^{n} (-)^k G(a_1, \cdots, \widehat{a}_k, \cdots, a_n; z)\,\mathrm{d}\log\frac{a_k - a_{k-1}}{a_k - a_{k+1}}.$$

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Function Integration

$$\mathcal{I} = \int \frac{\mathrm{d}t}{t+c} G(\vec{a}(t); z(t)), \quad \text{need } G(\vec{a}(t); z(t)) = \sum G(\vec{a}\,'; t).$$

This is possible when $a_k(t) - a_\ell(t)$ contain linear factors of t only.

Efficient algorithms to deal with **linearly reducible** integrals:
HyperInt [1403.3385] PolyLogTools [1904.07279] NumPolyLog [link] ...

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What about integrals that are not linearly reducible?

$$\int \frac{\mathrm{d}t}{\sqrt{At^2+Bt+C}} G(\vec{a}(t); z(t)), \quad \vec{a}(t) \supset \sqrt{At^2+Bt+C}.$$

When not Linearly Reducible ...

Strategy 1: rationalization. For example,

$$t' = \frac{\sqrt{At^2 + Bt + C} - \sqrt{C}}{t} \implies t = \frac{2\sqrt{C}t' - B}{A - t'^2}.$$

...And hope that the integral is linearly reducible wrt. t'.

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Almost certainly fails when multiple square roots are present.

Strategy 2: symbol integration.

> Given $\mathrm{d}F(t) = H(t)\,\mathrm{d}w(t)$, how to compute $\mathrm{d}\left(\int F(t)\right)$?

Symbol Integration: Linearly Reducible

$$\mathrm{d}F(t) = H(t)\,\mathrm{d}\log(t+d), \quad \mathcal{I} = \int_a^b F(t)\,\mathrm{d}\log(t+c).$$

Integrating by parts and using partial fractions [1112.1060],

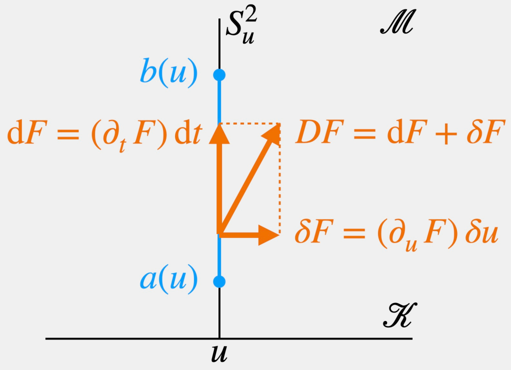
$$\mathcal{S}[\,\mathcal{I}\,] = \mathcal{S}[\,F(b)\,] \otimes (b+c) - \mathcal{S}[\,F(a)\,] \otimes \log(a+c)$$
$$+\,\mathcal{S}\left[\int_a^b H(t)\,\mathrm{d}\log\frac{t+c}{t+d}\right] \otimes (c-d).$$

We provide a new proof that readily generalizes.

Line Integral as a Linear Operator

Big space: $S^2 \hookrightarrow \mathcal{M} \twoheadrightarrow \mathcal{K}$.

$$\mathcal{I}(u) = \int_{a(u)}^{b(u)} F(t;u)\,\mathrm{d}\log(t + c(u)).$$



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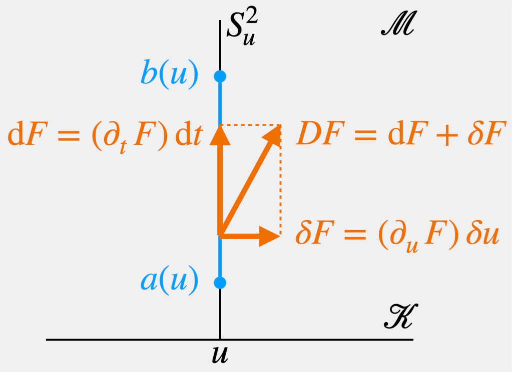

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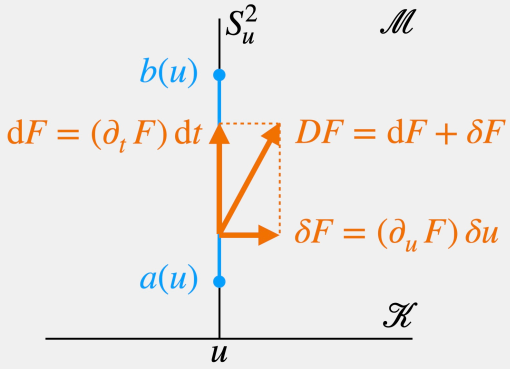

$$\int_{a(u)}^{b(u)} : \quad \Omega^1(S_u^2) \to \mathbb{C}$$

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Rules for Linearly Reducible Cases

Given: $DF(t) = H(t)\, D\log(t+d)$,

Compute: $\delta\mathcal{I} = \delta\left(\int_a^b F(t)\,\mathrm{d}\log(t+c)\right)$.

$$\delta\mathcal{I} = F(b)\,\frac{\delta b}{b+c} - F(a)\,\frac{\delta a}{a+c}$$
$$+ \left(\int_a^b F(t)\left(\partial_u \frac{1}{t+c}\right)\mathrm{d}t\right)\delta u$$
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Rules for Linearly Reducible Cases

Integrating the second term by parts,

$$\int_a^b F(t)\,\mathrm{d}\delta\log(t+c) = F(b)\,\frac{\delta c}{b+c} - F(a)\,\frac{\delta c}{a+c} - \int_a^b \underbrace{H(t)\,\mathrm{d}\log(t+d)\wedge\delta\log(t+c)}_{\mathrm{d}F(t)\wedge\delta\log(t+c)}\,.$$

Putting everything together,

$$\delta\mathcal{I} = F(b)\,\delta\log(b+c) - F(a)\,\delta\log(a+c) + \int_a^b H(t)\left[D\log(t+c)\wedge D\log(t+d)\right]^{(1,1)}\,.$$

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$$[D\log(t+c)\wedge D\log(t+d)]^{(1,1)}\in\Omega^1(S^2)\otimes\Omega^1(\mathcal{K}).$$

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Completely determined by poles and residues!

$$[D\log(t+c)\wedge D\log(t+d)]^{(1,1)}=\mathrm{d}\log\frac{t+c}{t+d}\wedge\delta\log(c-d).$$

$$\begin{aligned}\delta\mathcal{I}=&\,F(b)\,\delta\log(b+c)-F(a)\,\delta\log(a+c)\\&+\left(\int_a^b H(t)\,\mathrm{d}\log\frac{t+c}{t+d}\right)\delta\log(c-d).\end{aligned}$$

Summary

Given $DF(t) = H(t)\,Dw(t)$, integrating by parts implies

$$\delta\mathcal{I} = \delta\left(\int F(t)\,\mathrm{d}\kappa(t)\right) = \text{boundary} + \int H(t)\,\underbrace{D\kappa(t)\wedge Dw(t)}_{=:\,\omega}.$$

To obtain the last entry, separate the t-dependence of $\omega^{(1,1)}$:

$$\omega^{(1,1)} = \mathrm{d}\mu(t)\wedge\delta W \implies \delta\mathcal{I} = \text{boundary} + \left(\int H(t)\,\mathrm{d}\mu(t)\right)\delta W.$$

To compute $\delta\mathcal{I}$ purely algebraically, match residues! [Demo]

"Net" Square Roots

Suppose $\deg \Delta(t) \leq 2$ and $\deg R(t)$ arbitrary.

$$\omega = D\log \frac{A(t) + \sqrt{R(t)}}{A(t) - \sqrt{R(t)}} \wedge D\log \frac{B(t) + \sqrt{R(t)}\sqrt{\Delta(t)}}{B(t) - \sqrt{R(t)}\sqrt{\Delta(t)}}.$$

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$\sqrt{R(t)}$ is “spurious”. $\tilde{\omega}^{(1,1)} := \sqrt{\Delta(t)}\,\omega^{(1,1)}$ is single-valued on S^2.

$$\omega^{(1,1)} = \sum_{t_0 \in \{\text{poles of } \tilde{\omega}^{(1,1)}\}} \underbrace{\frac{\sqrt{\Delta(t_0)}\,\mathrm{d}t}{(t-t_0)\sqrt{\Delta(t)}}}_{\mathrm{d}\log \frac{C(t)+\sqrt{\Delta(t)}}{C(t)-\sqrt{\Delta(t)}}} \wedge \operatorname*{Res}_{t=t_0} \tilde{\omega}^{(1,1)}.$$

"Net" Square Roots

Similarly, with $\deg R(t)$ arbitrary,

$$\omega = D\log\frac{A(t)+\sqrt{R(t)}}{A(t)-\sqrt{R(t)}} \wedge D\log\frac{B(t)+\sqrt{R(t)}}{B(t)-\sqrt{R(t)}}.$$

There are no "net" square roots.

$$\omega^{(1,1)} = \sum_{t_0\in\{\text{poles of } \omega^{(1,1)}\}} \mathrm{d}\log(t-t_0)\wedge \operatorname*{Res}_{t=t_0}\omega^{(1,1)}.$$

[Demo]

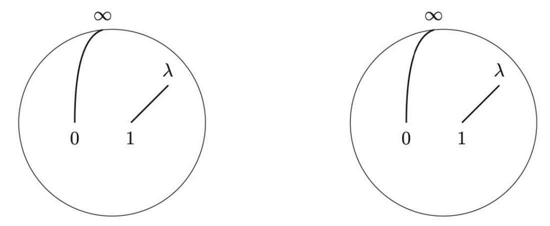
Elliptic Multiple Polylogarithms [1712.07089] [1803.10256]

Feature: $\sqrt{P(t)}$ with $\deg P(t) = 3, 4$ irreducible. Simplest example:

$$\mathcal{I} = \frac{1}{\omega_1} \int_a^b F(t) \frac{\mathrm{d}t}{\sqrt{P(t)}}, \qquad DF(t) = H(t)\, D \log \frac{B(t) + \sqrt{P(t)}}{B(t) - \sqrt{P(t)}}.$$

$\sqrt{P(t)}$ cannot be rationalized, because it involves the elliptic curve

$$\mathcal{E} = \{(t, y) \,|\, y^2 = P(t)\} \subset \mathbb{CP}^2.$$



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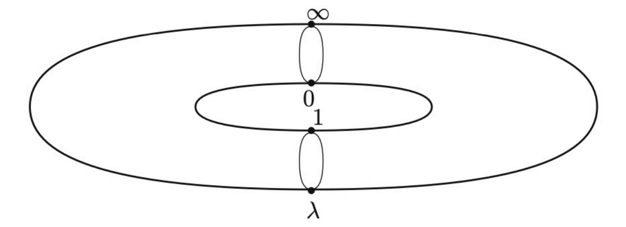

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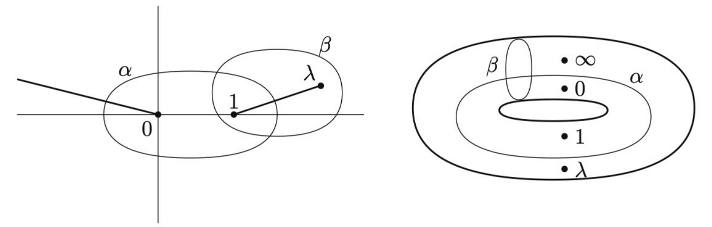

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$$\omega_{1,2} := \oint_{\alpha,\beta} \frac{\mathrm{d}t}{\sqrt{P(t)}}, \qquad \tau := \omega_2/\omega_1, \qquad \mathcal{E} \cong \mathbb{C}/\langle \omega_1, \omega_2 \rangle \cong \mathbb{C}/\langle 1, \tau \rangle.$$

2-form (Simplest Case)

The integration kernel is not $\mathrm{d}\log$, but

$$\frac{1}{\omega_1}\frac{\mathrm{d}t}{\sqrt{P(t)}} = \mathrm{d}W(t), \quad W(t) := \frac{1}{\omega_1}\int_*^t \frac{\mathrm{d}t'}{\sqrt{P(t')}}.$$

Through a bi-rational change of variables $(t, y) \mapsto (T, Y)$,

$$\mathcal{E} = \{(T, Y)\,|\,Y^2 = 4T^3 - g_2 T - g_3\} \subset \mathbb{CP}^2.$$

Using the isomorphism $(\wp, \wp') : \mathbb{C}/\langle\omega_1, \omega_2\rangle \mapsto \mathcal{E}$,

$$W(t) = \wp^{-1}(T; g_2, g_3)/\omega_1, \quad \text{defined up to } \langle 1, \tau\rangle.$$

Differential up to $\delta\tau$

We need to separate the t-dependence of

$$\omega = DW(t) \wedge D \log \frac{B(t) + \sqrt{P(t)}}{B(t) - \sqrt{P(t)}}.$$

$\sqrt{P(t)}$ is spurious, but $DW(t)$ is not single-valued on S^2.

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Restrict to the subspace $\underline{\mathcal{K}} \subset \mathcal{K}$ defined by $\delta\tau = 0$.

Rules for Elliptic Symbol Integration

In other words, we consider $\underline{\delta}\mathcal{I}$ instead of $\delta\mathcal{I}$, and consider

$$\omega = \underline{D}W(t) \wedge \underline{D} \log \frac{B(t) + \sqrt{P(t)}}{B(t) - \sqrt{P(t)}}.$$

$\omega^{(1,1)}$ is an $\Omega^1(\underline{\mathcal{K}})$-valued meromorphic single-valued 1-form on S^2.

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$\underline{D}W(t)$ has no poles. Define $t_\pm$ such that $B(t_\pm) \mp \sqrt{P(t_\pm)} = 0$.

$$\omega^{(1,1)} = \sum_{t_\pm} \pm \mathrm{d}\log(t-t_\pm) \wedge \underline{\delta}W(t_\pm).$$

Rules for Elliptic Symbol Integration

Similarly, if there is a "net" square root $\sqrt{B(t)}$ with $\deg B(t) \leq 2$,

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$$\omega^{(1,1)} = \sum_{t_\pm} \pm \frac{\sqrt{\Delta(t_\pm)}\,\mathrm{d}t}{(t - t_\pm)\sqrt{\Delta(t)}} \wedge \underline{\delta} W(t_\pm),$$

$$\underline{\delta}\mathcal{I} = \text{boundary} + \sum_{t_\pm} \pm \left(\int_a^b H(t)\, \frac{\sqrt{\Delta(t_\pm)}\,\mathrm{d}t}{(t - t_\pm)\sqrt{\Delta(t)}} \right) \underline{\delta} W(t_\pm).$$

[Demo]