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On the canonical basis for single-parameter elliptic integral family

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Contents

Introduction

Canonical Feynman integrals

- canonical basis: bring a differential equation system to eps-form [Henn 2013, Henn 2015].

$$\mathrm{d}\vec{I} = \epsilon \mathbf{d} A \vec{I} \tag{1}$$

For single parameter

$$\partial_x \vec{I}(x) = \epsilon A(x) \vec{I}(x) \tag{2}$$

Then the integrals can be solved to be iterated integrals.

- If the integral family belongs to the MPL(multiple polylogarithms), then we can introduce the degree of transcendentality (transcendental weight) $\mathcal{T}(f)$ which equals to the number of iteration in the iterated integral.

$$\begin{aligned} \mathcal{T}(1) &= 0, \\ \mathcal{T}(\log(z)) &= 1, \ \log(z) = \int_1^z \frac{\mathrm{d}x}{x} \end{aligned} \tag{3}$$

- In elliptic case, transcendental weight is not so clear [Broedel et al. 2018, Frellesvig and Weinzierl 2023].

How can we get the canonical basis?

- In the MPL case, there are many algorithms to do this. There are basically two kinds of method:
  - transformation of basis:

$$
\begin{aligned}
&\partial_x \vec{I} = A(x, \epsilon)\vec{I}, \vec{I'} \equiv T \cdot \vec{I} \Rightarrow \\
&\partial_x \vec{I'} = (\partial_x T \cdot T^{-1} + T \cdot A \cdot T^{-1})\vec{I'}
\end{aligned}
\tag{4}
$$

    programs based on above idea: FUCHSIA [Gituliar and Magerya 2017], EPSILON [Prausa 2017], CANONICA [Meyer 2018], Libra [Lee 2021], INITIAL [Dlapa, Henn, and Yan 2020]

$$
\begin{aligned}
&A = A_0 + \epsilon A_1 \\
&\partial_x T \cdot T^{-1} + T \cdot A_0 \cdot T^{-1} = 0
\end{aligned}
\tag{5}
$$

  - based on properties of integrand: leading singularities, dlog forms [Chicherin et al. 2019, Herrmann and Parra-Martinez 2020, Henn et al. 2020, Chen et al. 2020, Dlapa, Li, and Zhang 2021]

- In the elliptic case, no general algorithms like above. And mostly the canonical basis is obtained by the first way [Frellesvig 2021, Dlapa, Henn, and Wagner 2022].
  As for the second way, the generalization of $\mathrm{d}\log$-form is not known.
- Recently, an ansatz for banana family has been proposed [Pögel, Wang, and Weinzierl 2022] and it seems working fine for single-parameter integral family with Calabi-Yau manifold, including elliptic case.

Banana family shows the math structure, however, we want to generalize this analysis to more general and phenomenal examples.

Eps form for a non-planar triangle with one elliptic curve

The integral family studied

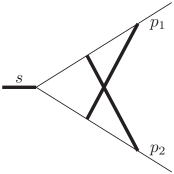

Figure 1: Two-loop three-point electroweak form factor

$$I[\nu_1, \ldots, \nu_7] = e^{2\epsilon\gamma} \int \frac{\mathrm{d}^d l_1}{i\pi^{d/2}} \frac{\mathrm{d}^d l_2}{i\pi^{d/2}} \frac{D_7^{-\nu_7}}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_6^{\nu_6}} \tag{6}$$

The top sector has been studied in [Broedel et al. 2019]. Now we want to study this whole family in a different manner. This family has a single parameter

$$y \equiv -\frac{m^2}{s} \tag{7}$$

The integral family studied

$$
\begin{aligned}
&D_1 = (k_1 - p_1)^2,\ D_2 = (k_2 - p_1)^2 - m^2,\ D_3 = (k_1 + p_2)^2, \\
&D_4 = (k_1 - k_2 + p_2)^2 - m^2,\ D_5 = (k_1 - k_2)^2,\ D_6 = k_2^2,\ D_7 = k_1^2\,. \qquad (8) \\
&(p_1 + p_2)^2 = s
\end{aligned}
$$

- LiteRed[Lee 2013] and Kira[Klappert et al. 2020] find 18 master integrals in this family, 3 in top sector and 15 in sub-sectors.
- Integrals in sub-sectors are all MPLs, so we first construct UT basis for all sub-sectors by constructing dlog-form basis [Chen et al. 2020, Chen et al. 2022] then we tackle the top sector.

Canonical basis for sub-sectors

Constructing dlog form in the integrand level with Baikov representation. For example, the most complicated case is sub-sector $\{1,1,1,1,0,0\}$:

$$\begin{aligned} I(\varphi) =& \frac{e^{2\epsilon\gamma_E} s^{\epsilon-1}}{4\pi^2\Gamma(2-2\epsilon)} \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_7 \, u(\mathbf{x})\varphi(\mathbf{x}) \,, \\ u(\mathbf{x}) =& P_1^{-1+\epsilon} P_2^{-\epsilon} P_3^{1/2-\epsilon} \,. \end{aligned} \tag{9}$$

$x_i = D_i$ and $\varphi(\mathbf{x})$ is a rational function for $\mathbf{x}$.

- Find some $\varphi(\mathbf{x})$ such that

$$u(\mathbf{x})\varphi(x) = P_1^{\epsilon} P_2^{-\epsilon} P_3^{-\epsilon} \mathrm{d}\log\alpha_1 \wedge \ldots \wedge \mathrm{d}\log\alpha_5 \tag{10}$$

- We can find two such items (2 MIs in this sector)

$$\varphi_1 = \frac{\epsilon^3(1-2\epsilon) s P_1}{x_1 x_2 x_3 x_4 P_3} \,, \; \varphi_2 = \frac{\epsilon^3(1-2\epsilon) s \sqrt{s(s+4m^2)} P_1^2}{x_1 x_2 x_3 x_4 P_2 P_3} \tag{11}$$

In above example, we have

$$P_1 = v,\ P_2 = sv - (v + x_1)(v + x_3),\ P_3 = v^2 + 2v(x_2 + x_4 + 2m^2) + (x_2 - x_4)^2\,. \tag{12}$$

where

$$v \equiv x_7 - x_1 - x_3 + s \tag{13}$$

There are some freedom in the definition of ISP and this will bring some complexity.

Anyway, sub-sectors can be easily solved in this family. We next turn to the top sector.

Elliptic integrals in the top sector

Elliptic integrals exist in the top sector. How can we identify this?

- If we try to construct dlog form in the top sector we will encounter

$$\int P(\mathbf{x})^{\epsilon} \frac{\mathrm{d}x_7}{\sqrt{x_7(s+x_7)(4m^4+4m^2x_7+sx_7+x_7^2)}} \wedge \mathrm{d}\log\alpha_1 \ldots \wedge \mathrm{d}\log\alpha_6 \tag{14}$$

- More efficient way to detect this is to perform maximal cut to extract the most hardcore information in a sector.
  Choose the simplest integral $I_1 = I[1,1,1,1,1,1,0]$ in top sector and perform the maximal cut

$$I_{1,mc} \propto \int P(x_7)^{\epsilon} \frac{\mathrm{d}x_7}{\sqrt{x_7(s+x_7)(4m^4+4m^2x_7+sx_7+x_7^2)}} \tag{15}$$

We first study this integral in 4 dimension, that is, $\epsilon \to 0$.

Under the condition of maximal cut and 4 dimension, we have

$$I_{1,mc}^{(4)} = \frac{iy^2}{4\pi^3 m^4} \int_{\mathcal{C}} \frac{\mathrm{d}z}{\sqrt{z(z+1)(z^2+(1-4y)z+4y^2)}} \tag{16}$$

where $z \equiv -yx_7$. And the elliptic curve is

$$Y^2 = X(X+1)(X^2+(1-4y)X+4y^2) = (X-a_1)(X-a_2)(X-a_3)(X-a_4) \tag{17}$$

where

$$a_1 = -1,\; a_2 = -\frac{(1+\sqrt{1-8y})^2}{4},\; a_3 = -\frac{(1-\sqrt{1-8y})^2}{4},\; a_4 = 0\,. \tag{18}$$

Here we assume y is around 0 and we will have

$$a_1 < a_2 < a_3 < a_4 \tag{19}$$

The integration contour $\mathcal{C}$ can be deformed like in the figure

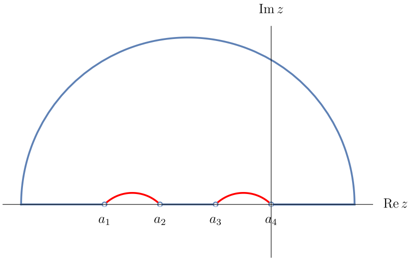

Figure 2: Integration contour for $I^{(4)}_{1,mc}$

This contour requires that $\sqrt{Y^2} = i\sqrt{|Y|^2}$ on the two line segments $[a_1, a_2] \cup [a_3, a_4]$.

Performing two conformal transformations and one variable replacement

$$t^2 = T_1(z) = \frac{z - a_1}{z - a_4} \frac{a_2 - a_4}{a_2 - a_1}, \; t^2 = T_2(z) = \frac{z - a_3}{z - a_2} \frac{a_4 - a_2}{a_4 - a_3} \tag{20}$$

Above integral can be mapped to the first kind complete elliptic integrals

$$\begin{aligned} & i \int_{\mathcal{C}} \frac{\mathrm{d}z}{\sqrt{z(z+1)(z^2 + (1-4y)z + 4y^2)}} = \\ & \frac{4}{\sqrt{(a_3 - a_1)(a_4 - a_2)}} \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2 t^2)}} = \frac{4K(k^2)}{\sqrt{(a_3 - a_1)(a_4 - a_2)}} \, . \end{aligned} \tag{21}$$

where

$$k^2 = 1/T_1(a_3) = 1/T_2(a_1) = \frac{(a_1 - a_2)(a_3 - a_4)}{(a_1 - a_3)(a_2 - a_4)} = \frac{1 - 4y - 8y^2 - \sqrt{1 - 8y}}{1 - 4y - 8y^2 + \sqrt{1 - 8y}}. \tag{22}$$

Above $I_{1,mc}^{(4)}$ is just one period of the elliptic function corresponding to the elliptic curve multiplied by some constant coefficients. We define the two periods to be

$$\psi_0(y) = \frac{y^2}{\sqrt{(a_3 - a_1)(a_4 - a_2)}} \frac{2K(k^2)}{\pi}, \ \psi_1(y) = \frac{y^2}{\sqrt{(a_3 - a_1)(a_4 - a_2)}} \frac{iK(1 - k^2)}{3\pi}. \tag{23}$$

the coefficients are set to match the solution from Frobenius method.
From IBP relation, we can derive the following differential equation for I_1.

$$L_{3,y}^{(\epsilon)} I_{1,mc} = 0\,. \tag{24}$$

where $L_{3,y}^{(\epsilon)}$ is the Picard-Fuchs operator. In 4 dimension,

$$L_{3,y}^{(0)} I_{1,mc}^{(4)} = 0 \tag{25}$$

Frobenius method for the q expansion

Why we need $L_{3,y}^{(0)} I_{1,mc}^{(4)} = 0$?

- This is equivalent to solve the inhomogeneous term in linear differential equation system:
$$\partial_y \vec{I} = (A_0 + \epsilon A_1 + \ldots)\vec{I} \tag{26}$$
- Get 3 solution at one time
- Get the series expansion results.
- A good guidance to the math structure underneath.

Frobenius method for the q expansion

$$
\begin{aligned}
L_{3,y}^{(0)} =& \frac{\mathrm{d}^3}{\mathrm{d}y^3} - \frac{3\,(1-2y)}{y(y+1)(1-8y)}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{-7+6y+30y^2+8t^3}{y^2(y+1)^2(1-8y)}\frac{\mathrm{d}}{\mathrm{d}x} \\
&+ \frac{4(-2+y+8y^2+2y^3)}{y^3(y+1)^2(1-8y)} \\
=& \left[\frac{\mathrm{d}}{\mathrm{d}y} + \frac{8}{8y-1}\right] \underbrace{\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\frac{8}{8y-1} + \frac{1}{y+1} - \frac{3}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y} + \frac{4(2y^2+4y-1)}{y^2(y+1)(8y-1)}\right]}_{\text{elliptic operator}}.
\end{aligned}
\tag{27}
$$

First line is derived from IBP. Second line can be performed by `Maple` command `DFactor`.

There are 4 regular singular point

$$
y = \frac{1}{8}, 0, -1, \infty \tag{28}
$$

Then we solve this differential equation by Frobenius method.

Frobenius method for the q expansion

- First, we expand around the point $y = 0$. $y = -m^2/s$, this corresponds to the high energy limit. Why this point?
  - Its indical equation is
  $$(r-2)^3 = 0 \tag{29}$$
  which means that it is a MUM (Maximal Unipotent Monodromy) point. This property will allow us to use the general framework developed for banana family. $y = \infty$ is also a MUM point.
- The solution will be in the form
$$\psi_k = \frac{1}{(2\pi i)^k} \sum_{j=0}^{k} \frac{\ln^j y}{j!} \sum_{n=0}^{\infty} a_{k-j,n}\, y^{n+2}\,, \quad (k = 0, 1, 2)\,. \tag{30}$$

here the power starts from y^2 because the solution of indical equation is 2. This can be solved by ansatz or by command `AsymptoticDSolveValue` in `Mathematica`.

Frobenius method for the q expansion

Then we get the series expansion solution (easily got up to very high orders)

$$\begin{aligned}\psi_0 =& y^2\left(1+2y+10y^2+56y^3+346y^4+2252y^5\right)+\mathcal{O}(y^8)\,,\\ \psi_1 =& \frac{1}{2\pi i}\left[y^3\left(3+\frac{33y}{2}+100y^2+\frac{2561y^3}{4}+\frac{42631y^4}{10}\right)+\psi_0\log y\right]+\mathcal{O}(y^8),\\ \psi_2 =& \frac{1}{(2\pi i)^2}\left[y^4\left(\frac{9}{4}+\frac{45y}{2}+\frac{2793y^2}{16}+\frac{10365y^3}{8}\right)+2\pi i\psi_1\log y+\frac{\psi_0}{2}\log^2 y\right]+\mathcal{O}(y^8).\end{aligned} \tag{31}$$

Here we introduce a new variable

$$\tau(y)=\frac{\psi_1(y)}{\psi_0(y)}\,,q\equiv e^{2\pi i\tau} \tag{32}$$

The relation between y and q can be calculated **locally** to be

$$\begin{aligned}q =& y(1+3y+15y^2+85y^3+522y^4+3366y^5+22450y^6)+\mathcal{O}(y^8)\,,\\ y =& q-3q^2+3q^3+5q^4-18q^5+15q^6+24q^7+\mathcal{O}(y^8)\,.\end{aligned} \tag{33}$$

Reconstruct the analtytic form by OEIS

$$y(\tau)=\frac{\eta(\tau)^3\,\eta(6\tau)^9}{\eta(2\tau)^3\,\eta(3\tau)^9}\,. \tag{34}$$

Frobenius method for the q expansion

To define the relation between y and q globally, we need to compensate possible discontinuity from $\psi_{1,0}$:

$$\tau(y) = \begin{cases} \frac{\psi_1(y)}{\psi_0(y)} = \frac{1}{6}\frac{I*K(1-k^2)}{K(k^2)}, & 0 \leq y < \frac{\sqrt{3}-1}{4} \\ \frac{\psi_1(y)}{\psi_0(y)} + \frac{1}{3} = \frac{1}{6}\left(\frac{I*K(1-k^2)}{K(k^2)} + 2\right), & \text{otherwise} \end{cases} \tag{35}$$

And under this definition we have

$$\tau(-1) = \frac{1}{2},\ \tau(0) = i\infty,\ \tau(1/8) = 0,\ \tau(\infty) = \frac{1}{3}\,. \tag{36}$$

The singularity are mapped to the cusps of modular form under $\Gamma_1(6)$.

Frobenius method for the q expansion

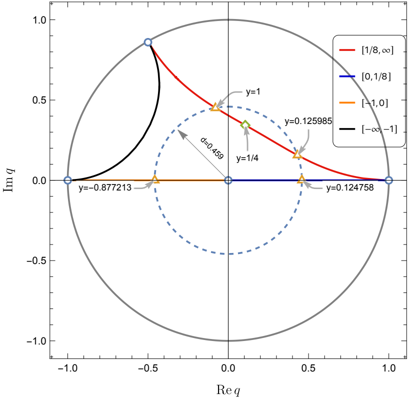

Figure 3: The relation between q and y for this family

We can see the relation between q and y is continuous. 3 of the 4 singularities lies on the unit circle. A benefit for using q variable.

The ansatz for top sector

Now we have all the elements for the following ansatz

$$
\begin{aligned}
M_1 &= \epsilon^4 \frac{I_1}{\psi_0(y)}\,, \\
M_2 &= \frac{J(y)}{\epsilon} \frac{d}{dy} M_1 - F_{11}(y) M_1\,, \\
M_3 &= \frac{1}{Y(y)} \left[\frac{J(y)}{\epsilon} \frac{d}{dy} M_2 - F_{21}(y) M_1 - F_{22}(y) M_2 \right].
\end{aligned} \tag{37}
$$

where

$$
\begin{aligned}
J(y) &\equiv \frac{1}{2\pi i} \frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{\mathrm{d}y}{\mathrm{d}\boldsymbol{q}} = \psi_0^2(y)(1+y)(1-8y)y^{-3}\,, \\
Y(y) &\equiv \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \frac{\psi_2}{\psi_0} = \frac{\eta(2\tau)^9}{\eta(6\tau)^3}.
\end{aligned} \tag{38}
$$

η is Dedekind eta function. F_{11}, F_{21}, F_{22} to be determined.

Explanation for this ansatz

First, we choose one "good" integral which we make as our first basis (seed), then we use its derivatives as basis (similar with INITIAL but more radical).

What is a "good" integral in elliptic case? Why I_1?

- dlog forms with only one last elliptic form, recall $I_1 \propto$

$$\int P(\mathbf{x})^{\epsilon} \frac{\mathrm{d}x_7}{\sqrt{x_7(s+x_7)(4m^4+4m^2x_7+sx_7+x_7^2)}} \wedge \mathrm{d}\log\alpha_1 \ldots \wedge \mathrm{d}\log\alpha_6 \qquad (39)$$

- degenerate to UT (MPL) in special kinematics limit. $I_1(y \to 0)$ is UT integral (actually this is our boundary condition for this system).

Explanation for this ansatz

It has been found that this ansatz works amazingly well in even more complicated geometry [Pögel, Wang, and Weinzierl 2022] and it shows the corresponding structure behind the integral. That is

$$L_{3,y}^{(0)} \propto \Theta_q \frac{1}{Y(q)} \Theta_q^2 \frac{1}{\psi_0(q)} \,. \tag{40}$$

where

$$\Theta_q = q\frac{\mathrm{d}}{\mathrm{d}q} = J(y)\frac{\mathrm{d}}{\mathrm{d}y}. \tag{41}$$

Hint: The study of (factorization property of) differential operator for 4 dimension integrals may also help understand some general structure in $4-2\epsilon$. Since it indicates the proper variables we should use, in this case, q.

The ansatz for top sector

The interesting thing is that if we require that the top sector satisfy eps-form differential equation

$$J(y)\frac{d}{dy}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}^{\mathrm{mc}} = \epsilon\, A^{\mathrm{mc}}(y)\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}^{\mathrm{mc}} . \tag{42}$$

$J(y)$ and $Y(y)$ defined above will just satisfy the non-trivial constraints automatically. We can further solve F_{11}, F_{21}, F_{22} to be

$$\begin{aligned} F_{11}(y) =& F_{22}(y) = \psi_0(y)^2\frac{1+2y+28y^2}{3y^4}\,, \\ F_{21}(y) =& \psi_0(y)^4\frac{(-1+2y)(-11+66y+84y^2+88y^3)}{3y^8}\,. \end{aligned} \tag{43}$$

The ansatz for top sector

You will find $J(y), Y(y), F_{ij}(y)$ all have a single form:

$$\psi_0(y)^m A(y) \tag{44}$$

$A(y)$ is a rational function of y (not arbitrary). Actually, this form corresponds to a special kind of integrals called modular form [Broedel et al. 2018].

- Modular group:

$$SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \tag{45}$$

it acts on a point on the upper half plane $\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup i\infty$ like [Broedel et al. 2018, GTM228]

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d} \tag{46}$$

- congruence subgroup of level N of modular group

$$\begin{aligned} \Gamma_0(N) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c = 0 \mod N \\ \Gamma_1(N) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c = 0 \mod N, a = d = 1 \mod N \\ \Gamma(N) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : b = c = 0 \mod N, a = d = 1 \mod N \end{aligned} \tag{47}$$

cusps: $\{\mathbb{Q} \cup i\infty\}/\Gamma$

Modular form: function

- modular function for Γ: A meromorphic function satisfying

$$f(\gamma(\tau)) = f(\tau), \quad \gamma \in \Gamma \tag{48}$$

- weakly modular of weight k for Γ: A meromorphic function satisfying

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma \tag{49}$$

- A modular form of weight k: a weakly modular function of weight k holomorphic on $\bar{\mathbb{H}}$. They are graded by their weight.

The differential matrix elements we study are modular forms or closely related to the modular forms.

Use `Sage` to loop up for the properties of some congruence group Γ and the corresponding modular forms.

Canonical differential equation

Then for top sector, we arrive at

$$A^{\mathrm{mc}} = \epsilon \begin{pmatrix} \frac{\left(28y^2+2y+1\right)\psi_0(y)^2}{3y^4} & 1 & 0 \\ \frac{(2y-1)\left(88y^3+84y^2+66y-11\right)\psi_0(y)^4}{3y^8} & \frac{\left(28y^2+2y+1\right)\psi_0(y)^2}{3y^4} & -\frac{(y+1)^2(8y-1)\psi_0(y)^3}{y^6} \\ \frac{64(y+1)^2(8y-1)\psi_0(y)^3}{27y^6} & 0 & \frac{2(y+1)(8y-1)\psi_0(y)^2}{3y^4} \end{pmatrix}. \tag{50}$$

This matrix is actually graded by modular weight (exchange M_2 and M_3):

$$A^{\mathrm{mc}} = \epsilon \begin{pmatrix} \frac{\left(28y^2+2y+1\right)\psi_0(y)^2}{3y^4} & 0 & 1 \\ \frac{64(y+1)^2(8y-1)\psi_0(y)^3}{27y^6} & \frac{2(y+1)(8y-1)\psi_0(y)^2}{3y^4} & 0 \\ \frac{(2y-1)\left(88y^3+84y^2+66y-11\right)\psi_0(y)^4}{3y^8} & -\frac{(y+1)^2(8y-1)\psi_0(y)^3}{y^6} & \frac{\left(28y^2+2y+1\right)\psi_0(y)^2}{3y^4} \end{pmatrix}. \tag{51}$$

Canoncial differential equation

Combining sub-sector UT integrals and we can get the eps-form for the whole integral family (tow methods):

- Naively, $(M_1, M_2, M_3) = T \cdot (I_1, I_2, I_3)$ where

$$I_1 = I[1,1,1,1,1,1,0], I_2 = I[1,1,1,2,1,1,0], I_3 = I[1,1,1,1,1,2,0] \tag{52}$$

  I_2, I_3 already depend on sub-sectors with ϵ factorized.

- More systematically, An ansatz for M_3

$$M_3 = \frac{1}{Y(y)} \left[\frac{J(y)}{\epsilon} \frac{d}{dy} M_2 - F_{21}(y) M_1 - F_{22}(y) M_2 \right] - \vec{g}(y) \cdot \vec{M}_{sub} \tag{53}$$

  Require the dependence on sub-sector UT integrals is ϵ factorized.

Canoncial differential equations

$$
\begin{aligned}
\frac{1}{2\pi i}\frac{d\,M_1}{d\tau} &= \varepsilon\big[\eta_{1,2}M_1 + M_2\big],\\
\frac{1}{2\pi i}\frac{d\,M_2}{d\tau} &= \varepsilon\Big[\eta_4 M_1 + \eta_{1,2}M_2 + \eta_{1,3}M_3 + 10\eta_{2,3}M_5 - 10\eta_{2,3}M_6 - 8\eta_{2,3}M_7 - 8\eta_{2,3}M_8\\
&\qquad + \varrho M_9 - 9\eta_{2,3}M_{10} + 10\eta_{2,3}M_{11} + 12\eta_{2,3}M_{12} + 8\eta_{2,3}M_{13} - 4\eta_{2,3}M_{14}\\
&\qquad - 7\eta_{2,3}M_{15} - 58\eta_{2,3}M_{16} - 30\eta_{2,3}M_{17} - 8\eta_{2,3}M_{18}\Big], \qquad (54)\\
\frac{1}{2\pi i}\frac{d\,M_3}{d\tau} &= \varepsilon\Big[-\frac{64}{27}\eta_{1,3}M_1 + \eta_{2,2}M_3 - 4\eta_{3,2}M_5 + 4\eta_{3,2}M_6 + 2\eta_{3,2}M_7 + 2\eta_{3,2}M_8\\
&\qquad - 4\vartheta M_9 - (4\eta_{3,2} + 6\varpi)M_{11} + 4\eta_{3,2}M_{13} - 2\eta_{3,2}M_{14} + \eta_{3,2}M_{15}\\
&\qquad - (8\eta_{3,2} + 24\varpi)M_{16} - (6\eta_{3,2} + 12\varpi)M_{17} + (5\eta_{3,2} + 9\varpi)M_{18}\Big].
\end{aligned}
$$

where

$$\varrho = \frac{7-8y}{\sqrt{1-4y}}\eta_{2,3}\,, \quad \vartheta = \frac{1+y}{\sqrt{1-4y}}\eta_{3,2}\,, \quad \varpi = \frac{\eta_{3,2}}{y-1}\,. \qquad (55)$$

η are all modular forms.

Solve all the master integral

Boundary condition

We have already got the differential equation system, to solve the master integrals, we still need the boundary condition.

They are chosen to be singular point in most cases. We choose $y \to 0$.

$$\epsilon^4 \frac{I_1}{y^2} = \epsilon^4 \left[\frac{1}{3}\log^4 y - \pi^2 \log^2 y - 40\zeta_3 \log y - \frac{49\pi^4}{90}\right] + \epsilon^5 \left[\frac{1}{5}\log^5 y - \frac{10\pi^2}{9}\log^3 y - 42\zeta_3 \log^2 y - \frac{29\pi^4}{90}\log y + \frac{32\pi^2}{3}\zeta_3 + 32\zeta_5\right] + \epsilon^6 \left[\frac{7}{90}\log^6 y - \frac{11\pi^2}{18}\log^4 y - 32\zeta_3 \log^3 y - \frac{23\pi^4}{30}\log^2 y + \frac{2}{3}\left(31\pi^2\zeta_3 - 354\zeta_5\right)\log y - \frac{8\pi^6}{27} + 246\zeta_3^2\right] + \mathcal{O}(\epsilon^7, y)\,. \tag{56}$$

where $L_q = Log[q]$.

Boundaries of $M_{2,3}$ can be derived from M_1. Sub-sectors are all solved to weight 6. The calculation is perform by Mellin-Barnes method and asymptotic expansion. There are tools like `MBTools` [Belitsky, Smirnov, and Smirnov 2022] and `XSummer` [Moch and Uwer 2006] can handle this.

The series expansion solutions

Now we can solve the system with q-expansion around $y = 0$, that is, $q = 0$.

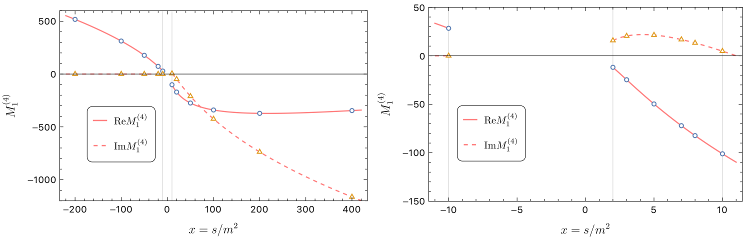

Figure 4: $M_1^{(4)}$ weight-4 part of M_1. The results agree well with `AMFlow` (the points on the graph) with q only expanded to q^8

The convergent region

The following figure shows the convergent region of M_1 with expansion around $y = 0$. It is constrained by the singularity $y = 1$ from subtopologies.

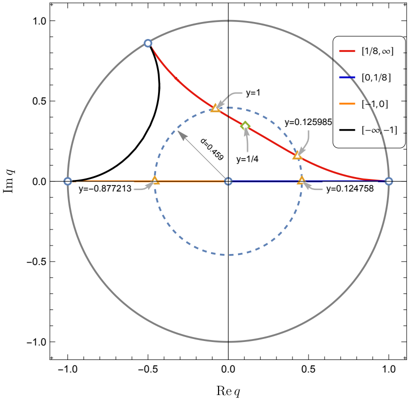

Figure 5: The relation between q and y for this family

summary

- We get the canonical basis for the non-planar triangle family with one elliptic curve and solve all the master integrals with q-expansion. (The sub-sector integrals are actually solved to weight-6 by HPL functions.)
- It seems the ansatz we use can be applied to general case.
- Elliptic integral family with more parameters or more than one elliptic curve?

Thank you!