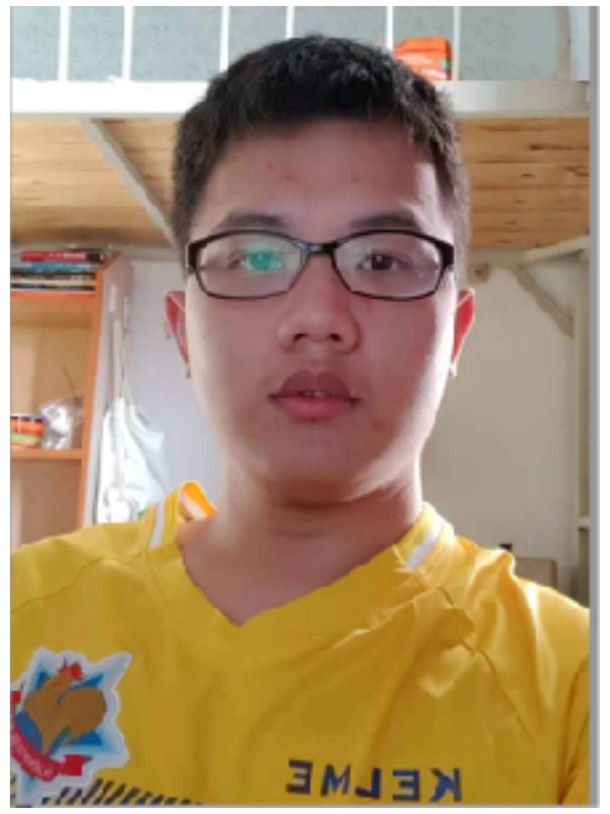
Scattering amplitudes, Feynman integrals & cluster algebras

何颂 (Institute of Theoretical Physics, CAS)

Based on works with
Z. Li, C. Zhang 1911.01290 (PRD), 2009.11471 (JHEP)
Z. Li, Y. Tang, Q. Yang   2012.13094 (JHEP)
Z. Li, Q. Yang, C. Zhang 2012.15092 (PRL)
Z. Li, Q. Yang 2103.02796, 2106.09314, 2108.07959 (JHEP)

Peng Huanwu center for fundamental theory, USTC



李振杰

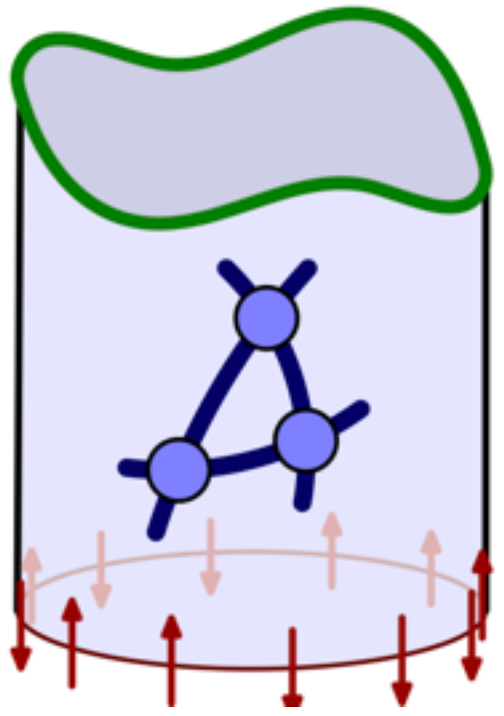
唐一朝

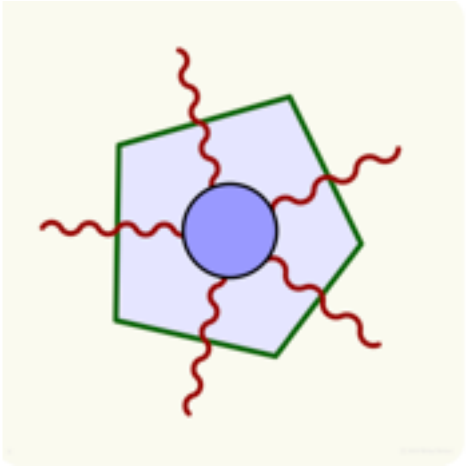
杨清霖

张驰

The simplest QFT

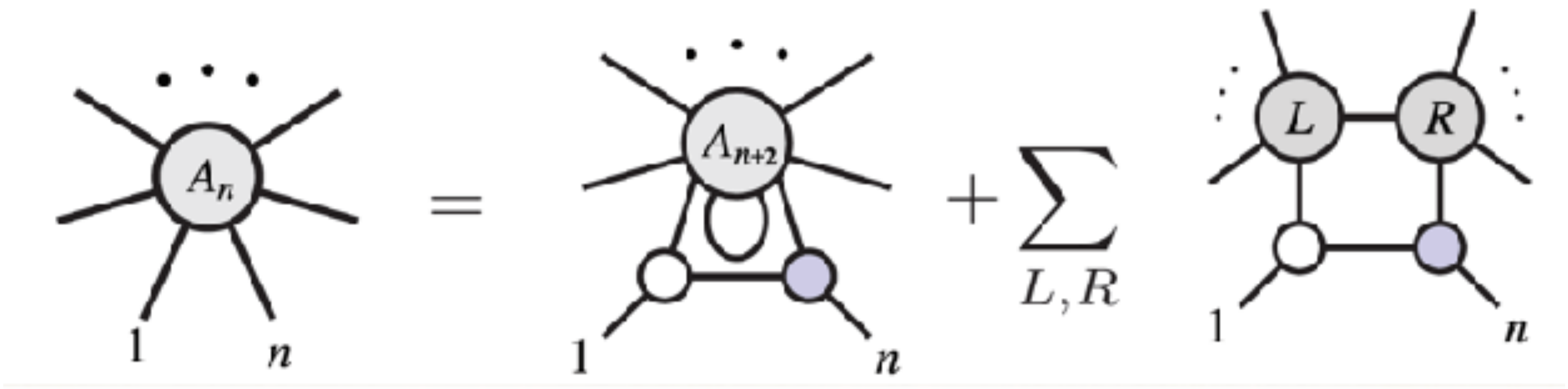
Harmonic oscillator of 21st century: hidden simplicity & structure in $\mathcal{N} = 4$ SYM (planar limit)

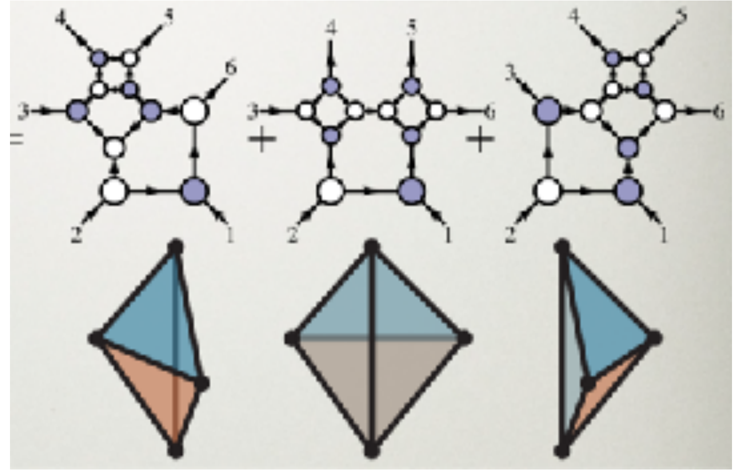




Integrability (planar limit): strong coupling via AdS/CFT, Wilson loops & OPE
Yangian symmetry … Ising model of gauge theories!

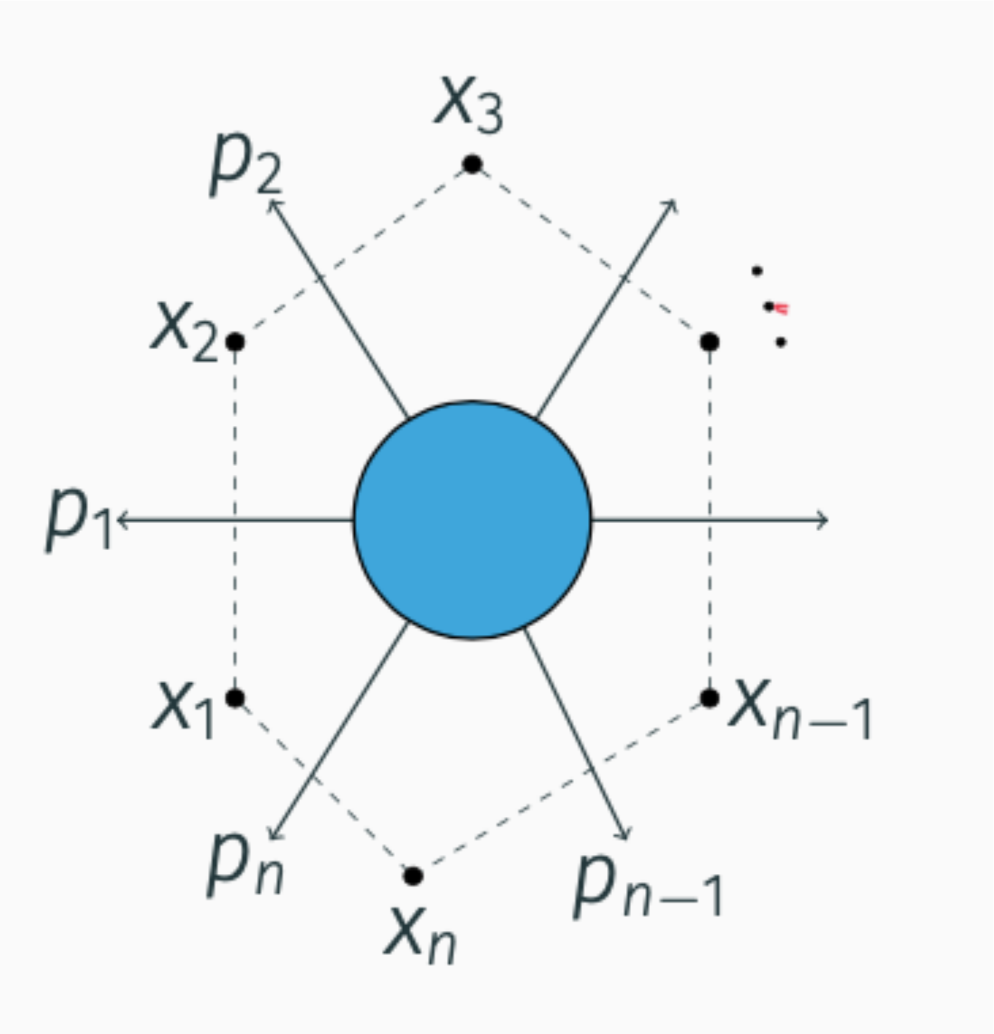
All-loop integrands <-> positive Grassmannian + amplituhedron [Arkani-Hamed, Trnka]





(Integrated) amplitudes + Feynman Integrals: extremely rich laboratory for perturbative QFT!
iterated integrals (polylogs & beyond), symbology, cluster algebras, differential eqs, bootstrap + Qbar,…

Amplitudes/WL in SYM



Amplitudes (MHV tree stripped) = null polygonal Wilson loops (strong+ weak coupling)

[Alday, Maldacena][Brandhuber, Heslop, Travaglini] [Drummond, Henn, Korchemski, Sokatchev][…]

$$A_n(p_1, p_2, \cdots, p_n) \leftrightarrow W_n(x_1, x_2, \cdots, x_n) \sim \langle \mathrm{Tr}\mathscr{P} \exp\,(i\oint \mathbf{A}\cdot dx)\rangle$$

dual space: $(x_{i+1}-x_i)^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}$, $(\theta_{i+1}-\theta_i)^{\alpha A} = \lambda_i^\alpha \eta_i^A$ (Yangian) symmetry broken by IR/UV divergences

BDS ansatz [Bern, Dixon, Smirnov]: $A_n^{\mathrm{BDS}} \sim \exp(\frac{1}{4}\Gamma_{\mathrm{cusp}} F_n^{1-\mathrm{loop}}) \implies$ BDS-normalized amps: $R_{n,k} = \mathscr{A}_{n,k}/A_n^{\mathrm{BDS}}$

- Dual conformal invariant (DCI) function of $3(n-5)$ cross-ratios ($n = 4{,}5$ trivial)
- natural separation into transcendental (w. discontinuities) & algebraic part (only poles)

$$R_{n,k} \sim \underline{\text{(Yangian invariants)}} \times \underline{\text{(Transcendental functions)}}$$

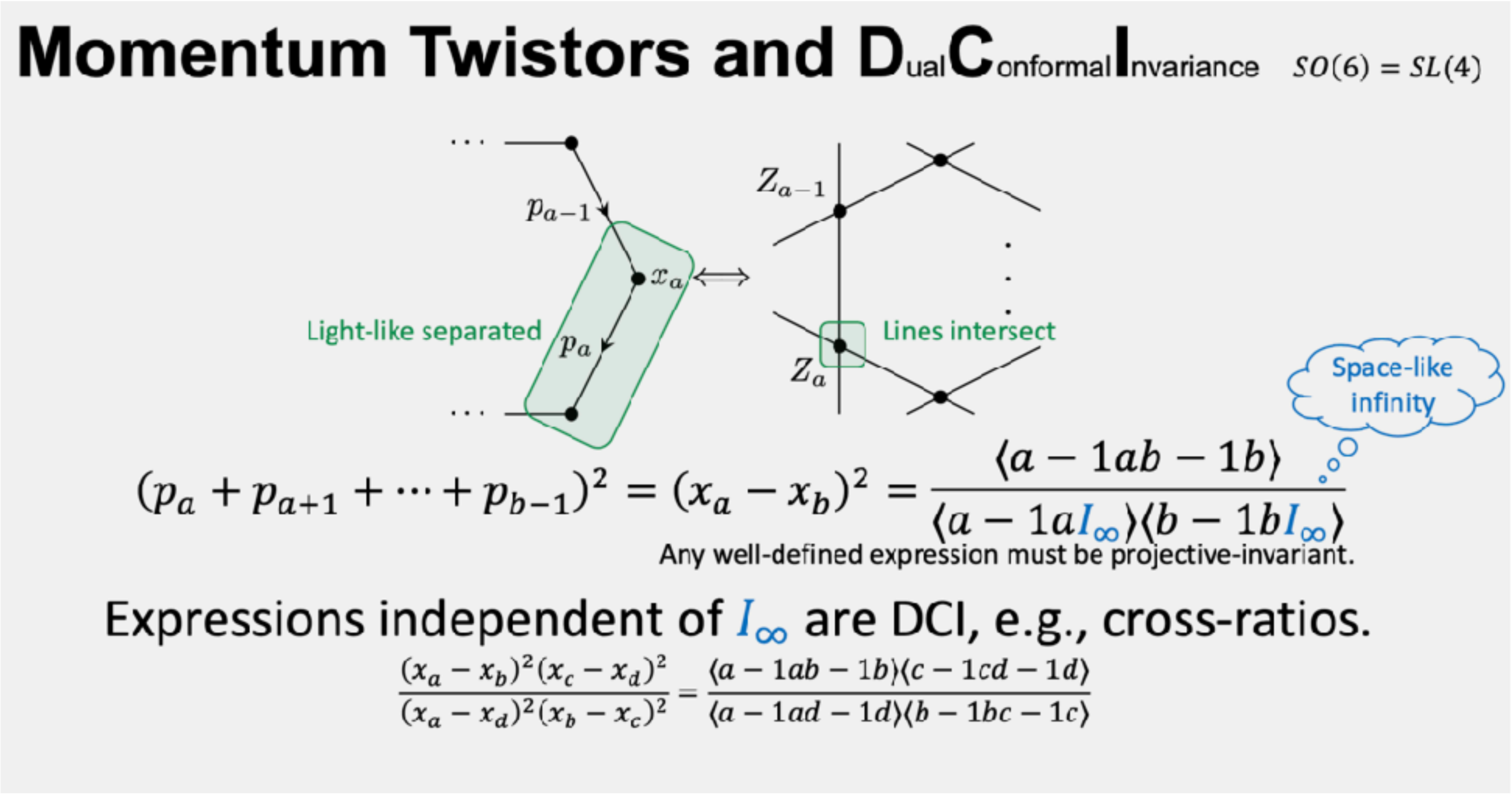
helicity,“rational/algebraic” DCI,“uniform weight”=2 L

Only MHV ($R_{n,0} \sim$ pure functions) & NMHV expected to be generalized polylogarithms!

$k \geq 2$ (NNMHV…): elliptic integrals etc. will appear [Bourjaily et al] [A. Kristensson, M.Wihelm, C. Zhang]

Momentum twistors [Hodges]

- Unconstrained variables for any massless kinematics (useful for QCD etc), naturally a planar ordering
- “Light-rays” of dual space, linearly realize dual symmetry SL(4|4): $\mathcal{Z}_i = (Z_i^a \mid \chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha,\dot\alpha}\lambda_{i,\alpha} \mid \theta_i^{\alpha,A}\lambda_i^\alpha)$
- Basic SL(4) invariant: 4-bracket $\langle i\,j\,k\,l\rangle := \epsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d$ e.g. $\langle i-1\, i\, j-1\, j\rangle \propto (x_i - x_j)^2$
- With DCI, only cross-ratios of invariants appear! configuration space: $G(4,n)/T$ (note $G(2,n)/T \sim \mathcal{M}_{0,n}$)



- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}$, $x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

$n = 6$ → precisely 3 ratios:

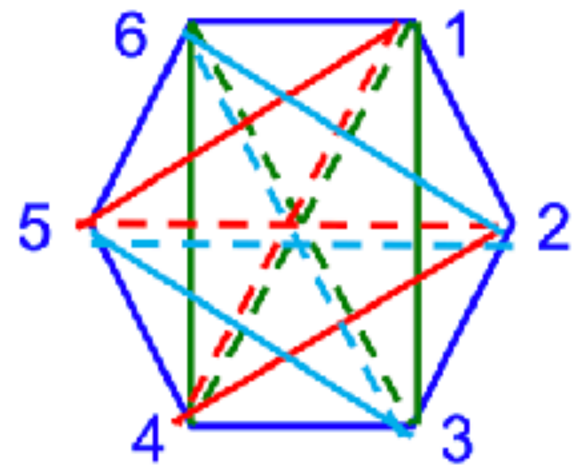
$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$$v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

$n = 7$ → 6 ratios.

In general, $3n$-15 ratios.



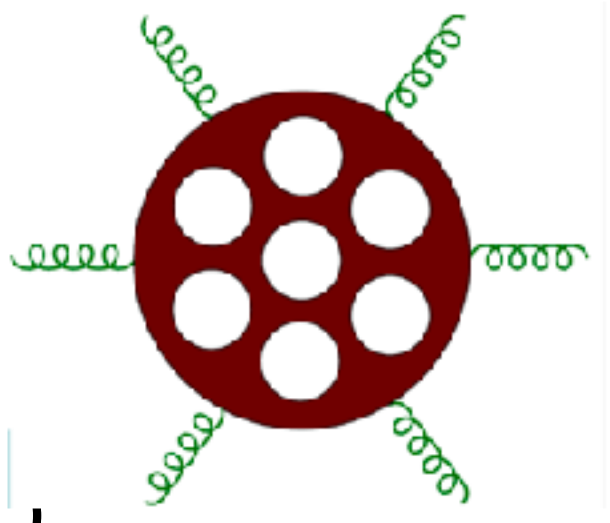
Symbology & bootstrap

Multiple polylogs: $G(\mathbf{a}, t_0) = \int_0^{t_0} \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_2}{t_2 - a_2} \cdots \int_0^{t_{w-1}} \frac{dt_w}{t_w - a_w}$ → symbol & letters [Goncharov, Spradlin, Vergu, Volovich]

$$dG^{(w)} = \sum_i G_i^{(w-1)} d\log x_i \implies \mathcal{S}(G^{(w)}) = \sum_i \mathcal{S}(G_i^{(w-1)}) \otimes x_i$$ e.g. $\mathcal{S}(\log(x)) = x,\ \mathcal{S}(\mathrm{Li}_2(x)) = -(1-x) \otimes x$

trivialize polylog relations; 1st entry: physical discontinuities, last entry: differential

For n=6,7: only 9 & 42 letters! conjecture: cluster variables of $G_+(4,n)$
finite-type (Dynkin) cluster algebras A_3 for n=6, E_6 for n=7[Golden et al][…]



hexagon/heptagon bootstrap: ansatz with alphabet + symmetry conditions (Qbar + collinear etc.)
-> unique answer to very high loops [Dixon et al] [Caron-Huot, Dixon, Dulat, McLeod, von Hippel, Papathanasiou][…]

starting n=8: infinite cluster algebra for G(4,n), how to obtain finite alphabet? square roots & more?

Generalized polylogarithms

Chen, Goncharov, Brown,…

- Can be defined as iterated integrals, e.g.

$$G(a_1, a_2, \dots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \dots, a_n, t)$$

- Or define differentially: $dF = \sum_{s_k \in \mathcal{S}} F^{s_k} \, d\ln s_k$

- There is a Hopf algebra that "co-acts" on the space of polylogarithms, $\Delta: F \to F \otimes F$
- The derivative dF is one piece of Δ: $\Delta_{n-1,1}F = \sum_{s_k \in \mathcal{S}} F^{s_k} \otimes \ln s_k$
- so we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F
- s_k **are letters in the symbol alphabet** $\mathcal{S}$

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

- Regular at $x = 0$, branch cut starts at $x = 1$.
- Iterated differentiation gives the symbol:

$$\mathcal{S}[Li_n(x)] = \mathcal{S}[Li_{n-1}(x)] \otimes x = \dots = -(1-x) \otimes x \otimes \dots \otimes x$$

- The $\{n\text{-}1,1\}$ coaction can be applied iteratively.
- Define the $\{n\text{-}2,1,1\}$ double coproducts, F^{s_k,s_j}, via the derivatives of the $\{n\text{-}1,1\}$ single coproducts F^{s_j}

$$dF^{s_j} \equiv \sum_{s_k \in \mathcal{S}} F^{s_k,s_j} \, d\ln s_k$$

- And so on for the $\{n\text{-}m,1,\dots,1\}$ m^{th} coproducts of F.
- The maximal iteration, n times for a weight n function, the symbol,

- Generalize the classical polylogs
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x) d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

- Symbol alphabet: $\mathcal{S} = \{x, 1-x\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$ is number of binary strings: 2^{2L}
- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol

Grassmannian cluster algebras

Symbol alphabet for n=6,7 (to high loops) given by cluster variables of G(4,n) [Golden et al][Dixon et al][Drummond et al]...

Cluster algebras: mutations from an initial quiver-> a_i (cluster variables) grouped into overlapping $\{a_1, \cdots, a_d\}$ (clusters) [Fomin, Zelevinski]
-> only finite for Dynkin-type quiver, $A_d(d(d+3)/2), B_d \sim C_d(d(d+1)), D_d(d^2), E_{6,7,8}(42,70,128), F_4(28), G_2(8)$!

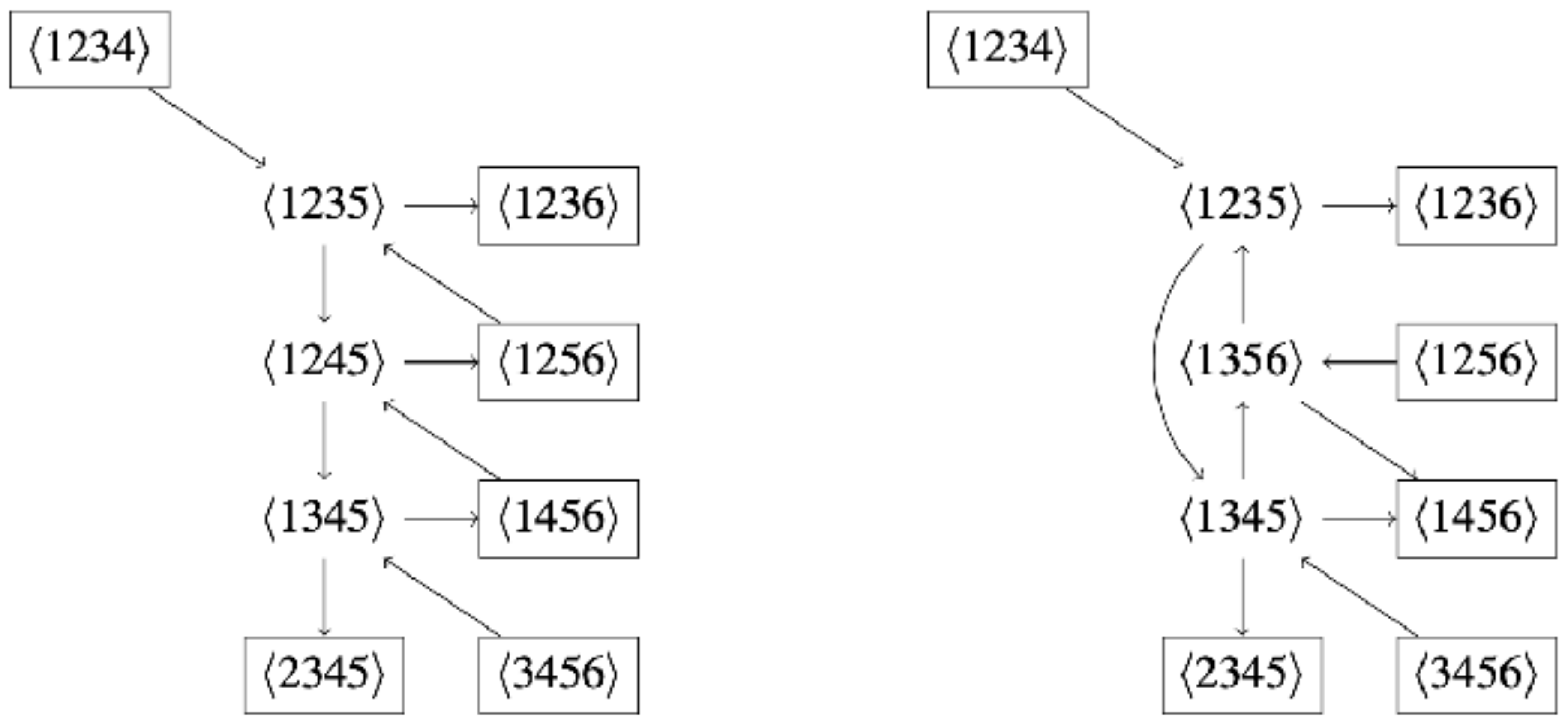
For G(4,n) only finite for n=6,7[Speyer, Williams]: A_3 & E_6 cluster-algebra alphabets!

$$\langle 1245\rangle \to \frac{\langle 1235\rangle\langle 1456\rangle + \langle 1345\rangle\langle 1256\rangle}{\langle 1245\rangle} = \langle 1356\rangle\,;$$

hexagon: 6 choose 4=15 <a b c d> (9 unfrozen + 6 frozen)

heptagon: 7 choose 4=35 <a b c d> (28 unfrozen + 7 frozen)
+ 14 degree-2: $\langle 1(23)(45)(67)\rangle$ + cyclic &$\langle 1(27)(34)(56)\rangle$ + cyclic

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

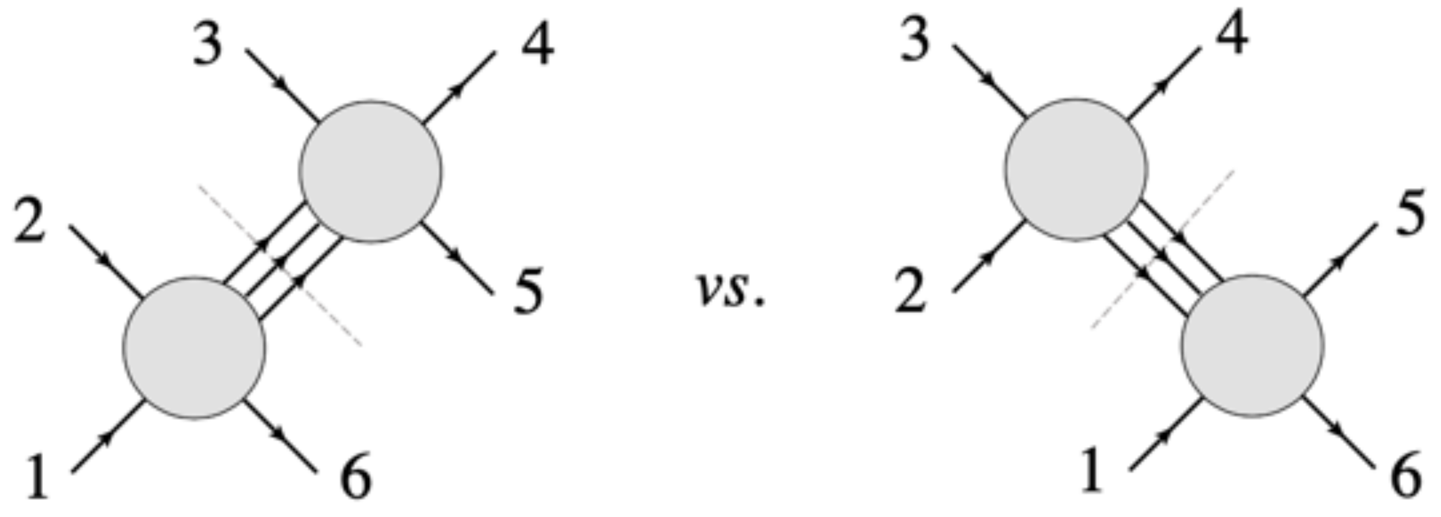


$\Longrightarrow$DCI letters: 9/42 X coordinates of A_3/E_6 cluster algebra

[from 2005.06735 by Caron-Huot et al]

(Extended) Steinmann relations

Basic properties of (massless) QFTs [Steinmann]: double discontinuities of any Feynman diagram (amplitude) vanish @ overlapping channels (no two overlapping $s_{a,\cdots,b-1}$ for planar) e.g. for 3-particle case:



$$\mathrm{Disc}_{s_{j,j+1,j+2}}\left(\mathrm{Disc}_{s_{i,i+1,i+2}}(A_{n,k})\right)=0\,,\quad \text{for } j=i\pm 1, i\pm 2\,,$$

[from 2005.06735 by Caron-Huot et al]

For n=6: $\langle 1245\rangle \nsim \langle 2356\rangle \nsim \langle 3461\rangle$ no pair for first 2 entries; even stronger constraints for n=7!

For divergent amps/integrals, need suitable subtraction due to 2-pt singularities (frozen)

Remarkably, data for n=6,7 reveals extended Steinmann relations: adjacent entries satisfy same constraints!

A new property of general QFT (for n=6,7) [Caron-Huot et al] more amps & integrals @ higher n/L?

For hexagon/heptagon, ES relations imply cluster adjacency: two cluster variables can appear consecutively only if there exist a cluster with both [Drummond et al]

$$a_1 = \frac{\langle 1245\rangle^2\langle 3456\rangle^2\langle 6123\rangle^2}{\langle 1234\rangle\langle 2345\rangle\ldots\langle 6123\rangle}\,,\quad m_1 = \frac{\langle 1356\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 3456\rangle}\,,\quad y_1 = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 1246\rangle\langle 3456\rangle}\,,$$

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i}\,, \qquad i \neq j,$$

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}\,, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}\,,$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}\,, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}\,, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

$$\sum_{\alpha,\beta=1}^{|\Phi|} D_{i\alpha\beta}\, F^{\phi_\alpha,\phi_\beta} = 0\,, \qquad i = 1,2,\ldots,l\,,$$

where we have again denoted the cluster $\mathscr{A}$-coordinates in blue, and

$$\text{First symbol entry of } A_{n,k} \in \begin{cases} a_i\,,\ i=1,\ldots,3\,, & \text{for } n=6\,,\\ a_{1i}\,,\ i=1,\ldots,7\,, & \text{for } n=7\,. \end{cases}$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle\,,$$

together with a_{ij} obtained from a_{i1} by cyclically relabeling the momentum twistors $Z_m \to Z_{m+j-1}$.

$$\text{Steinmann Relations:} \left\{ \begin{array}{ll} F^{a_i,a_{i+1}} = 0\,, 1 \leq i \leq 3\,, & \text{for } n=6 \\ F^{a_{1i},a_{1i+\delta}} = 0\,,\ \delta = 1,2\,,\ 1 \leq i \leq 7\,, & \text{for } n=7 \end{array} \right\} \text{if } F \text{ function of weight } 2\,.$$

$$\text{Extended Steinmann Relations:} \begin{cases} F^{a_i,a_{i+1}} = 0\,, 1 \leq i \leq 3\,, & \text{for } n=6\,.\\ F^{a_{1i},a_{1i+\delta}} = 0\,,\ \delta = 1,2\,,\ 1 \leq i \leq 7\,, & \text{for } n=7\,. \end{cases}$$

Steinmann cluster bootstrap

hexagon/heptagon bootstrap: compute amplitudes for n=6,7 without computing integrands/integrals at all

(1). hexagon/heptagon space: all integrable symbols (+ more) with A3/E6 alphabet
(2). first entry, Steinmann & ES relations/cluster adjacency -> huge reduction of the space
(3). physical constraints: final-entry, collinear, multi-Regge limits/OPE…

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
First entry	1	3	9	26	75	218	643	1929	5897	?	?	?	?	?
Steinmann	1	3	6	13	29	63	134	277	562	1117	2192	4263	8240	?
Ext. Stein.	1	3	6	13	26	51	98	184	340	613	1085	1887	3224	5431

Table 1: The dimensions of the hexagon, Steinmann hexagon, and extended Steinmann hexagon spaces at symbol level.

weight n	0	1	2	3	4	5	6	7
First entry	1	7	42	237	1288	6763	?	?
Steinmann	1	7	28	97	322	1030	3192	9570
Ext. Stein.	1	7	28	97	308	911	2555	6826

Table 2: The dimensions of the heptagon, Steinmann heptagon, and extended Steinmann heptagon spaces at symbol level.

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathcal{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

[from 2005.06735 by Caron-Huot et al]

Amp/WL from Qbar equations [Caron-Huot, SH]

$$\bar{Q}_a^A R_{n,k} = a \,\mathrm{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3}\mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R^{\rm tree}_{n+1,1}\right] + \text{cyclic},$$

loop parameter $$a := \frac{1}{4}\Gamma_{\rm cusp} = g^2 - \frac{\pi^2}{3}g^4 + \frac{11\pi^4}{45}g^6 + \ldots,$$

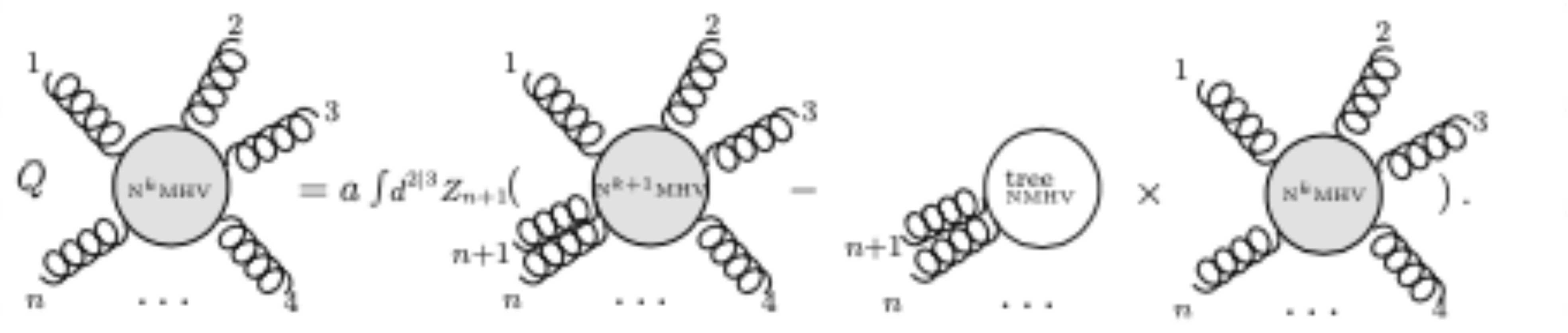


Figure 1. All-loop equation for planar $\mathcal{N}=4$ S-matrix.

1st order diff eqs for all-loop S-matrix: determine MHV & NMHV amps uniquely, given lower-loop amps

Last entries for all loops (important for bootstrap) MHV and NMHV (to all n) [SH, Z. Li, C. Zhang]

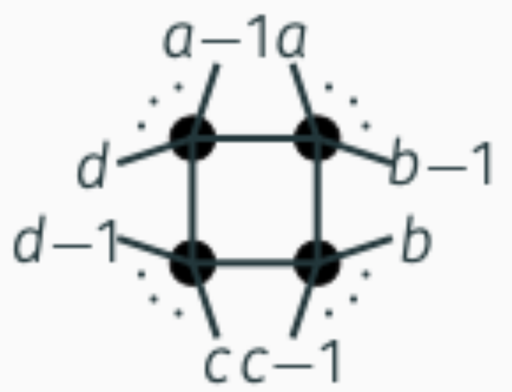
computed 2-loop NMHV heptagon -> 3-loop MHV hexagon [Caron-Huot, SH] + 2-loop MHV to all n

nice observation: even for n>7, 2-loop MHV have letters that are (simple) cluster variables of G(4,n)!

However, for n>7, k+l>2 (3-loop MHV, 2-loop NMHV, even 1-loop NNMHV): algebraic letters (square roots) -> rationalization on RHS -> new data for symbol alphabet & structures !

2-loop NMHV + 3-loop MHV [SH, Z. Li, C. Zhang]

$$
\begin{aligned}
&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{c-1}_{a,b,c,d} \otimes x^{c-1}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,b\,c-1]\\
-\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{c}_{a,b,c,d} \otimes x^{c}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,b\,c]\\
+\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{b-1}_{a,b,c,d} \otimes x^{b-1}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,c-1\,c]\\
-\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{b}_{a,b,c,d} \otimes x^{c}_{a,b,c,d}\,[a{-}1\,a\,b\,c{-}1\,c]\\
+\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{a-1}_{a,b,c,d} \otimes x^{a-1}_{a,b,c,d}\,[a{-}1\,b{-}1\,b\,c-1\,c]\\
-\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{a}_{a,b,c,d} \otimes x^{a}_{a,b,c,d}\,[a\,b{-}1\,b\,c{-}1\,c]\,,
\end{aligned}
$$



$$
\begin{cases}
u_{abcd} = \frac{x^2_{ab}x^2_{cd}}{x^2_{ac}x^2_{bd}},\ v_{abcd} = \frac{x^2_{ad}x^2_{bc}}{x^2_{ac}x^2_{bd}},\ \Delta_{abcd} = \sqrt{(1-u-v)^2 - 4uv}\\
z_{abcd} = \tfrac{1}{2}(1+u-v+\Delta),\ \bar{z}_{a,b,c,d} = \tfrac{1}{2}(1+u-v+\Delta),
\end{cases}
$$

$$
\mathcal{X}^*_{a,b,c,d} := \frac{(x^*_{a,b,c,d}+1)^{-1} - \bar{z}_{d,a,b,c}}{(x^*_{a,b,c,d}+1)^{-1} - z_{d,a,b,c}}\,, \qquad \tilde{\mathcal{X}}^*_{a,b,c,d} := \frac{(x^*_{a,b,c,d-1}+1)^{-1} - z_{d,a,b,c}}{(x^*_{a,b,c,d-1}+1)^{-1} - \bar{z}_{d,a,b,c}}
$$

with 6 choices $a{-}1, a, b{-}1, b, c{-}1, c$ of the superscript, where

$$
\begin{aligned}
x^a_{a,b,c,d} &= \frac{\langle \bar{d}(c{-}1\,c) \cap (a\,b{-}1\,b)\rangle}{\langle \bar{d}\,a\rangle\langle b{-}1\,b\,c{-}1\,c\rangle}\,, & x^{a-1}_{a,b,c,d} &= x^a_{a,b,c,d}|_{a\leftrightarrow a-1}\\
x^b_{a,b,c,d} &= \frac{\langle \bar{d}(c{-}1\,c) \cap (a{-}1\,a\,b)\rangle}{\langle \bar{d}(a{-}1\,a) \cap (b\,c{-}1\,c)\rangle}\,, & x^{b-1}_{a,b,c,d} &= x^b_{a,b,c,d}|_{b\leftrightarrow b-1}\\
x^c_{a,b,c,d} &= \frac{\langle \bar{d}\,c\rangle\langle a{-}1\,a\,b{-}1\,b\rangle}{\langle \bar{d}(a{-}1\,a) \cap (b{-}1\,b\,c)\rangle}\,, & x^{c-1}_{a,b,c,d} &= x^c_{a,b,c,d}|_{c\leftrightarrow c-1}
\end{aligned}
$$

Remarkably constrained & compact "algebraic part": 4-mass $\otimes$ algebraic^i $\otimes$ final^i $\times R_i$ (all correlated!)

$n=8:\ \Delta_{1,3,5,7}$ & $\Delta_{2,4,6,8}$, 9+9 independent algebraic letters (+180 rational letters)

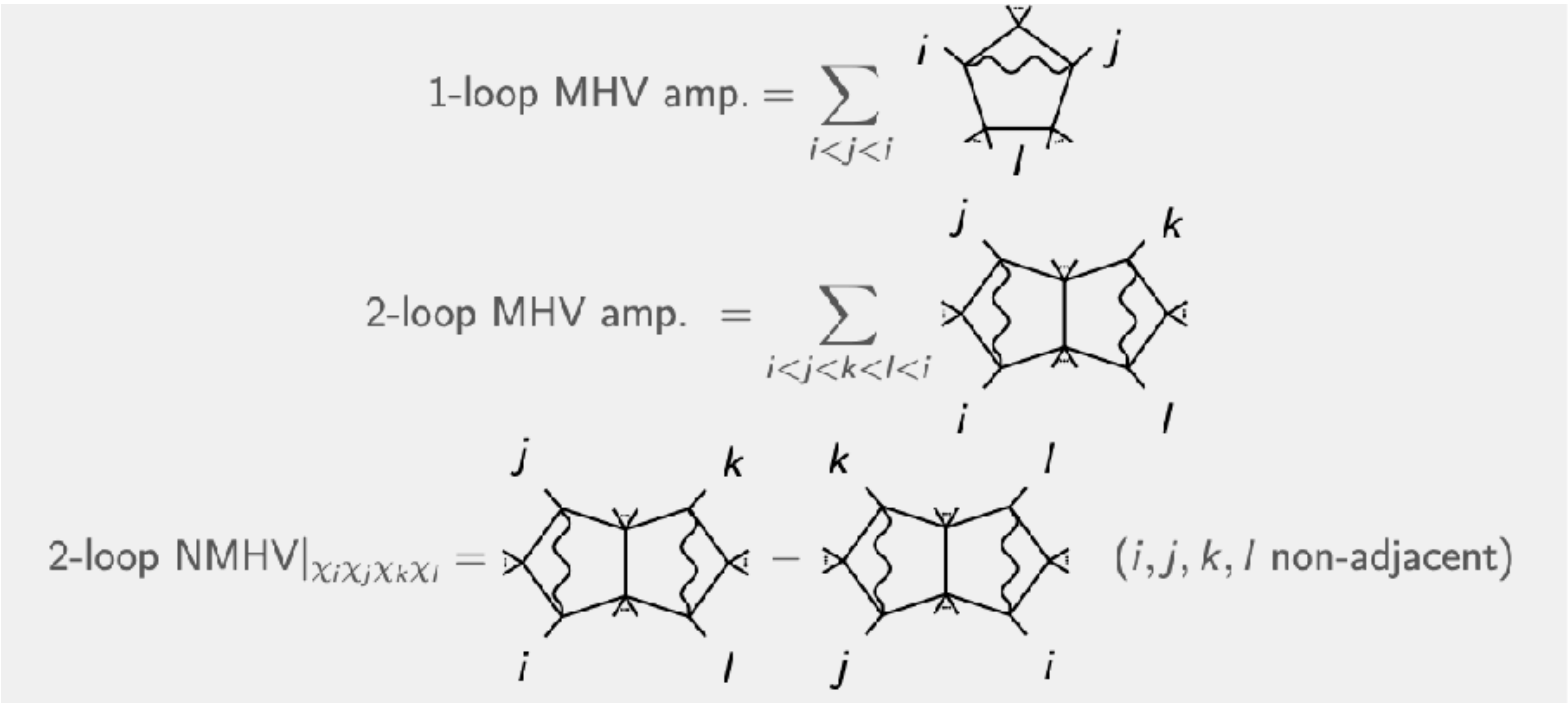
9 Δ for n=9, 11*9 independent algebraic letters… -> 3-loop n=8 MHV: still 9+9 (+204 rational)! [Z. Li, C. Zhang]

Origin of alphabet: Landau equations [Spradlin et al] tropical $G_+(4,8)$ etc. [Drummond, et al] [Arkani-Hamed, Lam, Spradlin]

poles/"letters" of Yangian invariants [Mago, Schreiber, Spradlin, Volovich] [SH, Z. Li] […]

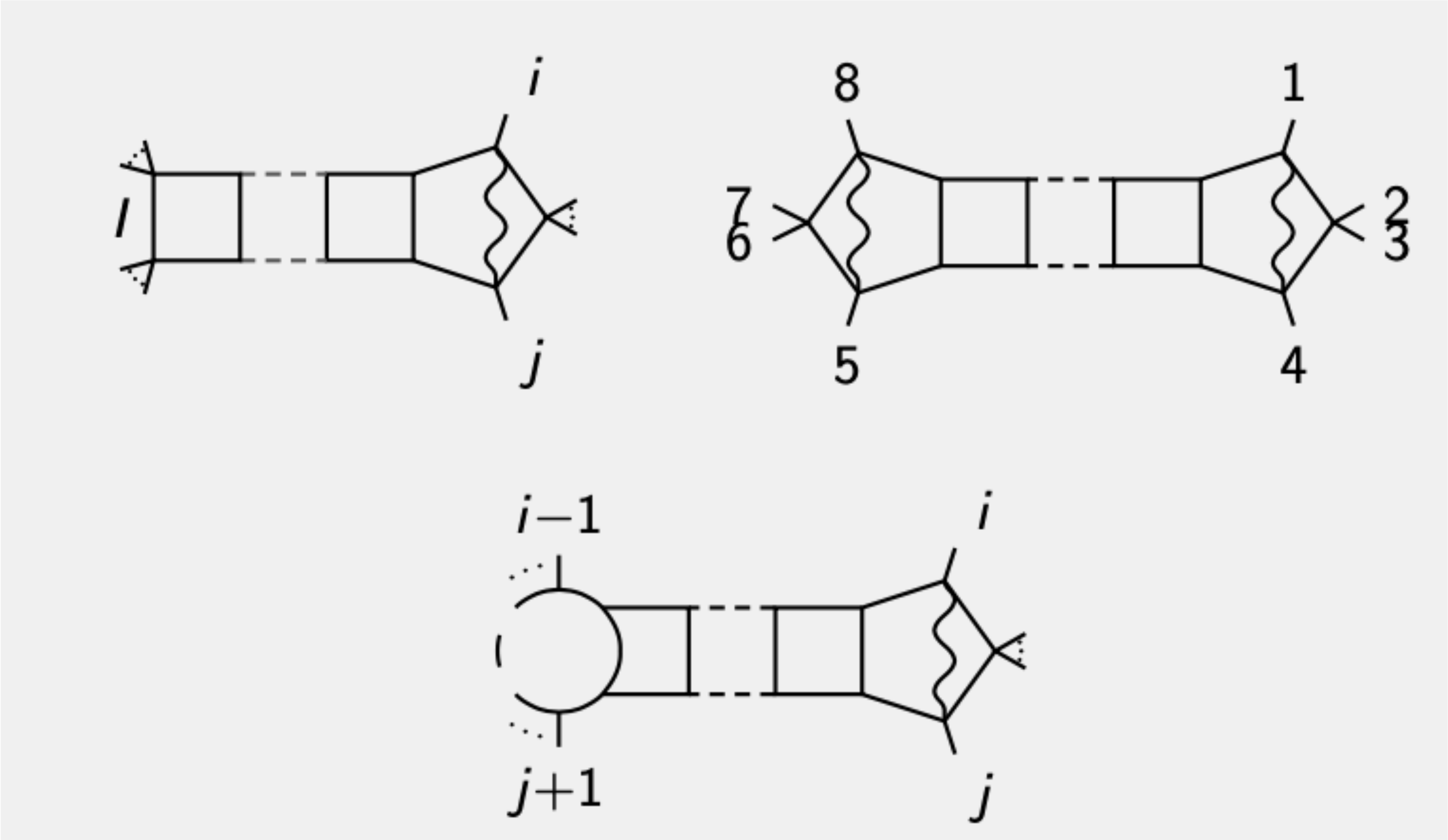
Feynman integrals from WLs

Uniform transcendental integrals



(double-) pentagons for MHV/NMHV amps [Arkani-Hamed et al]

(e.g. numerators $\langle \ell_1\, \bar{i} \cap \bar{j} \rangle$, $\langle \ell_2\, \bar{k} \cap \bar{l} \rangle$)



(double-) penta-ladder [Drummond, Henn, Trnka] & more

uniform transcendentality vs. dlog (focus on IR finite integrals e.g. $i < j - 1$ & $k < l - 1$ for dp)

-> a class of integrals in N=4 SYM, also play an important role in general (master integrals)

Feynman integrals from WL

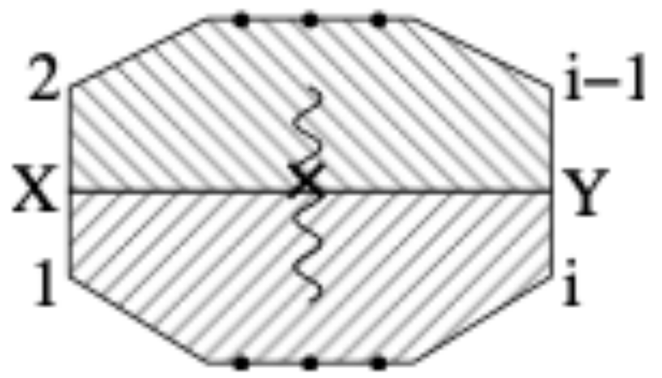
WL powerful for not only full amps, but also its building blocks: WL diagrams = Feynman integrals

How we originally computed 2-loop MHV: $dR_{n,0} = \sum_{i<j} C_{i,j} d\log\langle \bar{i} j\rangle$ w. $C_{i,j}$ (super-) WL diagrams

L=1:

$$C_{2,i} = \int_0^\infty d\tau_X d\tau_Y \frac{\langle \bar{2} i\rangle\langle \bar{i} 2\rangle}{\langle XY\rangle^2} = \log u_{2,i-1,i,1}$$

L=2: 1-d τ-integral of box integrals



simplest NMHV: difference of two WL diagrams=double pentagon!

$$\mathcal{W}^{(2)}_{n,k=1}\Big|_{\chi_i^A\chi_j^B\chi_k^C\chi_l^D} =$$

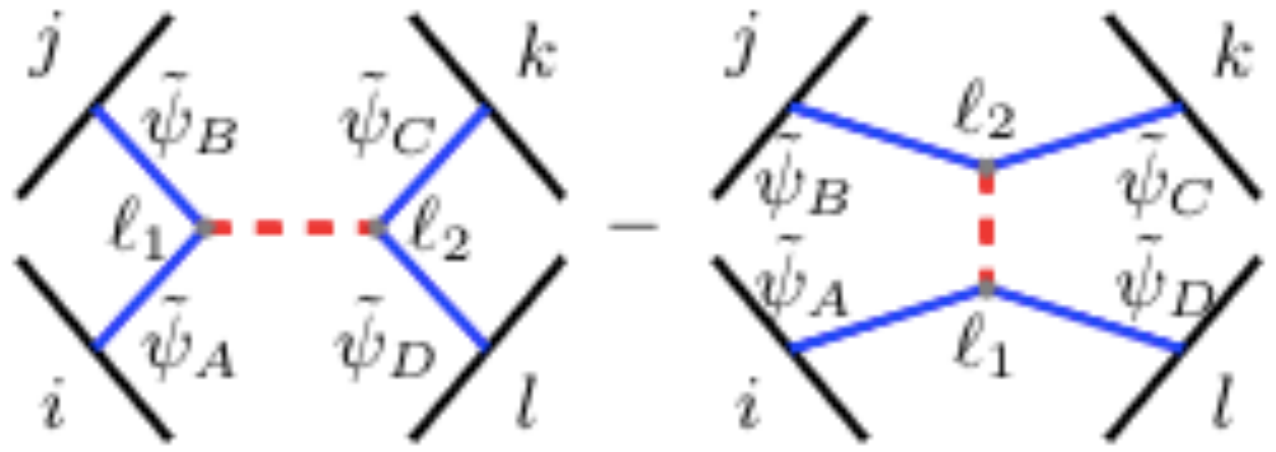


FIG. 2. NMHV component of super-WL as difference of two diagrams, each equals to a double-pentagon integral.

WL $d\log$: 1-loop examples [SH, Z. Li, Y. Tang, Q. Yang]

Why useful? swap order of integrals, left with simple line integrals ("smart parametrization")

chiral pentagon: $\frac{1}{\langle \ell i-1 i\rangle \, \langle \ell i i+1\rangle} = \int_0^\infty \frac{d\tau}{\langle \ell i X(\tau)\rangle^2}$, $X(\tau) := Z_{i-1} + \tau_X \, Z_{i+1}$ "fermions" at $x := (iX)$ & $y := (jY)$

$$\Longrightarrow I_{\text{pent.}} = \int d\tau_X d\tau_Y \int \frac{d^4\ell \langle \ell \bar{i} \cap \bar{j}\rangle \, \langle I\, i\, j\rangle}{\langle \ell\, i\, X\rangle^2 \langle \ell\, j\, Y\rangle^2 \langle \ell\, I\rangle} = \int d^2\tau \frac{\langle I\bar{i} \cap \bar{j}\rangle \langle I\, i\, j\rangle}{\langle I\, i\, X\rangle \langle I\, j\, Y\rangle \langle i\, X\, j\, Y\rangle}$$ (star-triangle identity)

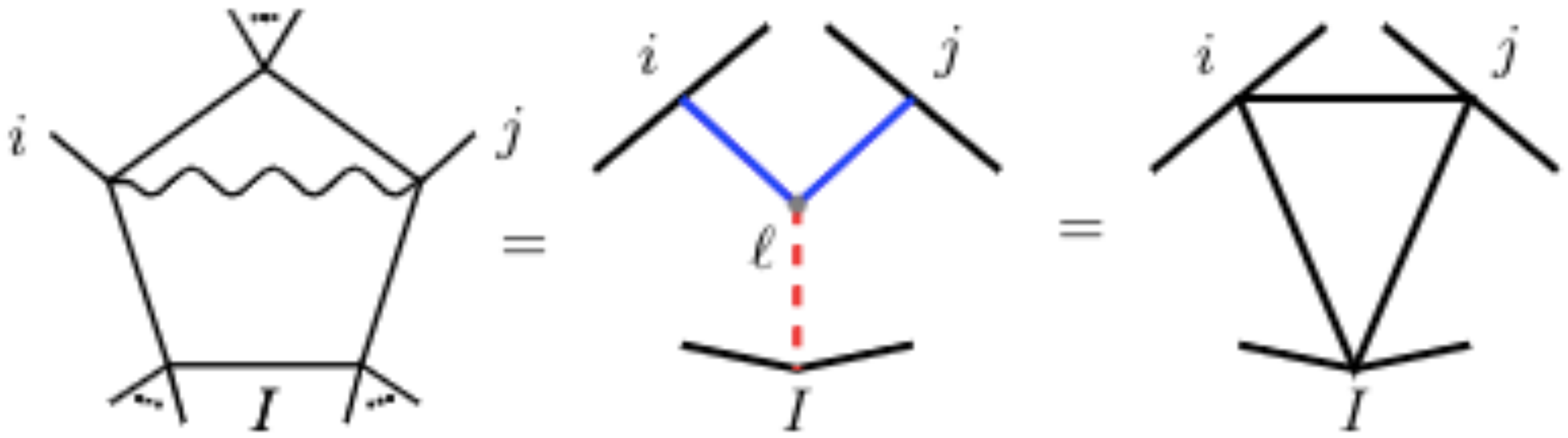


FIG. 3. The chiral pentagon written as a WL diagram, and loop integral performed using "star-triangle" identity.

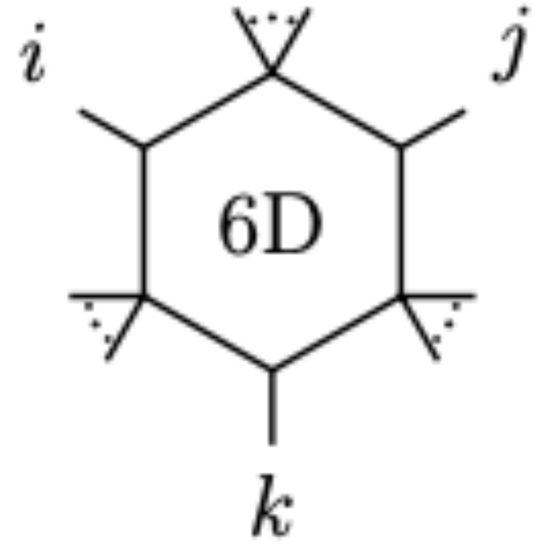
Nice $d\log$ 2-form: $\int_{(i,j)} d\log \frac{\langle IjY\rangle}{\langle \bar{i}(jY) \cap (iI)\rangle} d\log \frac{\langle iXjY\rangle}{\langle IiX\rangle}$

Trivially give well-known dilog (manifest DCI + weight-2)

Geometry: integrating $\Omega(\Delta')$ in Δ (similar to Aomoto)

6d 3-mass-easy hexagon [Del Duca et al]
(weight-3 polylog of 9 cross-ratios)

$$\Omega_1^{(6\mathrm{D})}(i,j,k) :=$$

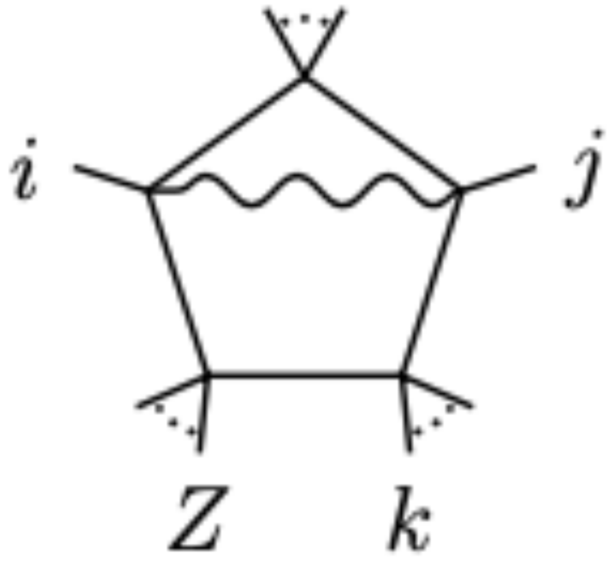


$$= \int \frac{\mathrm{d}^6 x_0}{\pi^3} \frac{x_{i,j+1}^2 x_{j,k+1}^2 x_{k,i+1}^2 \sqrt{\Delta_9}}{x_{0,i}^2 x_{0,i+1}^2 x_{0,j}^2 x_{0,j+1}^2 x_{0,k}^2 x_{0,k+1}^2}.$$

momentum twistors G(4,n): all square roots disappear

nice 3-fold dlog integral
(1-d integral of "deformed" pentagon)

$$\Omega_1^{(6\mathrm{D})}(i,j,k) = \int_0^\infty \mathrm{d}\tau_z \frac{\langle (ijk)\bar{i}\cap\bar{j}\cap\bar{k}\rangle}{\langle kZ\bar{i}\cap\bar{j}\rangle\langle kZij\rangle}$$



$$= \int_{\mathbb{R}^3_{\geq 0}} \mathrm{d}\log \frac{\langle kZij\rangle}{\langle kZ\bar{i}\cap\bar{j}\rangle} \left(\mathrm{d}\log \frac{\langle jYkZ\rangle}{\langle jYi(kZ)\cap\bar{i}\rangle} \mathrm{d}\log \frac{\langle iXjY\rangle}{\langle iXkZ\rangle} \right).$$

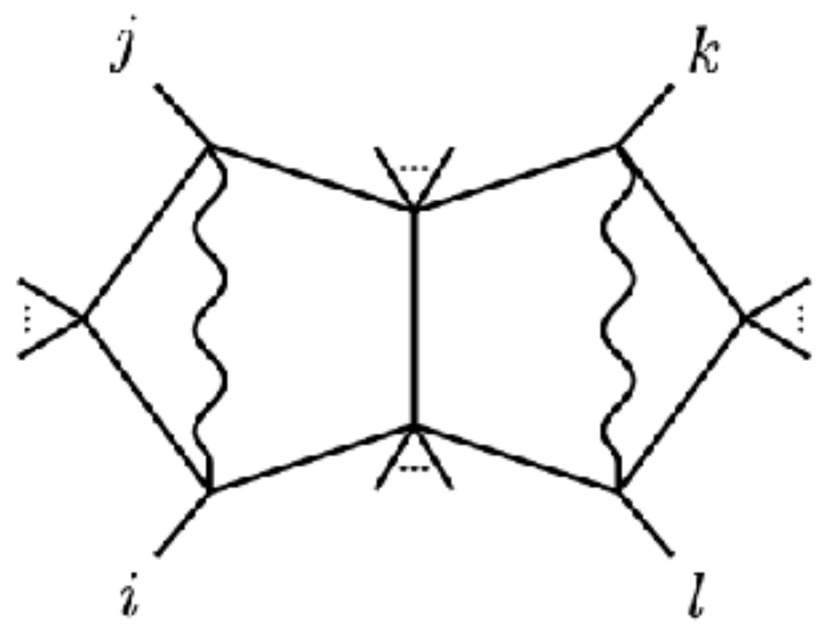
using symbol integration [Caron-Huot SH][…]
straightforward to get weight-3 symbol
without performing any integral

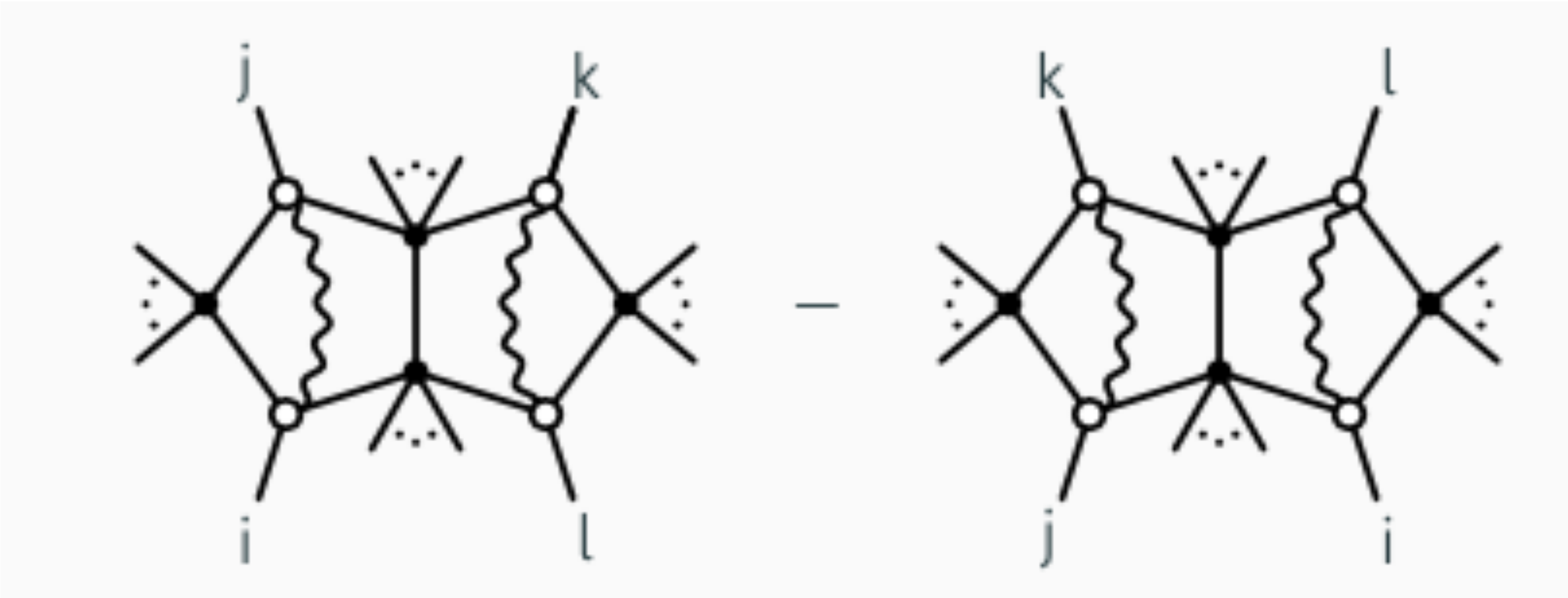
$$\int_a^b \mathrm{d}\log(t+c)\,(F(t)\otimes w(t)) \implies \{F(t)\otimes w(t)\otimes (t+c))|_{t=a}^{t=b}, \quad \left(\int_a^b \mathrm{d}\log(t+c)\,F(t)\right)\otimes w, \quad \left(\int_a^b \mathrm{d}\log\frac{t+c}{t+d}\,F(t)\right)\otimes(c-d)\}$$

Two-loop amps from integrals

Longstanding problem: compute generic double-pentagon analytically (12 legs, lots of square roots)

All we need for MHV: $A^{2-\text{loop}}_{n,\text{MHV}} = \sum_{i<j<k<l} I_{\text{dp}}(i,j,k,l)$; how to see cancellation of square roots?

$I_{dp}(i,j,k,l) =$ 



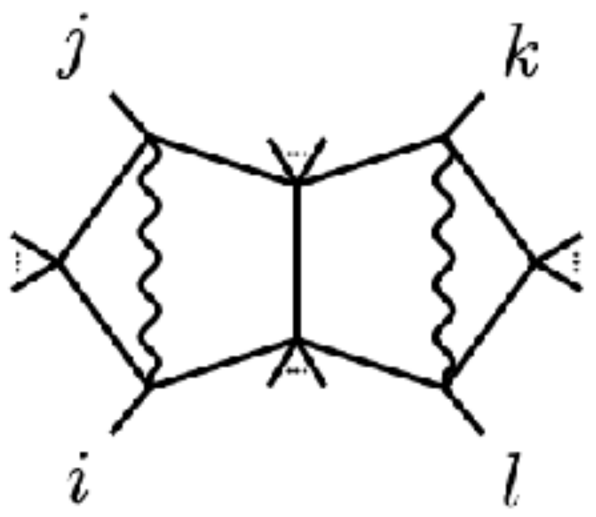
Component $\chi_i^1 \chi_j^2 \chi_k^3 \chi_l^4$ for non-adjacent i, j, k, l (vanishes for L=0,1) given by 2 double-pentagon integrals

Surprise: $\bar{Q}$ result for the components free of roots (algebraic words vanish)!
For n=8: also observed in [Bourjaily et al] by evaluating $I_{\text{dp}}(1,3,5,7)$ at a numeric point? Why cancel?

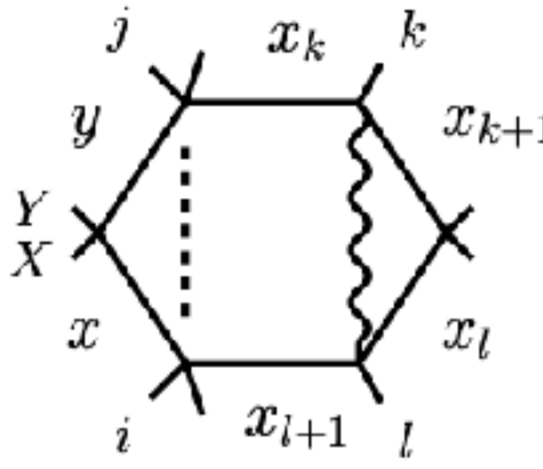
Generic double pentagon [SH, Z. Li, Q. Yang, C. Zhang]

Exactly the same method: 2d integral of a hexagon
($k = j+1,\ l = i-1$: chiral hexagon)

$$X := Z_{i-1} - \tau_X Z_{i+1}\,, \quad Y := Z_{j-1} - \tau_Y Z_{j+1}$$



$$= \int \frac{d^2\tau \langle ijkl\rangle}{\langle iXjY\rangle}$$



$$\int \left([x,x_k] I_{x,x_k} - (k{-}1 \leftrightarrow k{+}1)\right) - (\bar{k} \leftrightarrow \bar{l}) + [x,y] I_{x,y}$$

New: hexagon not “pure”, 15 boxes w. 2-form “leading singularities”;
need rationalization for computing τ-integrals

$$[x,y] = \mathrm{d}\log \frac{\langle iXkl\rangle}{\langle iXjY\rangle} \mathrm{d}\log \frac{\langle \bar{i}(jY)\cap(ikl)\rangle}{\langle jYkl\rangle}\,,$$

$$[x,x_k] = \mathrm{d}\log \frac{\langle jYil\rangle}{\langle jYkl\rangle} \mathrm{d}\log \frac{\langle iXjY\rangle}{\langle l(iX)(jY)(kk{+}1)\rangle}\,,$$

weight-3 symbol: “4-mass box ⊗ algebraic letter”
final-entry free of square roots (as 2-loop NMHV/3-loop MHV!)

$$\begin{aligned} I_{x,x_k} &:= \tilde{F}(x,y,x_{k+1},x_l) - \tilde{F}(x,y,x_{k+1},x_{l+1}) \\ &\quad - \mathrm{L}_2(l{+}1,x,y,l) + \mathrm{L}_2(l{+}1,x,k{+}1,l) \\ &\quad - \mathrm{L}_2(l{+}1,y,k{+}1,l) + \log u_{l+1,x,y,l} \log u_{x,y,k+1,l+1}\,, \\ I_{x,y} &:= \mathrm{L}_2(x,k,k{+}1,l) - \mathrm{L}_2(x,k,k{+}1,l{+}1) \\ &\quad - \mathrm{L}_2(l{+}1,x,k,l) + \mathrm{L}_2(l{+}1,x,k{+}1,l) \\ &\quad - \mathrm{L}_2(l{+}1,k,k{+}1,l) + \log u_{l+1,x,k,l} \log u_{x,k,k+1,l+1} \end{aligned}$$

compact algebraic words : sum of 16 simple blocks
(algebraic)$|_{(i,j,k,l)-(j,k,l,i)}$ =0 ! cancellation of square roots

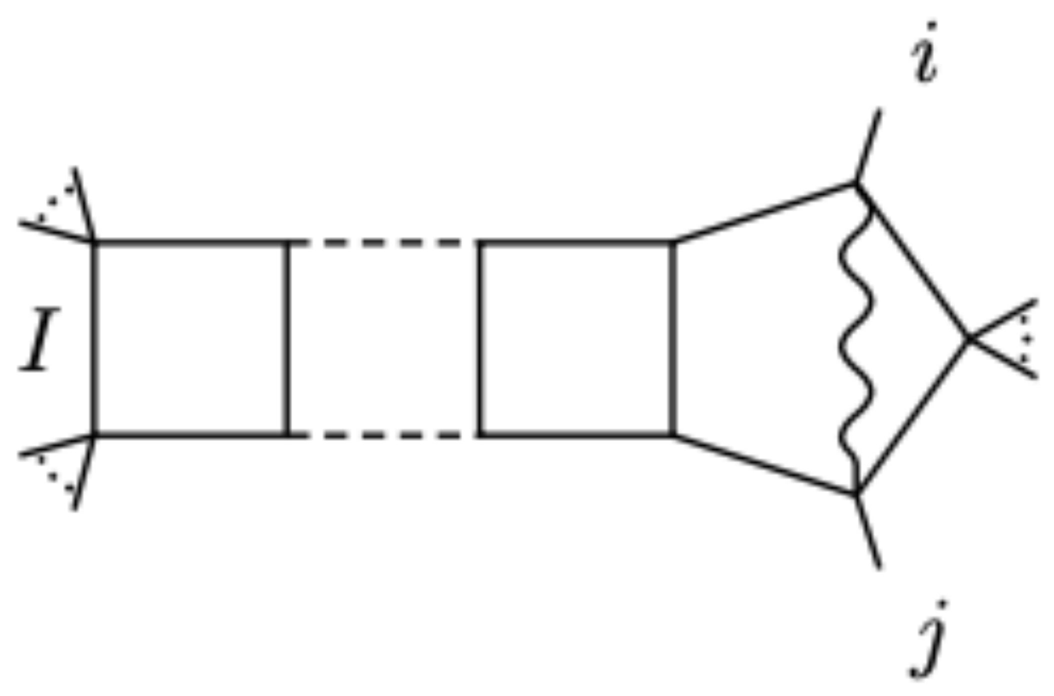
$$\mathrm{L}_2(a,b,c,d) := \mathrm{Li}_2(1 - u_{a,b,c,d})$$

$$\sum_{\sigma_a \in \{0,1\}} (-)^{\sigma_1+\sigma_2+\sigma_3+\sigma_4}\, S^{4-\mathrm{m}}(i+\sigma_1, j+\sigma_2, k+\sigma_3, l+\sigma_4) \otimes W^{i,j,k,l}_{\sigma_1,\cdots,\sigma_4}$$

Recursion for all-loop ladders [SH, Z. Li, Y. Tang, Q. Yang]

Simplest multi-loop application: penta-box ladder integral

$$\Psi_L(i,j;I) :=$$



recursion: L-loop as 2-fold dlog integral of deformed (L-1)-loop
1-loop pentagon=2 dlog -> 2L-fold dlog integral

$$u=\frac{\langle i{-}1 i I\rangle\langle j j{+}1 i i{+}1\rangle}{\langle i{-}1 i j j{+}1\rangle\langle I i i{+}1\rangle},\quad v=\frac{\langle j j{+}1 I\rangle\langle i{-}1 i j{-}1 j\rangle}{\langle j j{+}1 i{-}1 i\rangle\langle I j{-}1 j\rangle},\quad w=\frac{\langle i{-}1 i j j{+}1\rangle\langle j{-}1 j i i{+}1\rangle}{\langle i{-}1 i j{-}1 j\rangle\langle j j{+}1 i i{+}1\rangle}.$$

$$\Psi_L(i,j,I)=\int\left[\prod_{a=1}^{L-1}\mathrm{d}\log\langle i{-}1 i j Y_a\rangle \mathrm{d}\log\frac{\langle i X_a j Y_a\rangle}{\tau_{X_a}}\right]\mathrm{d}\log\frac{\langle j Y_L I\rangle}{\langle j Y_L i I\cap \bar{i}\rangle}\mathrm{d}\log\frac{\langle i X_L j Y_L\rangle}{\langle i X_L I\rangle}.$$

nice DCI form: simple deform & "odd" weight objects in between (tree=$1-u-v+uvw$)

$$\Psi_{L+\frac{1}{2}}(u,v,w)=\int \mathrm{d}\log\frac{\tau_X+1}{\tau_X}\,\Psi_L\left(\frac{u(\tau_X+w)}{\tau_X+uw},v,\frac{w(\tau_X+1)}{\tau_X+w}\right)$$

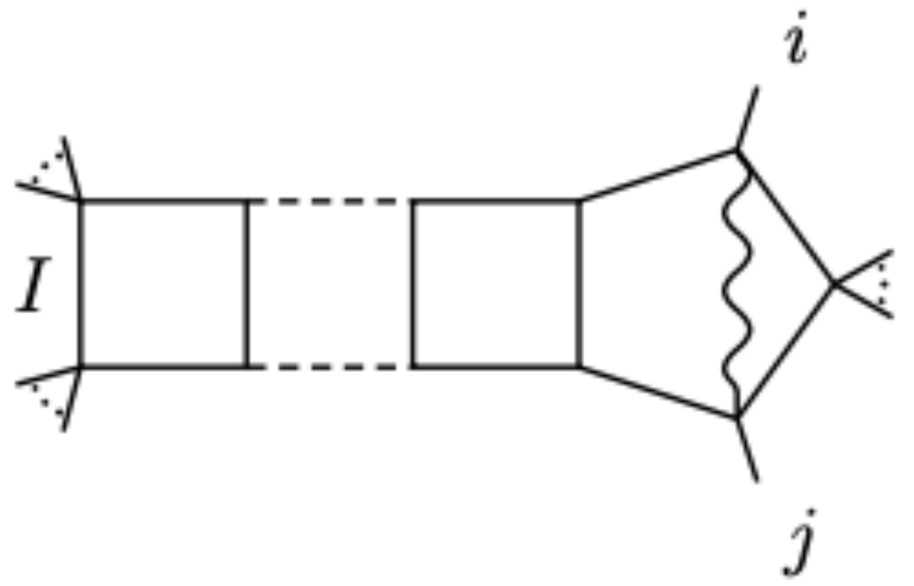
possible to get symbol to all loops w. 9 letters
$\{u,v,w,1-u,1-v,1-w,1-uw,1-vw,1-u-v+uvw\}$

$$\Psi_{L+1}(u,v,w)=\int \mathrm{d}\log(\tau_Y+1)\,\Psi_{L+\frac{1}{2}}\left(u,\frac{v(\tau_Y+1)}{v\tau_Y+1},\frac{\tau_Y+w}{\tau_Y+1}\right)$$

Differential eq. & resummation

easy to show the recursion satisfy beautiful diff. eq. [Drummond, Henn, Trnka]

$$(1-u-v+uvw)uv\partial_u\partial_v\Psi_{L+1}(u,v,w)=\Psi_L(u,v,w)$$

$\Psi_L(i,j;I) :=$ 

recursion helps to resum the ladders: define $\Psi_g := \sum_{L=1}^{\infty} g^{2L}\Psi_L$ (w. coupling const.), it satisfies

$$\Psi_g(u,v,w)=g^2\Psi_1(u,v,w)+g^2\int \mathrm{d}\log(\tau_Y+1)\,\mathrm{d}\log\frac{\tau_X+1}{\tau_X}\Psi_g(\tilde{u},\tilde{v},\tilde{w})$$

this can be solved by series expansion of kinematic var. $x=1-u^{-1}, y=1-v^{-1}, z=1-w$

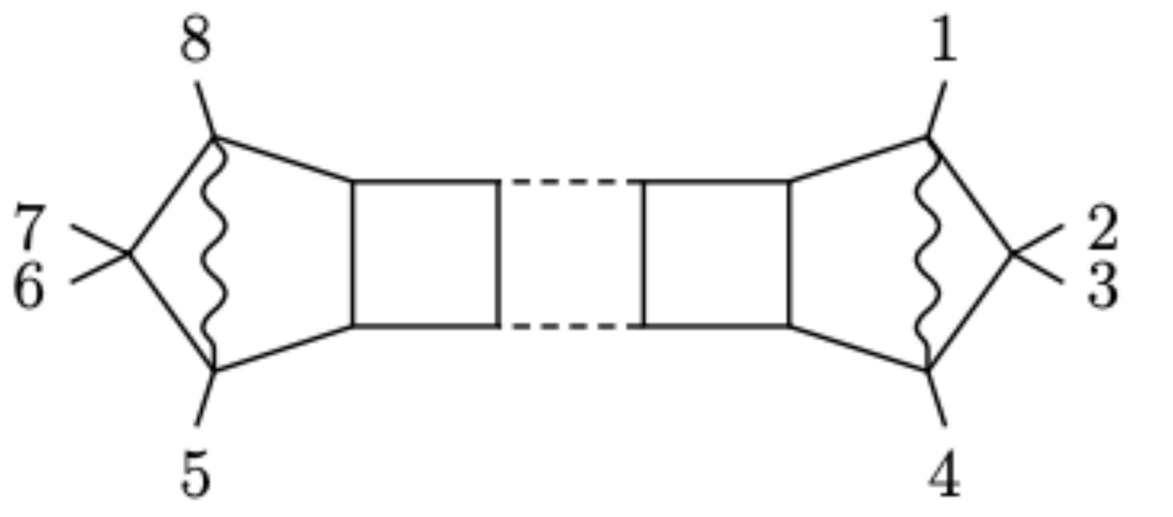
$$\Psi_g=g^2\sum_{k,l=1}^{\infty}\frac{x^ky^l}{kl+g^2}-g^2\sum_{k,l=0,m=1}^{\infty}\frac{x^ky^lz^m}{kl+g^2}\frac{g^2}{(k+m)(l+m)}\prod_{n=1}^{m}\frac{(k+n)(l+n)}{(k+n)(l+n)+g^2}\,.$$

Generalized penta ladders [SH, Z. Li, Y. Tang, Q. Yang]

applicable to a large class of integrals w. “pentagon handles” (or similar 1-loop)—> reduce loop orders
in particular a recursion for “generalized penta ladders”: 2(L-1)-fold dlog of some 1-loop integrals

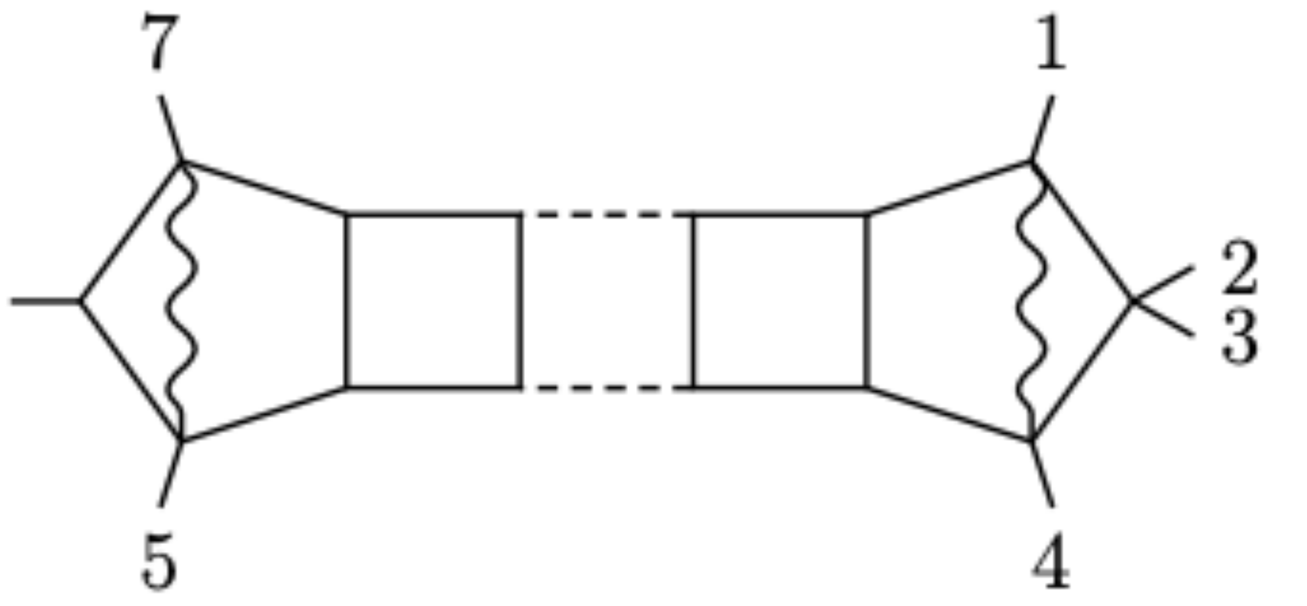
straightforward to obtain symbol if no square roots involved
but need “rationalization” otherwise

e.g. $\Omega_L(1,4,5,8)$ involves square root $\Delta_{2,4,6,8} = \sqrt{(1-u-v)^2 - 4uv}$

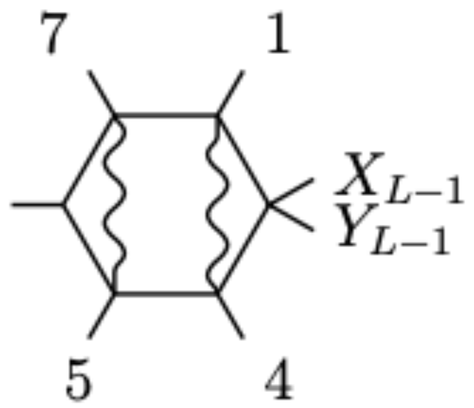


focus on $\Omega_L(1{,}4{,}5{,}7)$: 2(L-1) fold d log integral of 1-loop (7-pt) hexagon

$\Omega_L(1,4,5,7) = 6$



$$\Omega_L(1,4,5,7) = \int \prod_{a=1}^{L-1} \mathrm{d}\log\langle 147 Y_a\rangle \mathrm{d}\log\frac{\langle 1X_a 4Y_a\rangle}{\tau_{X_a}} \times$$



$$u_1 = \frac{\langle 1245\rangle\langle 5671\rangle}{\langle 1256\rangle\langle 4571\rangle},\quad u_2 = \frac{\langle 3471\rangle\langle 4567\rangle}{\langle 3467\rangle\langle 4571\rangle},\quad u_3 = \frac{\langle 1267\rangle\langle 3456\rangle}{\langle 1256\rangle\langle 3467\rangle},\quad u_4 = \frac{\langle 1234\rangle\langle 4571\rangle}{\langle 1245\rangle\langle 3471\rangle}.$$

beautiful DCI form: (rescaled) deformation

$$\Omega_{L+\frac{1}{2}}(u_1,u_2,u_3,u_4) = \int \mathrm{d}\log\frac{\tau_X+1}{\tau_X}\Omega_L\left(\frac{u_1(\tau_X+u_4)}{\tau_X+u_1u_4}, u_2, \frac{\tau_X u_3}{\tau_X+u_1u_4}, \frac{u_4(\tau_X+1)}{\tau_X+u_4}\right),$$

$$\Omega_{L+1}(u_1,u_2,u_3,u_4) = \int \mathrm{d}\log(\tau_Y+1)\Omega_{L+\frac{1}{2}}\left(u_1, \frac{u_2(\tau_Y+1)}{u_2\tau_Y+1}, \frac{u_3}{1+\tau_Y u_2}, \frac{\tau_Y+u_4}{\tau_Y+1}\right),$$

-> symbol to all loops w. 16 letters

$u_1, u_2, u_3, u_4, 1-u_1, 1-u_2, 1-u_3, 1-u_4,$

$1-u_1u_4, 1-u_2u_4, 1-u_3-u_1u_4, 1-u_3-u_2u_4; y_1, y_2, y_3, y_4$

nicely, alphabet of D_4 cluster algebra [SH, Z. Li, Q. Yang] also appear for 6d 1-mass hexagon [Chicherin, Henn, Papathanasiou]

w. $u_3 \to 0$ back to the 9 letters of Ψ_L : sub-algebra $D_3 = A_3$

Cluster algebras & adjacency

Cluster algebras for integrals [SH, Z. Li, Q. Yang]

Not only (n=6,7) amplitudes, but a large class of integrals (beyond N=4 SYM) have alphabet of cluster algebra!

e.g. all-loop n=6 ladder integrals -> A_3, certain n=7 integral -> E_6 , …

Systematically studied in a general setting: [Chicherin, Henn, Papathanasiou] e.g. 4-pt with a off-shell leg:

C_2 "folded" from A_3 :$\{z_1, z_2, z_3, 1-z_1, 1-z_2, 1-z_3\}$ $(z_1+z_2+z_3=1)$ 2d harmonic multi-polylogs

various 1-loop integrals w. A, C, D cluster algebras & 5-pt alphabet from limit of G(4,8)/T!

Natural question: more evidence for higher n (still DCI) & higher loops? algebraic letters?

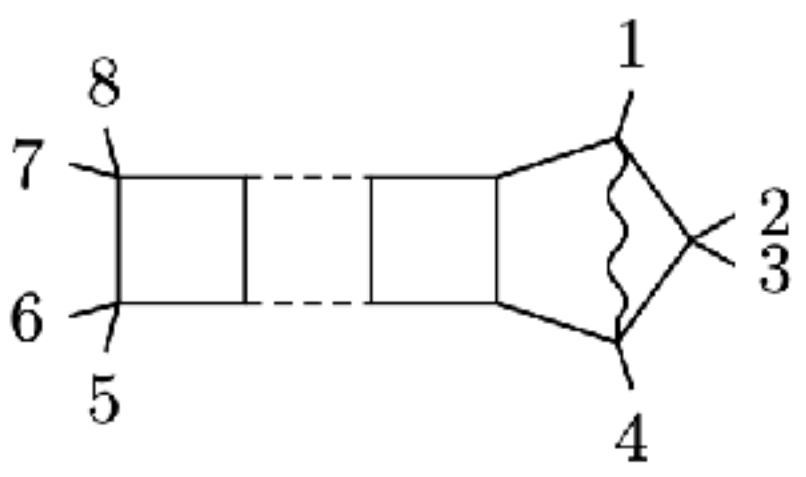
(trivial) example: 8-pt box ladders: $\{z, \bar{z}, 1-z, 1-\bar{z}\}$ $\sim$ $D_2 = A_1^2$

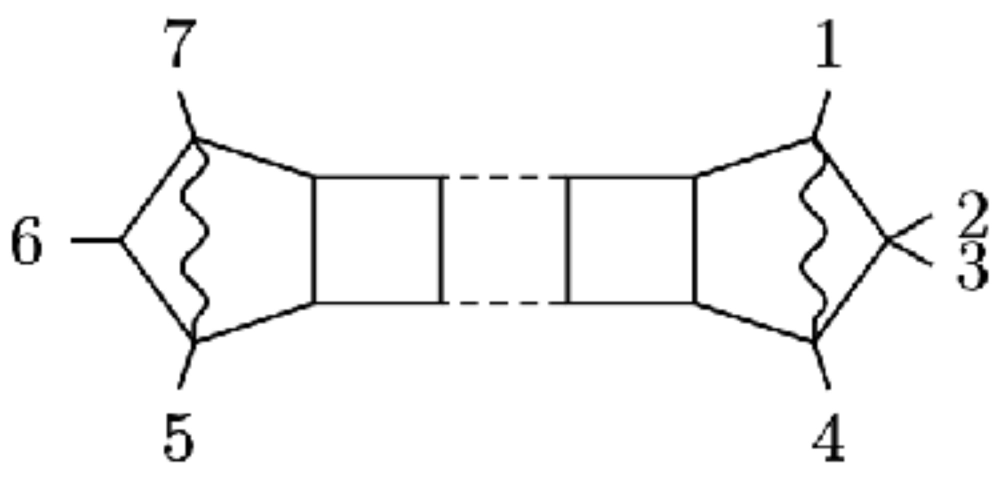
n=8 amplitudes/integrals in $\mathbb{R}^{1,1}$ kinematics: $C_2 = \text{overlap } A_2 + A_2$ $\{v, w, 1+v, 1+w, v-w, 1-vw\}$

much simpler than 4d → used to bootstrap octagons up to 3 loops! [w. Z. Li, Y. Tang, Q. Yang]

Observation: at least for ladder integrals, alphabet does not grow beyond e.g. 2 loops, independent of details (numerators…) → possibly fixed by kinematics (external dual points)!

e.g. 8-pt penta-box ladder $D_3 = A_3$ (not hexagon A_3), 7-pt double-penta ladder D_4 (emb. in heptagon E_6)





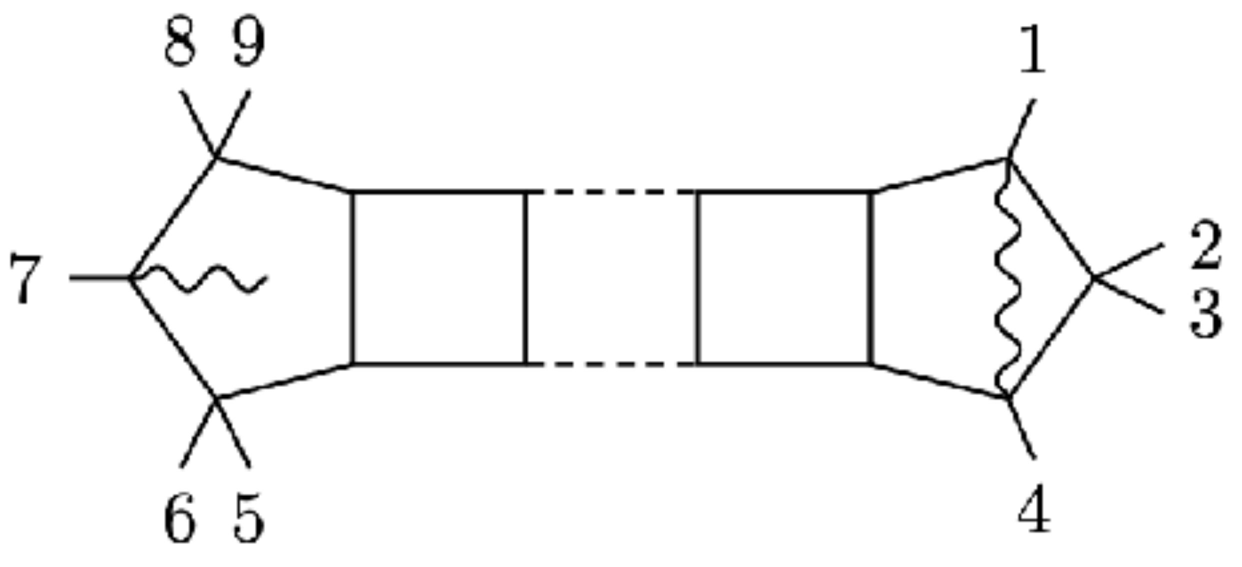
"good" variables: $z_1 = 1 - u^{-1}, z_2 = 1 - v^{-1}, z_3 = 1 - w$ for D3

$$D_3 = \{z_1, z_2, z_3, 1 + z_1, 1 + z_2, 1 + z_3, z_1 - z_2, z_1 - z_3, z_1 + z_2 z_3\}$$

similarly for D4: introduce $z_1, \cdots z_4$, same para. as [Chicherin, Henn, Papathanasiou] :

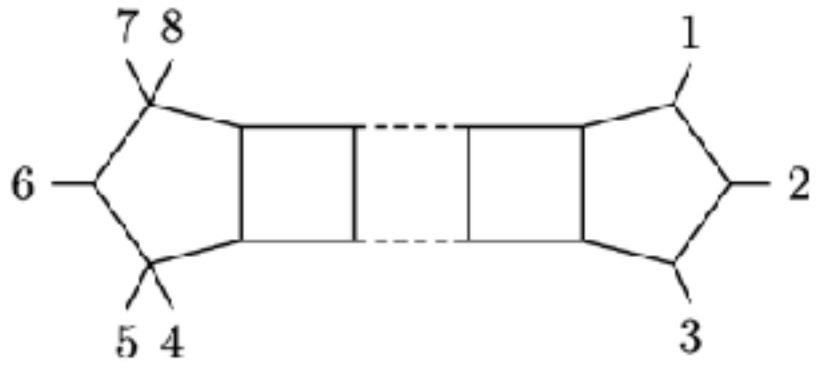
$$D_4 = \{z_1, \cdots, z_4, 1 + z_1, \cdots, 1 + z_4, z_1 - z_2, z_1 - z_3, z_1 - z_4, z_2 - z_3, z_2 - z_4, z_1 + z_3 z_4, z_2 + z_3 z_4, \cdots\}$$

many more cases: D6 (n=9) & deg. to D5, D4 etc.



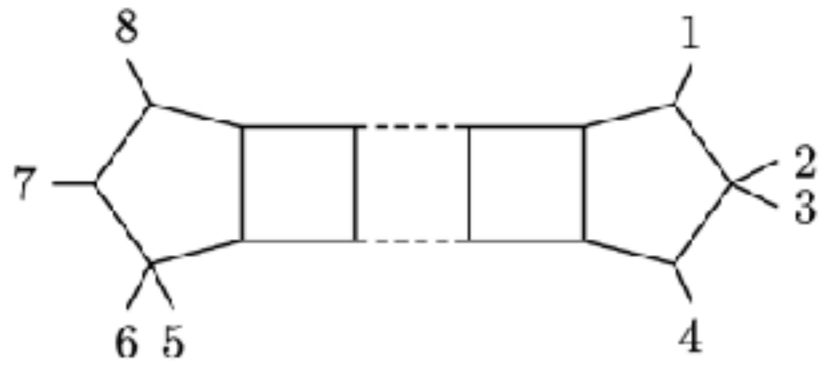
D_6

D_5

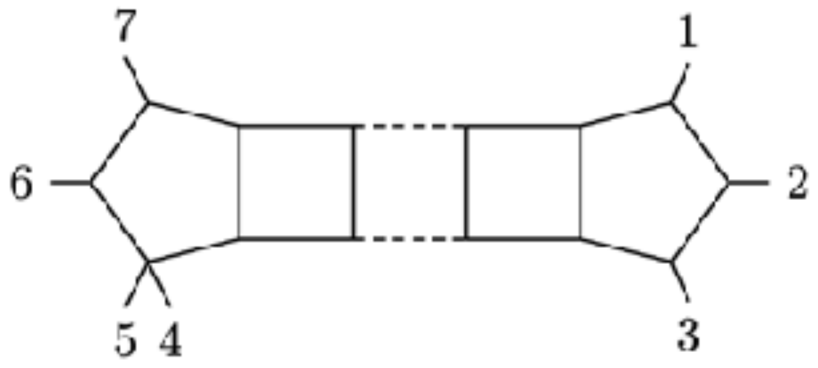


(a) Octagon A

D_5



(b) Octagon C



D_4

Cluster algebras from “kinematic quiver” [SH, Z. Li, Q. Yang]

A possible origin: (truncated) cluster algebra from mutating quivers associated with the kinematics

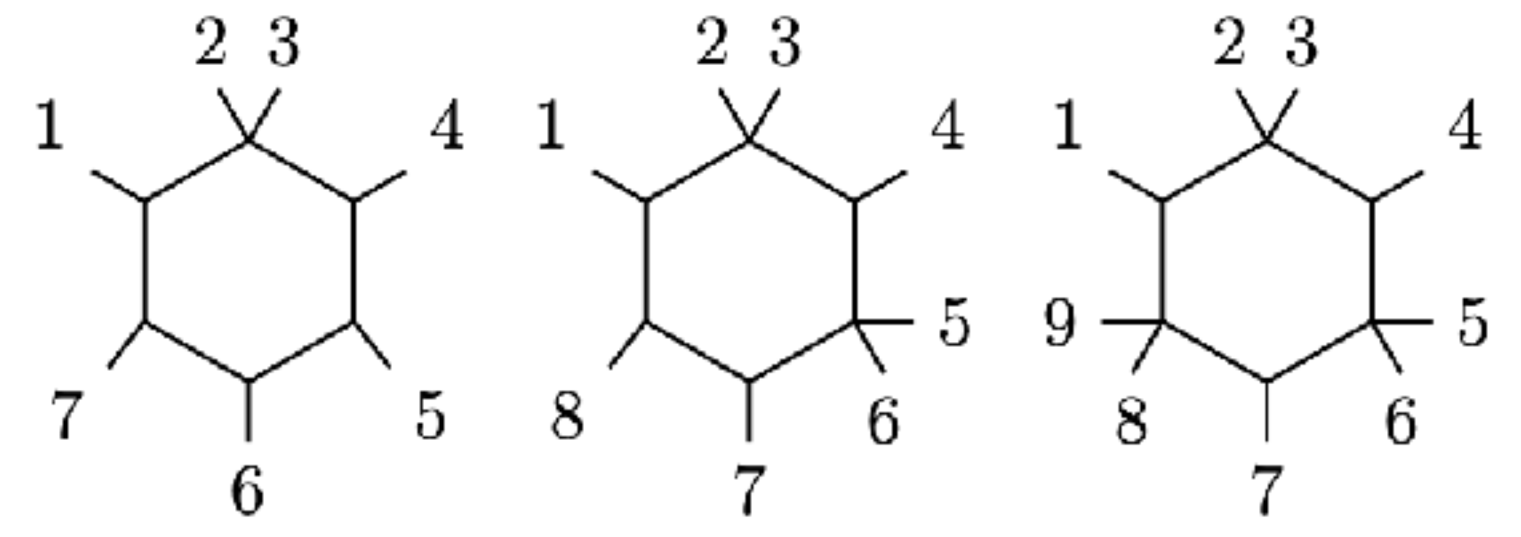
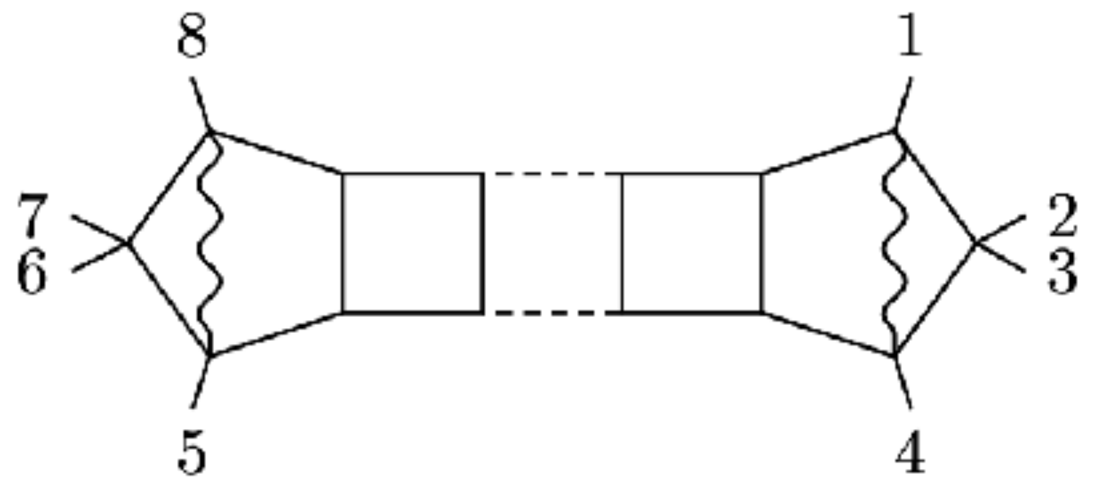
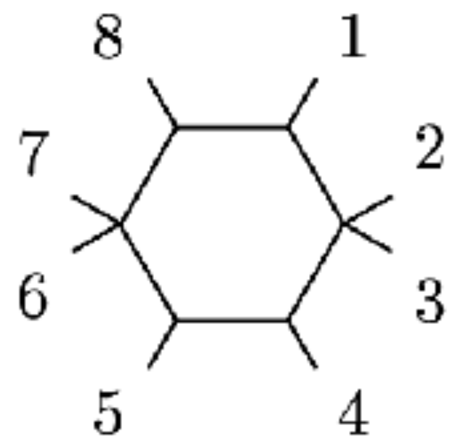
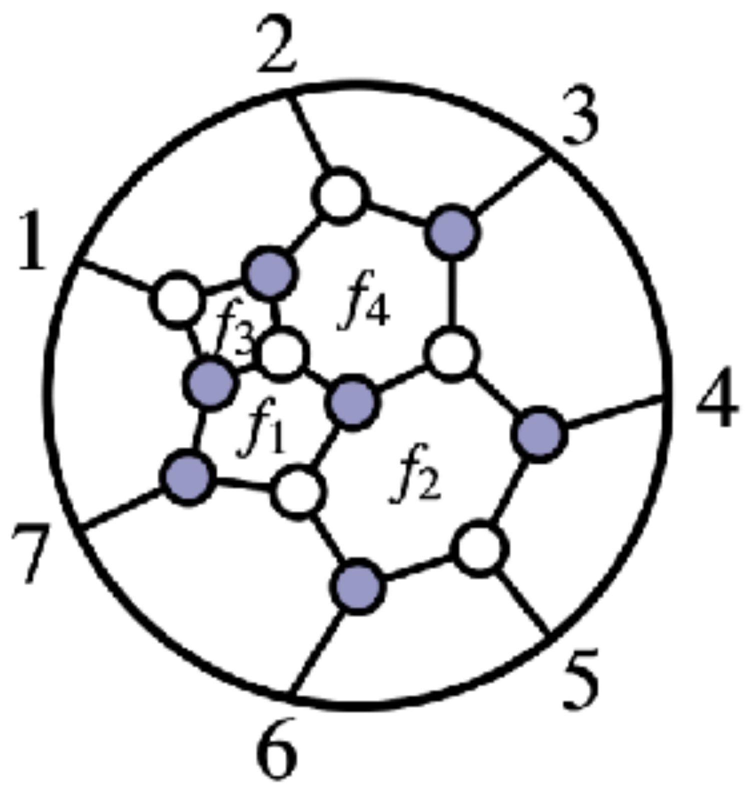


Figure 1: One-, two-, three-mass-easy hexagon kinematics with $n = 7, 8, 9$ legs





nicely, our cases correspond to lower-dim cell of G+(4,n)/T described by plabic graph

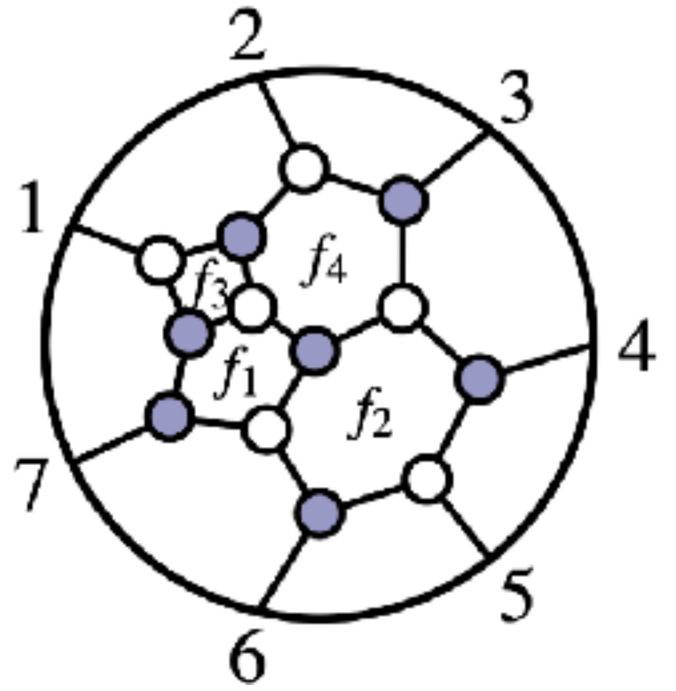


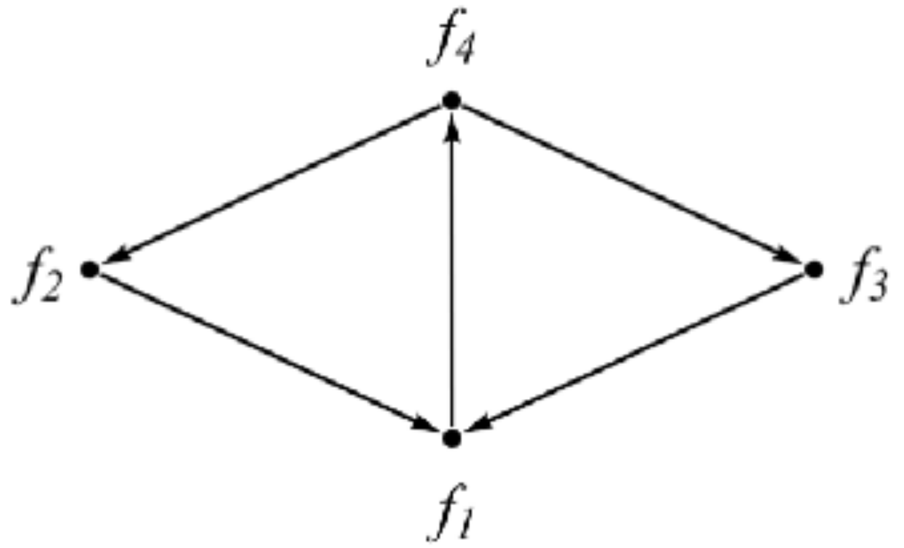
Algorithm: 1. find plabic graph for the kinematics (labelled by face var.)

2. draw the dual quiver, apply mutations -> cluster algebras

3. if it’s finite type -> the alphabet, otherwise need truncation!

4. similar to tropical G(4,n) for n>8: Minkowski sum from Pluckers etc.?

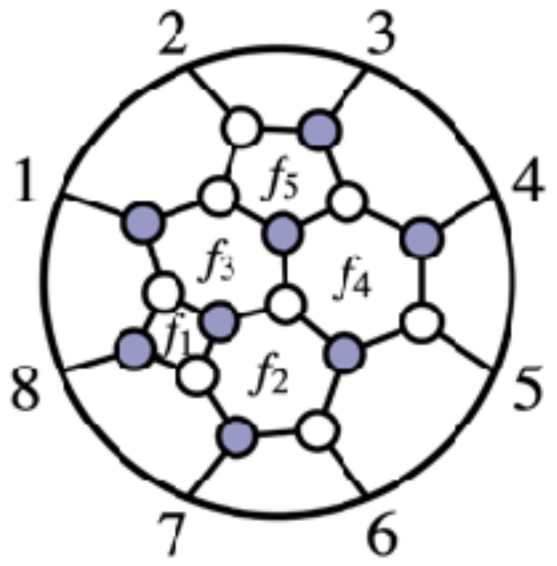


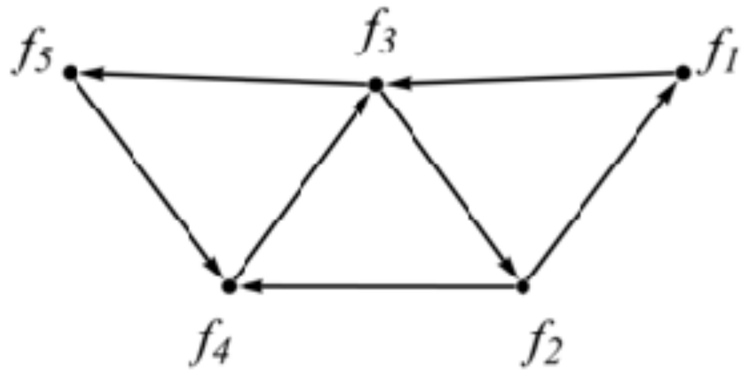


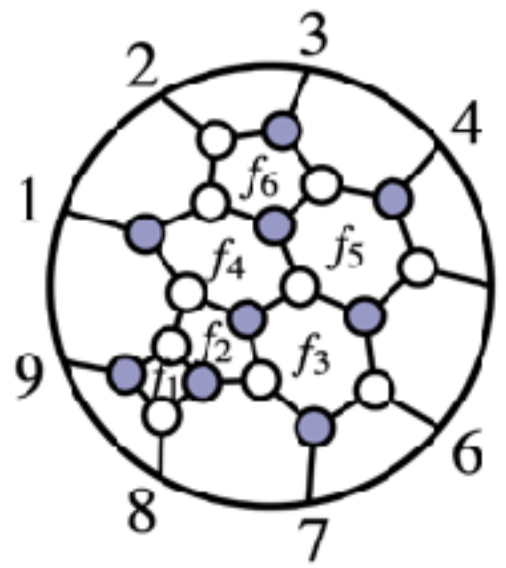
$$\begin{pmatrix} f_3f_4 & (1+f_3)\,f_4 & 1+f_4+f_3f_4 & 1 & 0 & 0 & 0 \\ 0 & f_1f_2f_4 & f_2\,(1+f_1+f_1f_4) & 1+f_2+f_1f_2 & 1 & 0 & 0 \\ 0 & 0 & f_2 & 1+f_2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

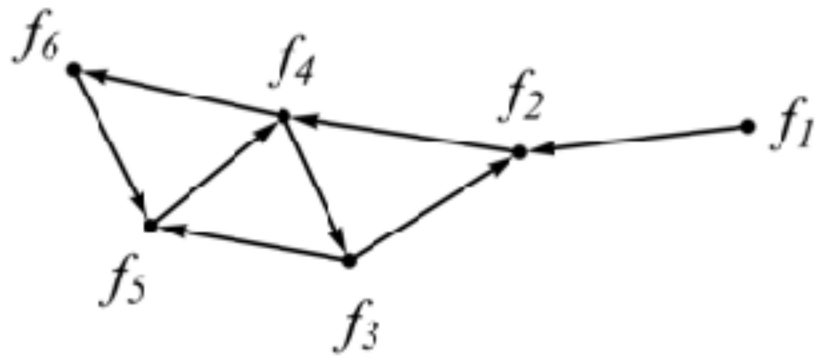
$\{1+f_1, 1+f_2, 1+f_3, 1+f_4, 1+f_2+f_1f_2, 1+f_3+f_1f_3, 1+f_1+f_1f_4, 1+f_4+f_2f_4,$
$1+f_4+f_3f_4, 1+f_2+f_3+f_2f_3+f_1f_2f_3, 1+f_4+f_2f_4+f_3f_4+f_2f_3f_4+f_1f_2f_3f_4\}$

D4: 16 letters









$$\begin{pmatrix} f_5 & f_5 & 1+f_5 & 1 & 0 & 0 & 0 & 0 \\ 0 & f_1f_2f_3f_4f_5 & f_2f_4\,(1+f_1+f_1f_3+f_1f_3f_5) & 1+f_2+f_1f_2+f_2f_4+f_1f_2f_4+f_1f_2f_3f_4 & 1+f_2+f_1f_2 & 1 & 0 & 0 \\ 0 & 0 & f_2f_4 & 1+f_2+f_2f_4 & 1+f_2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

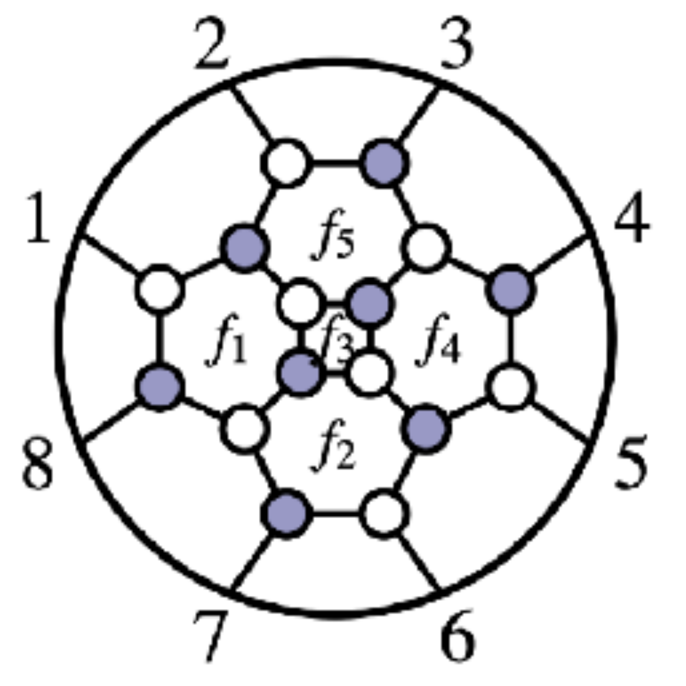
$$\begin{pmatrix} f_6 & f_6 & 1+f_6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_1f_2f_3f_4f_5f_6 & f_3f_5\,(1+f_1+f_1f_2+f_1f_2f_4+f_1f_2f_4f_6) & * & 1+f_1+f_3+f_1f_3+f_1f_2f_3 & 1+f_1 & 0 & -1 & 0 \\ 0 & 0 & f_3f_5 & 1+f_3+f_3f_5 & 1+f_3 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

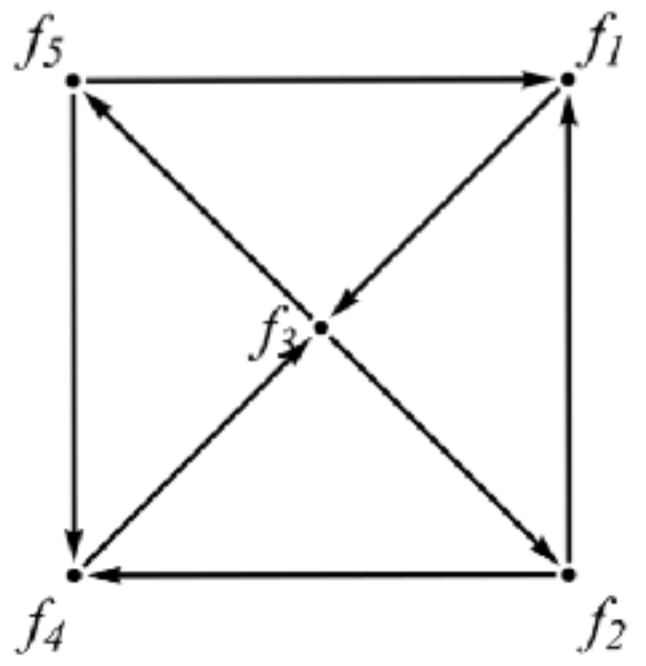
D5: 25 letters

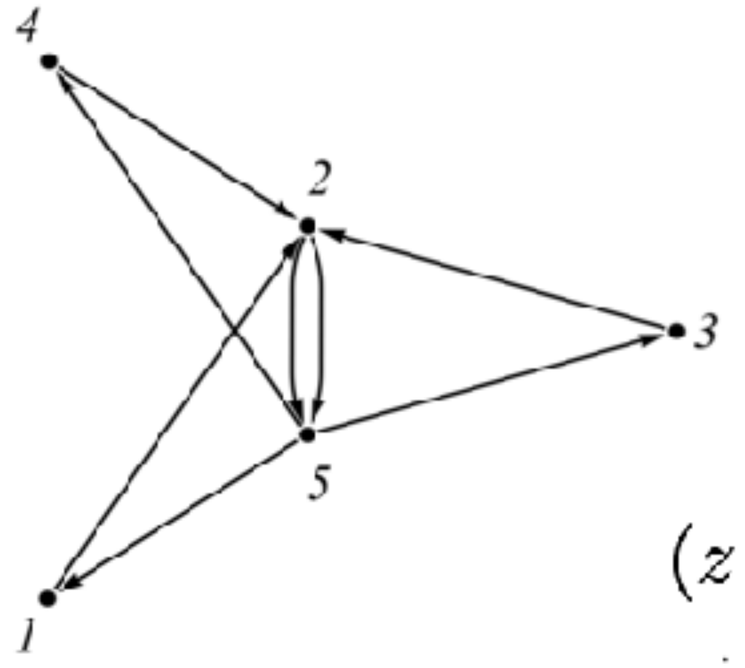
D6: 36 letters

Bootstrap Feynman integrals [SH, Z. Li, Q. Yang]

For $\Omega_L(1,4,5,8)$: affine D4 infinite cluster algebra! natural truncation -> 38 cluster variables + 1 limit ray







5-dim space of algebraic letters

$$u=\frac{x_{24}^2x_{68}^2}{x_{26}^2x_{48}^2}=z\bar{z},\quad v=\frac{x_{46}^2x_{82}^2}{x_{26}^2x_{48}^2}=(1-z)(1-\bar{z}).$$

$$(z-a)/(\bar{z}-a)\qquad a=0,\ 1,\ \frac{\langle 1234\rangle\langle 1568\rangle}{\langle 1256\rangle\langle 1348\rangle},\ \frac{\langle 1234\rangle\langle 4578\rangle}{\langle 1245\rangle\langle 3478\rangle},\ \left(1-\frac{\langle 1578\rangle\langle 3456\rangle}{\langle 1345\rangle\langle 5678\rangle}\right)^{-1}$$

Use this 38+5 alphabet -> bootstrap to weight 8 (without Steinmann): nicely locate integrals by DE!

conditions	# free parameters
weight-6 function space	3585
last entry	257
symmetry $z_3 \leftrightarrow z_4$	146
symmetry $z_1 \leftrightarrow z_2$	56
DE	3
boundary conditions	0

unique algebraic building block @ two-loop (weight-4):

$$\mathcal{S}_{2,4,6,8}:=\mathcal{S}(F_{2,4,6,8})\otimes\left(\frac{L_2L_5}{L_1L_3}\otimes z_1+\frac{L_2L_5}{L_1L_4}\otimes z_2+\frac{L_5}{L_1^2L_3L_4}\otimes z_3+\frac{L_5}{L_1}\otimes z_4+\frac{L_1^2L_3L_4}{L_2L_5^2}\otimes z_5\right)+\text{rational}$$

conjecturally all-loop algebraic structure (checked to 4 loops):

$$\sum_{i=1}^{5}\mathcal{S}(F(2,4,6,8))\otimes L_i\otimes\mathcal{S}(F_i)$$

Extended Steinmann relations universally [SH, Z. Li, Q. Yang]

Remarkably, ES relations hold for all known finite integrals, NMHV finite amps & MHV (w. subtraction)!!!

e.g. all ladder integrals (n=8,9, etc., up to 4 loops): cancellation between rational & algebraic letters

$$\mathrm{Disc}^{s}_{x_{ij}^2=0}(\mathrm{Disc}^{s}_{x_{kl}^2=0}(\mathcal{S}(F)))=0$$

for any $1 \leq s \leq w-1$ and any two overlapping channels x_{ij}^2, x_{kl}^2, *i.e.* $(ij) \not\sim (kl)$.

similarly, first finite amplitudes (NMHV components) @ 2 loops satisfy ES relations for all n

after subtraction, even 3-loop MHV octagon [Z. Li, C. Zhang] satisfies ES relations!

$$\mathcal{S}(E_8^{(3)}) = \mathcal{S}\left(R_8^{(3)} + \frac{F_8^3}{6} + R_8^{(2)} F_8\right)$$

can also check rational letters satisfy cluster adjacency of G(4,8) using Sklyanin brackets [Golden et al]

possible to bootstrap for n=8 (with alphabet of ~200 rational letters + 18 algebraic ones):
easy to get 2-loop MHV octagon (no alg. letters), what about 3 loops etc.? [Z. Li et al, WIP]