LECTURES

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1. Landau equations

Main references :

(1) L.D. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys. 13 (1959) 181.
(2) J. Collins, A new and complete proof of the Landau condition for pinch singularities of Feynman graphs and other integrals, 2007.04085
(3) R.K. Ellis, G. Zanderighi, Scalar one-loop integrals for QCD, JHEP, 0802, 002

Feynman loop integrals or amplitudes are special in the sense that they contain divergences, which prevent us from performing naive numerical evaluation (branch cut is another problem). These divergences can be categorized to ultra-violet (UV) and infra-red (IR) divergences.

The UV divergences appear because the loop momenta are integrated up to infinitely large region. These can be estimated by power counting. Notice that the UV divergences have different forms in different regulators. For example,

$$\int_1^\Lambda \frac{k^n dk}{k^m} = \begin{cases} \Lambda^{n+1-m} - 1, & n+1-m \neq 0 \\ \ln \Lambda, & n+1-m = 0 \end{cases} \tag{1}$$

$$\int_1^\infty \frac{k^{n+\alpha} dk}{k^m} = -\frac{1}{n+\alpha+1-m}, \quad \alpha = d-4 \tag{2}$$

From eq. (1) , we know that the integral is divergent if $n+1-m \geq 0$, while it is divergent only if $n+1-m=0$ in dimensional regulator as shown in eq. (2) .

Usually, in Lorentz invariant field theories, each loop contributes a $1/\epsilon$ UV divergence. But higher divergences occur in effective field theory.

The IR divergences arise from the (usually at least two) on-shell propagators in a loop diagram, when it is not possible to use integration contour deformation to bypass the pole. Take the simple real example,

$$\int_{-\infty}^{+\infty} d^d l \frac{1}{l^2 + i0^+} \frac{1}{(p-l)^2 + i0^+} \frac{1}{(l+...)^2 + i0^+} \cdots, \quad p^2 = 0 \tag{3}$$

The first two denominators have poles at

$$l^0 = |\vec{l}| - i0^+, \quad l^0 = p^0 - |\vec{p} - \vec{l}| + i0^+, \tag{4}$$

If the loop momentum l is parallel to the external momentum p. Then the two poles are located at the same place of the real l^0-axis and can not be bypassed via a contour around

the pole. Now, the integration contour is pinched between the two poles. In this case, the integral is divergent.

The systematic way to check whether there is a pinched pole in an integral is to use Landau equation. A generic Feynman integral with trivial numerators takes the form

$$G = \int \prod_{j=1}^{L} d^d k_j \int_0^1 \prod_{i=1}^{P} d\alpha_i \frac{\delta(1-\sum_i \alpha_i)}{\psi^P}\,, \tag{5}$$

where L are the number of loops, P the number of internal lines and the denominator reads

$$\psi = \sum_{j=1}^{P} \alpha_j (l_j^2 - m_j^2) + i\epsilon\,, \tag{6}$$

where the $l_j^\mu(k,p)$ is the momentum of the j-th propagator that depends linearly on the loop momenta k_i and external momenta p_i.

Our main interest is the positions of all poles and branch points of G as a function of the external momenta p_i. They arise from zeros of the denominator ψ. But the mere presence of a zero of D is not enough to produce a singularity in G because the integral is in complex (k,α) space. Singularities can appear in two ways. (i) End-point singularities. We can omit the constraint $\delta(1-\sum_i \alpha_i)$ in analyzing the singularities because we can replace it with $\rho\delta(\rho-\sum_i \alpha_i)$ with arbitrary positive ρ. [digression]: Cheng-Wu theorem

$$\begin{aligned}
&\int_0^\infty \prod_i d\alpha_i \frac{\delta(1-\sum_i \alpha_i)}{(\sum \alpha_i D_i)^3} \\
=&\int_0^\infty \prod_i d\alpha_i \frac{\rho\delta(\rho-\sum_i \alpha_i)}{(\sum \alpha_i D_i)^3} \\
&\qquad \alpha_i = \eta\alpha_i', \quad \eta = \alpha_1+\alpha_2 \\
=&\int d\eta \int_0^\infty \prod_i d\alpha_i' \frac{\rho\delta(\rho-\eta\sum_i \alpha_i')}{\eta^3(\sum \alpha_i' D_i)^3}\eta^2\delta(1-\alpha_1'-\alpha_2') \\
=&\int_0^\infty \prod_i d\alpha_i' \frac{1}{(\sum \alpha_i' D_i)^3}\delta(1-\alpha_1'-\alpha_2') \qquad (7)
\end{aligned}$$

Therefore, we only need to consider

$$G' = \int \prod_{j=1}^{L} d^d k_j \int_0^\infty \prod_{i=1}^{P} d\alpha_i \frac{1}{\psi^P}\,, \tag{8}$$

We should find out the points in the multiple dimensional space of k_j and α_i where the integral becomes singular.

The integration range of k is infinite. There are no end-point singularities in the k integration. The only possible end-point singularities are at

$$\alpha_i = 0 \tag{9}$$

(ii) Pinch singularities, which means that ψ has two poles at the same point. These are so-called *leading singularities*. These can be found by solving

$$\frac{\partial\psi}{\partial\alpha_i}=0\,,\qquad \frac{\partial\psi}{\partial k_j}=0\,. \tag{10}$$

The first condition puts the internal legs on-shell, $l_i^2=m_i^2$, while the second condition relates momenta belonging to the same loop, l, as

$$\sum_{i\in l}\alpha_i l_i=0\,. \tag{11}$$

Here l_i should be aligned in the same direction, i.e., in the form $k_j+....$

The above three equations are the famous Landau equations. Note that $\psi=0$ is automatically satisfied given these equations. The physical picture is simple. To obtain a singularity, the propagators should be either on-shell, corresponding to $l_i^2=m_i^2$, or contracted, corresponding to $\alpha_i=0$. And thus, the Laudau equations are closely related to the unitary cut of Feynman diagrams, and can be used to determine symbols [2304.02629].

The integration over the loop momentum can then be readily done and yields

$$G=\int_0^1\prod_{j=1}^P d\alpha_j\,\frac{\delta(1-\sum_{j=1}^P\alpha_j)\,U^{P-(L+1)d/2}}{F^{P-Ld/2}}\,, \tag{12}$$

with

$$U=\det a_{ij}\,,\qquad F=\det\begin{pmatrix}a_{ij}&-b_j\\-b_j&c\end{pmatrix}, \tag{13}$$

where $i,j=1,\ldots,L$ and

$$a_{ij}=\frac{1}{2}\frac{\partial^2\psi}{\partial k_i\partial k_j}\bigg|_{k=0}\,,\quad b_j=\frac{1}{2}\frac{\partial\psi}{\partial k_j}\bigg|_{k=0}\,,\quad c=\psi|_{k=0}\,. \tag{14}$$

Note that a_{ij} is only a function of α_i. U is non-negative. b_j and c depend on both α_i and kinematic invariants. In general, F can be both positive and negative depending on the value of α_i. $F=0$ is the necessary condition for the existence of divergence in the integral.

An equivalent form of the Landau equations is obtained for representation (12),

$$F=0\,,\qquad \frac{\partial F}{\partial\alpha_i}=0\,. \tag{15}$$

Note that since $F\propto\alpha_i\frac{\partial F}{\partial\alpha_i}$ is homogeneous, $F=0$ is automatically satisfied.

Simple examples:

Two-point integral :

$$G(p,m)=\int d^dk\frac{1}{[k^2-m^2+i0^+][(k-p)^2-m^2+i0^+]} \tag{16}$$

It Landau equations are

$$k^2-m^2=0,\quad (k-p)^2-m^2=0,\quad \alpha_1k^\mu+\alpha_2(k-p)^\mu=0 \tag{17}$$

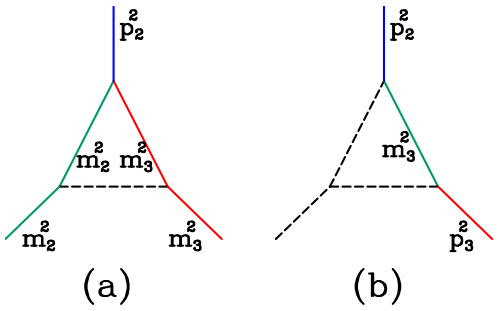

FIGURE 1. Examples of triangle diagrams with divergences.

Here we have omitted the case in which only one propagator is on-shell. One can solve the above equations and obtains

$$\alpha_1 = \alpha_2, \quad k^\mu = \frac{1}{2}p^\mu, \quad p^2 = 4m^2 \tag{18}$$

The contour can be described by setting $k \to k + i\lambda v(k)$. The imaginary parts of the denominators are

$$2\lambda v(k).k, \quad 2\lambda v(k).(k-p) \tag{19}$$

At the singular point, they have opposite signs. Therefore, the singularity can be avoided by contour deformation. Eq. (18) shows the classical threshold for the two-point integrals, and explains the appearance of $\sqrt{p^2 - 4m^2}$ in the solution.

In Feynman parameter representation,

$$\begin{aligned} F &= -p^2\alpha_1\alpha_2 + m^2(\alpha_1+\alpha_2)^2 \\ \frac{\partial F}{\partial \alpha_1} &= -p^2\alpha_2 + 2m^2(\alpha_1+\alpha_2) = 0 \\ \frac{\partial F}{\partial \alpha_2} &= -p^2\alpha_1 + 2m^2(\alpha_1+\alpha_2) = 0 \end{aligned} \tag{20}$$

The solution gives

$$\alpha_1 = \alpha_2, \quad p^2 = 4m^2 \tag{21}$$

which are the same as Eq. (18) . But the information of k^μ is missing.

If the masses are vanishing, $m^2 = 0$. The Landau equations require

$$k^\mu = \frac{\alpha_2}{\alpha_1+\alpha_2}p^\mu, \quad k^2 = p^2 = 0 \tag{22}$$

This is a typical collinear singularity.

Three-point integral : We consider the triangle shown in Fig. (1a) which contains a soft singularity. In this case the denominator is given by

$$F = (m_2^2 + m_3^2 - p_2^2)a_2a_3 + m_2^2a_2^2 + m_3^2a_3^2\,. \tag{23}$$

This expression satisfies the Landau conditions for $a_2 = a_3 = 0$ and a_1 arbitrary. The triangle shown in Fig. (1b) which contains a collinear singularity. In this case the denominator reads

$$F = (m_3^2 - p_2^2)a_2a_3 + (m_3^2 - p_3^2)a_1a_3 + m_3^2a_3^2\,, \tag{24}$$

which satisfies the Landau conditions for $a_3 = 0$ and a_1, a_2 arbitrary. In these two cases, the external momenta are arbitrary. The corresponding Landau singularities correspond to soft and collinear divergences.

The IR divergences appear also in real corrections in gauge theories. And each loop may have $1/\epsilon^2$ divergences, which can be seen simply from

$$\int \frac{l_0^{-2\epsilon} dl_0}{l_0} \frac{(\sin\theta)^{-2\epsilon} d\theta}{1-\cos\theta} \tag{25}$$

The IR divergences at the scattering amplitudes are usually simpler than those in each integrals.

2. Dimensional regularization

Main references :

(1) G. t Hooft and M. Veltman, Regularization and Renormalization of Gauge Fields, Nucl.Phys. B44 (1972) 189.
(2) C. Gnendiger et al, To d, or not to d: Recent developments and comparisons of regularization schemes, Eur.Phys.J.C 77 (2017) 7, 1705.01827
(3) F. Jegerlehner, Facts of life with gamma(5), Eur. Phys. J. C 18 (2001), hep-th/0005255

The Feynman integrals contain UV and IR divergences. The UV divergence is caused by the infinitely large values of the loop momenta. A naive regulator is to impose an upper limit. But this cutoff breaks Lorentz invariance, e.g., it violates the Ward identity in QED. The Pauli-Villars regularization introduces massive fictitious particles so that the loop integrands drop fast enough. This regulator preserves Lorentz invariance but is cumbersome.

Dimensional regularization, "dim reg," is by far the most common regularization procedure in QFT, almost always used in conjunction with the modified minimal subtraction (MS) renormalization scheme. It regularizes both UV and IR divergences, and is gauge invariant. No additional scales are introduced and thus it is simple to calculate at higher orders.

The basic idea of all dimensional regularization schemes is changing the space-time dimension from 4 to $d = 4 - 2\epsilon$. For example,

$$\int \frac{d^4k}{(2\pi)^4} \cdots \to \int \frac{d^dk}{(2\pi)^d} \cdots \tag{26}$$

In order to keep the coupling dimensionless,

$$g_s^2 \to g_s^2 \mu_R^{2\epsilon} \tag{27}$$

so that the RG equation in d-dimension

$$\frac{d\ln g_s}{d\ln \mu} \to -\epsilon + \mathcal{O}(\alpha_s) \tag{28}$$

This equation is useful in deriving the divergences of the amplitudes from their anomalous dimensions.

The questions are whether we regularize every thing, such as the internal and external momenta, g^μ_μ, polarizations, or only part of them. Keeping some of them 4 dimensional is important to be compatible with supersymmetry or helicity methods.

The original 4-dimensional metric tensor is denoted by $\bar{g}^{\mu\nu}$ The d-dimensional analog is $\hat{g}^{\mu\nu}$. The space of polarizations is represented by $g^{\mu\nu}$. Then it is defined that

$$\bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = 4, \quad \hat{g}^{\mu\nu}\hat{g}_{\mu\nu} = d, \quad g^{\mu\nu}g^{\mu\nu} = 4 \tag{29}$$

The projection relations are given by

$$\bar{g}^{\mu\nu}\hat{g}_{\nu\rho} = \bar{g}^\mu_\rho, \quad \bar{g}^{\mu\nu}g_{\nu\rho} = \bar{g}^\mu_\rho,, \quad \hat{g}^{\mu\nu}g_{\nu\rho} = \hat{g}^\mu_\rho \tag{30}$$

These schemes are listed in table 1.

TABLE 1. Different dimensional schemes. Here 'internal' means those in loop or soft/collinear.

	CDR	HV	FDH	DRED
Internal	$\hat{g}^{\mu\nu}$	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$	$\bar{g}^{\mu\nu}$
External	$\hat{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$

The Dirac algebra is generalized to d dimensional

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{31}$$

However, it is fine to still use $\mathrm{Tr}1 = 4$.

Conversions between results in CDR, HV, FDH, and DRED can be made for individual parts contributing to a cross section. For the virtual contributions this is known to NNLO and can be elegantly described solely through the scheme dependence of β functions and anomalous dimensions. For real corrections and initial-state factorization terms the explicit scheme dependence is only known to NLO. These results have been used to explicitly demonstrate the scheme independence of a cross section at NLO.

FDH and DRED are perfectly consistent regularizations schemes, at least up to NNLO. However, they require the introduction of additional (evanescent) couplings with (in general) different counterterms.

The definition of γ_5 does not exist in d dimensions. It is defined in 4 dimension by $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, which satisfies

$$\begin{aligned} \gamma_5^2 &= 1, \\ \{\gamma_5, \gamma^\mu\} &= 0, \\ \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) &= 4i\varepsilon_{\mu\nu\rho\sigma} \end{aligned} \tag{32}$$

Consider

$$\kappa^{\mu\nu\rho\sigma}\mathrm{Tr}(\gamma_\tau\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma^\tau\gamma_5) \tag{33}$$

with $\kappa^{\mu\nu\rho\sigma}=0$ if two indices are equal. Then we find

$$2(d-4)\kappa^{\mu\nu\rho\sigma}\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5)=0 \tag{34}$$

In HV scheme, γ_5 is taken as a 4-dimensional object.

$$\begin{aligned}\{\gamma^\mu,\gamma_5\}&=0, \quad \mu\in\{0,1,2,3\}\\ [\gamma^\mu,\gamma_5]&=0, \quad \text{otherwise}\end{aligned} \tag{35}$$

The axial Ward identities are broken in this scheme due to the non-anticommuting property, which can be fixed by an additional finite renormalization of the axial current, e.g.,

$$Z_5^{\mathrm{ns}}=1+\frac{\alpha_s}{4\pi}(-4C_F)+\left(\frac{\alpha_s}{4\pi}\right)^2\left(22C_F^2-\frac{107}{9}C_FC_A+\frac{2}{9}C_Fn_f\right) \tag{36}$$

3. Strategy of regions and asymptotic expansion

Main references :

(1) M. Beneke, V.A. Smirnov, Asymptotic expansion of Feynman integrals near threshold. Nucl. Phys. B522, 321344 (1998) [hep-ph/9711391].
(2) V.A. Smirnov, Applied Asymptotic Expansions in Momenta and Masses. Springer Tracts in Modern Physics
(3) T. Becher, A. Broggio, A. Ferroglia, Introduction to Soft-Collinear Effective Theory, Lect. Notes Phys. **896** (2015), pp.1-206 Springer

The strategy of regions is a technique which allows one to carry out asymptotic expansions of loop integrals in dimensional regularization around various limits. The expansion is obtained by splitting the integration in different regions and appropriately expanding the integrand in each case. In the effective theory, the different regions will be represented by different effective theory fields. The expanded integrals obtained by means of the strategy of regions technique are in one-to-one correspondence to the Feynman diagrams of effective field theories regularized in dimensional regularization.

If one is simply interested to expand some perturbative result in a small parameter, one can therefore work directly with the strategy of regions technique, without constructing an effective Lagrangian. However, the use an effective field theory offers some important advantages when one is interested in deriving all-order statements. In particular, one can use the effective Lagrangian

- to derive factorization theorems and
- to resum logarithmically enhanced contributions at all orders in the coupling constant using Renormalization Group (RG) techniques.

In addition, in the effective field theory gauge invariance is manifest at the Lagrangian level, while this is not the case for individual diagrams. The effective Lagrangian also provides a systematic way to organize higher power corrections, by including subleading terms in the

effective Lagrangian. (In a collider physics context, higher-power contributions are also called higher twist corrections.)

3.1. **A Simple Example.** In order to illustrate the main idea of the strategy of regions we start by considering a simple integral, which we will expand using different methods, first using a cutoff to separate two different regions and then with dimensional regularization. The integral we will consider is

$$I = \int_0^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln\frac{M}{m}}{M^2-m^2}\,. \tag{37}$$

This corresponds to a self-energy one-loop integral with two different particle masses at zero external momentum, evaluated in $d = 2$. We will assume a large hierarchy between the masses, for example $m^2 \ll M^2$, and will discuss the expansion of the integral around the limit of small m. Since we know the full result, we can obtain the expansion simply by expanding the denominator on the r.h.s. of Eq. (37)

$$I = \frac{\ln\frac{M}{m}}{M^2}\left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots\right). \tag{38}$$

Note that the integral is not analytic in the expansion parameter m/M because of the presence of the logarithm. Expansions of functions around points where they have essential singularities are also called asymptotic expansions. Our goal in the following is to obtain the expansion in Eq. (38) by expanding the integrand in Eq. (37) before carrying out the integral. This is important in cases where the full result is not available. It will also tell us what kind of degrees of freedom the effective theory will contain.

A naive expansion of the integrand leads to trouble, because it gives rise to IR divergent integrals. In fact

$$\frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) \tag{39}$$

cannot be used in the integrand of Eq. (37):

$$I \neq \int_0^\infty dk\, \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right). \tag{40}$$

This was to be expected: If it had been legitimate to simply Taylor expand the integrand in m/M and integrate term by term, the result would necessarily be an analytic function of m in the vicinity of $m = 0$ because none of the integrals on the r.h.s. of Eq. (40) depend on m and so the integrals would simply give the Taylor coefficients of the expansion in m. But the result for I is not analytic in m/M, as we stressed above. So just from the form of the result in Eq. (38), it is clear that expansion and integration do not commute. The reason is simply that the series expansion in Eq. (39) is valid only for $k \gg m^2$, while the integration domain in Eq. (37) includes a region in which $k^2 \sim m^2$, which contributes to the integral. To account for this fact, we should split the integration into two regions. We can do this by introducing a new scale Λ such that $m \ll \Lambda \ll M$. We will call the scale

Λ a cutoff, even though the name is misleading, since we do not cut away any part of the integral. The role of Λ is just to separate the two momentum regions. We then obtain

$$I = \underbrace{\int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{I_{(I)}} + \underbrace{\int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{I_{(II)}} . \tag{41}$$

We call the region $[0,\Lambda]$ the *low-energy* region. In this region $k \sim m \ll M$, and therefore one can expand the integrand in the integral $I_{(I)}$ as follows

$$I_{(I)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)M^2}\left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right) . \tag{42}$$

The scale Λ acts as an ultraviolet cutoff for the integrals on the r.h.s. of the Eq. (42).

The region $[\Lambda,\infty]$ is referred to as the *high-energy* region; in that region $m \ll k \sim M$, and one can expand the integrand according to

$$I_{(II)} = \int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_\Lambda^\infty dk \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) . \tag{43}$$

In the equation above, Λ acts as an infrared cutoff.

By integrating the first two terms on the r.h.s. of Eq. (42) one finds

$$I_{(I)} \approx \frac{M^2+m^2}{2M^4}\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{2M^4} = -\frac{1}{M^2}\ln\left(\frac{m}{\Lambda}\right) - \frac{\Lambda^2}{2M^4} + \mathcal{O}\left(\frac{\Lambda^4}{M^6}, \frac{m^2}{M^4}\log\left(\frac{\Lambda}{m}\right)\right) , \tag{44}$$

since it was assumed above that $\Lambda \gg m$. Similarly, by integrating the first term on the r.h.s. of Eq. (43) one obtains

$$I_{(II)} \approx \frac{1}{2M^2}\ln\left(1+\frac{M^2}{\Lambda^2}\right) = -\frac{1}{M^2}\ln\left(\frac{\Lambda}{M}\right) + \frac{\Lambda^2}{2M^4} + \mathcal{O}\left(\frac{\Lambda^4}{M^6}\log\left(\frac{M}{\Lambda}\right)\right) . \tag{45}$$

Adding up the Eq. (44) and (45) one finally obtains

$$I = I_{(I)} + I_{(II)} = -\frac{1}{M^2}\ln\left(\frac{m}{M}\right) + \mathcal{O}\left(\frac{m^2}{M^4}\log\left(\frac{M}{m}\right)\right) , \tag{46}$$

which is the expected result (see Eq. (38)). When summing the results for the low-energy and high-energy regions, the terms which depend on the cutoff Λ cancel out; this has to happen, since the scale Λ is not present in the original integral and was only introduced in order to split the original integral in a sum of two different terms. Since the final result cannot depend on Λ, there should be a way to obtain the expansion without introducing this additional scale. Our ultimate goal is to apply a similar technical expedient to the calculation of loop diagrams and it is well known that the use of hard cutoffs is impractical in such calculations. Fortunately it is possible to separate the low- and high-energy regions using dimensional regularization. To see this, let us rewrite the original integral as follows

$$I = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)(k^2+M^2)} , \tag{47}$$

where we will eventually send $\varepsilon \to 0$ at the end of the calculation. (For simplicity, we did not introduce the d-dimensional angular integration so this is not exactly dimensional regularization.)

The integral in the low-energy region $k \sim m \ll M$ will be

$$I_{(I)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right) . \tag{48}$$

In Eq. (48) the integral is infrared safe in the region in which $k \to 0$, the dimensional regulator ε can be chosen positive, so that the integrand is also ultraviolet finite.

The integral in the high-energy region will be

$$I_{(II)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{k^2(k^2+M^2)} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) . \tag{49}$$

The integral is ultraviolet safe, and we consider $\varepsilon < 0$, so that the integrand does not give rise to an infrared singularity in the region where $k \to 0$.

By integrating the first term on the r.h.s. of Eq. (48) one finds, at leading power in the expansion around m/M,

$$I_{(I)} = \frac{m^{-\varepsilon}}{2M^2} \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{1}{M^2} \left(\frac{1}{\varepsilon} - \ln m + \mathcal{O}(\varepsilon)\right) . \tag{50}$$

The integral of the first term on the r.h.s. of Eq. (49) is

$$I_{(II)} = -\frac{M^{-\varepsilon}}{2M^2} \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{1}{M^2} \left(-\frac{1}{\varepsilon} + \ln M + \mathcal{O}(\varepsilon)\right) . \tag{51}$$

The poles in ε cancel in the sum of Eqs. (50,51), and the final result is again the one obtained by means of the cutoff method in Eq. (46).

The reader might be worried that we choose $\varepsilon < 0$ in the low-energy region and $\varepsilon > 0$ in the high-energy region and then combine the two. It is important to remember that the integrals in dimensional regularization are defined for arbitrary ε: we only choose $\varepsilon < 0$ to be able to evaluate $I_{(I)}$ as a standard integral, but by analytic continuation the resulting function on the right-hand side is uniquely defined for any complex-valued ε and can be combined with $I_{(II)}$.

Also, the fact that in both Eq. (48) and Eq. (49) the integration domain coincides with the full integration domain of the original integral might seem disturbing at first sight. Since we integrate the high-energy part over the low-energy region (and vice versa), one could fear that this leads to additional contributions which are already accounted for in the low-energy part. To see that this does not happen and that the two parts lead a life of their own, one should observe that the two integrals scale differently. The low-energy integral $I_{(I)}$ factors out $m^{-\varepsilon}$, while the high-energy integral $I_{(II)}$ factors out $M^{-\varepsilon}$. This statement remains true even if we consider the subleading terms. When keeping the complete dependence on m and M the result is

$$I = \frac{1}{2} \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) \frac{m^{-\varepsilon} - M^{-\varepsilon}}{M^2 - m^2} . \tag{52}$$

The result clearly displays the low-energy and the high-energy part. Expanding in one region, one loses the other part and the full integral is recovered after adding the two contributions. Even though we integrate twice over the full integration domain, there is no double counting, since the two pieces scale differently: the low-energy integrals can never produce a term $M^{-\varepsilon}$ since they depend analytically on the large scale, and vice-versa.

To demonstrate directly from the integral that there is indeed no double counting, let us now see what happens if we insist in restricting the integration domain of the low- and high-energy region integrals when using dimensional regularization. The integral in the low-energy region would become in this case

$$\begin{aligned} I^{\Lambda}_{(I)} &= \int_0^{\Lambda} dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\cdots\right) \\ &= \left[\int_0^{\infty} dk - \int_{\Lambda}^{\infty} dk\right] k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\cdots\right) \\ &= I_{(I)} - R_{(I)}\,. \end{aligned} \tag{53}$$

The first integral in the second line of the equation above is the same as the one in Eq. (48). In the integrand of $R_{(I)}$, which depends on the cutoff Λ, one can use the fact that $k \geq \Lambda \gg m^2$ to expand in the small m limit:

$$\begin{aligned} R_{(I)} &= \int_{\Lambda}^{\infty} dk k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\cdots\right) \\ &= \int_{\Lambda}^{\infty} dk k^{-\varepsilon} \frac{k}{k^2 M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\cdots\right). \end{aligned} \tag{54}$$

For the remainder part $R_{(I)}$, we thus have performed two expansions. First the low-energy expansion, which is equivalent to expanding the integrand in the limit $M \to \infty$. Then we have expanded the result around $m \to 0$, which is equivalent to the high-energy expansion.

At this point it is sufficient to observe that for dimensional reasons the integrals in the equation above must behave as follows

$$\int_{\Lambda}^{\infty} dk\, k^{n-\varepsilon} \sim \Lambda^{n+1-\varepsilon}\,. \tag{55}$$

So the cutoff pieces scale as fractional powers of the cutoff. Since the Λ dependent terms must cancel out completely in the calculation of I, one can as well drop the Λ dependent integrals from the start. Therefore, when regulating divergences by means of dimensional regularization one can integrate over the complete integration domain, in this case $k \in [0,\infty]$.

We can explicitly verify that the cutoff pieces vanish if we also consider the high-energy integral $I_{(II)}$ in Eq. (49) with a lower cutoff Λ on the integration. Proceeding in the same way as before, we can rewrite the high-energy integral as the expanded integral without a

cutoff and a remainder which depends on the cutoff

$$\begin{aligned} R_{(II)} &= \int_0^\Lambda dk k^{-\varepsilon} \frac{k}{k^2(k^2+M^2)} \left(1 - \frac{m^2}{k^2} + \cdots\right) \\ &= \int_0^\Lambda dk k^{-\varepsilon} \frac{k}{k^2 M^2} \left(1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \cdots\right) . \end{aligned} \tag{56}$$

In this remainder, we have again expanded the integrand in both the limit of small m and also in the limit of large M, but in the opposite order as in $R_{(I)}$. However, the two expansions commute so that the integrands of $R_{(I)}$ and $R_{(II)}$ are identical. Adding up the two pieces, we find that

$$R = R_{(I)} + R_{(II)} = \int_0^\infty dk k^{-\varepsilon} \frac{k}{k^2 M^2} \left(1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \cdots\right) . \tag{57}$$

This is manifestly independent of the cutoff. It is also manifestly zero, because it is given by a series of scaleless integrals. In the context of SCET, the overlap contribution R is usually referred to as the "zero-bin" contribution. There are two ways of obtaining the full overlap R. One can either expand the integrand of the high-energy integral $I_{(II)}$ around the low-energy limit, or the integrand of the low-energy integral $I_{(I)}$ around the high-energy limit. Since the overlap is obtained by expanding the single-scale integrals $I_{(I)}$ or $I_{(II)}$ it is given by scaleless integrals which vanish in dimensional regularization.

The example considered had the purpose of illustrating some common features of the expansion of Feynman diagrams in the simplest possible setting. The general strategy to obtain the expansion of a given Feynman integral in a given kinematic limit is the following:

i) Identify all regions of the integrand which lead to singularities in the limit under consideration,
ii) Expand the integrand in each region and integrate each expansion over the full phase space.
iii) Add the result of the integrations over the different regions to obtain the expansion of the original full integral.

In order for the procedure to work, it is necessary to make sure that all of the expanded integrals are properly regularized. Sometimes dimensional regularization alone is not sufficient to regularize the integrals in every region, and one might need to employ additional analytic regulators or to perform subtractions. Below, we will discuss the massive Sudakov form factor, which is an example where this is necessary. It is also important to consider each region only once to avoid double counting. As stated above, one needs to identify all regions of integration which lead to singularities. Often, this is a simple task and the regions which one encounters at one loop are the same which are relevant at higher order. However, there are examples in which new regions must be added to the list when increasing the number of loops present in the diagram [Phys.Lett.B 465 (1999) 226]. We also stress that there is so far no general proof that the above procedure always produces the correct result. Recent work towards such a proof can be found in JHEP 12 (2011) 076.

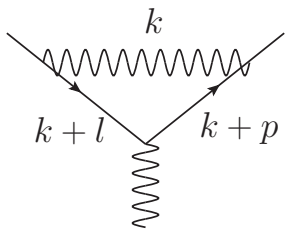

FIGURE 2. One-loop vertex corrections. The Feynman diagram is here shown in terms of fermions and photons, however, the spin structure is neglected in this section.

3.2. **The Sudakov Problem.** We want now to consider the simplest possible example relevant in the context of SCET, namely a one-loop vertex diagram. We neglect complications related to the spin of the particles, since the momentum regions that one finds in the calculation of the tensor integrals are the same that one finds in the calculation of the scalar integral considered below. With reference to Figure 2, the vertex correction requires the evaluation of the following Feynman integral (all the internal propagators are considered massless):

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,[(k+l)^2+i0]\,[(k+p)^2+i0]}\,, \tag{58}$$

where $d = 4-2\varepsilon$ is the dimensional regulator. The 't Hooft scale μ has been introduced to make the mass dimension of I independent of the value of d. We introduce the following notation:

$$L^2 \equiv -l^2 - i0\,, \qquad P^2 \equiv -p^2 - i0\,, \qquad Q^2 \equiv -(l-p)^2 - i0\,. \tag{59}$$

The goal is to calculate the integral in Eq. (58) in the limit in which $L^2 \sim P^2 \ll Q^2$ that is, in the case in which the external legs carrying momenta l and p have large energies but small invariant masses.

Before going any further, we now need to introduce some basic notation used in SCET. We choose two light-like reference vectors in the direction of the momenta p and l in the frame in which[1] $\vec{Q} = 0$:

$$n_\mu = (1,0,0,1) \quad \text{and} \quad \bar{n}_\mu = (1,0,0,-1)\,. \tag{60}$$

It is immediate to verify that

$$n^2 = \bar{n}^2 = 0\,, \quad \text{and} \quad n\cdot\bar{n} = 2\,. \tag{61}$$

[1]In this lectures we employ the "mostly minuses" metric, and the components of a generic four-vector x^μ are (t,x,y,z).

Any vector can be then decomposed in a component proportional to n, a part proportional to $\bar{n}$, and a remainder perpendicular to both

$$p^\mu = (n\cdot p)\frac{\bar{n}^\mu}{2} + (\bar{n}\cdot p)\frac{n^\mu}{2} + p_\perp^\mu \equiv p_+^\mu + p_-^\mu + p_\perp^\mu \,. \tag{62}$$

Splitting the vectors into their light-cone components is useful to organize the expansion, since the different components scale differently. For the square of the vector p one then finds

$$p^2 = (n\cdot p)(\bar{n}\cdot p) + p_\perp^2 \,, \tag{63}$$

while the scalar product between two vectors p and q becomes

$$p\cdot q = p_+\cdot q_- + p_-\cdot q_+ + p_\perp\cdot q_\perp \,. \tag{64}$$

In the following we will often identify a vector by means of its components in the n, $\bar{n}$, and $\perp$ basis, with the notation

$$p^\mu = (\underbrace{\quad n\cdot p \quad}_{\text{“+ comp.”}}, \underbrace{\quad \bar{n}\cdot p \quad}_{\text{“− comp.”}}, p_\perp^\mu) \,. \tag{65}$$

We warn the reader that in certain situations it is convenient to work with the scalar quantities $p_+ \equiv n\cdot p$ and $p_- \equiv \bar{n}\cdot p$, which should not be mixed up with the related vector quantities $p_\pm^\mu$ introduced above. In the following we explicitly indicate what we mean by the symbols $p_\pm$ whenever the notation can give rise to ambiguities.

We now introduce an expansion parameter λ which vanishes in the limit in which we are interested in:

$$\lambda^2 \sim \frac{P^2}{Q^2} \sim \frac{L^2}{Q^2} \,, \qquad \text{and} \qquad p^2 \sim l^2 \sim \lambda^2 Q^2 \,. \tag{66}$$

We choose the reference vectors in the directions of large momentum flow $p^\mu \approx Qn^\mu/2$ and $l^\mu \approx Q\bar{n}^\mu/2$. The components of p and l will then typically scale as follows

$$p^\mu \sim \left(\lambda^2, 1, \lambda\right) Q \,, \qquad \text{and} \qquad l^\mu \sim \left(1, \lambda^2, \lambda\right) Q \,, \tag{67}$$

but the scaling is not completely unique. We could, for example, choose the reference vector n^μ such that the perpendicular components of p^μ are zero, which is compatible with Eq. (67), but also with $\left(1, \lambda^2, \lambda^n\right) Q$ for any $n > 1$. However, when computing the loop diagram via the strategy of regions, one finds that only scalings $k^\mu \sim (\lambda^a, \lambda^b, \lambda^c)Q$, with $a+b=2c$ are important. For $c > 0$, these describe particles which go on shell as $\lambda \to 0$. In later sections, we will see that the corresponding propagators are associated with particles in the low-energy theory. Specifically, upon expanding the integrals, one finds that only the following four regions give non-vanishing contributions:

- **Hard Region** (denoted by h in the following) where the components of the integration momentum scale as $k^\mu \sim (1,1,1)\, Q$,
- **Region Collinear to p** (denoted by c) where k scales as $k^\mu \sim (\lambda^2, 1, \lambda)\, Q$,
- **Region Collinear to l** (denoted by $\bar{c}$) where k scales as $k^\mu \sim (1, \lambda^2, \lambda)\, Q$,
- **Soft Region** (denoted by s) where k scales as $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)\, Q$.

All of the other possible scalings of the integration momentum, of the form $k^\mu \sim (\lambda^a, \lambda^b, \lambda^c)\, Q$ and with a, b, c not matching one of the four cases listed above, give rise upon expanding to scaleless integrals only, and therefore they do not contribute to the final result.

In the following, we will compute the contribution of each of the non-vanishing regions in turn, but it is instructive to start by considering an example of a scaling which does not contribute for the case of the form factor, namely a soft scaling $k^\mu \sim (\lambda, \lambda, \lambda)Q$, which we will call semi-hard in order to distinguish it from the standard soft region, whose components scale as λ^2. The expansion of the propagator denominators takes the form

$$(k+l)^2 = \overbrace{k^2}^{\mathcal{O}(\lambda^2)} + 2(\overbrace{k_+ \cdot l_-}^{\mathcal{O}(\lambda^3)} + \overbrace{k_- \cdot l_+}^{\mathcal{O}(\lambda)} + \overbrace{k_\perp \cdot l_\perp}^{\mathcal{O}(\lambda^2)}) + \overbrace{l^2}^{\mathcal{O}(\lambda^2)} = 2k_- \cdot l_+ + \mathcal{O}(\lambda^2)\,, \tag{68}$$

and analogously

$$(k+p)^2 = 2k_+ \cdot p_- + \mathcal{O}(\lambda^2)\,, \tag{69}$$

after which the hypothetical semi-hard contribution becomes

$$I_{sh} = i\pi^{-d/2}\mu^{4-d} \int d^dk \frac{1}{(k^2+i0)\,(2k_- \cdot l_+ + i0)\,(2k_+ \cdot p_- + i0)}\,. \tag{70}$$

This integrals vanishes: $I_{sh} = 0$.[2] As an exercise, we invite the reader to show that also the Glauber region $k^\mu \sim (\lambda^2, \lambda^2, \lambda)Q$ gives a vanishing contribution to the form factor integral.

In order to determine the integral that one needs to evaluate when the integration momentum is considered hard, we consider the way in which the terms in the propagators in Eq. (58) scale. Clearly $k^2 \sim \lambda^0 Q^2$; for the other two propagators one finds

$$(k+l)^2 = \overbrace{k^2}^{\mathcal{O}(1)} + 2(\overbrace{k_+ \cdot l_-}^{\mathcal{O}(\lambda^2)} + \overbrace{k_- \cdot l_+}^{\mathcal{O}(1)} + \overbrace{k_\perp \cdot l_\perp}^{\mathcal{O}(\lambda)}) + \overbrace{l^2}^{\mathcal{O}(\lambda^2)} = k^2 + 2k_- \cdot l_+ + \mathcal{O}(\lambda)\,, \tag{71}$$

and, similarly

$$(k+p)^2 = k^2 + 2k_+ \cdot p_- + \mathcal{O}(\lambda)\,. \tag{72}$$

The contribution of the **hard region** to the integral I is therefore given by

$$I_h = i\pi^{-d/2}\mu^{4-d} \int d^dk \frac{1}{(k^2+i0)\,(k^2+2k_- \cdot l_+ + i0)\,(k^2+2k_+ \cdot p_- + i0)}\,; \tag{73}$$

it coincides with the form factor integral with on shell external legs (i.e. calculated by setting $p^2 = l^2 = 0$ from the start). The integral evaluates to

$$\begin{aligned} I_h &= \frac{\Gamma(1+\varepsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\frac{\mu^2}{2l_+ \cdot p_-}\right)^\varepsilon \\ &= \frac{\Gamma(1+\varepsilon)}{Q^2} \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6}\right) + \mathcal{O}(\varepsilon)\,. \end{aligned} \tag{74}$$

[2] The calculation proceeds through the same steps as the evaluation of the soft integral when the external legs are put on-shell, $I_{sh} = I_s(p^2 = 0, l^2 = 0) = 0$, which also vanishes, as discussed below.

The poles in ε are of infrared origin.

In the region collinear to p the integration momentum scales as $k^\mu \sim (\lambda^2, 1, \lambda)Q$. In this region $k^2 \sim \lambda^2 Q^2$, while

$$(k+l)^2 = 2k_- \cdot l_+ + \mathcal{O}(\lambda^2)\,, \qquad (k+p)^2 = \mathcal{O}(\lambda^2)\,. \tag{75}$$

The **collinear region** integral is obtained by keeping only the leading term in each propagator

$$\begin{aligned} I_c &= i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + i0)\,[(k+p)^2+i0]} \\ &= -\frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{P^2}\right)^\varepsilon \\ &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}\right)+\mathcal{O}(\varepsilon)\,. \end{aligned} \tag{76}$$

We observe that the integral scales as $P^{-2\varepsilon}$. The calculation of the integral in the region collinear to l is identical to the calculation of the integral in the region collinear to p, Eq. (76), except that one needs to replace P^2 with L^2 in the final result.

In the soft region all of the components of the integration momentum are proportional to λ^2, therefore

$$k^2 = \mathcal{O}(\lambda^4)\,, \quad (k+l)^2 = 2k_-\cdot l_+ + l^2 + \mathcal{O}(\lambda^3)\,, \text{ and } \quad (k+p)^2 = 2k_+\cdot p_- + p^2 + \mathcal{O}(\lambda^3)\,, \tag{77}$$

and therefore the integral in the **soft region** is

$$\begin{aligned} I_s &= i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + l^2 + i0)\,(2k_+\cdot p_- + p^2 + i0)} \\ &= -\frac{\Gamma\,(1+\varepsilon)}{2l_+\cdot p_-}\Gamma(\varepsilon)\Gamma\,(-\varepsilon)\left(\frac{2l_+\cdot p_-\mu^2}{L^2P^2}\right)^\varepsilon \\ &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2}+\frac{\pi^2}{6}\right)+\mathcal{O}\,(\varepsilon)\,. \end{aligned} \tag{78}$$

The poles in the last line of Eq. (78) are of ultraviolet origin. As expected, the result depends on the "new" soft scale $\Lambda^2_{\rm soft} \sim P^2L^2/Q^2$.

One can now sum the results obtained in the different regions to obtain what was the original goal of the calculation: an analytic expression for the integral in Eq. (58) in the

limit in which $L^2 \sim P^2 \ll Q^2$. One finds

$$
\begin{aligned}
I_h &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_c &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_{\bar{c}} &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}-\frac{1}{2}\ln^2\frac{\mu^2}{L^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_s &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
\hline
I \equiv I_h+I_c+I_{\bar{c}}+I_s &= \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2}+\frac{\pi^2}{3}+\mathcal{O}(\lambda)\right). \qquad (79)
\end{aligned}
$$

The final result does not depend on the dimensional regulator ε and the reader is invited to check that it coincides with the one that would be obtained by evaluating directly the integral in Eq. (58) and then expanding the result in the $\lambda \to 0$ limit. We stress the fact that the infrared divergences found in the hard region cancel out against the ultraviolet divergences found in the sum of the soft and collinear contributions. This feature is general and requires a nontrivial interplay of the logarithms found in the various integrals:

$$
-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2}=-\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}\,. \qquad (80)
$$

The requirement that infrared divergences of the hard region should cancel against the ultraviolet divergences of the soft and collinear regions leads to constraints that must be satisfied by the infrared pole structure of a generic amplitude.