Intersection Theory
Rules
Symbology

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arxiv: 2305.01283

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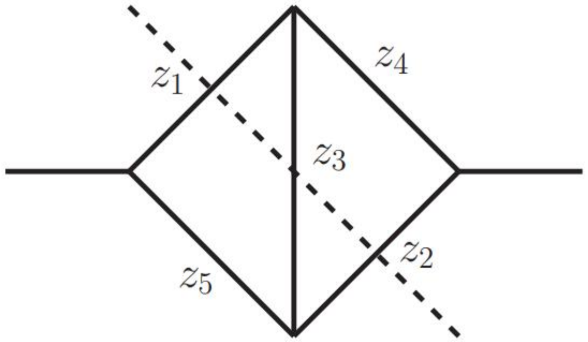
Part 01

Background & Motivation

Canonical Differential Equation (CDE) & Symbology



Example: Kite diagram (cut sunrise propagator)

$$z_1 = l_1^2 - m^2, \quad z_2 = (l_2 - p)^2 - m^2, \quad z_3 = (l_1 - l_2)^2,$$
$$z_4 = l_2^2, \quad z_5 = (l_1 - p)^2, \quad p^2 = s,$$

3 Master integrals: $\varphi_1, \varphi_2, \varphi_3$ DE given by $\partial_{s_i} \langle\varphi| = \Omega^{(s_i)} \langle\varphi|$

Reduction： Integration-By-Parts (IBP)

$$\boxed{\int d^d k \frac{\partial}{\partial k^\mu} q^\mu \frac{1}{(k^2 - m_1^2)^{\alpha_1}((k-p_1)^2 - m_2^2)^{\alpha_2} \cdots} = 0}$$

Canonical Differential Equation (CDE) & Symbology

UT basis gives CDE: $\hat{\mathrm{d}}\,\langle\varphi_I| = \left(\hat{\mathrm{d}}\Omega\right)_{IJ}\langle\varphi_J| \qquad \hat{\mathrm{d}}\Omega = \sum_i \Omega^{(s_i)}\hat{\mathrm{d}}s_i$

$$\Omega = \varepsilon \begin{pmatrix} -2\log(m^2) - \log(1-2m^2) & 0 & 4\left(\log(m^2) + 2\log(1-2m^2) - 3\log(1-m^2)\right) \\ 0 & 2\log(m^2) - 3\log(1-4m^2) & 6\tan^{-1}\left(\sqrt{4m^2-1}\right) \\ 0 & 4\tan^{-1}\left(\sqrt{4m^2-1}\right) & -3\log(m^2) \end{pmatrix}$$

Rescale:s→1

$$\langle\varphi_I|^{(i+1)} = \underbrace{B_I^{(i+1)}}_{\text{boundary}} + \underline{\int \mathrm{d}\Omega_{IJ}\,\langle\varphi_J|^{(i)}}$$

Symbol: $S[W_1, \underline{W_2}, \cdots] \equiv \int \mathrm{d}\log W_1 \int \mathrm{d}\log W_2 \cdots$

Symbol Letter (whose complete set is known as the **symbol alphabet**)

⇒

Bootstrap Amplitude

Canonical Differential Equation (CDE)

$$
\begin{aligned}
\langle\varphi_1| \quad & \frac{1}{4}\,\mathrm{eps}^2\,\big(G[1,\{-1,0,1,1,1\}] + \mathrm{m2}\,G[1,\{-1,1,1,1,1\}] + G[1,\{0,-1,1,1,1\}] - G[1,\{0,0,0,1,1\}] - \\
& G[1,\{0,0,1,0,1\}] - G[1,\{0,0,1,1,0\}] - G[1,\{0,0,1,1,1\}] + 4\,\mathrm{m2}\,G[1,\{0,0,1,1,1\}] + G[1,\{0,1,0,0,1\}] - \\
& G[1,\{0,1,0,1,1\}] - \mathrm{m2}\,G[1,\{0,1,0,1,1\}] - G[1,\{0,1,1,0,0\}] - \mathrm{m2}\,G[1,\{0,1,1,0,1\}] + G[1,\{0,1,1,1,0\}] - \\
& \mathrm{m2}\,G[1,\{0,1,1,1,0\}] - \mathrm{m2}\,G[1,\{0,1,1,1,1\}] + 3\,\mathrm{m2}^2\,G[1,\{0,1,1,1,1\}] + \mathrm{m2}\,G[1,\{1,-1,1,1,1\}] + \\
& G[1,\{1,0,0,1,0\}] - G[1,\{1,0,0,1,1\}] - \mathrm{m2}\,G[1,\{1,0,0,1,1\}] - G[1,\{1,0,1,0,0\}] + G[1,\{1,0,1,0,1\}] - \\
& \mathrm{m2}\,G[1,\{1,0,1,0,1\}] - \mathrm{m2}\,G[1,\{1,0,1,1,0\}] - \mathrm{m2}\,G[1,\{1,0,1,1,1\}] + 3\,\mathrm{m2}^2\,G[1,\{1,0,1,1,1\}] + \\
& G[1,\{1,1,-1,1,1\}] - G[1,\{1,1,0,0,0\}] - G[1,\{1,1,0,0,1\}] + \mathrm{m2}\,G[1,\{1,1,0,0,1\}] - G[1,\{1,1,0,1,0\}] + \\
& \mathrm{m2}\,G[1,\{1,1,0,1,0\}] + G[1,\{1,1,0,1,1\}] - 2\,\mathrm{m2}\,G[1,\{1,1,0,1,1\}] - \mathrm{m2}^2\,G[1,\{1,1,0,1,1\}] + G[1,\{1,1,1,-1,0\}] + \\
& G[1,\{1,1,1,0,-1\}] - G[1,\{1,1,1,0,0\}] - 2\,\mathrm{m2}\,G[1,\{1,1,1,0,0\}] + \mathrm{m2}\,G[1,\{1,1,1,0,1\}] - \mathrm{m2}^2\,G[1,\{1,1,1,0,1\}] + \\
& \mathrm{m2}\,G[1,\{1,1,1,1,0\}] - \mathrm{m2}^2\,G[1,\{1,1,1,1,0\}] - \mathrm{m2}^2\,G[1,\{1,1,1,1,1\}] + 2\,\mathrm{m2}^3\,G[1,\{1,1,1,1,1\}]\big)
\end{aligned}
$$

$$\langle\varphi_2| \quad \frac{1}{8}\sqrt{-1+4\,m^2}\;\epsilon^2\,G[1,\{1,1,1,0,0\}] \qquad\qquad \langle\varphi_3| \quad \frac{1}{4}\,\epsilon^2\,\big(G[1,\{1,1,1,-1,0\}] - m^2\,G[1,\{1,1,1,0,0\}]\big)$$

$$\boxed{\partial_{m^2}\langle\varphi_1| = \varepsilon\,\left[\,\frac{2}{1-2\,m^2} - \frac{2}{m^2}\,\langle\varphi_1| + 4\left(\frac{3}{1-m^2} + \frac{1}{m^2} - \frac{4}{1-2\,m^2}\right)\langle\varphi_3|\,\right]}$$

Reduction

Why it is so simple?

Previous Research

Specific cases: N=4 SYM (generalized to d dim massless propagator integral), one-loop, first, second, and final entry conditions, etc (many of these methods are algebraic).

General cases: diagrammatic coaction (integration on discontinuity 1703.05064 Abreu, et.al. and its development), direct integration after expansion in the order of ε (recently developed to be algebraic 2304.01776 Tan, He), etc.

The challenges general cases could face:

Divergence (e.g. when expanding with respect to ε)

High order of ε

Complicated contour

Non-rational symbol (Rationalize all variables→GPL)

…

(many of which emerge in the integration method)

Our starting point

Challenge: High order of ε → Solved by CDE, iteratively giving symbols to arbitrary order of ε.

Our previous one-loop practice: Integrate the expansion of d log integrand (UT) to get CDE element (also symbol letters)

2201.12998 Chen, et al. (d log integrand gives UT 1212.5605 Nima, et al, construction in Baikov rep. see 2008.03045 & 2202.08127 Chen, et al)

Observation: despite the complicated contours and trick plus distribution expansion for divergent cases the calculation could involve, letters are simply just the d log integrand take the pole values! (Roughly)

Suggestion: a simpler algebraic method should exist, where the multivariate poles play the key roles.

Intersection theory
+
d log integrand

Show the Role of multivariate poles

CDE
(Symbology)

Intersection number: an IBP invariant inner product

Hypergeometric-like functions:

$$\langle\varphi_L| \equiv \int u\,\varphi_L\,, \quad |\varphi_R\rangle \equiv \int u^{-1}\,\varphi_R\,,$$

u-factor

$$u = \prod_i [\underline{P_i(\boldsymbol{z})}]^{\beta_i}\,, \quad \varphi \equiv \hat{\varphi}(\boldsymbol{z})\bigwedge_j \mathrm{d}z_j = \frac{Q(\boldsymbol{z})}{\prod_i \underline{P_i^{a_i}}}\bigwedge_j \mathrm{d}z_j\,.$$

Baikov rep. & Schwinger parameterization & e.t.c. all take this form.

Intersection number:

$$\langle\varphi_L|\varphi_R\rangle = \sum_{\boldsymbol{p}} \boxed{\mathrm{Res}_{\boldsymbol{z}=\boldsymbol{p}}}\,(\psi_L\hat{\varphi}_R)\,,$$

1810.03818 Mastrolia, Mizera
1907.02000 Frellesvig, et al
2002.10476 Mizera
…

$$\boxed{\nabla_n\cdots\nabla_1\psi_L = \varphi_L}$$

only need information around the poles → solving ψ by Laurent series around poles (to finite order) → legal expansion needs the pole to be factorized → Sector-Decomposition-like transformation factorizes the pole and could divide the region around p into several pieces.

2209.01997 Chestnov, Frellesvig, Gasparotto, Mandal, Mastrolia
Intersection Numbers from Higher-order Partial Differential Equations

Challenge: Complicated contour →
Challenge: Non-rational symbol →
algebraic in intersection theory **but still tricky**. (to be continued)

Factorization of poles

Notation: $\boldsymbol{\rho}^{(\alpha)} : \boldsymbol{z} \longrightarrow \boldsymbol{x}^{(\alpha)}$ α denote the index of a factorized region

$$u(\boldsymbol{x}^{(\alpha)})\big|_{\boldsymbol{x}^{(\alpha)}\to\boldsymbol{\rho}^{(\alpha)}} = \underbrace{\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})}_{\text{non-zero}} \prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\underbrace{\gamma_i^{(\alpha)}}_{\text{u-power}}}$$

factorized

Example (of a Degenerate pole):

p=(0,0) $z_1^{\beta_1} z_2^{\beta_2} \underline{(z_1+z_2)}^{\beta_3}$ $z_1 = x_1$ $z_2 = x_1(x_2-1)$

$$u = x_1^{\beta_1+\beta_2+\beta_3} x_2^{\beta_3} \underbrace{(x_2-1)}_{\text{non-zero}}^{\beta_2}$$

factorized

Challenge: Divergence → addressed locally by factorization of poles and intersection theory.

People do not need to handle all poles at the same time, but only factorize one pole for each time.

Part 02

Universal formula for first- and second-order intersection number

Intersection number

$$\nabla_i = dz_i \wedge (\underline{\partial_{z_i}} + \underline{\hat{\omega}_i})$$

$$\omega = \sum_i \hat{\omega}_i \mathrm{d}z_i\,, \quad \hat{\omega}_i = \underline{\partial_{z_i} \log(u)}$$

only simple pole

$$\underline{\nabla_n \cdots \nabla_1} \psi_L = \varphi_L$$

rank **-1**

$$\langle \varphi_L | \varphi_R \rangle = \sum_{\boldsymbol{p}} \boxed{\mathrm{Res}_{\boldsymbol{z}=\boldsymbol{p}}} (\psi_L \hat{\varphi}_R)\,,$$

$$\varphi^{(\boldsymbol{b})} = C^{(\boldsymbol{b})} \bigwedge_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)} \right]^{\boxed{b_i}} \mathrm{d}x_i^{(\alpha)}$$

Non-zero contributions arise only when $b_{L,i} + b_{R,i} \leq -2$ for all i

CDE of dlog: only first- and second-order contributions to intersection number

d log integrand building block

rational type

$$\mathrm{d}\log(z-c)=\frac{dz}{z-c}$$

sqrt type

$$\mathrm{d}\log(\underline{\tau[z,c;c_\pm]})=\frac{\sqrt{(c-c_+)(c-c_-)}dz}{(z-c)\sqrt{(z-c_+)(z-c_-)}}$$

$$\equiv\mathrm{d}\log\frac{\sqrt{c-c_+}\sqrt{z-c_-}+\sqrt{c-c_-}\sqrt{z-c_+}}{\sqrt{c-c_+}\sqrt{z-c_-}-\sqrt{c-c_-}\sqrt{z-c_+}}$$

CDE in intersection theory

$$\langle\dot{\varphi}_I|\equiv\hat{\mathrm{d}}\,\langle\varphi_I|=\left(\hat{\mathrm{d}}\Omega\right)_{IJ}\langle\varphi_J|$$

$$\left(\hat{\mathrm{d}}\Omega\right)_{IK}=\boxed{\langle\dot{\varphi}_I|\varphi_J\rangle}\,\boxed{\left(\eta^{-1}\right)_{JK}}$$

$$\eta_{IJ}=\langle\varphi_I|\varphi_J\rangle$$

dlog-form only involves multivariate simple pole: $b_i\geq-1$

Non-zero contributions arise only when $b_{L,i}+b_{R,i}\leq-2$ for all i

First-order contribution all $b_{L,i}+b_{R,i}=-2$

Second-order contribution $b_{L,i}+b_{R,i}=-2$ except one $b_{L,j}+b_{R,j}=-3$

Universal formula of First- & Second-order contribution

Directly solving ψ $\nabla_n \cdots \nabla_1 \psi_L = \varphi_L$ $\varphi^{(\boldsymbol{b})} = C^{(\boldsymbol{b})} \bigwedge_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{b_i} \mathrm{d}x_i^{(\alpha)}$

First-order contribution

all $b_{L,i} + b_{R,i} = -2$

$$\frac{C_L^{(\boldsymbol{b}_L)} C_R^{(\boldsymbol{b}_R)}}{\tilde{\gamma}_1^{(\alpha)} \cdots \tilde{\gamma}_n^{(\alpha)}}$$

$$\tilde{\gamma}_i^{(\alpha)} = \gamma_i^{(\alpha)} - b_{R,i} - 1$$

Second-order contribution

$b_{L,i} + b_{R,i} = -2$

$b_{L,j} + b_{R,j} = -3$ for one j

$$-\frac{C_L^{(\boldsymbol{b}_L)} C_R^{(\boldsymbol{b}_R)}}{\tilde{\gamma}_1^{(\alpha)} \cdots \tilde{\gamma}_n^{(\alpha)}} \partial_{\rho_j^{(\alpha)}} \log\left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})\right)$$

Challenge: Complicated contour → →
Challenge: Non-rational symbol → →

Algebraically. Simple and universal formula.

Part 03 CDE Selection Rule

(n-1)- and n-SP contribution

$$(\hat{\mathrm{d}}\Omega)_{IK} = \boxed{\langle \dot{\varphi}_I | \varphi_J \rangle} \, (\eta^{-1})_{JK}$$

φ$_I$ and φ$_J$ share a (n-1)-SP $\boldsymbol{x}_{\underline{\hat{k}}}^{(\alpha)}$
(n-1)-variable Simple Pole

$$b_{I,i} + b_{J,i} = -2 \text{ except } b_{I,\underline{k}} + b_{J,\underline{k}} = -1$$

$\partial_{\rho_{\underline{k}}^{(\alpha)}} \Longrightarrow$ first-order contribution

$$\boxed{-\frac{\gamma_{\underline{k}}^{(\alpha)}}{\boldsymbol{\gamma}^{(\alpha)}} \, \hat{\mathrm{d}} \underline{\int C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)} \, \hat{\mathrm{d}}\rho_{\underline{k}}^{(\alpha)}}}$$

$$\boldsymbol{\gamma}^{(\alpha)} \equiv \gamma_1^{(\alpha)} \cdots \gamma_n^{(\alpha)}$$

We will discuss why it always is a simple dlog-form in the reduced univariate problem later

φ$_I$ and φ$_J$ share a n-SP $\boldsymbol{x}^{(\alpha)}$
n-variable Simple Pole

$$\boldsymbol{b}_I = \boldsymbol{b}_J = \boldsymbol{-1}$$

$\partial_{\rho_i^{(\alpha)}} \Longrightarrow$ second-order contribution

For pure parameter factor $P_0^{\beta_0}$

$\partial_{P_0} \Longrightarrow$ first-order contribution

$$\boxed{\frac{C_I^{(\boldsymbol{-1})} C_J^{(\boldsymbol{-1})}}{\boldsymbol{\gamma}^{(\alpha)}} \, \hat{\mathrm{d}} \log \left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)}) \right)}$$

n-SP chain and non-zero η^{-1}_{JK}

$$(\hat{\mathrm{d}}\Omega)_{IK} = \langle \dot{\varphi}_I | \varphi_J \rangle \boxed{\left(\eta^{-1}\right)_{JK}} \qquad \eta_{IJ} = \langle \varphi_I | \varphi_J \rangle$$

Only when φ_I and φ_J share a n-SP, η_{IJ} could be non-zero.

n-SP chain: If φ_I and φ_J share an n-SP, we say that they are n-SP related (denoted as $\varphi_I \sim \varphi_J$) and belong to a n-SP chain. If $\varphi_J \sim \varphi_K$ and $\varphi_I \sim \varphi_J$, the three n-forms belong to an n-SP chain. Similar for more dlog-forms.

Only when φ_J and φ_K in the same n-SP chain, η^{-1}_{JK} could be non-zero.

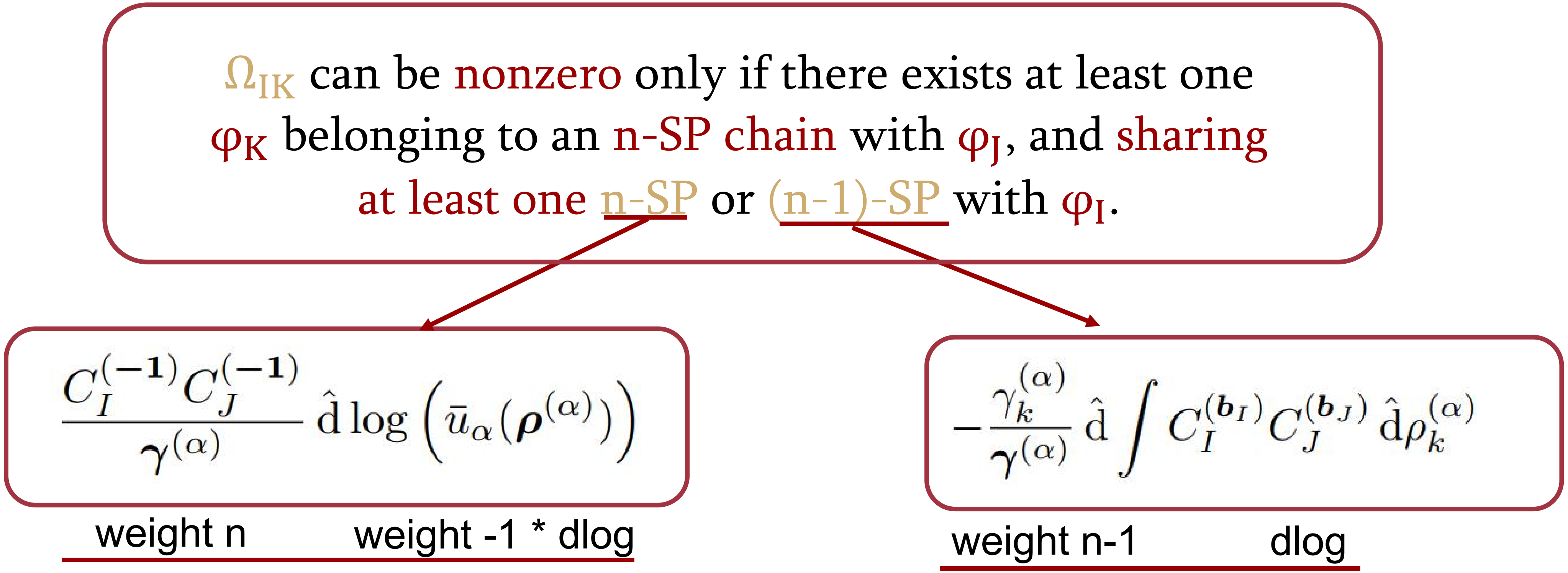
Example:

$\varphi_J \sim \varphi_1$, $\varphi_1 \sim \varphi_2$, $\varphi_2 \sim \varphi_K$,
then a term in the cofactor $\eta^{(JK)}$ could be non-zero.

$\eta^{-1}_{JK} = \eta^{(JK)}/|\eta|$

$$\begin{array}{c|cccccc} & K & 1 & 2 & 3 & 4 & 5 \\ \hline J & 0 & \eta_{J1} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \eta_{12} & 0 & 0 & 0 \\ 2 & \eta_{2K} & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & \eta_{33} & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & \eta_{44} & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & \eta_{55} \end{array}$$

CDE Selection Rules



Ω_{IK} can be nonzero only if there exists at least one φ_K belonging to an n-SP chain with φ_J, and sharing at least one n-SP or (n-1)-SP with φ_I.

$$\frac{C_I^{(-1)}C_J^{(-1)}}{\gamma^{(\alpha)}}\,\hat{\mathrm{d}}\log\left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})\right)$$

weight n

weight -1 * dlog

$$-\frac{\gamma_k^{(\alpha)}}{\gamma^{(\alpha)}}\,\hat{\mathrm{d}}\int C_I^{(\boldsymbol{b}_I)}C_J^{(\boldsymbol{b}_J)}\,\hat{\mathrm{d}}\rho_k^{(\alpha)}$$

weight n-1

dlog

all u-powers regarded as weight -1

$$\left(\hat{\mathrm{d}}\Omega\right)_{IK} = \langle\dot{\varphi}_I|\varphi_J\rangle\left(\eta^{-1}\right)_{JK}$$

weight -n

⇨ weight -1 coeff x dlog

We show how canonical form emerges!

Part 04

Reduce to univariate problem

Reduce to univariate problem

n-SP

$$\frac{C_I^{(-1)} C_J^{(-1)}}{\boxed{\boldsymbol{\gamma}^{(\alpha)}}} \hat{\mathrm{d}} \log\left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})\right)$$

(n-1)-SP

$$-\boxed{\frac{\gamma_k^{(\alpha)}}{\boldsymbol{\gamma}^{(\alpha)}}} \hat{\mathrm{d}} \int C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)} \, \hat{\mathrm{d}}\rho_k^{(\alpha)}$$

$$\boldsymbol{\gamma}^{(\alpha)} \equiv \gamma_1^{(\alpha)} \cdots \gamma_n^{(\alpha)}$$

Take the n-1 variable Residue on $\boldsymbol{x}_{\hat{k}}^{(\alpha)}$ gives the contribution $\prod_{i\neq k} \gamma_i^{(\alpha)}$

Then, the multivariate problem is reduced to a univariate one.

Which can be easily analyzed systematically!

Rational-type univariate problem

$$u = \underset{\overset{\uparrow}{\partial_{P_0}}}{P_0^{\beta_0}}(z - \underset{\overset{\uparrow}{\partial_{\rho_i^{(\alpha)}}}}{c_1})^{\beta_1}(z - c_2)^{\beta_2}(z - c_3)^{\beta_3}$$

Although the parameters here could involve the same original parameters, they are regarded as independent when we use a partial operator for each selected $\boldsymbol{x}_{\hat{k}}^{(\alpha)}$.

$$c_\alpha \in \{c_1, c_2, c_3, \infty\},$$

$$\gamma^{(\alpha)} \in \left\{\beta_1, \beta_2, \beta_3, -\sum_i \beta_i\right\}$$

dlog integrand: $\varphi_I \in \left\{\frac{\mathrm{d}z}{z - c_1}, \frac{\mathrm{d}z}{z - c_2}\right\}$

Each φ_I involves two poles, $\underline{c_I \text{ and } c_4 = \infty}$.

$$\Downarrow$$

$$\langle \dot{\varphi}_I | \varphi_I \rangle = \sum_{\alpha \neq I} \frac{\gamma^{(\alpha)}}{\gamma^{(I)}} \hat{\mathrm{d}} \log(c_I - c_\alpha) + \eta_{II}\beta_0 \, \hat{\mathrm{d}} \log P_0$$

$$\langle \dot{\varphi}_I | \varphi_J \rangle = -\hat{\mathrm{d}} \log(c_I - c_J) + \eta_{IJ}\beta_0 \, \hat{\mathrm{d}} \log P_0$$

$$\eta = \begin{pmatrix} \frac{1}{\gamma^{(1)}} + \frac{1}{\gamma^{(4)}} & \frac{1}{\gamma^{(4)}} \\ \frac{1}{\gamma^{(4)}} & \frac{1}{\gamma^{(2)}} + \frac{1}{\gamma^{(4)}} \end{pmatrix}$$

All letters are pole distance and pure parameter factor!

Sqrt-type univariate problem

$$u = (z - c_1)^{\beta_1}(z - c_2)^{\beta_2}(z - c_+)^{\beta_3}(z - c_-)^{\beta_4}$$

$$c_\alpha \in \{c_1, c_2, \infty, c_+, c_-\}\,,$$

$$\gamma^{(\alpha)} \in \left\{\beta_1, \beta_2, -\sum_i \beta_i, \beta_3, \beta_4\right\}$$

dlog integrand

$$\mathrm{d}\log\tau[z, c_1; c_\pm],\ \mathrm{d}\log\tau[z, c_2; c_\pm],\ \mathrm{d}\log\tau[z, \infty; c_\pm]$$

$$\begin{aligned}\langle\dot{\varphi}_I|\varphi_I\rangle &= \frac{1}{\gamma^{(I)}}\,\hat{\mathrm{d}}\log(\bar{u}_I(c_I)) - \hat{\mathrm{d}}\log(c_+ - c_-)\\ &\quad + \hat{\mathrm{d}}\log(c_I - c_+) + \hat{\mathrm{d}}\log(c_I - c_-)\,,\\ \langle\dot{\varphi}_I|\varphi_J\rangle &= \langle\dot{\varphi}_J|\varphi_I\rangle = -\hat{\mathrm{d}}\log\underline{\tau[c_I, c_J; c_\pm]}\,.\end{aligned}$$

pole distance
except for the τ

By univariate rationalization, all conclusions of rational-type apply!

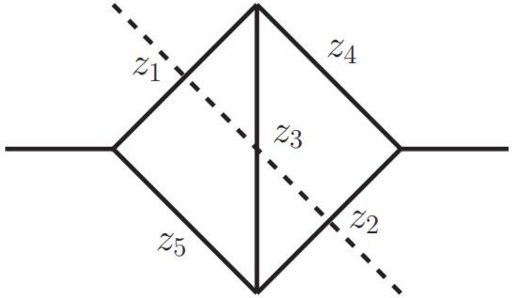
Challenge:

Non-rational symbol → → →By merely univariate rationalization

→ Letters are pole distance and parameter factors

Part 04 Newton polytope

2-loop pedagogical example



Kite diagram (cut sunrise propagator)

$$z_1 = l_1^2 - m^2, \quad z_2 = (l_2 - p)^2 - m^2, \quad z_3 = (l_1 - l_2)^2,$$
$$z_4 = l_2^2, \quad z_5 = (l_1 - p)^2, \quad p^2 = s,$$

$$\boxed{u = z_4^{\delta_1} z_5^{\delta_2} [\mathcal{G}(z_4, z_5)]^{-\epsilon}}$$

$$\mathcal{G} \equiv \mathrm{G}(l_1, l_2, p)\Big|_{z_1=z_2=z_3=0} = -2m^6 + m^4(s + z_4 + z_5) + m^2(2z_4 z_5 - sz_4 - sz_5) + z_4 z_5(s - z_4 - z_5)$$

Example: $\mathrm{degenerate\ pole}\ (\infty, 0)$ $\qquad z_4 = 1/t_4$

$$\boxed{u = t_4^{2\epsilon - \delta_1} z_5^{\delta_2} \mathcal{G}_{\infty 0}^{-\epsilon}} \qquad \mathcal{G}_{\infty 0} \equiv t_4^2 \mathcal{G}(1/t_4, z_5) \equiv t_4[r_+(t_4) - z_5]\underline{[z_5 - r_-(t_4))]}$$

$$z_5 - m^2(m^2 - s)t_4 + \mathcal{O}(t_4^2)$$

Another factor can provide a pole at (∞,0)

→degenerate pole

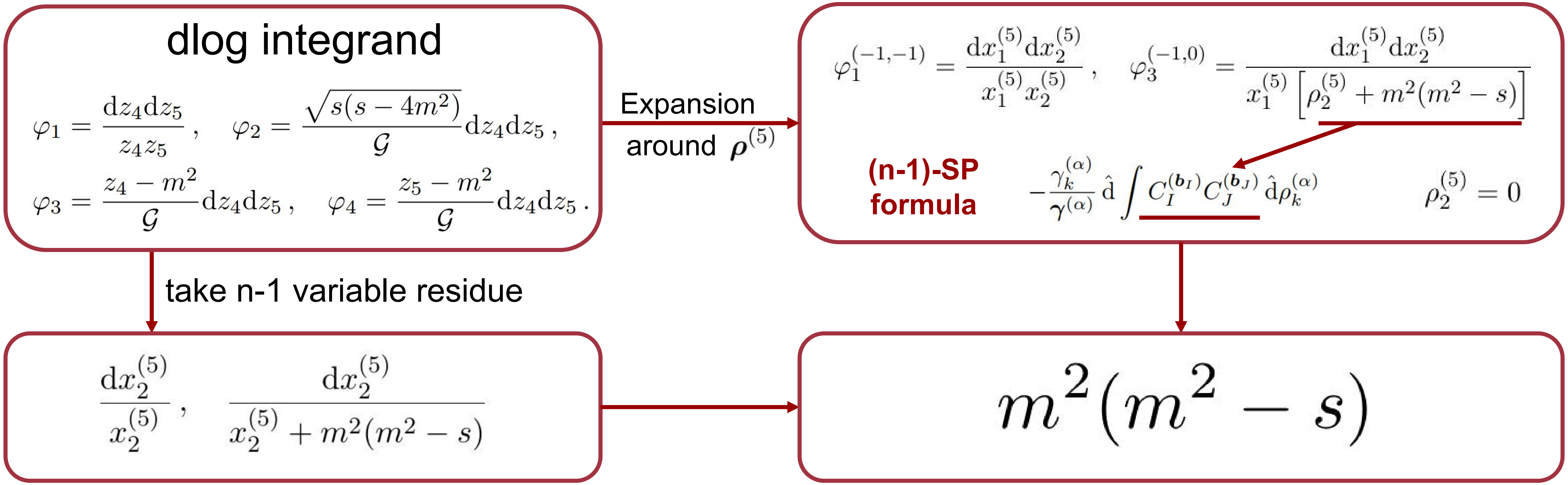
$\boldsymbol{\rho}^{(5)}$ in degenerate pole $(\infty, 0)$

Denote 3 factorized regions at (∞,0) as

$$\boldsymbol{x}^{(4)} : (\{t_4\}, \{z_5, z_5 - r_-(t_4)\})\,,$$
$$\boldsymbol{x}^{(5)} : (\{t_4, z_5 - r_-(t_4)\}, \{z_5\})\,,$$
$$\boldsymbol{x}^{(6)} : (\{z_5 - r_-(t_4)\}, \{t_4, z_5\})\,.$$

$$\xrightarrow[\text{transformation}]{\text{Factorization}} \quad t_4 = x_1^{(5)}\,, \qquad z_5 = x_1^{(5)} x_2^{(5)}$$

letter in $\langle \dot{\varphi}_1 | \varphi_3 \rangle$ from (n-1)-SP $\boldsymbol{x}^{(5)}_{\hat{2}}$



dlog integrand

$$\varphi_1 = \frac{\mathrm{d}z_4 \mathrm{d}z_5}{z_4 z_5}\,, \qquad \varphi_2 = \frac{\sqrt{s(s-4m^2)}}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5\,,$$
$$\varphi_3 = \frac{z_4 - m^2}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5\,, \qquad \varphi_4 = \frac{z_5 - m^2}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5\,.$$

Expansion around $\boldsymbol{\rho}^{(5)}$

$$\varphi_1^{(-1,-1)} = \frac{\mathrm{d}x_1^{(5)} \mathrm{d}x_2^{(5)}}{x_1^{(5)} x_2^{(5)}}\,, \qquad \varphi_3^{(-1,0)} = \frac{\mathrm{d}x_1^{(5)} \mathrm{d}x_2^{(5)}}{x_1^{(5)} \underline{\left[\rho_2^{(5)} + m^2(m^2 - s)\right]}}$$

(n-1)-SP formula

$$-\frac{\gamma_k^{(\alpha)}}{\gamma^{(\alpha)}} \hat{\mathrm{d}} \int \underline{C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)}}\, \hat{\mathrm{d}} \rho_k^{(\alpha)} \qquad \rho_2^{(5)} = 0$$

take n-1 variable residue

$$\frac{\mathrm{d}x_2^{(5)}}{x_2^{(5)}}\,, \qquad \frac{\mathrm{d}x_2^{(5)}}{x_2^{(5)} + m^2(m^2 - s)}$$

$$m^2(m^2 - s)$$

Newton polytope

$$\mathcal{G}_{\infty 0} \equiv t_4^2\,\mathcal{G}(1/t_4, z_5) \equiv t_4[r_+(t_4)-z_5][z_5-r_-(t_4))]$$

$$\boxed{z_5 - \underline{m^2(m^2-s)}t_4 + \mathcal{O}(t_4^2)}$$

The Letter

The high-order terms will automatically vanish in the calculation.

$$\mathcal{G}_{\infty 0} = (m^4 s - 2m^6)t_4^2 + \underline{(m^4 - m^2 s)t_4}$$
$$+ m^4 t_4^2 z_5 - m^2 s t_4^2 z_5 + 2m^2 t_4 z_5 + s t_4 z_5$$
$$- t_4 z_5^2 \underline{- z_5},$$

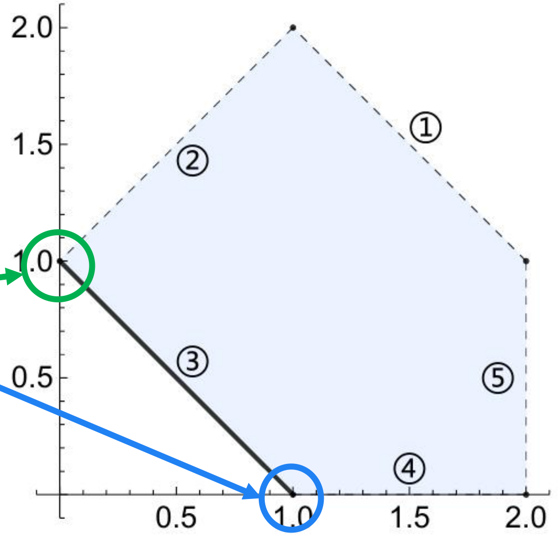


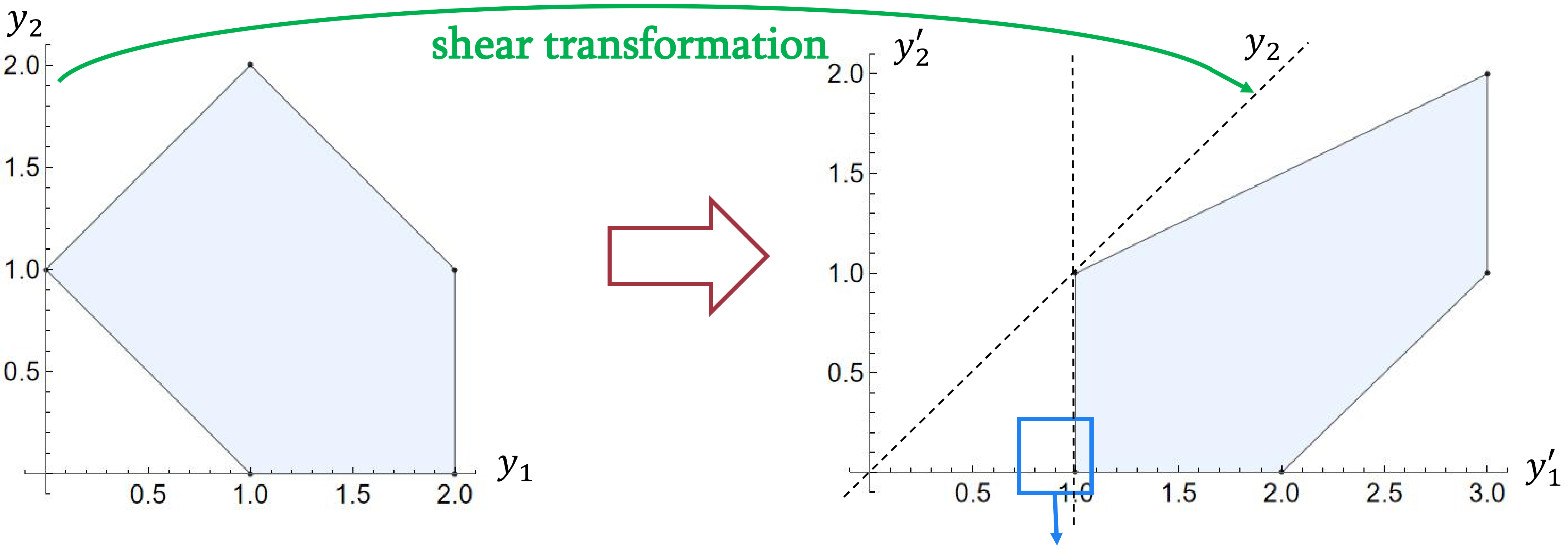
FIG. 1. The Newton polytope of $\mathcal{G}_{\infty 0}$. Horizontal and vertical axis are the power of t_4 and z_5. The solid line represents the zero facet of $(\infty, 0)$.

Facet ③ is a degenerate facet for (∞,0).
(The normal vector points towards the fourth quadrant.)

Letters are related to the coefficient of the vertices!

Factorization in Newton polytope

Factorization transformation $$t_4 = x_1^{(5)}\,, \quad z_5 = x_1^{(5)} x_2^{(5)}$$



a z_4^1 is factorized out, and no degenerate factor remained.

Ω_{13}

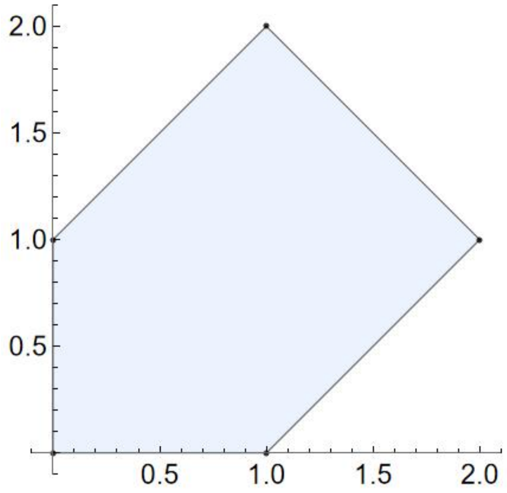
$$\left(\hat{\mathrm{d}}\Omega\right)_{IK} = \langle\dot{\varphi}_I|\varphi_J\rangle\left(\eta^{-1}\right)_{JK}$$

$$\left(\eta^{-1}\right)_{JK} = \begin{pmatrix} \frac{\delta_1\delta_2(-\delta_1-\delta_2+\epsilon)}{\epsilon} & 0 & -2\delta_1\delta_2 & -2\delta_1\delta_2 \\ 0 & \epsilon^2 & 0 & 0 \\ -2\delta_1\delta_2 & 0 & -2\epsilon\left(\delta_2+\epsilon\right) & -\epsilon\left(\delta_1+\delta_2+\epsilon\right) \\ -2\delta_1\delta_2 & 0 & -\epsilon\left(\delta_1+\delta_2+\epsilon\right) & -2\epsilon\left(\delta_1+\epsilon\right) \end{pmatrix}$$

$\langle\dot{\varphi}_1|\varphi_1\rangle$ $\langle\dot{\varphi}_1|\varphi_3\rangle$ $\langle\dot{\varphi}_1|\varphi_4\rangle$ need to be considered.

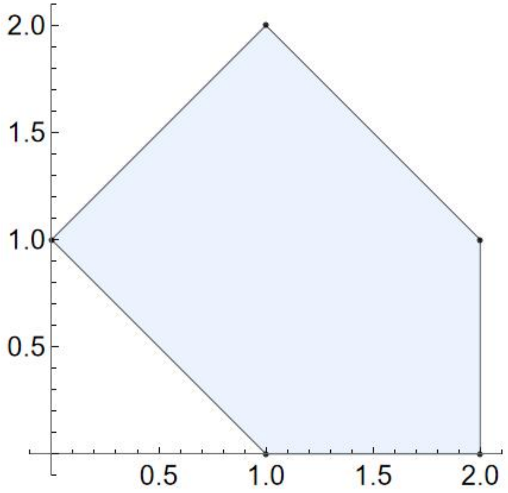
Newton polytopes related to Ω_{13}

$(0, 0)$



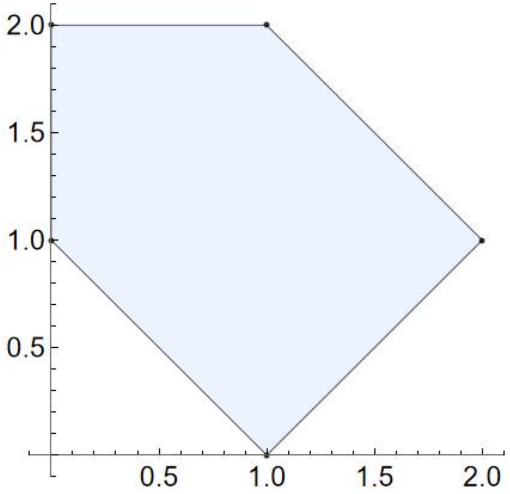
Non-degenerate

$(\infty, 0)$



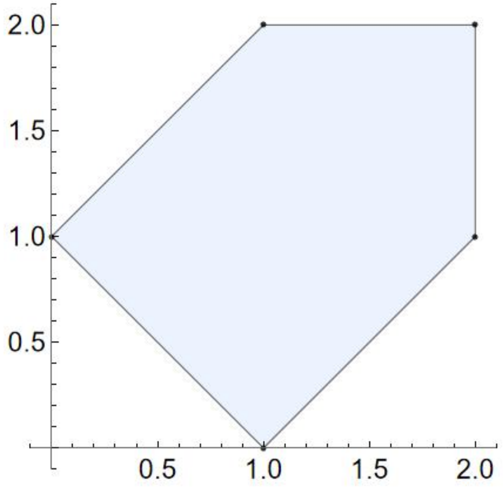
degenerate

$(0, \infty)$



degenerate

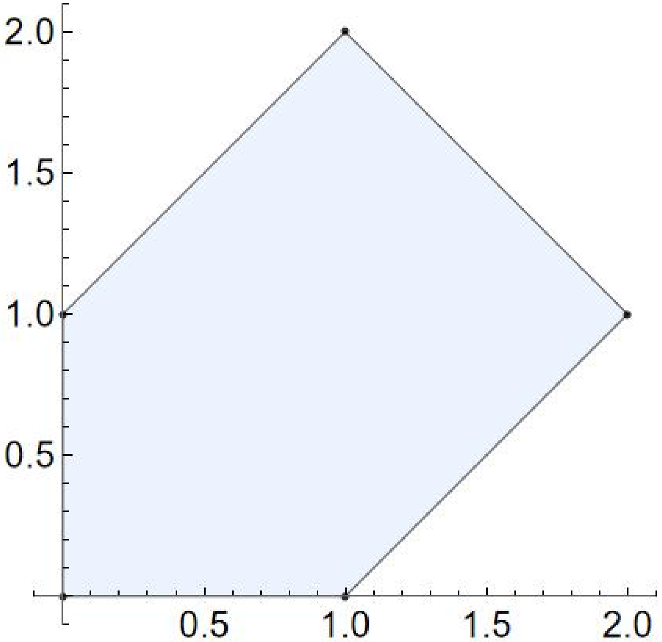
(∞, ∞)



degenerate

Ω_{13}

$$\mathcal{G} = -2m^6 + m^4(s + z_4 + z_5) + m^2(2z_4z_5 - sz_4 - sz_5) + z_4z_5(s - z_4 - z_5)$$



$\boxed{(0,0):\ -2m^6 + m^4}$

$\boxed{(1,0):\ m^4 - sm^2}$

$(2,1):\ -1$

$(0,1):\ m^4 - sm^2$

$(1,2):\ -1$

letters in Ω_{13}

$-2m^6 + m^4$

$m^4 - sm^2$

maximal cut

$$\begin{pmatrix} -2\log(m^2) - \log(1-2m^2) & 0 & \underline{4\left(\log(m^2) + 2\log(1-2m^2) - 3\log(1-m^2)\right)} \\ 0 & 2\log(m^2) - 3\log(1-4m^2) & 6\tan^{-1}\left(\sqrt{4m^2-1}\right) \\ 0 & 4\tan^{-1}\left(\sqrt{4m^2-1}\right) & -3\log(m^2) \end{pmatrix}$$

univariate

$$u_{\text{sunrise}} = (z - m^2)^{-\epsilon}\, z^{\epsilon}\, \lambda(z, m^2, s)^{-\frac{1}{2}-\epsilon}$$

For general cases, one Newton polytope will be not enough.
For sqrt-type, we also can read a part of the information of the letters for the coefficient of vertices.

Part 05

Summary and outlook

Summary: how we handle the challenges in general cases

Challenge: High order of ε → Solved by CDE, iteratively giving symbols to arbitrary order of ε.

Challenge: Divergence → Addressed locally by factorization of poles and intersection theory.

Challenge: Complicated contour → Algebraic formulas in intersection theory.

Challenge: Non-rational symbol → Intersection theory, n-1 variables residue → univariate

Intersection theory plays an essential role to transform the geometric integration contour problem into an algebraic problem

Summary

- With only universal formula for the first- and second-contribution of intersection number, CDE selection rules and the formula for n-SP and (n-1)-SP are derived. We show how CDE emerges for dlog integrand.
- This problem can be transformed into a univariate problem, then all letters are pole distance or parameter factor after merely univariate rationalization.
- We take a first glance at applying Newton polytope which can help people get a geometric and intuitive view of this problem.
- Since CDE also are reduction relations, intersection theory shows its true power to explicitly relate the algebraic structure of reduction and the analytic structure of Feynman integral together!

Outlook

$$u = z_4^{\delta_1} z_5^{\delta_2} [\mathcal{G}(z_4, z_5)]^{-\epsilon}$$

For the reduction aspect:

- CDE are also reduction relations. The reduction relation we get keeps the power of the propagator as a parameter, so it may serve as iterative reduction relation by taking different values of the powers.
- People can investigate elliptic (and beyond) cases.
- The formula beyond first- and second-order contribution to intersection number can be investigated. People could consider avoiding redundant calculations in intersection numbers since they may have the same formula.

For the symbology aspect:

- The role of Newton polytope can be investigated systematically.
- How to read out the symbol alphabet before the construction of dlog integrand could be considered.
- With the development of elliptic UT integrand (such as using the period matrix), elliptic symbology could be analyzed in a similar way.

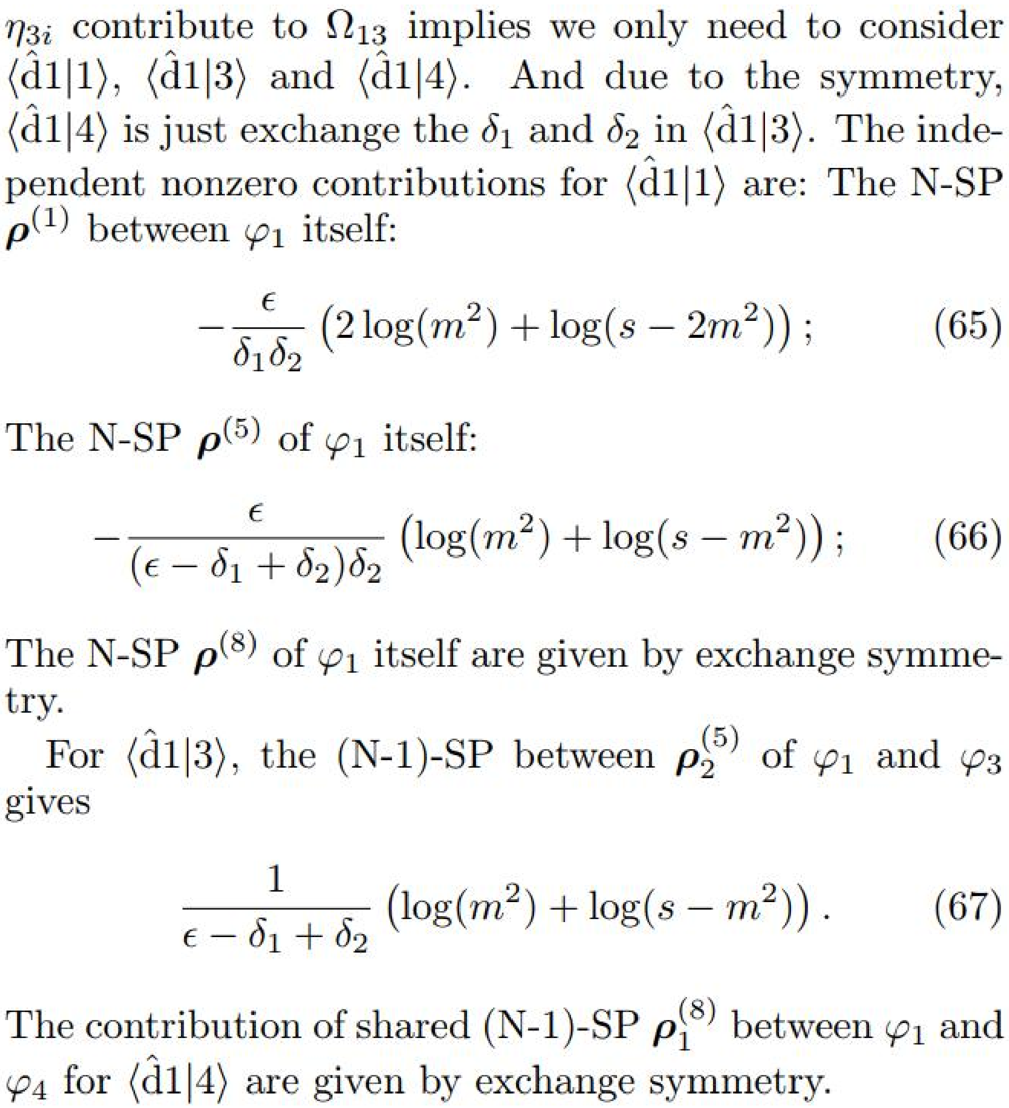
Thank you for listening

Backup

dlog form of master integrals

$$
\begin{aligned}
\varphi_1 &= \mathrm{d}\log(z_4) \wedge \mathrm{d}\log(z_5) \\
\varphi_2 &= \mathrm{d}\log(\tau[z_4, m^2; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_5 - r_{5+}}{z_5 - r_{5-}}\right) \\
\varphi_3 &= -\mathrm{d}\log(\tau[z_4, \infty; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_5 - r_{5+}}{z_5 - r_{5-}}\right), \\
\varphi_4 &= -\mathrm{d}\log(\tau[z_5, \infty; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_4 - r_{4+}}{z_4 - r_{4-}}\right), \\
& r_{1;\pm} \equiv r_{\pm}[\mathcal{G}_1; z_5], \qquad \mathcal{G}_1(z_5) \equiv \mathrm{G}(l_1, p) \\
& r_{4\pm}(z_5) \equiv r_{\pm}[\mathcal{G}; z_4], \quad r_{5\pm}(z_4) \equiv r_{\pm}[\mathcal{G}; z_5], \\
& r_{5+}(\infty) = \infty, \quad r_{5-}(\infty) = 0, \quad r_{5\pm}(m^2) = m^2.
\end{aligned}
$$

Details in Ω_{13}



η_{3i} contribute to Ω_{13} implies we only need to consider $\langle \hat{\mathrm{d}}1|1\rangle$, $\langle \hat{\mathrm{d}}1|3\rangle$ and $\langle \hat{\mathrm{d}}1|4\rangle$. And due to the symmetry, $\langle \hat{\mathrm{d}}1|4\rangle$ is just exchange the δ_1 and δ_2 in $\langle \hat{\mathrm{d}}1|3\rangle$. The independent nonzero contributions for $\langle \hat{\mathrm{d}}1|1\rangle$ are: The N-SP $\boldsymbol{\rho}^{(1)}$ between φ_1 itself:

$$-\frac{\epsilon}{\delta_1\delta_2}\left(2\log(m^2)+\log(s-2m^2)\right); \qquad (65)$$

The N-SP $\boldsymbol{\rho}^{(5)}$ of φ_1 itself:

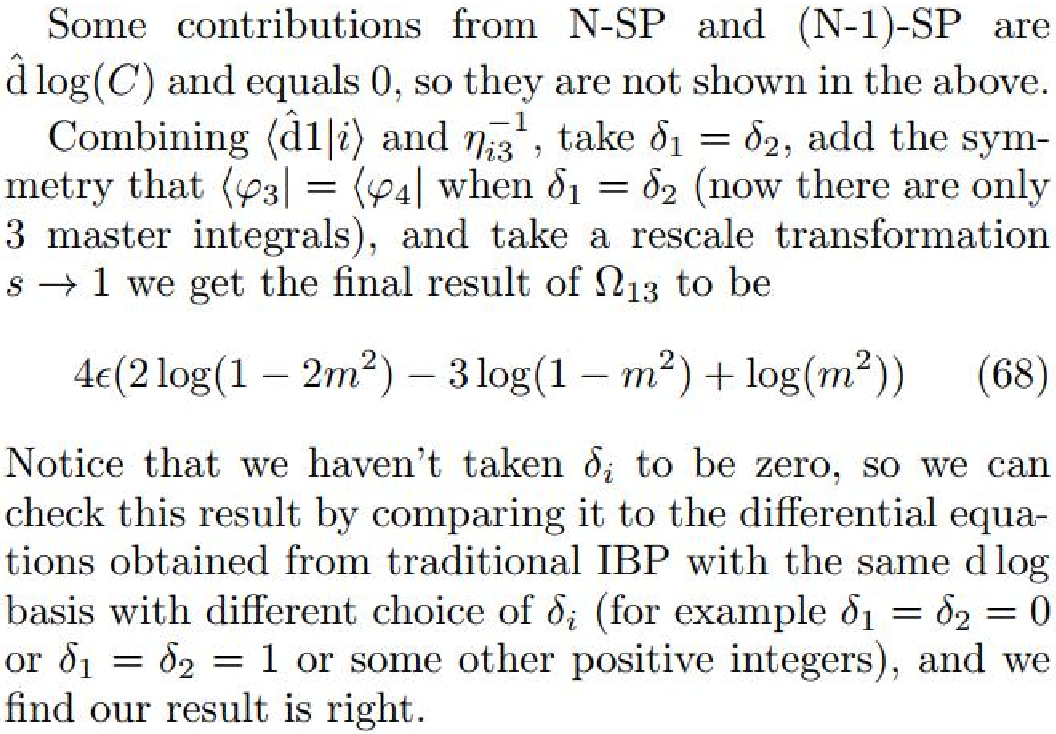
$$-\frac{\epsilon}{(\epsilon-\delta_1+\delta_2)\delta_2}\left(\log(m^2)+\log(s-m^2)\right); \qquad (66)$$

The N-SP $\boldsymbol{\rho}^{(8)}$ of φ_1 itself are given by exchange symmetry.

For $\langle \hat{\mathrm{d}}1|3\rangle$, the (N-1)-SP between $\boldsymbol{\rho}_2^{(5)}$ of φ_1 and φ_3 gives

$$\frac{1}{\epsilon-\delta_1+\delta_2}\left(\log(m^2)+\log(s-m^2)\right). \qquad (67)$$

The contribution of shared (N-1)-SP $\boldsymbol{\rho}_1^{(8)}$ between φ_1 and φ_4 for $\langle \hat{\mathrm{d}}1|4\rangle$ are given by exchange symmetry.



Some contributions from N-SP and (N-1)-SP are $\hat{\mathrm{d}}\log(C)$ and equals 0, so they are not shown in the above.

Combining $\langle \hat{\mathrm{d}}1|i\rangle$ and η_{i3}^{-1}, take $\delta_1=\delta_2$, add the symmetry that $\langle\varphi_3| = \langle\varphi_4|$ when $\delta_1=\delta_2$ (now there are only 3 master integrals), and take a rescale transformation $s\to 1$ we get the final result of Ω_{13} to be

$$4\epsilon(2\log(1-2m^2)-3\log(1-m^2)+\log(m^2)) \qquad (68)$$

Notice that we haven't taken δ_i to be zero, so we can check this result by comparing it to the differential equations obtained from traditional IBP with the same $\mathrm{d}\log$ basis with different choice of δ_i (for example $\delta_1=\delta_2=0$ or $\delta_1=\delta_2=1$ or some other positive integers), and we find our result is right.