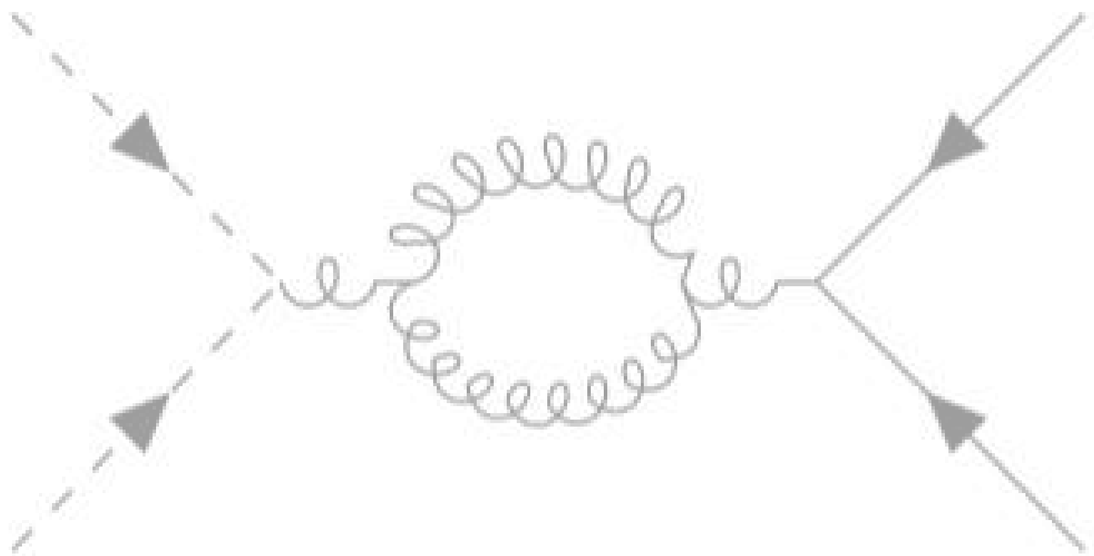
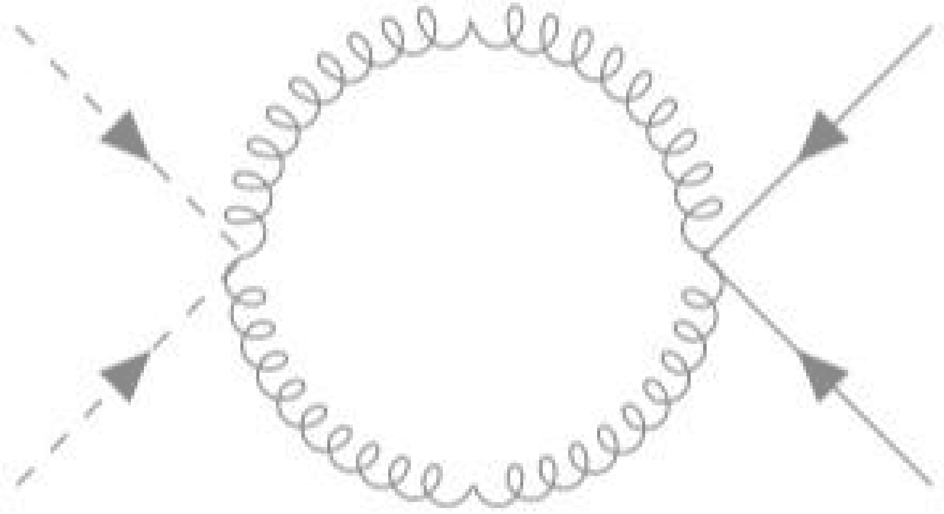
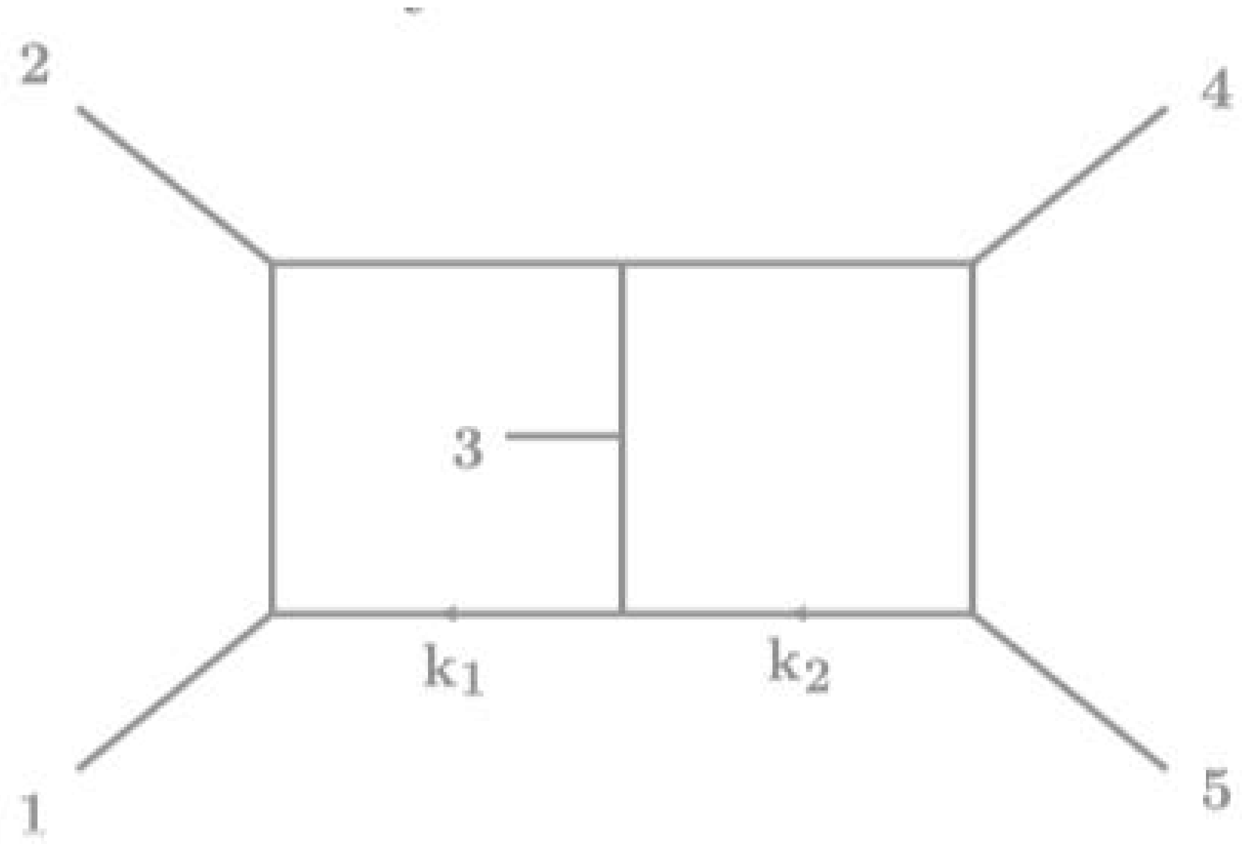
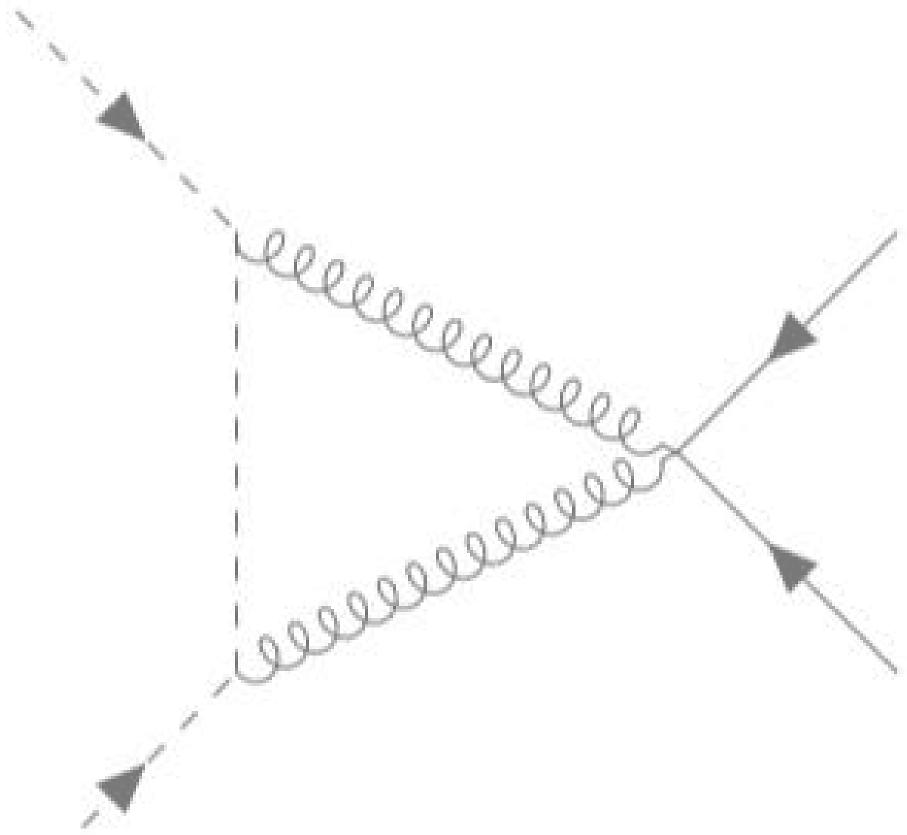


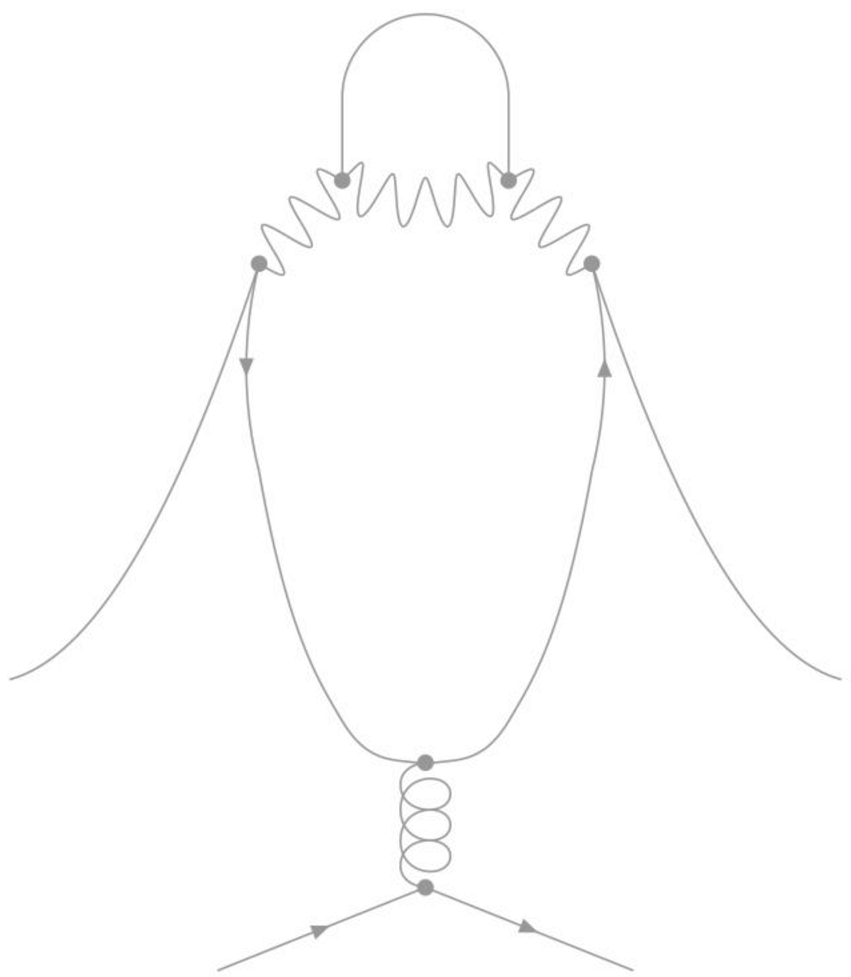


北京工業大學
BEIJING UNIVERSITY OF TECHNOLOGY

无自旋双星系统韧致辐射的潮汐效应NLO阶修正计算







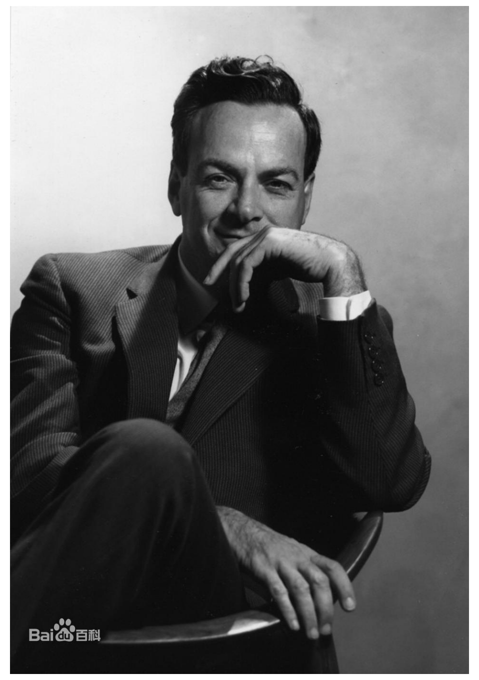
报告人：孙宇轩
指导老师：曾定方

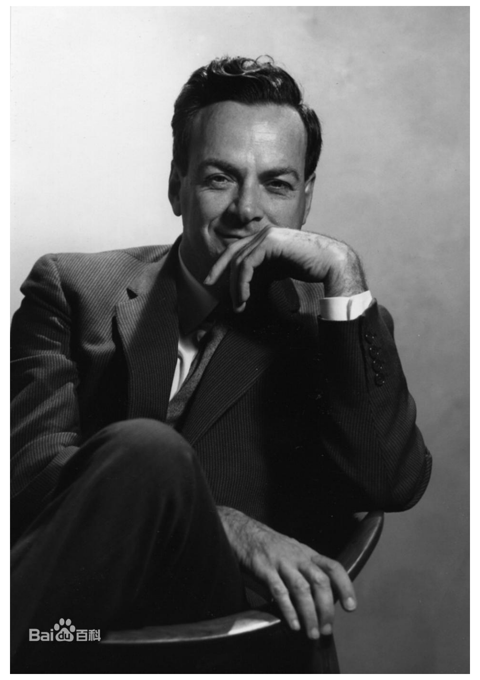
Picture (Scattering System)

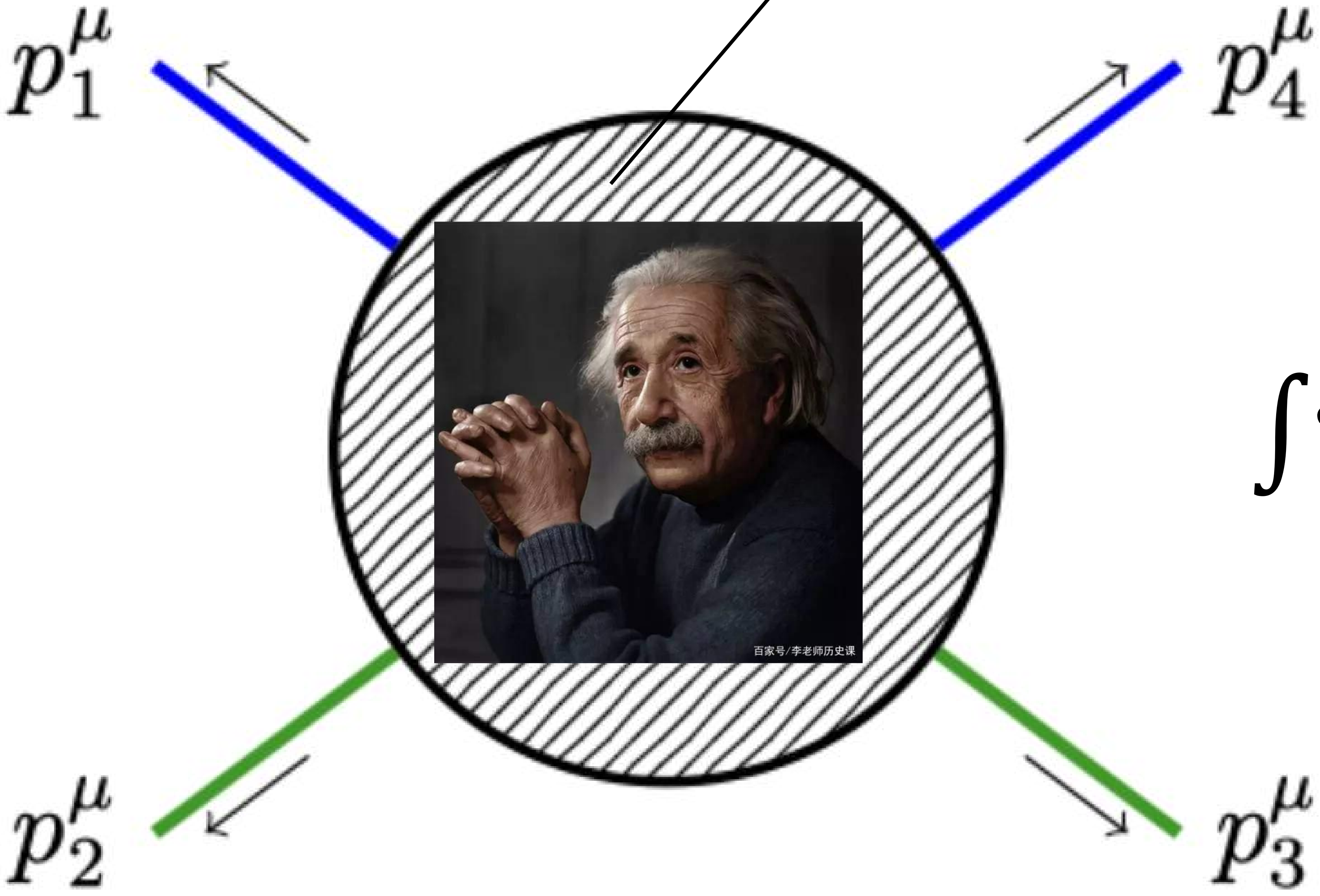
Elastic Scattering

$$p_1+p_2+p_3+p_4=0$$

$$R_{\mu\nu}+\frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

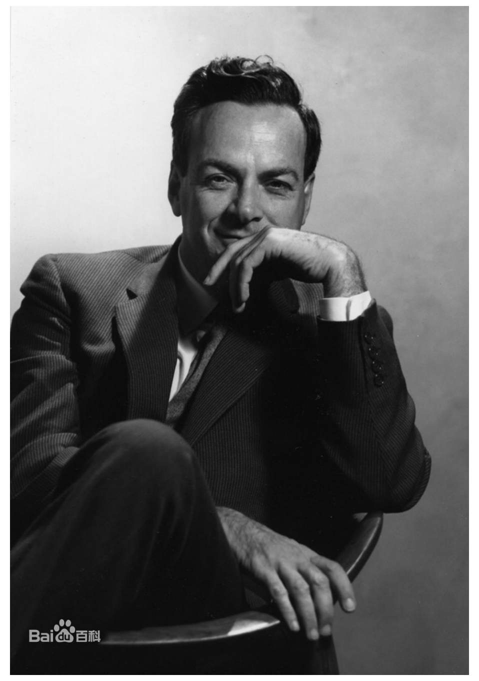


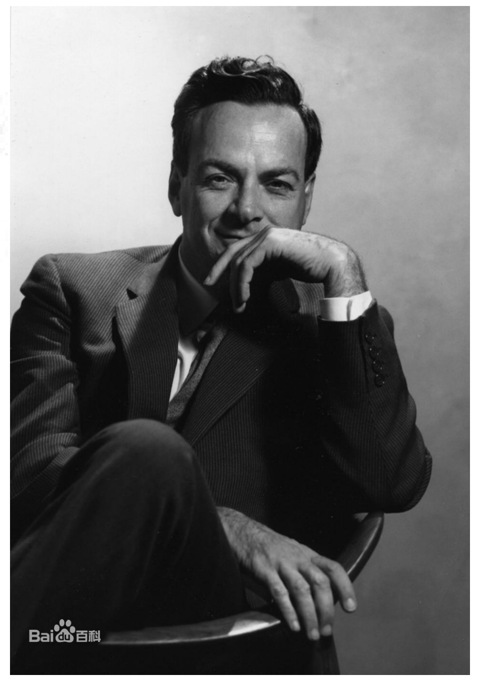




$$\int \mathcal{D}h\mathcal{D}\cdots e^{iS_{\text{Gra}}+iS_{\text{matter}}}$$

$\underbrace{}_{\text{DoF}}$



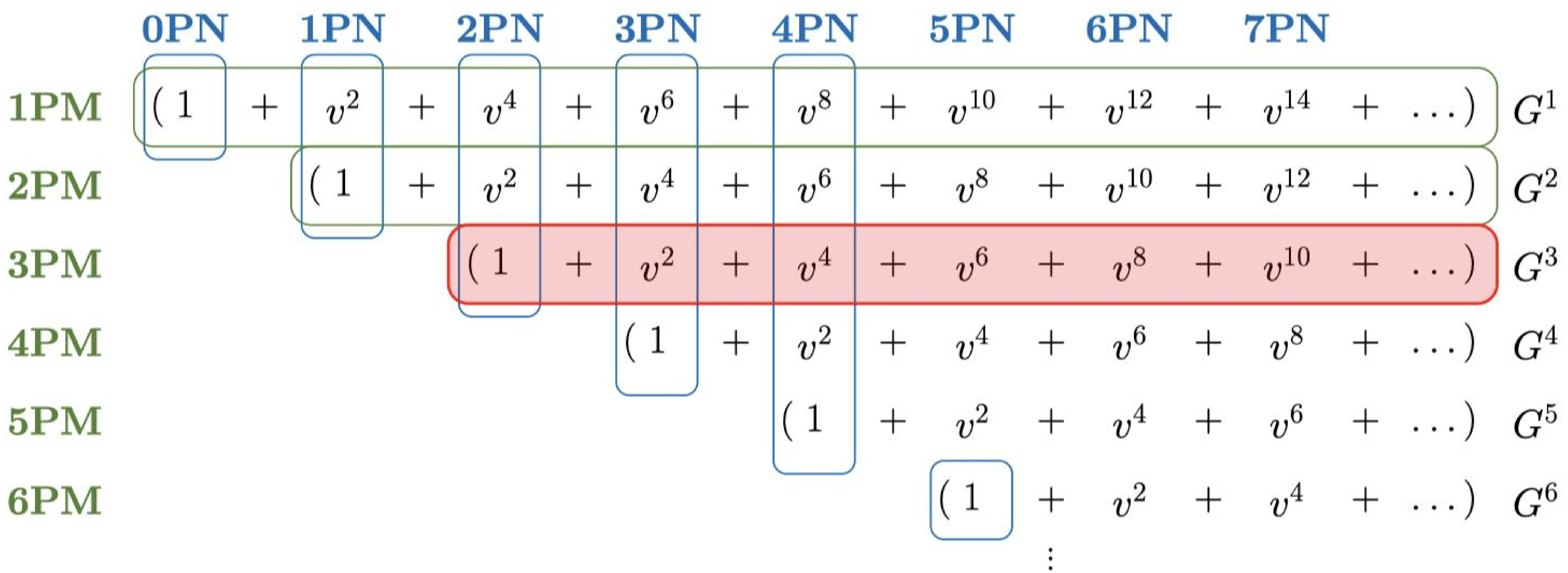


$r\to-\infty$

$r\to+\infty$

$$S=\langle\psi_{r\to-\infty}|\psi_{r\to+\infty}\rangle$$

Motivation



- No Spin Binary System
- 1PN 1917 (GR-PN) Lorentz, Droste, 1917
- 4PN 2005 (EOB) Damour et al. 2005
- 6PN 2020 (EOB+EFT) Damour et al. 2020
- 3PM 2022 (EFT) Zvi.Bern et al. 2020
- 4PM 2022 (EFT) Zvi.Bern et al. 2021

Why need Analytic?

Parameter oder estimation

Inspiral: v<<1

$$v^2 \sim \frac{Gm}{r} \equiv \frac{r_s}{2r} \quad t \sim \frac{2\pi r}{v} \quad \omega \sim \frac{v}{2\pi r}$$

Radiation power (LO), time and cycles

$$E \sim \frac{1}{2} m v^2$$

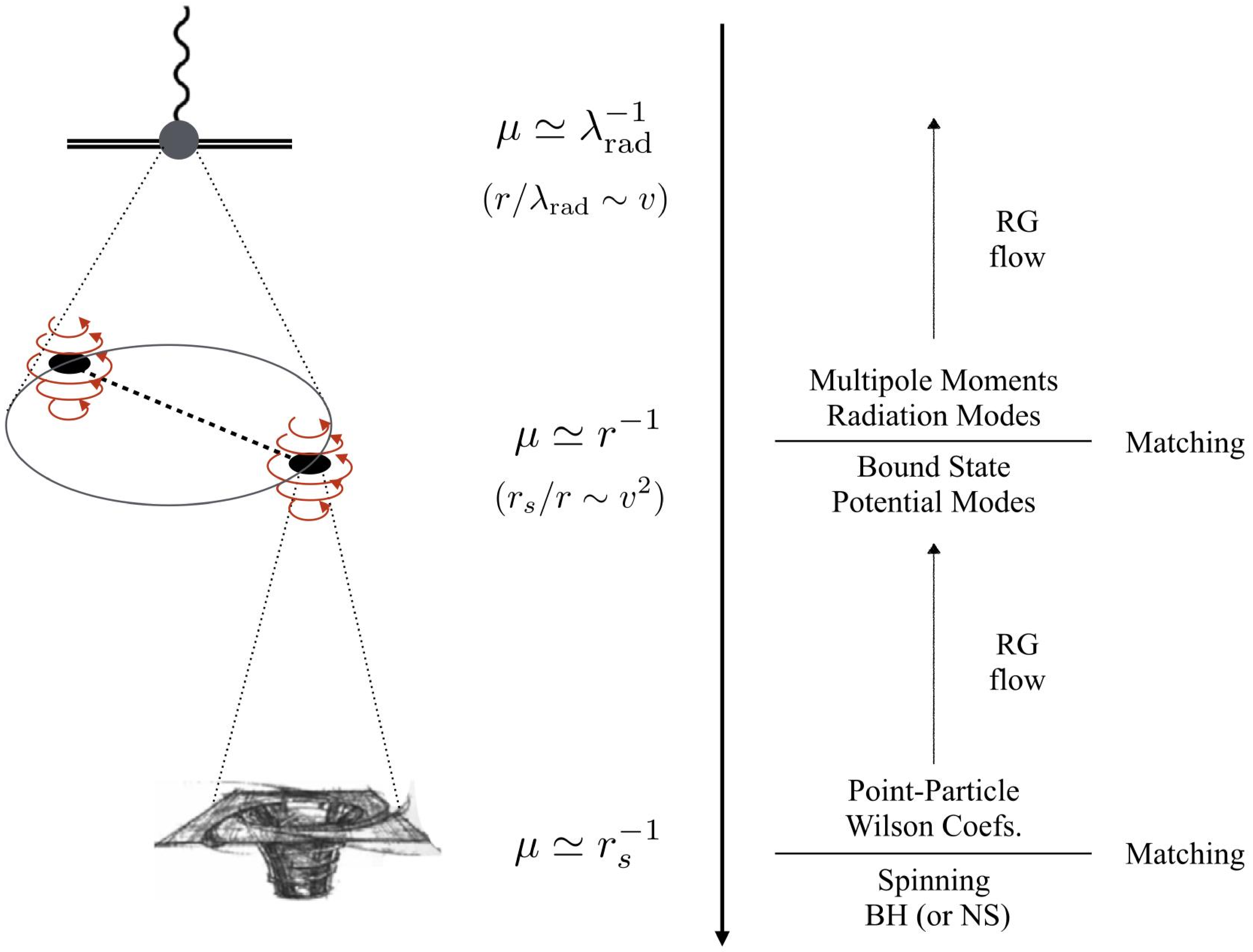
$$\frac{dE}{dt} = \frac{32}{5} G^{-1} v^{10} \quad \frac{dv}{dt} = \frac{32}{5} \frac{v^9}{Gm} \quad \Delta t = \frac{5Gm}{256} \left(\frac{1}{v_i^8} - \frac{1}{v_f^8} \right)$$

$$N \sim \int_{t_i}^{t_f} \omega(t)\, dt = \frac{1}{32} \left[\frac{1}{v_i^5} - \frac{1}{v_f^5} \right]$$

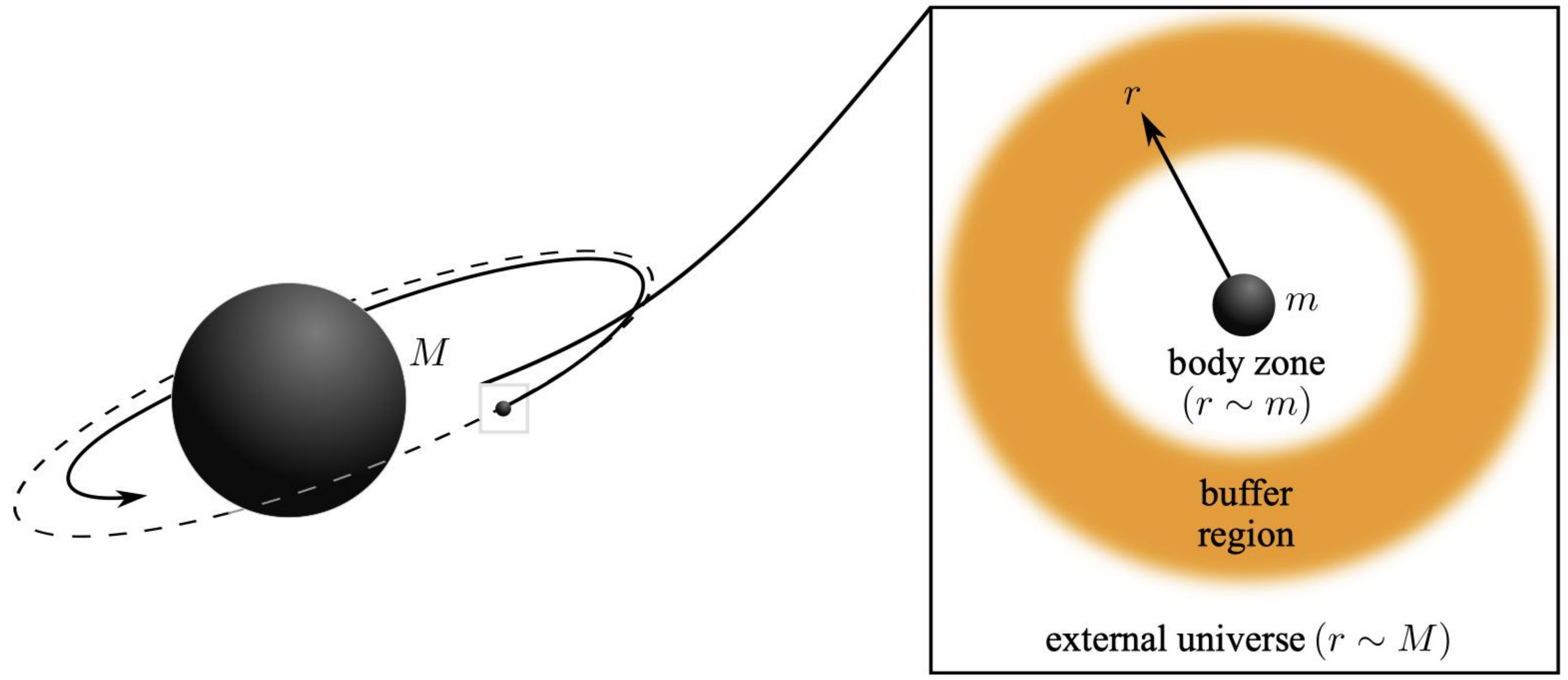
Impossible to use Numerical GR !!

Limitation: Weak field

Star Mass Inspiral—LIGO

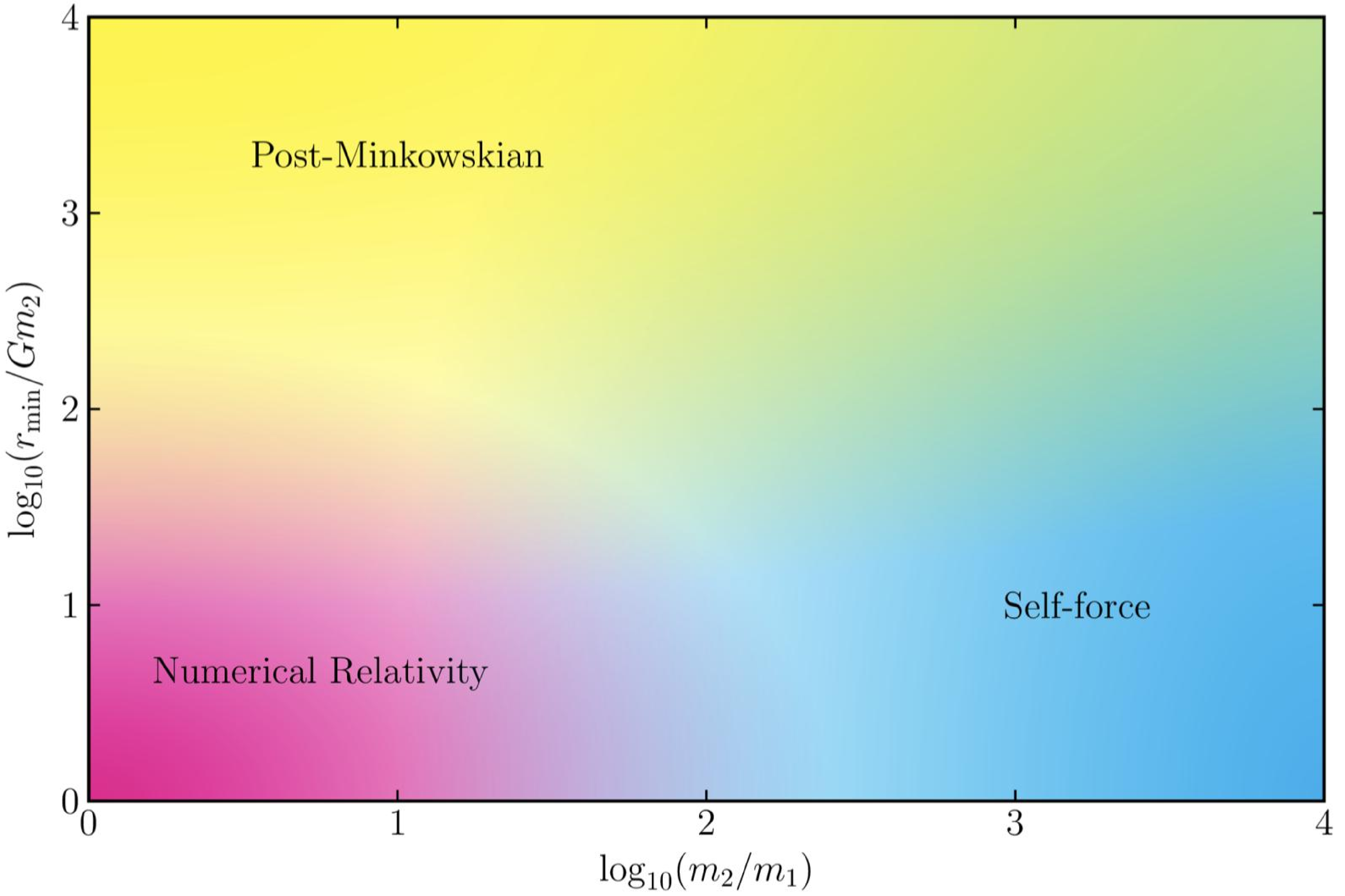


Extrem Mass-ratio Inspiral (EMI)—LISA

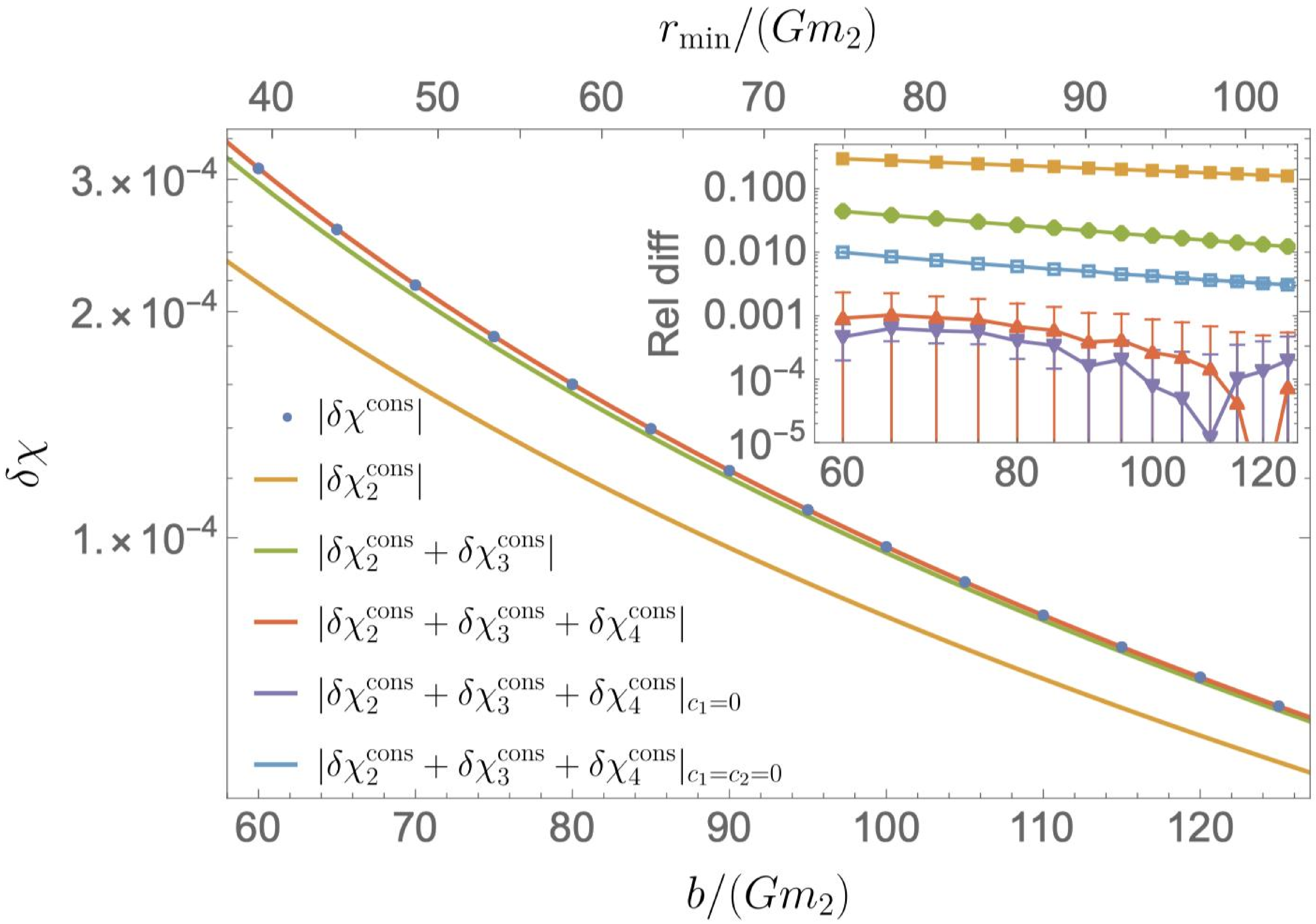


Self-Force

Ambition: Extend to EMI

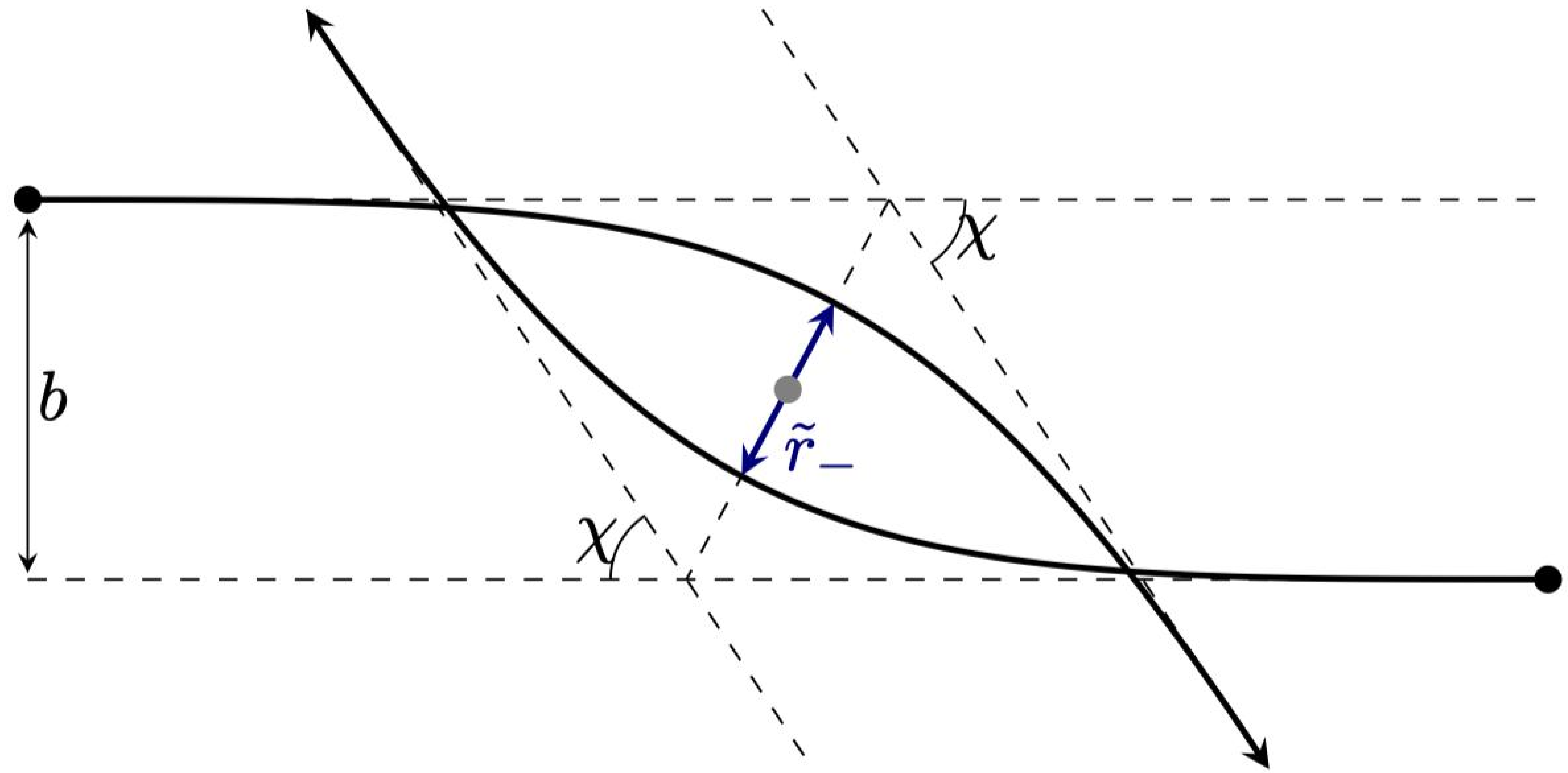


Z.Bern et al. 2023



Observable: Scattering angle (gauge invariant)

In COM frame



Eikonal approch:

$$i\frac{\mathcal{A}}{4pE} = \int d^{D-2}b\left(e^{2i\delta} - 1\right)e^{ib\cdot Q}$$

Saddle point

$$\tan\frac{\chi}{2} = -\frac{1}{2p}\frac{\partial \mathrm{Re}2\delta}{\partial b_J}$$

Vecchia, Veneziano et al. 2021

Scattering angle to Potential

Kalin, proto
2020, 2021

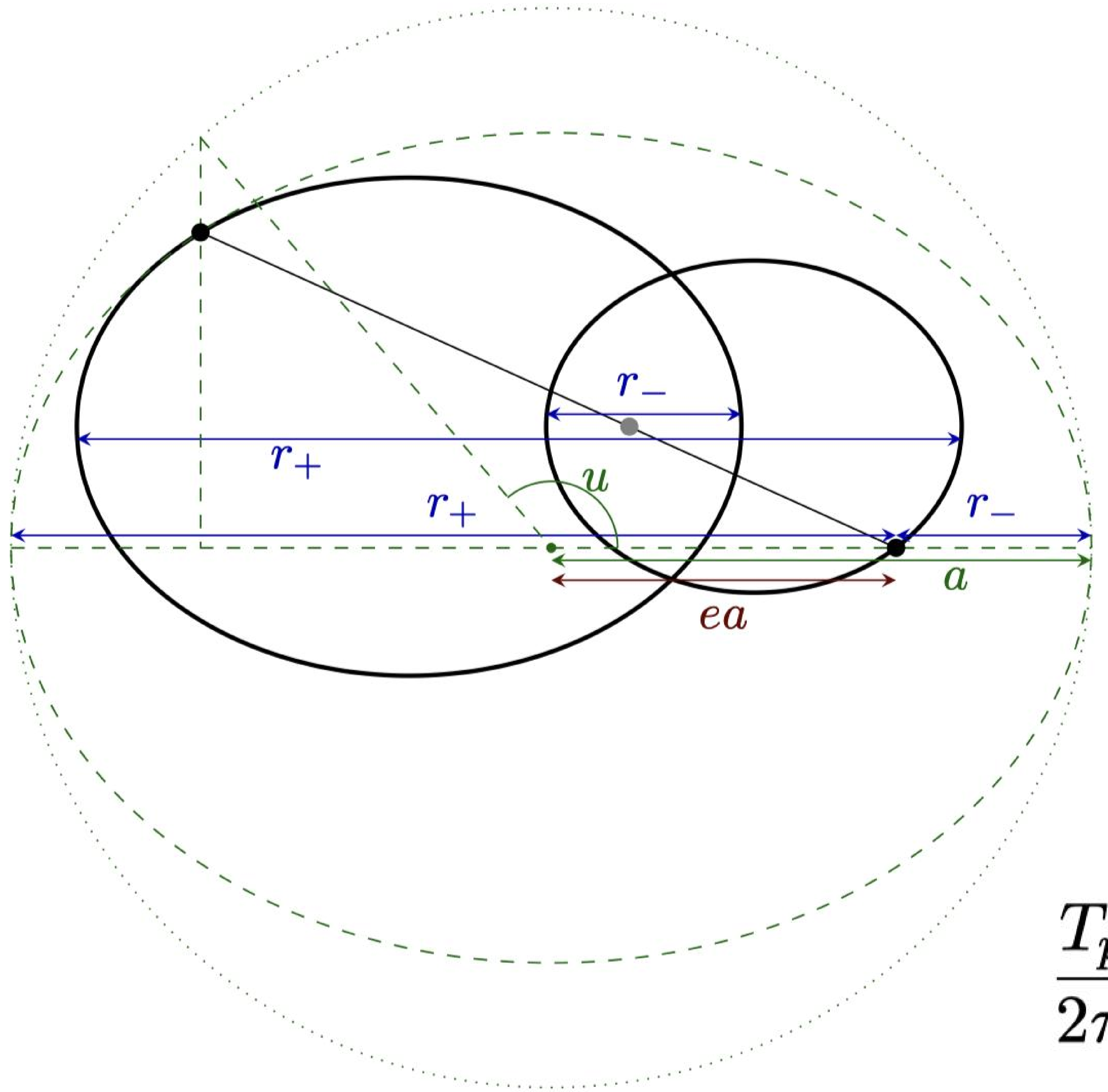
$\chi \Leftrightarrow V$

$$\bar{p}^2(r,E) = \exp\left[\frac{2}{\pi}\int_{r|\bar{p}(r,E)|}^{\infty} \frac{\chi_b(\tilde{b},E)\mathrm{d}\tilde{b}}{\sqrt{\tilde{b}^2 - r^2\bar{p}^2(r,E)}}\right],$$

$$\sqrt{\boldsymbol{p}^2 - \sum_{i=1}^{\infty} P_i(E)\left(\frac{G}{r}\right)^i + m_1^2} + \sqrt{\boldsymbol{p}^2 - \sum_{i=1}^{\infty} P_i(E)\left(\frac{G}{r}\right)^i + m_2^2} = \sum_{i=0}^{\infty} \frac{c_i(\boldsymbol{p}^2)}{i!}\left(\frac{G}{r}\right)^i .$$

Boundary (Scattering) to Bound

Kalin, proto
2020, 2021



$$\mathcal{S}_r(J,\mathcal{E}) = \frac{1}{\pi}\int_{r_-}^{r_+} \mathrm{d}r \sqrt{Q(J,\mathcal{E},r) + \lambda \sum_{\ell=1}^{\infty} \frac{D_\ell(\mathcal{E})}{r^{\ell+2}}}\,,$$

$$Q(J,\mathcal{E},r) \equiv A(\mathcal{E}) + \frac{2B(\mathcal{E})}{r} + \frac{C(J,\mathcal{E})}{r^2}$$

$$A(\mathcal{E}) \equiv p_\infty^2(\mathcal{E})\,,$$

$$2B(\mathcal{E}) \equiv \widetilde{M}_1(\mathcal{E})G$$

$$C(J,\mathcal{E}) \equiv \widetilde{M}_2(\mathcal{E})G^2 - J^2\,,$$

$$D_n(\mathcal{E}) \equiv \widetilde{M}_{n+2}(\mathcal{E})G^{n+2}\,,$$

Damour et al.
2000

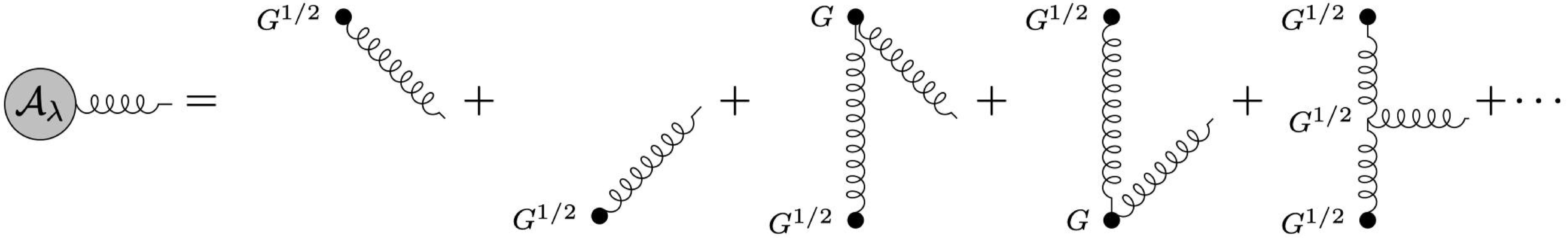
$$\frac{T_p}{2\pi} \equiv \frac{1}{\mu}\frac{\partial \mathcal{S}_r(J,\mathcal{E})}{\partial \mathcal{E}}\,, \qquad \frac{\Phi}{2\pi} = 1 + \frac{\Delta\Phi}{2\pi} = -\frac{\partial \mathcal{S}_r(J,\mathcal{E})}{\partial J}\,.$$

Radiation?

Mougiakakos, Riva et al. 2021

Reverse Unitarity

$$P^{\mu}_{\text{rad}} = \sum_{\lambda} \int_{k} \hat{\delta}_{+}(k^2) k^{\mu} \left| \mathcal{A}_{\lambda}(k) \right|^2 \ ,$$



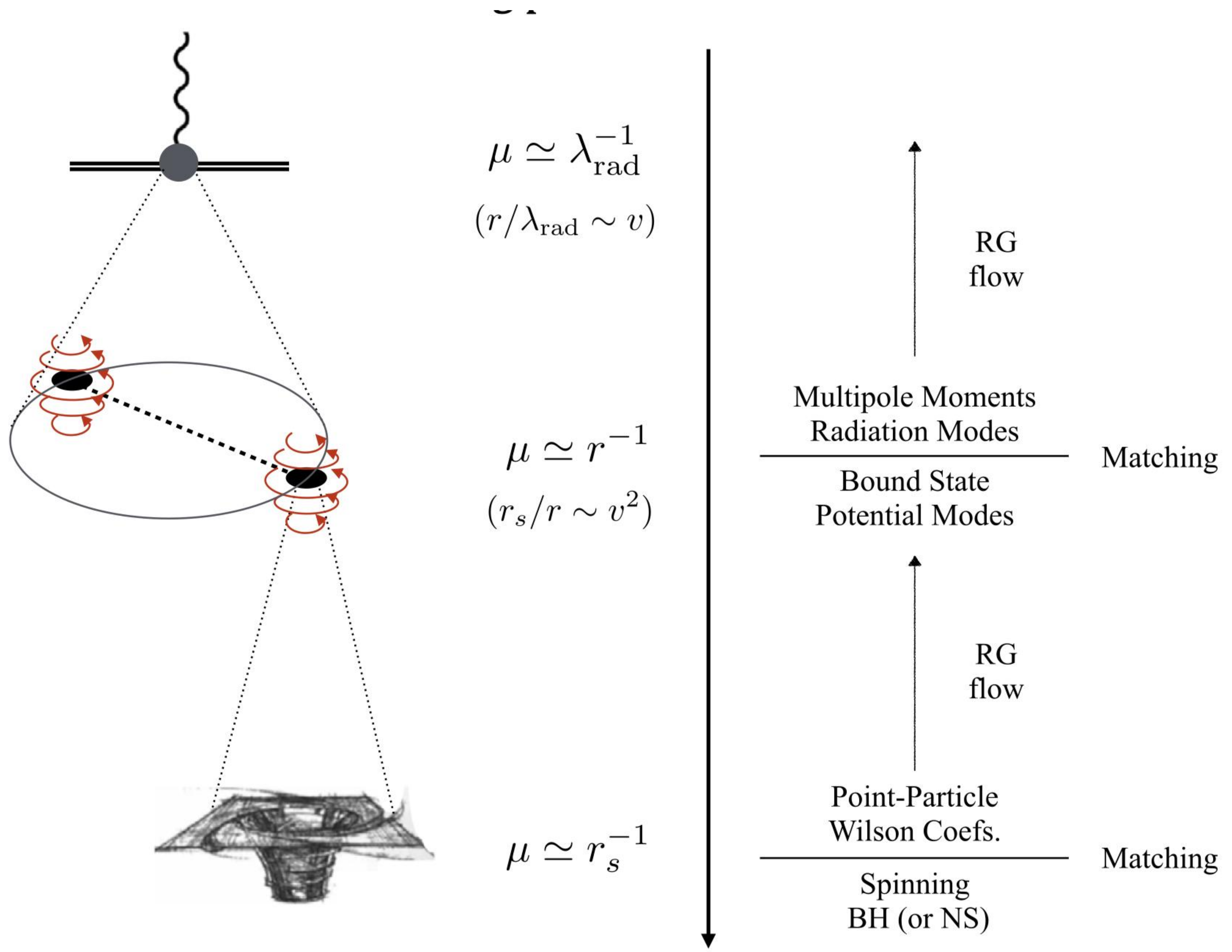
How to calculate Amplitude?

1. Hierarchy
2. PN-EFT
3. PN&PM-QFT
4. PM-EFT

Hierarchy

$$\mathcal{O}(1/J) \sim \mathcal{O}(\boldsymbol{q}) \quad \text{(classical expansion)},$$
$$\mathcal{O}(\boldsymbol{v}) \sim \mathcal{O}(\boldsymbol{p}) \quad \text{(nonrelativistic expansion)}$$

$$\text{hard} \;:(\omega, \boldsymbol{\ell}) \sim (m, m)$$
$$\text{soft}: \; (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}|, |\boldsymbol{q}|) \sim J^{-1}(m|\boldsymbol{v}|, m|\boldsymbol{v}|),$$
$$\text{potential}: \; (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}|) \sim J^{-1}\left(m|\boldsymbol{v}|^2, m|\boldsymbol{v}|\right),$$
$$\text{radiation}: \; (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}||\boldsymbol{v}|) \sim J^{-1}\left(m|\boldsymbol{v}|^2, m|\boldsymbol{v}|^2\right)$$



PN-EFT

Goldberger et al. 2008

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Theory

$$S_{\mathrm{EH}} = -2m_{\mathrm{Pl}}^2 \int d^4x \sqrt{g} R(x)$$

$$S_{\mathrm{pp}} = \; -\sum_a m_a \int d\tau_a + \sum_a c_R^{(a)} \int d\tau_a R(x_a) + \sum_a c_V^{(a)} \int d\tau_a R_{\mu\nu}(x_a) \dot{x}_a^\mu \dot{x}_a^\nu + \cdots$$

Potential mode (NR):

$$\langle H_{\mathbf{k}\mu\nu}(t) H_{\mathbf{q}\alpha\beta}(0) \rangle = -(2\pi)^3 \delta^3(\mathbf{k}+\mathbf{q}) \frac{i}{\mathbf{k}^2} \delta(t) P_{\mu\nu;\alpha\beta}$$

Radiation mode (Relative):

$$\langle \bar{h}_{\mu\nu}(x) \bar{h}_{\alpha\beta}(y) \rangle = D_F(x-y) P_{\mu\nu;\alpha\beta}$$

PN&PM-QFT (Based on S-matrix)

Z.bern et al. 2020

Theory

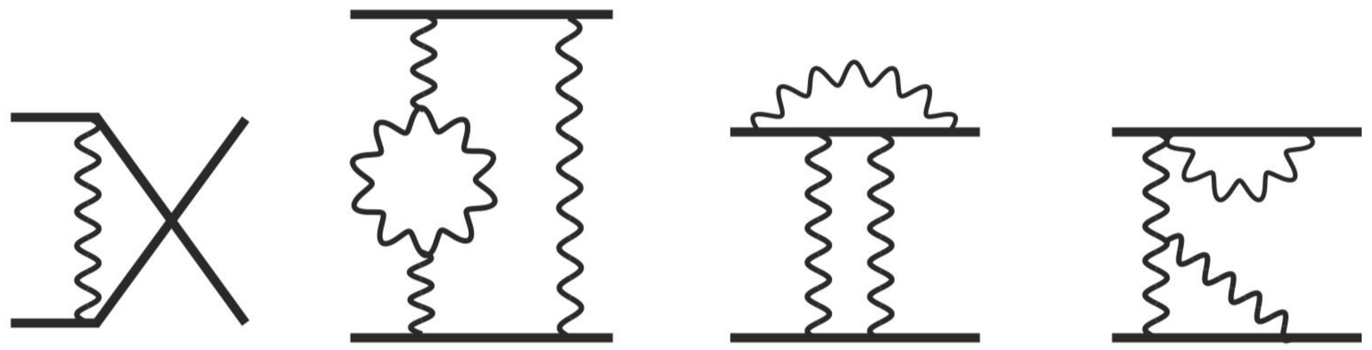
$$S_{\mathrm{GR}} = \int d^D x \sqrt{-g} \left[-\frac{1}{16\pi G} R + \frac{1}{2} \sum_{i=1,2} \left(D^\mu \phi_i D_\mu \phi_i - m_i \phi_i^2 \right) \right]$$

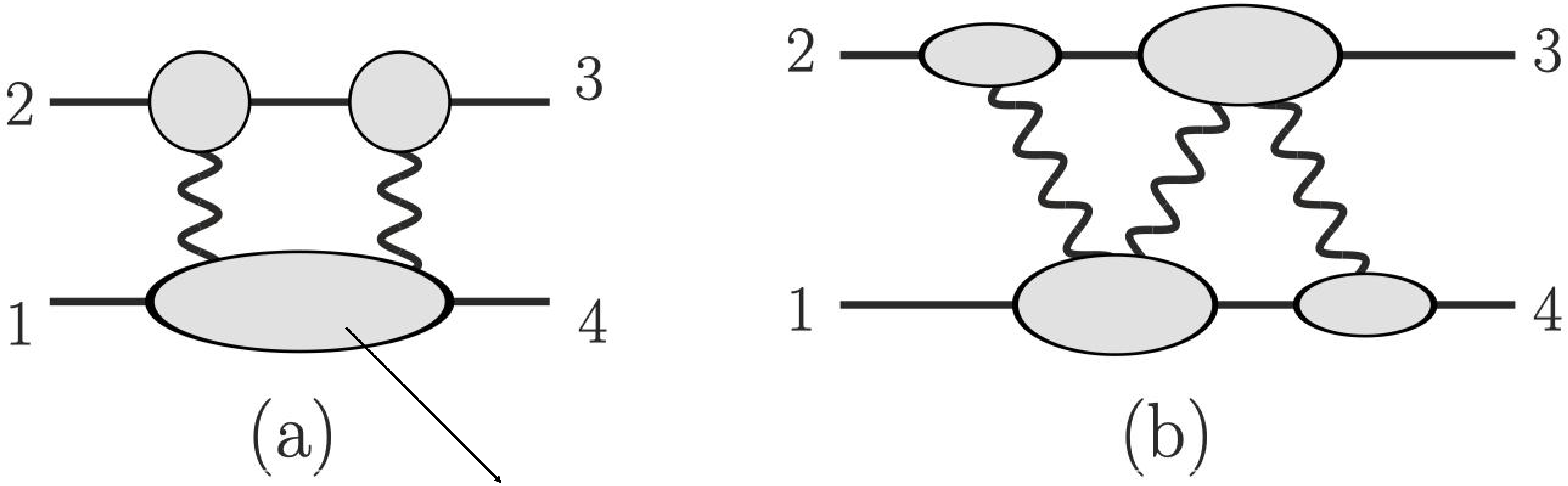
In S-matrix, forget the feynman rules!

How to get diagram?

Unitarity Cuts

1. No matter contact diagram (Hard mode)
2. No internal graviton loop (Hard mode)
3. No start & end point both at the same side (Scaleless Quantum)





2PM

Tree diagram

3PM

With 1,2 <-> 3,4

How to get Feynman Integrand?

Double Copy

KLT relation (D=4)

Color ordered

$$M_3^{\text{tree}}(1,2,3) = iA_3^{\text{tree}}(1,2,3)\, A_3^{\text{tree}}(1,2,3)\,,$$
$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)\,,$$
$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\, A_5^{\text{tree}}(3,1,4,2,5)$$

Color dressed

BCJ duality (D=4)

$$\mathbb{A}_m^{\text{tree}} = g^{m-2}\sum_j \frac{c_j n_j}{D_j}$$

Color&kinetic Jacobi Identity

$$c_i = c_j - c_k \Rightarrow n_i = n_j - n_k$$

$$M_m^{tree} = i\sum_j \frac{\tilde{n}_j(A-gauge)n_j(B-gauge)}{D_j(B-gauge)}$$
$$\mathbb{A}_m^{\text{tree}} = g^{m-2}\sum_j \frac{c_j n_j}{D_j}$$

(In potential region)

How to integrate?

Graviton pole k^0-> Soft-part(positive energy) + Hard-part(negative energy)

$$\frac{1}{(E_1+\omega)^2-(\boldsymbol{p}+\boldsymbol{\ell})^2-m_1^2}=\frac{1}{(\omega-\omega_{P_1})(\omega-\omega_{A_1})}, \quad \omega_{P_1},\omega_{A_1}=-E_1\pm\sqrt{E_1^2+2\boldsymbol{p\ell}+\boldsymbol{\ell}^2}.$$

2. Integrate soft-pole, expand hard-pole

$$\int\frac{dk_0}{2\pi}(\cdot)=\frac{i}{2}\left[\sum_{k_*\in\mathbb{H}^+}\operatorname*{Res}_{k_0=k_*}(\cdot)-\sum_{k_*\in\mathbb{H}^-}\operatorname*{Res}_{k_0=k_*}(\cdot)\right]$$

C.Cheung et al. 2019

Contour Integral — Expand by v (PN-QFT)

3. Integrate residual 3-dim feynman intgral (PN)

4. Resummation (PM)

Or IBP+ODE/ Mellin Barnes

PM-EFT

Porto, Zhengwen Liu et. 2020

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\rm Pl}}$$

$$e^{iS_{\rm eff}[x_a]} = \int \mathcal{D}h_{\mu\nu}\, e^{iS_{\rm EH}[h]+iS_{\rm GF}[h]+iS_{\rm pp}[x_a,h]}$$

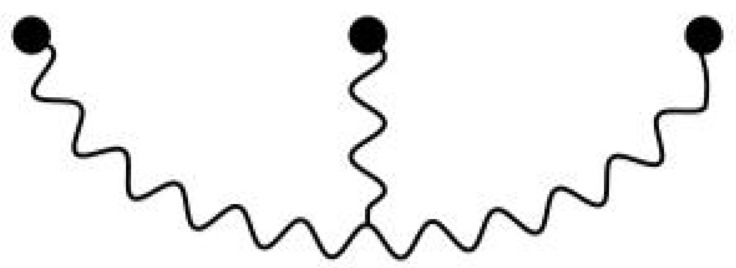
$$S_{\rm EH} = -2M_{\rm Pl}^2 \int {\rm d}^4x \sqrt{-g}\, R[g]$$

$$S_{\rm pp} = -\sum_{a=1,2} \frac{m_a}{2} \int {\rm d}\sigma_a\, e_a \left(\frac{1}{e_a^2} g_{\mu\nu}(x_a^\alpha(\sigma)) v_a^\mu(\sigma_a) v_a^\nu(\sigma_a) + 1 \right)$$

$$e_a^2 = g_{\mu\nu}(x_a(\tau_a)) v_a^\mu(\tau_a) v_a^\nu(\tau_a) = 1$$

Saddle Point

Only connected tree level diagrams, no graviton quantum loop

No ghost, No  (Scaleless integral)

Effective Lagrangian

Particle 1

$$S_{\text{eff}} = \sum_n \int d\tau_1 \, \mathcal{L}_n[x_1(\tau_1), x_2(\tau_2)]$$

$$\delta S = 0 \quad S_{\text{pp}} = -\sum_{a=1,2} \frac{m_a}{2} \int d\sigma_a \, e_a \left(\frac{1}{e_a^2} g_{\mu\nu}(x_a^\alpha(\sigma)) v_a^\mu(\sigma_a) v_a^\nu(\sigma_a) + 1 \right)$$

Equation of Motion

$$-\eta^{\mu\nu} \frac{d}{d\tau_1} \left(\frac{\partial \mathcal{L}_0}{\partial v_1^\nu} \right) = m_1 \frac{dv_1^\mu}{d\tau_1} = -\eta^{\mu\nu} \left(\sum_{n=1}^{\infty} \frac{\partial \mathcal{L}_n}{\partial x_1^\nu(\tau_1)} - \frac{d}{d\tau_1} \left(\frac{\partial \mathcal{L}_n}{\partial v_1^\nu} \right) \right)$$

Damour 2018

$$\langle h_{\mu\nu}(x) h_{\alpha\beta}(y) \rangle = \frac{i}{k^2} P_{\mu\nu\alpha\beta} e^{ik\cdot(x-y)}$$

$$v_a^\mu(\tau_1) = u_a^\mu + \sum_n \delta^{(n)} v_a^\mu(\tau_a) \, ,$$

Correction

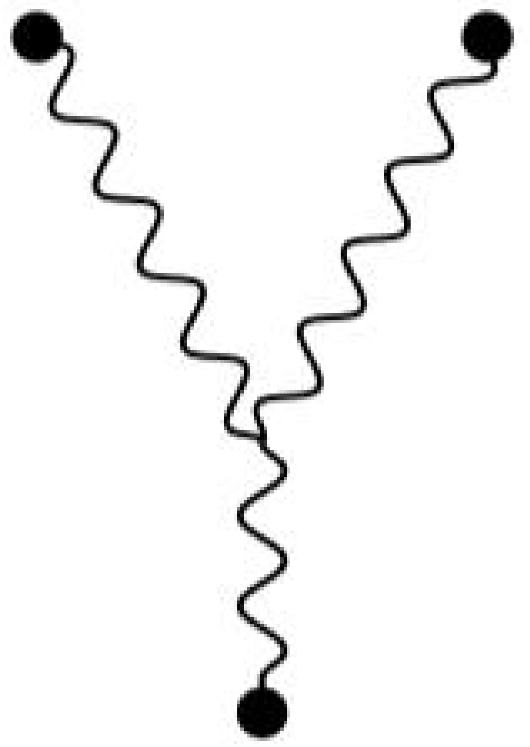
$$-\frac{m_a}{2} \int_{\tau_a} h_{\mu\nu} u^\mu(\tau_a) u^\nu(\tau_a)$$

$$x_a^\mu(\tau_1) = b_a^\mu + u_a^\mu \tau_a + \sum_n \delta^{(n)} x_a^\mu(\tau_a) \, ,$$

Example

$$= -i\frac{-i}{2}\frac{-i}{2}\int_{-\infty}^{+\infty}\mathrm{d}\tau_2\int_k \frac{iP_{\alpha\beta\mu\nu}}{k^2}v_1^{\alpha}(\tau_1)v_1^{\beta}(\tau_1)v_2^{\mu}(\tau_2)v_2^{\nu}(\tau_2)e^{ik\cdot(x_1(\tau_1)-x_2(\tau_2))}$$

$$= -\frac{m_1 m_2}{8M_{\mathrm{Pl}}^2}\int_{-\infty}^{+\infty}\mathrm{d}\tau_2\left(2(v_1(\tau_1)\cdot v_2(\tau_2))^2 - v_1^2(\tau_1)v_2^2(\tau_2)\right)\int_k \frac{1}{k^2}e^{ik\cdot(x_1(\tau_1)-x_2(\tau_2))}$$



$$= -\frac{m_1 m_2^2}{16M_{\mathrm{Pl}}^3}v_1^{\alpha}(\tau_1)v_1^{\beta}(\tau_1)\int\mathrm{d}\tau_2\int\mathrm{d}\tilde{\tau}_2 v_2^{\gamma}(\tau_2)v_2^{\rho}(\tau_2)v_2^{\sigma}(\tilde{\tau}_2)v_2^{\kappa}(\tilde{\tau}_2)P_{\gamma\rho\tilde{\gamma}\tilde{\rho}}(k_1)P_{\sigma\kappa\tilde{\sigma}\tilde{\kappa}}(k_2)P_{\alpha\beta\tilde{\alpha}\tilde{\beta}}$$

$$\times\int_{k_{1,2,3}} e^{ik_1\cdot x_1(\tau_1)}e^{ik_2\cdot x_2(\tau_2)}e^{ik_3\cdot x_2(\tilde{\tau}_2)}\frac{V_{hhh}^{\tilde{\gamma}\tilde{\rho}\tilde{\sigma}\tilde{\kappa}\tilde{\alpha}\tilde{\beta}}(k_1,k_2,k_3)}{k_1^2k_2^2k_3^2}\delta^4(k_1+k_2+k_3)+(1\leftrightarrow 2),$$

Bremsstrahlung (NLO Tidal effect)

Decoupling theorem

Encode heavy degree of freedom

$$S_{\mathrm{pp}} = \sum_{a=1,2} \int \mathrm{d}\tau_a \left(-\frac{m_a}{2} g_{\mu\nu} u_a^{\mu}(\tau) u_a^{\nu}(\tau) + c_{E^2}^{(a)} E_{\mu\nu} E^{\mu\nu} \right.$$

$$\left. + c_{B^2}^{(a)} B_{\mu\nu} B^{\mu\nu} - c_{\ddot{E}^2}^{(a)} E_{\mu\nu\alpha} E^{\mu\nu\alpha} - c_{\ddot{B}^2}^{(a)} B_{\mu\nu\alpha} B^{\mu\nu\alpha} + \cdots \right)$$

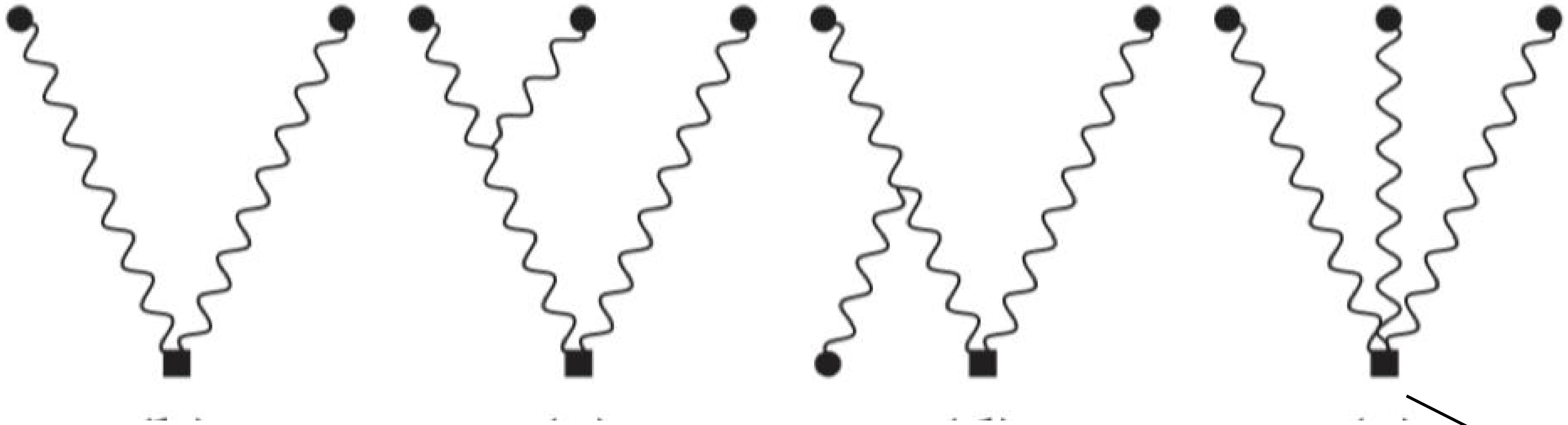
$$E_{\alpha\beta} = R_{\mu\alpha\nu\beta} u^{\mu}(\tau) u^{\nu}(\tau), \qquad B_{\alpha\beta} = R^{\star}_{\mu\alpha\nu\beta} u^{\mu}(\tau) u^{\nu}(\tau)$$

$$E_{\alpha\beta\gamma} = \nabla^{\perp}_{\{\alpha} R_{\beta\rho\gamma\}\nu} u^{\rho}(\tau) u^{\nu}(\tau), \quad B_{\alpha\beta\gamma} = \nabla^{\perp}_{\{\alpha} R^{\star}_{\beta\rho\gamma\}\nu} u^{\rho}(\tau) u^{\nu}(\tau)$$

$$-C_V \int d\tau\, R + C_R \int d\tau\, R_{\mu\nu} u^{\mu} u^{\nu} + \cdots \qquad \delta S_{\mathrm{EH}} \to \left(\epsilon_1 + \frac{\epsilon_2}{2} \right) \int d\tau\, R - \epsilon_2 \int d\tau\, R_{\mu\nu} u^{\nu} u^{\nu}$$

Point Vertex

Porto et al. 2021



$\longrightarrow S_{\text{eff}}$

Tidal Vertex

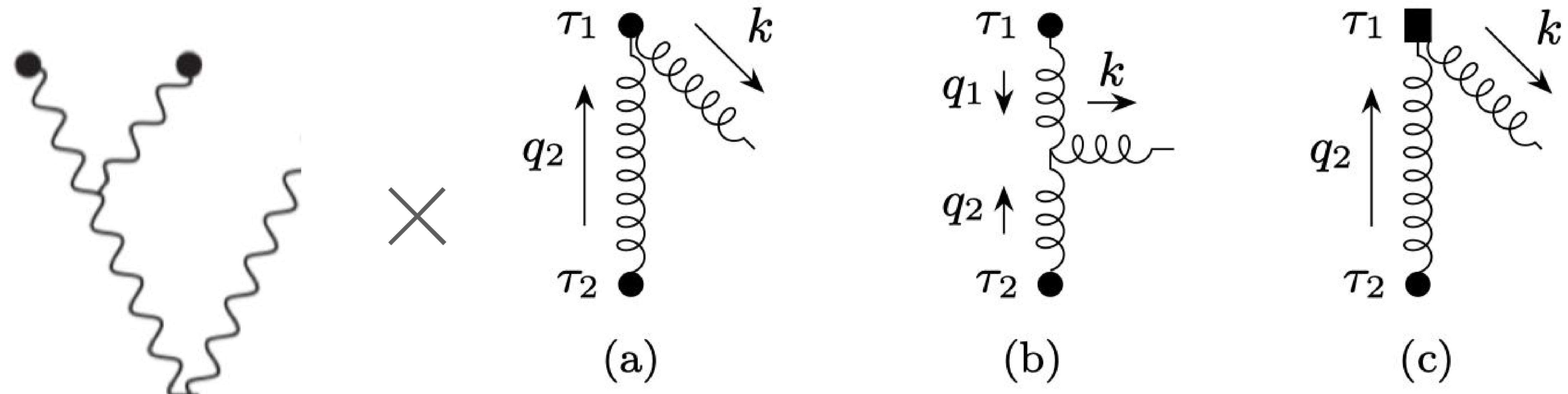
$$\mathcal{T}_{\mu\nu} = \frac{\delta S_{\text{eff}}}{\delta h_{\mu\nu}}$$

$$T^{\mu\nu}_{\text{C1}}(k) = \frac{m_2^2}{2M_{\text{pl}}^2} \int_{q_{123},\tau_{122'},x} V_{h^3}^{\mu_1\nu_1;\mu_2\nu_2;\mu\nu} \frac{P_{\mu_1\nu_1;\alpha_1\beta_1} P_{\mu_2\nu_2;\alpha_2\beta_2}}{q_2^2 q_1^2} \times e^{iq_1(x - x_1(\tau_1))} e^{iq_2\left(x - x_2(\tau_2')\right)} e^{ikx} \left[u_2^{\alpha_2}(\tau_2') u_2^{\beta_2}(\tau_2')\right] V_{E^2}^{\alpha_1\beta_1;\mu_3\nu_3} \times \frac{1}{q_3^2} P_{\mu_3\nu_3;\alpha_3\beta_3} e^{iq_3(x_2(\tau_2) - x_1(\tau_1))} \left[u_2^{\alpha_3}(\tau_2) u_2^{\beta_3}(\tau_2)\right]$$

IR finite

IR divergence

$$P^{\mu}_{\text{rad}} = \sum_{\lambda} \int_k \hat{\delta}_+(k^2) k^{\mu} \left|\mathcal{A}_{\lambda}(k)\right|^2 = 2 \int_k \hat{\delta}_+\left(k^2\right) k^{\mu} T^{*\mu\nu}_{\text{2PM}}(k)\, P_{\mu\nu;\rho\sigma} T^{\rho\sigma}_{\text{tidal-NLO}}(k)$$



M. Maria Riva et al. 2022

$$
\begin{aligned}
P^{\mu}_{\rm C1rad} =& 2\int_k \hat{\delta}_+\left(k^2\right) k^{\mu} T^{*\mu\nu}_{\rm 2PM}\left(k\right) P_{\mu\nu;\rho\sigma} T^{\rho\sigma}_{\rm tidal-NLO}\left(k\right) \\
=& 2\sum_{\lambda}\int_k \hat{\delta}_+\left(k^2\right) k^{\mu} \frac{P_{\mu\nu;\rho\sigma}}{4M_{\rm pl}^2}\frac{m_2^2}{2M_{\rm pl}}\int_{q_{123},q}\mu_{\rm C1}\left(k\right) V_{h^3}^{\mu_1\nu_1;\mu_2\nu_2;\mu\nu} \\
& \frac{P_{\mu_1\nu_1;\alpha_1\beta_1}P_{\mu_2\nu_2;\alpha_2\beta_2}P_{\mu_3\nu_3;\alpha_3\beta_3}}{q_2^2 q_1^2 q_3^2} u_2^{\alpha_2} u_2^{\beta_2} V_{E^2}^{\alpha_1\beta_1;\mu_3\nu_3} u_2^{\alpha_3} u_2^{\beta_3} \\
& \left\{\frac{m_1 m_2}{4m_{\rm Pl}^2}\hat{\delta}\left(q\cdot u_1\right)\hat{\delta}\left(q\cdot u_2 - k\cdot u_2\right)\frac{e^{iq\cdot b}e^{ik\cdot b_2}}{q^2(q-k)^2}\right. \\
& \left.\left[t^{\mu\nu}_{\uparrow}(q,k)+t^{\mu\nu}_{\downarrow}(q,k)+t^{\mu\nu}_{\vdash}(q,k)\right]\right\}^* + \cdots
\end{aligned}
$$

No internal graviton loop!

$$
\mu_{\rm C1}\left(k\right) = \int_l \hat{\delta}^4\left(l-q_1-q_3\right)\hat{\delta}^4\left(q_1+q_2+k\right)\hat{\delta}\left(l\cdot u_1\right)\hat{\delta}\left(q_2\cdot u_2\right)\hat{\delta}\left(q_3\cdot u_2\right)e^{-ilb}e^{ikb_2}
$$

Reverse Unitarity

$$\hat{\delta}_+\left(k^2\right) \to \frac{1}{\underline{k^2}},\ \hat{\delta}\left(q \cdot u_1\right) \to \frac{1}{\underline{q \cdot u_1}},\ \hat{\delta}\left(q \cdot u_2 - k \cdot u_2\right) \to \frac{1}{\underline{(q-k) \cdot u_2}}$$

$$\hat{\delta}\left(l \cdot u_1\right) \to \frac{1}{\underline{l \cdot u_1}},\ \hat{\delta}\left(q_2 \cdot u_2\right) \to \frac{1}{\underline{-q_2 \cdot u_2}},\ \hat{\delta}\left(q_3 \cdot u_2\right) \to \frac{1}{\underline{q_3 \cdot u_2}}$$

Then we meet 3-loop feynman integral

$$\int_p \hat{\delta}\left(p \cdot u_1\right) \hat{\delta}\left(p \cdot u_2\right) e^{-ipb} \int_{k,q_{23}} \frac{1}{\underline{k^2}\underline{(-q_2 \cdot u_2)(q_3 \cdot u_2)}\left(q_2+k\right)^2 q_2^2 q_3^2}$$

$$\times \frac{1}{\underline{\left(\left(p+q_2+k-q_3\right) \cdot u_1\right)}\left(p+q_2+k-q_3\right)^2\left(p-q_3+q_2\right)^2}$$

Feynman Integral Calculation

- IBP
- ODE
- Epsilon-form ODE (static boundary condition)
- Cutosky Rule

IBP

Define denominator basis

$$0=\int\prod_{i=1}^{L}d^dk_i\,\frac{\partial}{\partial k_i^{\mu}}\left(q_j^{\mu}\mathcal{I}\right)$$

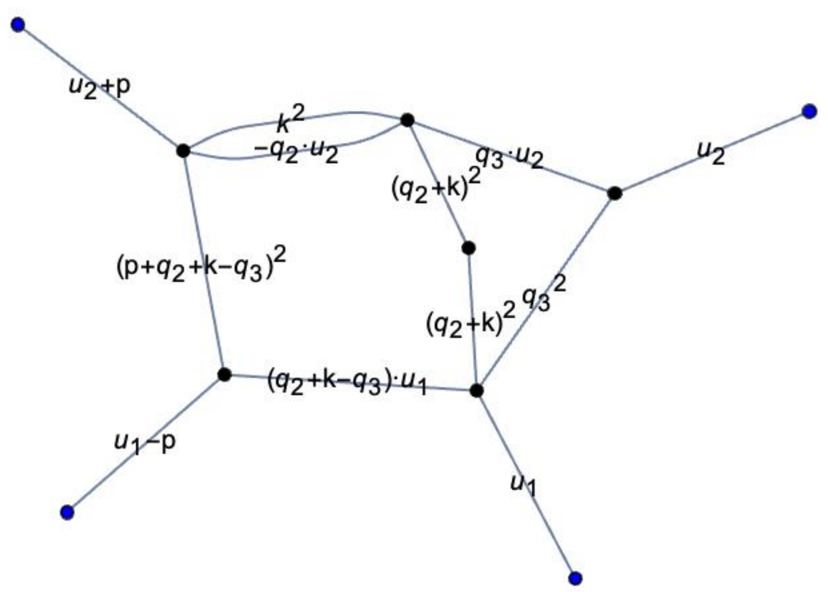
$$\rho_1=k\cdot u_1,\ \rho_2=k\cdot u_2,\ \rho_3=q_2\cdot u_1,\ \rho_4=-q_2\cdot u_2,\ \rho_5=(q_2+k-q_3)\cdot u_1$$

$$\rho_6=q_3\cdot u_2,\ \rho_7=k^2,\ \rho_8=q_2^2,\ \rho_9=q_3^2,\ \rho_{10}=q_2\cdot k,\ \rho_{11}=k\cdot p,$$

$$\rho_{12}=k\cdot q_3,\ \rho_{13}=p\cdot q_2,\ \rho_{14}=p\cdot q_3,\ \rho_{15}=q_2\cdot q_3$$

$$\frac{(\rho_1+\rho_2)\,\mathcal{N}\left(q_2,q_3,k,p\right)}{\underline{\rho_4\rho_5\rho_6\rho_7}\rho_8\rho_9\rho_{10}\rho_{12}\rho_{15}}=\sum_i c_iI_i$$

Master Integral (MI)



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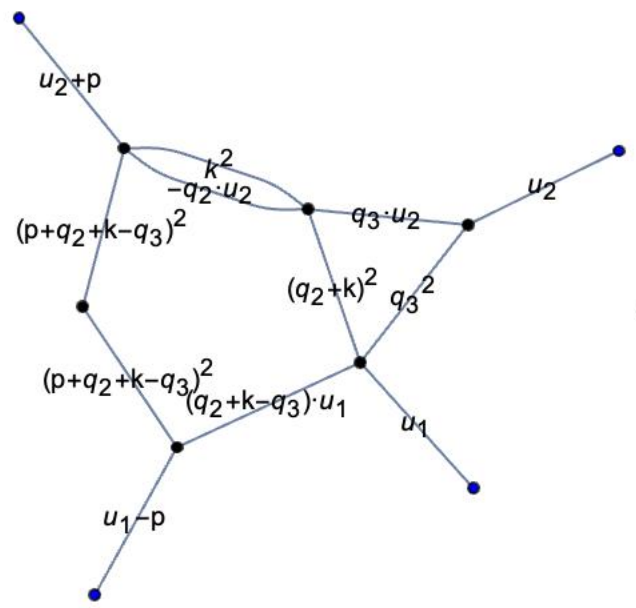

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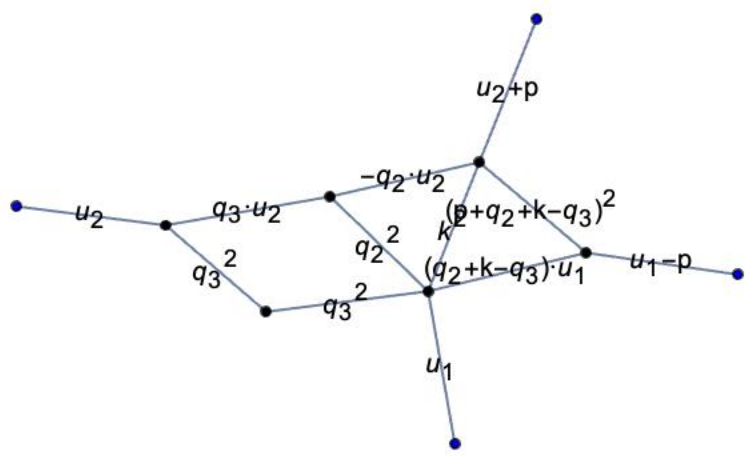

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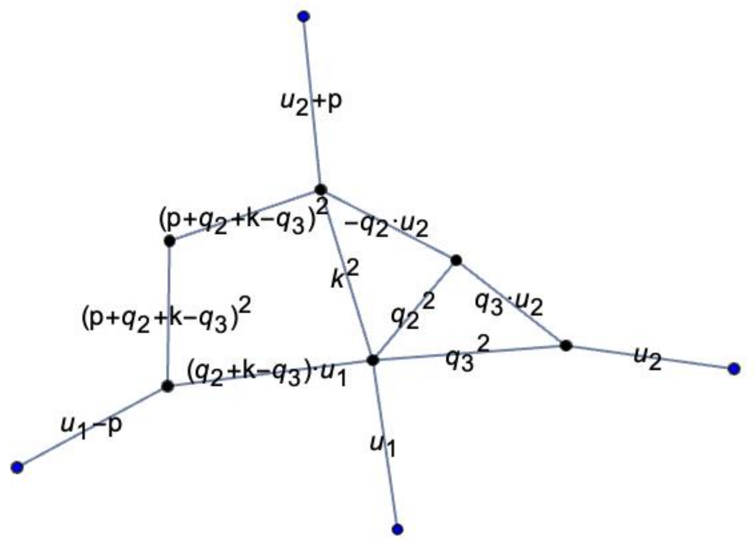

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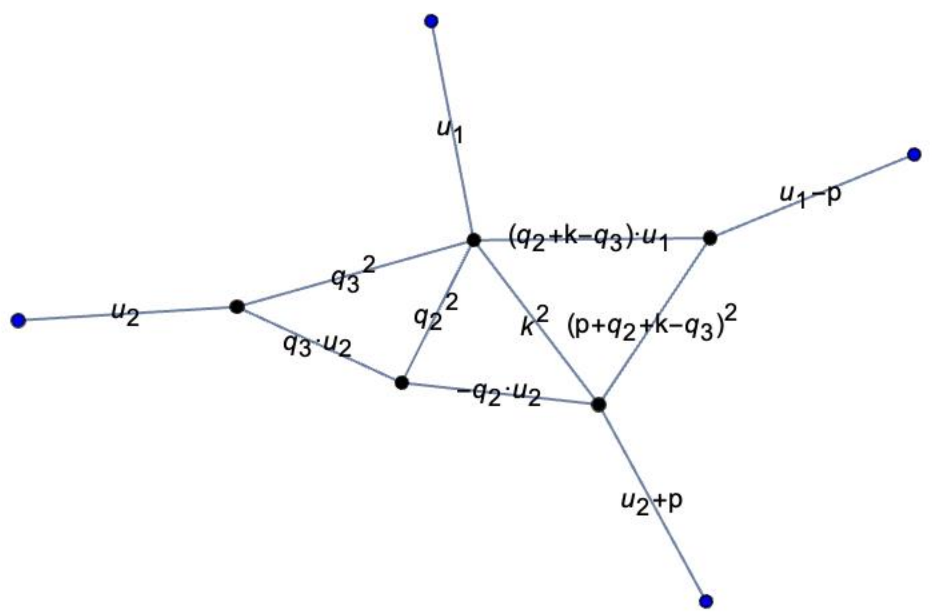
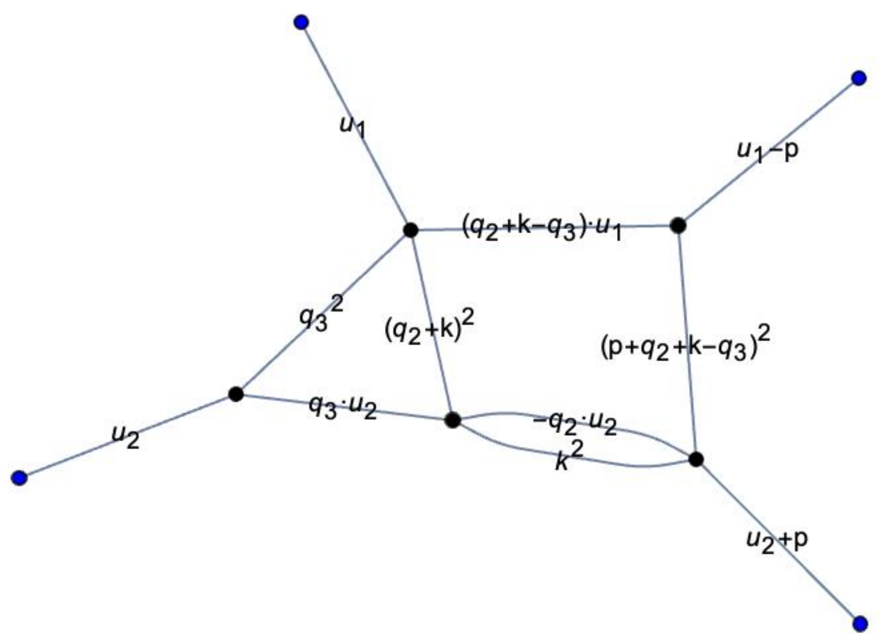




ODE

$$\gamma = u_1 \cdot u_2 \qquad x = \gamma - \sqrt{\gamma^2 - 1}$$

$$\partial_x \boldsymbol{I} = A(x,\epsilon)\,\boldsymbol{I}$$

$$A(x,\epsilon) = A_1 \oplus A_2$$

$$A_1 = \begin{pmatrix} \frac{4\epsilon\left(2\epsilon\left(x^4-10x^2+1\right)-\left(x^2-1\right)^2\right)}{(2\epsilon-1)x(x^4-1)} & \frac{64\epsilon^2 x}{(2\epsilon-1)(x^4-1)} & -\frac{2\epsilon(6\epsilon-1)(8\epsilon-1)\left(x^2-1\right)}{(2\epsilon+1)(x^3+x)} \\ -\frac{4(2\epsilon+1)(6\epsilon-1)x}{(2\epsilon-1)(x^4-1)} & \frac{(2\epsilon+1)\left(2\epsilon\left(x^4+10x^2+1\right)-\left(x^2+1\right)^2\right)}{(2\epsilon-1)x(x^4-1)} & -\frac{(6\epsilon-1)(8\epsilon-1)\left(x^2-1\right)}{2(x^3+x)} \\ -\frac{8(2\epsilon x+x)}{(2\epsilon-1)(x^4-1)} & \frac{32\epsilon(2\epsilon+1)x}{(4\epsilon(3\epsilon-2)+1)(x^4-1)} & \frac{(6-32\epsilon)x^2+x^4+1}{x-x^5} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \frac{2\epsilon\left(-8\epsilon\left(x^4+4x^2+1\right)+x^4+6x^2+1\right)}{(6\epsilon-1)x(x^4-1)} & \frac{64\epsilon^2 x}{(6\epsilon-1)(x^4-1)} & \frac{2\epsilon(4\epsilon-1)(8\epsilon-1)\left(x^2-1\right)}{(6\epsilon-1)x(x^2+1)} \\ -\frac{(4\epsilon-1)\left(x^2-1\right)}{2(x^3+x)} & \frac{2\epsilon\left(x^4-6x^2+1\right)+\left(x^2+1\right)^2}{x(x^4-1)} & -\frac{(4\epsilon-1)(8\epsilon-1)\left(x^2-1\right)}{2(x^3+x)} \\ -\frac{(4\epsilon-1)\left(x^2-1\right)}{(6\epsilon-1)(x^3+x)} & -\frac{32\epsilon x}{(6\epsilon-1)(x^4-1)} & \frac{(4\epsilon-1)\left(4\epsilon\left(x^4+10x^2+1\right)-\left(\left(x^2+6\right)x^2\right)-1\right)}{(6\epsilon-1)x(x^4-1)} \end{pmatrix}$$

Epsilon-form ODE

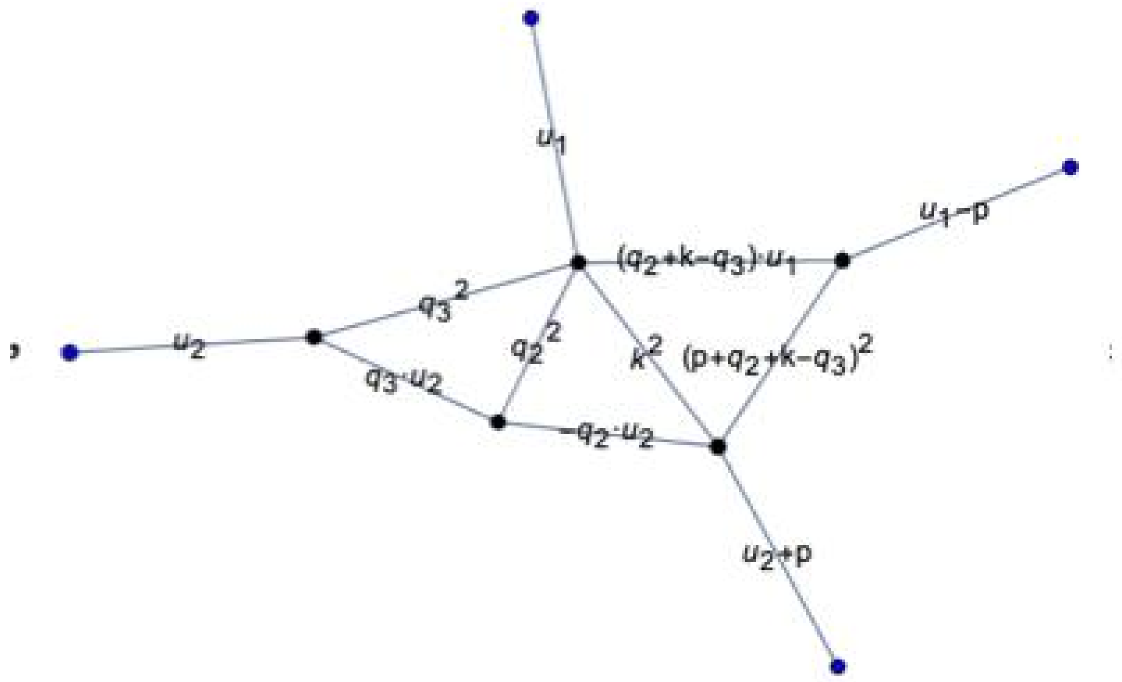
$$\partial_x \boldsymbol{I}' = \epsilon A'(x)\, \boldsymbol{I}' \qquad\qquad \boldsymbol{I}' = T^{-1}\boldsymbol{I}$$

$$d\boldsymbol{I}' = \epsilon\left(A'_{-1} d\log(x-1) + A'_0 d\log(x) + A'_1 d\log(x+1)\right)\boldsymbol{I}'$$

$$A'_{-1} = \begin{pmatrix} -\frac{128}{9} & \frac{160}{9} & \frac{35}{27} & 0 & 0 & 0 \\ -\frac{463}{27} & \frac{578}{27} & \frac{553}{324} & 0 & 0 & 0 \\ -\frac{544}{63} & \frac{704}{63} & -\frac{32}{27} & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & \frac{21}{4} & -2 & \frac{9}{4} \\ 0 & 0 & 0 & 11 & 0 & 7 \end{pmatrix}, \quad A'_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 32 & 0 \end{pmatrix}$$

$$A'_1 = \begin{pmatrix} \frac{200}{9} & -\frac{160}{9} & -\frac{35}{27} & 0 & 0 & 0 \\ \frac{2377}{135} & -\frac{382}{27} & -\frac{287}{324} & 0 & 0 & 0 \\ \frac{736}{315} & -\frac{64}{63} & -\frac{56}{27} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & -\frac{75}{4} & -2 & \frac{9}{4} \\ 0 & 0 & 0 & -35 & 0 & 1 \end{pmatrix}$$

Calculate MI (No Cut)



$$\int_{kq_2q_3} \frac{1}{k^2 q_2^2 (q_2 \cdot u_2)((k+q_2-q_3)\cdot u_1)(k+p+q_2-q_3)^2 q_3^2 (q_3 \cdot u_2)}$$

$$= \int_{kq_2q_3} \int_{s_{1-7}} \mathrm{Exp}(-)\left[s_4\left(k^2+p^2+q_2^2+q_3^2+2k\cdot p+2k\cdot q_2-2k\cdot q_3\right)\right.$$
$$s_4(+2p\cdot q_2-2p\cdot q_3-2q_2\cdot q_3)+s_1k^2+s_2q_2^2+$$
$$\left.s_3q_3^2+s_5u_2\cdot q_2+s_6u_2\cdot q_3+s_7(k\cdot u_1+q_2\cdot u_1-q_3\cdot u_1)\right]$$

Schwinger parameterization

Exp part $S_5 \sim S_7$ $I = \int_{s_{5-7}} \mathrm{Exp}(-)\left\{as_5^2+bs_6^2+cs_7^2+As_5s_6+Bs_6s_7+Cs_5s_7\right\}$

$$a=\frac{s_{13}+s_{14}+s_{34}}{4T}=\frac{\hat{s}_2^2}{4T}, b=\frac{s_{12}+s_{14}+s_{24}}{4T}=\frac{\hat{s}_3^2}{4T}, c=\frac{s_{12}+s_{13}+s_{23}}{4T}=\frac{\hat{s}_4^2}{4T}$$

$$T=s_1s_2s_3+s_1s_2s_4+s_1s_3s_4+s_2s_3s_4, \qquad s_{ik}=s_is_k \qquad \hat{s}_i^2=\sum_{1\sim 4,\mathrm{no}\,i}^{\mathrm{pair}} s_{lm}$$

$$A=\frac{s_{14}}{2T} \qquad B=-\frac{\gamma s_{12}}{2T} \qquad C=\frac{\gamma s_{13}}{2T}$$

In static limit (ODE boundary $u_1 \cdot u_2 \to -1$)

$$
\begin{aligned}
I = \frac{\sqrt{\pi}}{4}\frac{1}{\sqrt{\delta}\Delta_1\xi_1}\Bigg[& i\pi\Delta_1\xi_1 + 2\Delta_1\xi_1\text{ArcCoth}\left(\frac{2\sqrt{a}\sqrt{\delta}}{2aB-AC}\right) \\
& -\left(\sqrt{a}\left(AB-2bC\right)+A\sqrt{\delta}\right)\sqrt{\Xi-2\sqrt{a}\sqrt{\delta}\xi_2} \\
& \times\text{ArcCoth}\left(\frac{2\sqrt{b}\sqrt{\Xi-2\sqrt{a}\sqrt{\delta}\xi_2}}{A^2B-2AbC-4\sqrt{a}b\sqrt{\delta}}\right) \\
& -\left(-\sqrt{a}\left(AB-2bC\right)+A\sqrt{\delta}\right)\sqrt{\Xi+2\sqrt{a}\sqrt{\delta}\xi_2} \\
& \times\text{ArcCoth}\left(\frac{2\sqrt{b}\sqrt{\Xi+2\sqrt{a}\sqrt{\delta}\xi_2}}{A^2B-2AbC+4\sqrt{a}b\sqrt{\delta}}\right) \\
& -\left(\sqrt{a}\left(AB-2bC\right)+A\sqrt{\delta}\right)\sqrt{\Xi-2\sqrt{a}\sqrt{\delta}\xi_2} \\
& \times\text{ArcCoth}\left(\frac{2\sqrt{c}\sqrt{\Xi-2\sqrt{a}\sqrt{\delta}\xi_2}}{-2A^2c+ABC+2\sqrt{a}B\sqrt{\delta}-2a\Delta_2}\right) \\
& -\left(-\sqrt{a}\left(AB-2bC\right)+A\sqrt{\delta}\right)\sqrt{\Xi+2\sqrt{a}\sqrt{\delta}\xi_2} \\
& \times\text{ArcCoth}\left(\frac{2\sqrt{c}\sqrt{\Xi+2\sqrt{a}\sqrt{\delta}\xi_2}}{-2A^2c+ABC-2\sqrt{a}B\sqrt{\delta}-2a\Delta_2}\right)\Bigg]
\end{aligned}
$$

$$\delta = A^2c + a\left(B^2-4bc\right) - ABC + bC^2$$

$$\Xi = 2a\left(A^2\left(B^2-2bc\right)-2AbBC+2b^2C^2\right) + A^2\left(A^2c-ABC+bC^2\right)$$

$$\Delta_1 = A^2-4ab,\ \Delta_2 = B^2-4bc,\ \xi_1 = A^2c-ABC+bC^2,\ \xi_2 = A^2B-2AbC$$

$$F_- = \Xi-2\sqrt{a}\sqrt{\delta}\xi_2 = \frac{s_1^2}{64T^3}\left(\hat{s}_2^2-s_4^2+2is_4\hat{s}_2\right) = \frac{s_1^2}{64T^3}\left(\hat{s}_2+is_4\right)^2$$

$$F_+ = \Xi+2\sqrt{a}\sqrt{\delta}\xi_2 = \frac{s_1^2}{64T^3}\left(\hat{s}_2-is_4\right)^2$$

$$G_1^- = A^2B-2AbC-4\sqrt{a}b\sqrt{\delta} = -\frac{s_1^2s_4+i\hat{s}_3^2\hat{s}_2}{8T^2}$$

$$G_1^+ = A^2B-2AbC+4\sqrt{a}b\sqrt{\delta} = -\frac{s_1^2s_4-i\hat{s}_3^2\hat{s}_2}{8T^2}$$

$$G_2^- = -2A^2c+ABC-2\sqrt{a}B\sqrt{\delta}-2a\Delta_2 = \frac{\left(s_1^2s_3+is_1s_2\hat{s}_2\right)+T}{8T^2}$$

$$G_2^+ = -2A^2c+ABC+2\sqrt{a}B\sqrt{\delta}-2a\Delta_2 = \frac{\left(s_1^2s_3-is_1s_2\hat{s}_2\right)+T}{8T^2}$$

$$\tau^+ = \sqrt{a}\left(AB-2bC\right)+A\sqrt{\delta} = -\frac{s_1}{8T^{5/2}}\left(s_1\hat{s}_2\hat{s}_1^2+s_4\left(s_2\hat{s}_2s_3-iT\right)\right) = -\frac{s_1\left(\hat{s}_2-is_4\right)}{8T^{3/2}}$$

$$\tau^- = -\sqrt{a}\left(AB-2bC\right)+A\sqrt{\delta} = \frac{s_1}{8T^{3/2}}\left(\hat{s}_2+is_4\right)$$

Conjugate Relation $F_- = (F_+)^*, \qquad G^-_{1,2} = \left(G^+_{1,2}\right)^*, \qquad \tau^+ = -\left(\tau^-\right)^*$

$$I = \sqrt{\pi T}\left\{\pi + 2\text{ArcCot}\left(\frac{\hat{s}_2}{s_1}\right) + 2\text{ImArcCoth}\left(\frac{\hat{s}_3 s_1\left(\hat{s}_2 - i s_4\right)}{-s_1^2 s_4 + i\hat{s}_3^2\hat{s}_2}\right)\right.$$
$$\left. +2\text{ImArcCoth}\left(\frac{\hat{s}_4 s_1\left(\hat{s}_2 - i s_4\right)}{\left(s_1^2 s_3 + i s_1 s_2 \hat{s}_2\right) + T}\right)\right]$$

Exp part $S_1 \sim S_4$

Introduce scale λ $\quad 1 = \int d\lambda^3\, \delta\left(\lambda^3 - T\right), \qquad s_{1-4} \to \lambda\left(s_{1-4}\right)$

With Ordinary scale behavior $s_i \sim \mathcal{O}(\tau^0), \;\; \tau \to 0$

$$= \frac{\Omega_d^3}{2^3\left(2\pi\right)^{12-6\epsilon}} \int_{s_{1-4}} \text{Exp}\left(-\right)\left\{\frac{s_1 s_2 s_3 s_4 p^2}{T}\right\} T^{-2+\epsilon}\sqrt{\pi T}$$
$$\times\left[\pi + 2\text{ArcCot}\left(\frac{\hat{s}_2}{s_1}\right) + 2\text{ImArcCoth}\left(\frac{\hat{s}_3 s_1\left(\hat{s}_2 - i s_4\right)}{-s_1^2 s_4 + i\hat{s}_3^2\hat{s}_2}\right)\right.$$
$$\left. +2\text{ImArcCoth}\left(\frac{\hat{s}_4 s_1\left(\hat{s}_2 - i s_4\right)}{\left(s_1^2 s_3 + i s_1 s_2 \hat{s}_2\right) + T}\right)\right]$$

$$= \frac{3\Omega_d^3\sqrt{\pi}\,\Gamma\left(2-\epsilon\right)^3\Gamma\left(-\frac{1}{2}+3\epsilon\right)}{2^3\left(2\pi\right)^{12-6\epsilon}\left(p^2\right)^{1/2-3\epsilon}} \int_{s_{1-4}} \frac{1}{\left(s_1 s_2 s_3 s_4\right)^{1/2-3\epsilon}}\delta\left(1 - sss\right)\cdots$$

- We found cycle sum

$$\sum_{\text{permutations}}^{\text{Num}\neq\text{Deno}} \text{ArcCot}\left(\frac{\hat{s}_2}{s_1}\right) = 2\pi$$

$$\sum_{\text{permutations}}^{1,2,3,4} \text{Im}\left\{\text{ArcCoth}\left(\frac{\hat{s}_3 s_1 \left(\hat{s}_2 - i s_4\right)}{-s_1^2 s_4 + i \hat{s}_3^2 \hat{s}_2}\right)\right\} = 8\pi$$

$$\sum_{\text{permutations}}^{1,2,3,4} \text{Im}\left\{\text{ArcCoth}\left(\frac{\hat{s}_4 s_1 \left(\hat{s}_2 - i s_4\right)}{\left(s_1^2 s_3 + i s_1 s_2 \hat{s}_2\right) + 1}\right)\right\} = 8\pi$$

With $1 - s_1 s_2 s_3 - s_1 s_2 s_4 - s_1 s_3 s_4 - s_2 s_3 s_4 = 0$

Use symmetry, finally

$$G^{(o)} = \int_{k q_2 q_3} \frac{1}{k^2 q_2^2 \left(q_2 \cdot u_2\right)\left(\left(k + q_2 - q_3\right) \cdot u_1\right)\left(k + p + q_2 - q_3\right)^2 q_3^2 \left(q_3 \cdot u_2\right)}$$

$$= \frac{8}{3} \frac{1}{(4\pi)^{6-3\epsilon}} \cdot \pi^{3/2} e^{-2i\pi\epsilon} \cdot \frac{\Gamma\left(\frac{1}{2} - \epsilon\right)^4 \Gamma\left(-\frac{1}{2} + 3\epsilon\right)}{\Gamma\left(2 - 4\epsilon\right)} \left(p^2\right)^{\frac{1}{2} - 3\epsilon}$$

- In fact we need extra contribution at $s_i \to \infty$
- With nontrivial singular scale behav

$$s_1 \sim \frac{s_1}{\tau^2}$$

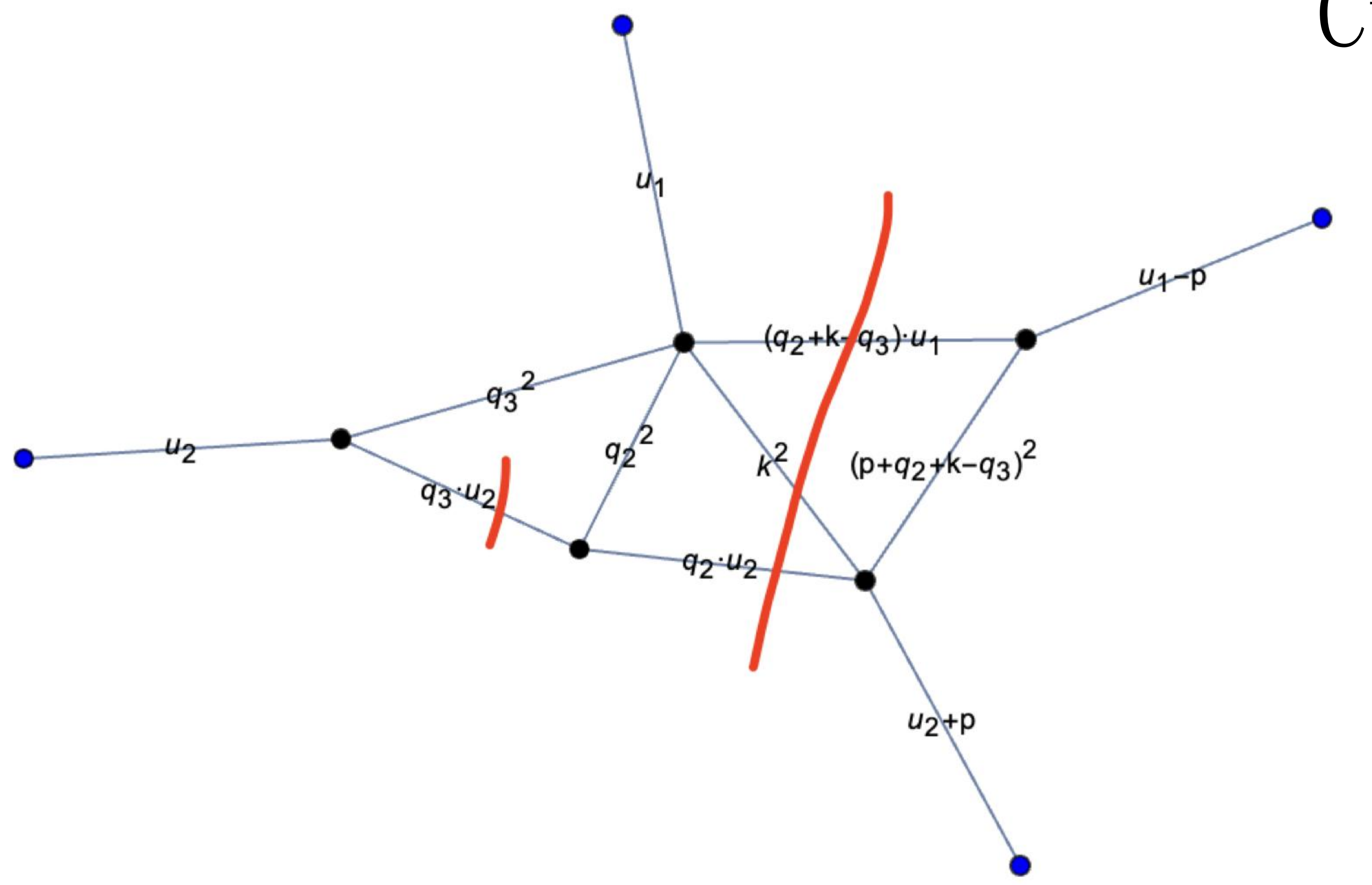
$$s_{2\sim 4} \sim \tau^0 s_{2\sim 4}$$

Finally

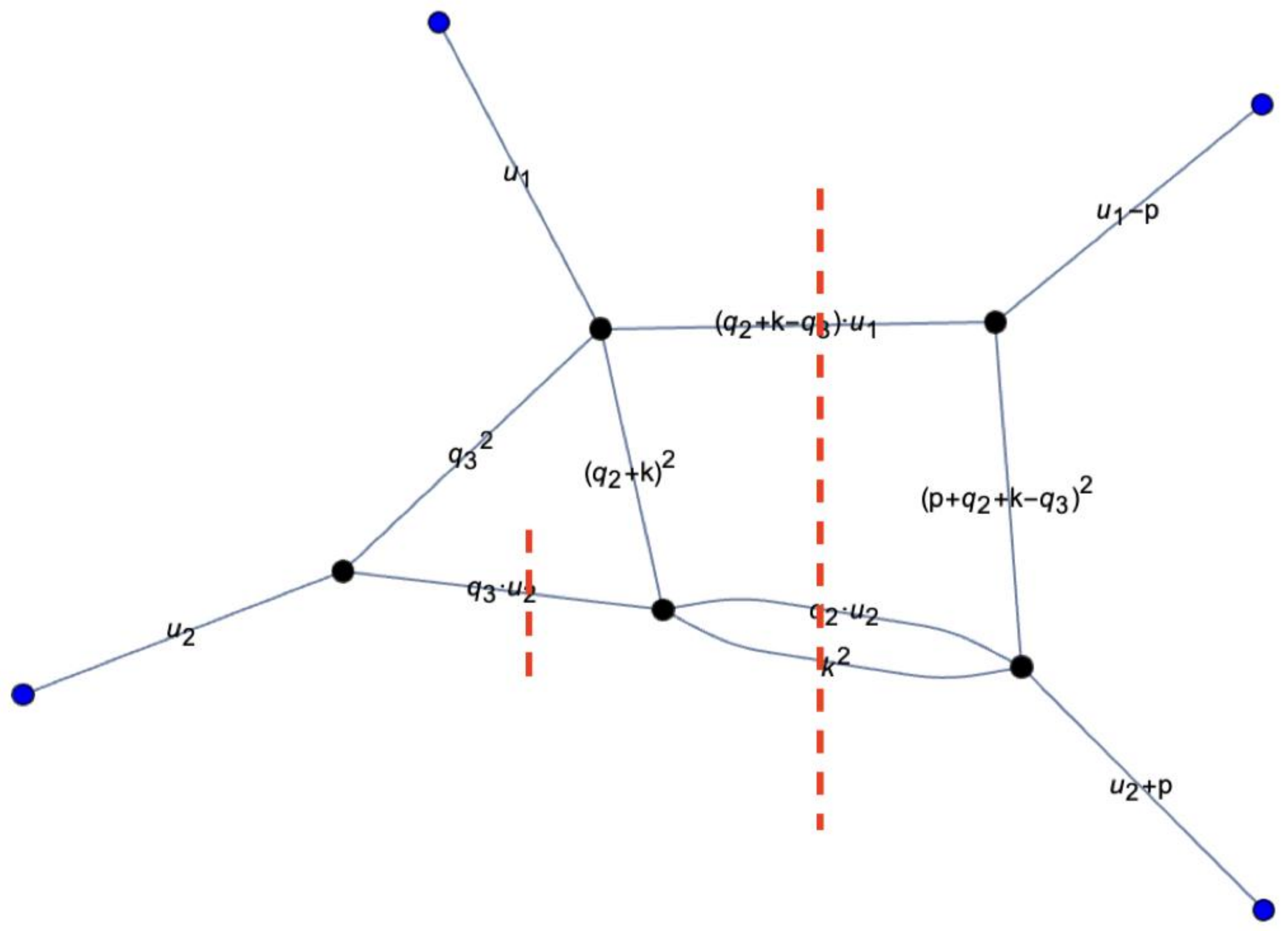
$$G^{(s)}_{k+1,j+1} = \frac{\pi^3 (1-i)\,\Omega_d^3 \Gamma\left(-\frac{1}{2}+j+k+3\epsilon\right)}{2^3 (2\pi)^{12-6\epsilon}} \frac{\left(\mathrm{Csc}(\pi\epsilon) + i\mathrm{Sec}(\pi\epsilon)\right)}{\Gamma\left(\frac{3}{2}-\epsilon\right)\Gamma(\epsilon)} \tau^{1-\epsilon}$$
$$\times \frac{\Gamma\left(\frac{3}{2}-k-3\epsilon\right)\Gamma(1-j-2\epsilon)\Gamma\left(\frac{1}{2}-\epsilon\right)\Gamma\left(\frac{1}{2}-k-\epsilon\right)}{\Gamma\left(\frac{5}{2}-j-k-5\epsilon\right)\Gamma(1-k-2\epsilon)} \left(p^2\right)^{1/2-j-k-3\epsilon}$$

Calculate MI (With Cut)

Cutosky Rule is simple, but …



$$
=\frac{\Omega_{3-2\epsilon}}{(2\pi)^{3-2\epsilon}}\Gamma\left(\frac{3}{2}-\epsilon\right)\Gamma\left(\frac{1}{2}+\epsilon\right)\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)^2}{\Gamma(1-2\epsilon)}\left(p^2\right)^{\frac{1}{2}-3\epsilon}
\frac{4^{-4+3\epsilon}\pi^{-1+2\epsilon}\tau^{1-2\epsilon}\Gamma(1-2\epsilon)}{\Gamma\left(\frac{5}{2}-5\epsilon\right)\Gamma\left(\frac{3}{2}-\epsilon\right)\Gamma(2\epsilon)}\mathrm{Sec}(3\pi\epsilon)\,\mathrm{Csc}(\pi\epsilon)
$$



$$\sim 0$$

- Baikov rep. is not available

$$= \frac{1}{4} + \frac{z1^2}{2} - \frac{z10}{2} - \frac{3\, z1^2\, z10}{4} + \frac{z10^2}{4} + \frac{z1^2\, z10^2}{4} - \frac{z11}{2} - \frac{z1^2\, z11}{2} + \frac{z10\, z11}{2} + \frac{1}{4} z1^2\, z10\, z11 + \frac{z11^2}{4} + \frac{z12}{4} + \frac{z1^2\, z12}{4} - \frac{z10\, z12}{4} - \frac{z11\, z12}{4} + \frac{z12^2}{16} - \frac{z10\, z13}{4} + \frac{z10^2\, z13}{4} + \frac{1}{2} z1^2\, z11\, z13 + \frac{z10\, z11\, z}{4}$$

$$\frac{1}{4} z1^2\, z10\, z11\, z13 - \frac{z12\, z13}{4} - \frac{1}{2} z1^2\, z12\, z13 - \frac{z10\, z12\, z13}{8} + \frac{z11\, z12\, z13}{4} - \frac{z12^2\, z13}{8} + \frac{z1^2\, z13^2}{2} - \frac{1}{4} z1^2\, z10\, z13^2 + \frac{z10^2\, z13^2}{16} + \frac{1}{4} z1^2\, z12\, z13^2 - \frac{1}{8} z10\, z12\, z13^2 + \frac{z12^2\, z13^2}{16} - \frac{z14}{4} -$$

$$\frac{z1^2\, z14}{4} + \frac{z10\, z14}{2} + \frac{1}{4} z1^2\, z10\, z14 - \frac{z10^2\, z14}{4} + \frac{z11\, z14}{2} + \frac{1}{4} z1^2\, z11\, z14 - \frac{z10\, z11\, z14}{2} - \frac{z11^2\, z14}{4} - \frac{z12\, z14}{8} + \frac{z10\, z12\, z14}{8} + \frac{z11\, z12\, z14}{8} - \frac{1}{2} z1^2\, z13\, z14 + \frac{z10\, z13\, z14}{8} + \frac{1}{4} z1^2\, z10\, z13\, z14 -$$

$$\frac{1}{8} z10^2\, z13\, z14 - \frac{1}{4} z1^2\, z11\, z13\, z14 - \frac{1}{8} z10\, z11\, z13\, z14 + \frac{z12\, z13\, z14}{8} + \frac{1}{8} z10\, z12\, z13\, z14 - \frac{1}{8} z11\, z12\, z13\, z14 - \frac{1}{4} z1^2\, z13^2\, z14 + \frac{z14^2}{16} - \frac{z10\, z14^2}{8} + \frac{z10^2\, z14^2}{16} - \frac{z11\, z14^2}{8} + \frac{1}{8} z10\, z11\, z14^2 +$$

$$\frac{z11^2\, z14^2}{16} + \frac{1}{4} z1^2\, z13\, z14^2 - \frac{z15}{2} - \frac{3\, z1^2\, z15}{4} + \frac{3\, z10\, z15}{4} + \frac{1}{2} z1^2\, z10\, z15 - \frac{z10^2\, z15}{4} + \frac{3\, z11\, z15}{4} + \frac{1}{4} z1^2\, z11\, z15 - \frac{z10\, z11\, z15}{2} - \frac{z11^2\, z15}{4} - \frac{z12\, z15}{4} + \frac{z10\, z12\, z15}{4} + \frac{z11\, z12\, z15}{8} + \frac{z13\, z15}{4}$$

$$\frac{z11\, z13\, z15}{4} - \frac{1}{4} z1^2\, z11\, z13\, z15 + \frac{1}{8} z10\, z11\, z13\, z15 + \frac{3\, z12\, z13\, z15}{8} - \frac{1}{8} z11\, z12\, z13\, z15 - \frac{1}{4} z1^2\, z13^2\, z15 + \frac{1}{8} z10\, z13^2\, z15 - \frac{1}{8} z12\, z13^2\, z15 + \frac{z14\, z15}{4} + \frac{1}{4} z1^2\, z14\, z15 - \frac{z10\, z14\, z15}{4} -$$

$$\frac{3\, z11\, z14\, z15}{8} + \frac{1}{8} z10\, z11\, z14\, z15 + \frac{1}{8} z11^2\, z14\, z15 - \frac{z13\, z14\, z15}{8} + \frac{1}{4} z1^2\, z13\, z14\, z15 - \frac{1}{8} z10\, z13\, z14\, z15 + \frac{1}{8} z11\, z13\, z14\, z15 + \frac{z15^2}{4} + \frac{z1^2\, z15^2}{4} - \frac{z10\, z15^2}{4} - \frac{z11\, z15^2}{4} + \frac{z11^2\, z15^2}{16} -$$

$$\frac{z13\, z15^2}{4} + \frac{1}{8} z11\, z13\, z15^2 + \frac{z13^2\, z15^2}{16} - \frac{z1^2\, z2^2}{4} + \frac{z13\, z2^2}{2} + \frac{1}{2} z1^2\, z13\, z2^2 + \frac{z13^2\, z2^2}{2} - \frac{1}{4} z1^2\, z13^2\, z2^2 - \frac{3}{4} z13\, z14\, z2^2 - \frac{1}{4} z13^2\, z14\, z2^2 + \frac{1}{4} z13\, z14^2\, z2^2 - \frac{z15\, z2^2}{4} - \frac{3}{4} z13\, z15\, z2^2 +$$

$$\frac{1}{4} z14\, z15\, z2^2 + \frac{1}{4} z13\, z14\, z15\, z2^2 + \frac{z15^2\, z2^2}{4} + \frac{z1\, z3}{2} - \frac{z1\, z10\, z3}{2} - \frac{1}{2} z1\, z10\, z11\, z3 - \frac{1}{2} z1\, z11^2\, z3 + \frac{z1\, z12\, z3}{4} + \frac{1}{2} z1\, z10\, z12\, z3 + \frac{1}{4} z1\, z11\, z12\, z3 + \frac{z1\, z13\, z3}{2} - \frac{3}{4} z1\, z10\, z13\, z3 -$$

$$\frac{1}{2} z1\, z11\, z13\, z3 - \frac{1}{4} z1\, z10\, z11\, z13\, z3 + \frac{1}{2} z1\, z12\, z13\, z3 - \frac{1}{4} z1\, z11\, z12\, z13\, z3 - \frac{1}{4} z1\, z10\, z13^2\, z3 + \frac{1}{4} z1\, z12\, z13^2\, z3 - \frac{3\, z1\, z14\, z3}{4} + \frac{3}{4} z1\, z10\, z14\, z3 + \frac{1}{2} z1\, z11\, z14\, z3 + \frac{1}{4} z1\, z10\, z11\, z14\, z3 +$$

$$\frac{1}{4} z1\, z11^2\, z14\, z3 - \frac{1}{4} z1\, z12\, z14\, z3 - \frac{1}{4} z1\, z13\, z14\, z3 + \frac{1}{2} z1\, z10\, z13\, z14\, z3 + \frac{1}{4} z1\, z11\, z13\, z14\, z3 - \frac{1}{4} z1\, z12\, z13\, z14\, z3 + \frac{1}{4} z1\, z14^2\, z3 - \frac{1}{4} z1\, z10\, z14^2\, z3 - \frac{1}{4} z1\, z11\, z14^2\, z3 - z1\, z15\, z3 +$$

$$\frac{1}{2} z1\, z10\, z15\, z3 + \frac{1}{4} z1\, z11\, z15\, z3 + \frac{1}{4} z1\, z11^2\, z15\, z3 - \frac{1}{2} z1\, z12\, z15\, z3 - \frac{1}{4} z1\, z13\, z15\, z3 - \frac{1}{4} z1\, z13^2\, z15\, z3 + \frac{1}{2} z1\, z14\, z15\, z3 - \frac{1}{4} z1\, z11\, z14\, z15\, z3 + \frac{1}{4} z1\, z13\, z14\, z15\, z3 + \frac{1}{2} z1\, z15^2\, z3 -$$

$$z1\, z2^2\, z3 - \frac{1}{2} z1\, z13\, z2^2\, z3 - \frac{1}{2} z1\, z13^2\, z2^2\, z3 + \frac{1}{2} z1\, z14\, z2^2\, z3 + \frac{1}{2} z1\, z13\, z14\, z2^2\, z3 + z1\, z15\, z2^2\, z3 + \frac{z12\, z3^2}{2} - \frac{1}{2} z11\, z12\, z3^2 + \frac{z12^2\, z3^2}{4} + \frac{1}{4} z12\, z13\, z3^2 - \frac{1}{4} z11\, z12\, z13\, z3^2 -$$

$$\frac{1}{4} z12\, z14\, z3^2 + \frac{1}{4} z11\, z12\, z14\, z3^2 - \frac{z15\, z3^2}{4} + \frac{1}{4} z11^2\, z15\, z3^2 - \frac{1}{2} z12\, z15\, z3^2 - \frac{1}{4} z13\, z15\, z3^2 + \frac{1}{4} z11\, z13\, z15\, z3^2 + \frac{1}{4} z14\, z15\, z3^2 - \frac{1}{4} z11\, z14\, z15\, z3^2 + \frac{z15^2\, z3^2}{4} - z2^2\, z3^2 - z13\, z2^2\, z3^2 -$$

$$\frac{1}{4} z13^2\, z2^2\, z3^2 + z14\, z2^2\, z3^2 + \frac{1}{2} z13\, z14\, z2^2\, z3^2 - \frac{1}{4} z14^2\, z2^2\, z3^2 + z15\, z2^2\, z3^2 + \frac{z8}{4} + \frac{z1^2\, z8}{4} - \frac{z10^2\, z8}{4} - \frac{z10\, z11\, z8}{2} + \frac{1}{4} z1^2\, z10\, z11\, z8 - \frac{z11^2\, z8}{4} + \frac{1}{4} z1^2\, z11^2\, z8 - \frac{z12\, z8}{8} -$$

$$\frac{1}{2} z1^2\, z12\, z8 + \frac{3\, z10\, z12\, z8}{8} + \frac{3\, z11\, z12\, z8}{8} - \frac{z12^2\, z8}{8} + \frac{z13\, z8}{4} - \frac{1}{2} z1^2\, z13\, z8 - \frac{3\, z10\, z13\, z8}{8} + \frac{1}{2} z1^2\, z10\, z13\, z8 - \frac{1}{8} z10^2\, z13\, z8 - \frac{z11\, z13\, z8}{4} + \frac{1}{2} z1^2\, z11\, z13\, z8 - \frac{1}{8} z10\, z11\, z13\, z8 -$$

$$\frac{1}{2} z1^2\, z12\, z13\, z8 + \frac{1}{4} z10\, z12\, z13\, z8 + \frac{1}{8} z11\, z12\, z13\, z8 - \frac{1}{8} z12^2\, z13\, z8 + \frac{1}{4} z1^2\, z13^2\, z8 - \frac{1}{8} z10\, z13^2\, z8 + \frac{1}{8} z12\, z13^2\, z8 - \frac{3\, z14\, z8}{8} + \frac{z10\, z14\, z8}{4} - \frac{1}{4} z1^2\, z10\, z14\, z8 + \frac{1}{8} z10^2\, z14\, z8 +$$

$$\frac{z11\, z14\, z8}{4} - \frac{1}{4} z1^2\, z11\, z14\, z8 + \frac{1}{4} z10\, z11\, z14\, z8 + \frac{1}{8} z11^2\, z14\, z8 - \frac{1}{8} z10\, z12\, z14\, z8 - \frac{1}{8} z11\, z12\, z14\, z8 - \frac{z13\, z14\, z8}{8} + \frac{1}{4} z10\, z13\, z14\, z8 + \frac{1}{8} z11\, z13\, z14\, z8 - \frac{1}{8} z12\, z13\, z14\, z8 + \frac{z14^2\, z8}{8} -$$

Solve ODE

$$\boldsymbol{I}' = \left(p^2\right)^{-3\epsilon} \sum_n \epsilon^n \boldsymbol{I}'^{(\epsilon^n)}$$

$$\boldsymbol{I}'^{(\epsilon^n)}\left(x\right) = \int \epsilon A\left(x'\right) \boldsymbol{I}'^{(\epsilon^{n-1})}\left(x'\right) dx' + \boldsymbol{c}^{(\epsilon^n)}$$

$$\boldsymbol{I}'^{(\epsilon^n)}\left(x\right)|_{x\to 1} = \boldsymbol{I}'^{(\epsilon^n)}_{\text{boundary}}$$

$$I_1^{\prime(\epsilon^0)}{=}0, I_2^{\prime(\epsilon^0)}{=}-\frac{-23+\gamma_E-3\log\pi+2\log(1-x)-78\log x+2\log(1+x)}{6480\pi^3}$$

$$I_3^{\prime(\epsilon^0)}{=}\frac{2\left(-23+\gamma_E-3\log\pi+2\log(1-x)+3\log x+2\log(1+x)\right)}{945\pi^3}, I_{4\sim6}^{\prime(\epsilon^0)}{=}0$$

$$I_1^{\prime(\epsilon)}=-\frac{\pi^2+12\log x\log(1+x)+12\mathrm{Li}_2(1-x)+12\mathrm{Li}_2(-x)}{54\pi^2},$$

$$I_2^{\prime(\epsilon)}=\frac{1}{25920\pi^3}\Big[-16\mathrm{Li}_2\left(\frac{1}{2}-\frac{x}{2}\right)-3936\mathrm{Li}_2(-x)+7584\mathrm{Li}_2(x)-16\mathrm{Li}_2\left(\frac{x+1}{2}\right)$$

$$-184\log(x+1)-1595\pi^2+1328+18\log^2(\pi)-16\log^2(2)+8\log^2(x+1)+2\gamma$$

$$+8\left(\log(x+1)(\gamma_E-570\log(x)-3\log(\pi)+\log(4))\right)$$

$$-92\gamma_E-12(\gamma_E-23)\log(\pi)+8\log^2(1-x)+8\log(1-x)(\gamma_E+870\log(x))$$

$$-8\cdot39\log(x)(\gamma_E+\log(x)-23-3\log(\pi))$$

$$+8\log(1-x)(\gamma_E+870\log(x)-23-3\log(\pi)+\log(4))\Big]$$

$$I_3^{\prime(\epsilon)}=-\frac{1}{1890\pi^3}\Big[12\mathrm{Li}_2\left(x^2\right)-16\mathrm{Li}_2\left(\frac{1}{2}-\frac{x}{2}\right)-288\mathrm{Li}_2(x)-16\mathrm{Li}_2\left(\frac{x+1}{2}\right)$$

$$+8\log(1-x)\left(\gamma_E-30\log(x)-23+\log\left(\frac{4}{\pi^3}\right)\right)+2{\gamma_E}^2$$

$$+12\log(x)(\gamma_E+4\log(x+1)-23-3\log(\pi))$$

$$+8\log^2(1-x)+12\log^2(x)+43\pi^2+1328+18\log^2(\pi)-16\log^2(2)$$

$$+8\log(x+1)\left(\gamma_E+\log\left(\frac{4(x+1)}{\pi^3}\right)-23\right)-92\gamma_E-12(\gamma_E-23)\log(\pi)\Big]$$

$$I_{4\sim6}^{\prime(\epsilon)}=0$$

$$I_1^{\prime(1/\epsilon)}=0,\ I_2^{\prime(1/\epsilon)}=\frac{1}{6480\pi^3},\ I_3^{\prime(1/\epsilon)}=-\frac{2}{945\pi^3},\ I_{4\sim6}^{\prime(1/\epsilon)}=0$$

Result

$$P^{\mu}_{\mathrm{C1}} = \frac{u_1^{\mu} + u_2^{\mu}}{1+\gamma} \frac{4068 G^4 \pi^3 m_2^3 m_1}{|\boldsymbol{b}|^5} \left(c_E F_{3E} + c_B F_{3B}\right)$$

$$F_{3E} = -\frac{45197629\,\gamma}{3853516800\,\pi^3 x^2 (-1+\gamma^2)^4} + \frac{17579\,\gamma_E\,\gamma}{22020096\,\pi^3 x^2 (-1+\gamma^2)^4} + \frac{17579\,\gamma^2}{66060288\,\pi\, x^2 (-1+\gamma^2)^4} - \frac{535385773\,\gamma^3}{1926758400\,\pi^3 x^2 (-1+\gamma^2)^4} + \frac{2937499\,\gamma_E\,\gamma^3}{165150720\,\pi^3 x^2 (-1+\gamma^2)^4} + \cdots 1272 \cdots + \frac{2132477\,\gamma^7\,\mathrm{Li}_2[-1+2x\gamma]}{1486356480\,\pi^3 x^2 (-1+\gamma^2)^3} + \frac{1075073\,\gamma^9\,\mathrm{Li}_2[-1+2x\gamma]}{8918138880\,\pi^3 x^2 (-1+\gamma^2)^3} - \frac{32633\,\gamma^{11}\,\mathrm{Li}_2[-1+2x\gamma]}{8918138880\,\pi^3 x^2 (-1+\gamma^2)^3} + \frac{\gamma^{13}\,\mathrm{Li}_2[-1+2x\gamma]}{26542080\,\pi^3 x^2 (-1+\gamma^2)^3}$$

$$F_{3B} = -\frac{69953921\,\gamma}{1101004800\,\pi^3 x^2 (-1+\gamma^2)^4} + \frac{44403\,\gamma_E\,\gamma}{10485760\,\pi^3 x^2 (-1+\gamma^2)^4} + \frac{14801\,\gamma^2}{10485760\,\pi\, x^2 (-1+\gamma^2)^4} - \frac{44165850223\,\gamma^3}{69363302400\,\pi^3 x^2 (-1+\gamma^2)^4} + \frac{5185993\,\gamma_E\,\gamma^3}{132120576\,\pi^3 x^2 (-1+\gamma^2)^4} + \cdots 1196 \cdots + \frac{375437\,\gamma^7\,\mathrm{Li}_2[-1+2x\gamma]}{297271296\,\pi^3 x^2 (-1+\gamma^2)^3} + \frac{1369619\,\gamma^9\,\mathrm{Li}_2[-1+2x\gamma]}{17836277760\,\pi^3 x^2 (-1+\gamma^2)^3} - \frac{17011\,\gamma^{11}\,\mathrm{Li}_2[-1+2x\gamma]}{8918138880\,\pi^3 x^2 (-1+\gamma^2)^3} + \frac{11\,\gamma^{13}\,\mathrm{Li}_2[-1+2x\gamma]}{283115520\,\pi^3 x^2 (-1+\gamma^2)^3}$$

Summary

- We introduce the GW EFT based method
- We calculate the Integral with no cut at 3-loop about GW radiation NNLO correction
- The cut integral is so difficult, we try some naive method but result is not UT.
- More efficient cuts calculation method need to be studied.

Thanks !