Multi-loop Feynman Integrals

Lecture I: Basics of Feynman Integrals

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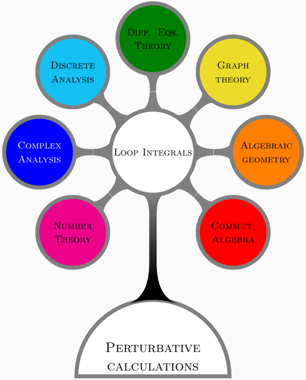
- High-precision theoretical description of Standard Model processes is of crucial importance. In particular, the New Physics — new particles and interactions — is likely to appear as small deviations from SM and therefore can be detected only with high precision of theoretical predictions at hand.
- From the computational point of view, our ability to obtain high-precision results depends crucially on multiloop calculation techniques. Complexity grows both qualitatively and quantitatively in an explosive way with the number of loops and/or scales.
- Besides these practical purposes, multiloop calculations provide a perfect polygon for trying the methods from various mathematical fields: differential equations, complex analysis, number theory, algebraic geometry etc.



Example: $g-2$ at one loop [Schwinger, 1948]

- Electron scattering in electromagnetic field is described by two form factors $F_{1,2}$:

$$j_\mu = \overline{u}(p')\left[\gamma_\mu F_1(q^2) - \frac{\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2)\right]u(p), \quad p' = p + q, \quad \sigma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\,.$$

- $F_1(0) = 1$ and it can be shown that $F_2(0) = \frac{1}{2}(g-2)$ is the **anomalous magnetic moment (AMM)**.
- NB: To calculate AMM it is sufficient to expand j_μ up to linear in q terms.
- In the leading approximation $j_\mu^{(0)} = \overline{u}(p')\gamma^\mu u(p) \implies F_1^{(0)} = 1, F_2^{(0)} = 0$.
- In the next-to-leading (NLO) approximation we have

$$j_\mu^{(1)} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\overline{u}(p')\gamma^\nu(\hat{p}' - \hat{k} + m)\gamma_\mu(\hat{p} - \hat{k} + m)\gamma_\nu u(p)}{[(p'-k)^2 - m^2 + i0]\,[k^2 + i0]\,[(p-k)^2 - m^2 + i0]}$$

This expression is already somewhat complicated, but we still can treat it manually if we use Feynman parametrization.

Feynman parametrization

$$j_\mu^{(1)} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\overline{u}(p')\gamma^\nu(\hat{p}' - \hat{k} + m)\gamma_\mu(\hat{p} - \hat{k} + m)\gamma_\nu u(p)}{[(p'-k)^2 - m^2 + i0]\,[k^2 + i0]\,[(p-k)^2 - m^2 + i0]}$$

We use FP to write (here and below $\bar{x} = 1 - x,\ \bar{z} = 1 - z$,)

$$\frac{1}{[(p'-k)^2-m^2+i0][k^2+i0][(p-k)^2-m^2+i0]} = 2\int_0^1\int_0^1 \frac{dxdz}{[k^2 - 2k\cdot(zp + \bar{z}p')x + i0]^3},$$

and make a shift $k \to k + (zp + \bar{z}p')x$. After some γ-matrix algebra we get

$$j_\mu^{(1)} = -2ie^2 \int_0^1\int_0^1 dxdz \int \frac{d^4k}{(2\pi)^4\left[-k^2 + (zp+\bar{z}p')^2 x^2 - i0\right]^3}$$
$$\times\, \overline{u}(p')\left\{\gamma_\mu\left[2x^2\left(m^2 + z\bar{z}q^2\right) - 2\bar{x}\left(2m^2 - q^2\right) - k^2\right] - \frac{\sigma_{\mu\nu}q^\nu}{2m}\left(4x\bar{x}m^2\right)\right\}u(p)$$

The highlighted parts contribute to $F_1^{(1)}$ and $F_2^{(1)}$, respectively.

Performing Wick rotation $k_0 \to i\tilde{k}_0$ and taking the integrals we obtain a celebrated result:

[Schwinger, 1948]

$$F_2^{(1)}(0) = 2\pi^2 e^2 \int_0^1 \int_0^1 dx x dz \int \frac{d\tilde{k}^2 \tilde{k}^2 \left(4x\bar{x}m^2\right)}{(2\pi)^4 \left[\tilde{k}^2 + m^2x^2 - i0\right]^3} = \frac{\alpha}{2\pi},$$

where $\alpha = e^2/4\pi \approx 1/137$ is a fine structure constant.

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Note that already in one loop we will encounter problems when calculating

$$F_1^{(1)}(q^2) = 2e^2 \int_0^1 \int_0^1 dx x dz \int \frac{d^4\tilde{k}}{(2\pi)^4} \frac{\tilde{k}^2 - 2\bar{x}\left(2m^2 - q^2\right) + 2x^2\left(m^2 + z\bar{z}q^2\right)}{\left[\tilde{k}^2 + (zp + \bar{z}p')^2 x^2 - i0\right]^3}.$$

F_1 diverges both at large (UV) and at small (IR) $\tilde{k}^2$ (the latter comes from small x region).

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$$F_1^{(1)}(q^2) \quad = \quad 2e^2 \int_0^1 \int_0^1 dx x dz \int \frac{d^d\tilde{k}}{(2\pi)^d} \frac{\tilde{k}^2 - 2\bar{x}\left(2m^2 - q^2\right) + 2x^2\left(m^2 + z\bar{z}q^2\right)}{\left[\tilde{k}^2 + (zp + \bar{z}p')^2 x^2 - i0\right]^3}.$$

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Both UV and IR divergencies are regularized within *dimensional regularization* $d = 4 - 2\epsilon$.

Next corrections

At two loops the calculations of $g-2$ get much more involved [Sommerfield, 1957]. Starting from 3 loops it is practically impossible to do calculations by hand. Current world records for analytical $g-2$ is 4 loops [Laporta, 2017] (earlier calculated numerically with impressive efforts by Kinoshita and collaborators).

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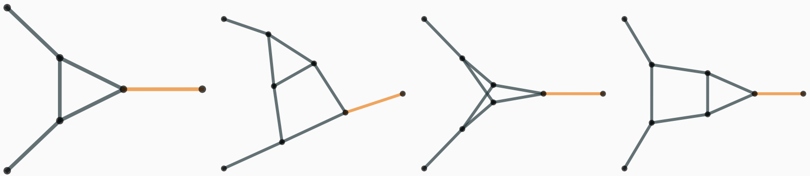
Incomplete list of modern multi-loop methods and tools

- Parametric representations
  - alpha- (or Feynman) representation (also LP repr.).
  - Baikov representation.
  - Mellin-Barnes representation.
- Expansion by regions.
  - In momentum representation
  - In Feynman representation.
- IBP reduction.
  - In momentum representation.
  - In parametric representations.
- Differential equations.
  - Reduction to ϵ-form.
  - Frobenius expansion near singular point.
  - Using ϵ-regular basis.
- Recurrence relations
  - with respect to dimensionality d.
  - with respect to powers of denominators.

2 loops:



- Dispersion relation
- Feynman parametrization

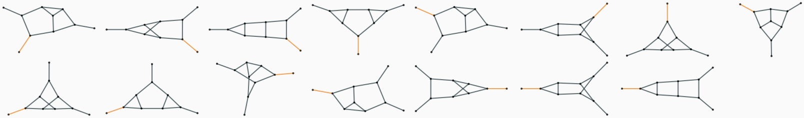
} [Matsuura, van der Marck, and van Neerven, 1989; Harlander, 2000]

- Mellin-Barnes parametrization
- ${}_pF_q$ expansion in indices, `HypExp`

} [Gehrmann, Huber, and Maitre, 2005]

3 loops:

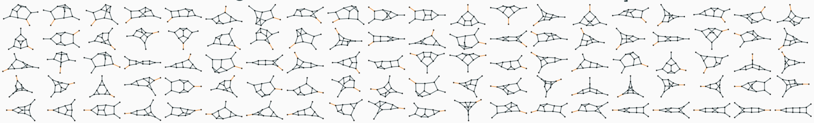
[Gehrmann, Heinrich, Huber, and Studerus, 2006; Heinrich, Huber, and Maître, 2008; RL, Smirnov, and Smirnov, 2010]



- Feynman parametrization
- Mellin-Barnes parametrization, MB, AMBRE [Czakon, 2006; Gluza et al., 2007]
- Recurrence+analyticity in *d*, [Tarasov, 1996; RL, 2010]
- PSLQ recognition [Ferguson et al., 1998]

4 loops:

[Henn, Smirnov, Smirnov, and Steinhauser, 2016; RL, Smirnov, Smirnov, and Steinhauser, 2019; RL, von Manteuffel, Schabinger, Smirnov, Smirnov, and Steinhauser, 2021b]



- $\sim$ 100 big topologies.
- Linear reducibility, `HyperInt` [Panzer, 2013]
- Parallelization for IBP reduction,
  finite fields reconstruction [von Manteuffel and Schabinger, 2015; Smirnov and Chuharev, 2020]
- Differential equations, reduction to ϵ-form [Henn, 2013; RL, 2015], `Libra` [RL, 2021]
- PSLQ recognition

5 loops:

- ~ 1000 big topologies.
- It looks like no available techniques can help.

- Massless form factors represent a traditional topic of the multiloop calculations where the “world records” are fixed. But from the experimental point of view less loops and more scales are more important.
- In particular, only very recently multiloop methods have grown to **NNLO differential cross section** calculations of $2 \to 2$ processes with massive particles. NNLO corrections to differential cross sections are not even known for basic QED process: $e^+e^- \to \gamma\gamma$, $e^+e^- \to \mu^+\mu^-$, etc. Partial results start to appear [Duhr, Smirnov, and Tancredi, 2021; Banerjee et al., 2020].
- The complexity of NNLO calculations with massive internal lines is connected with appearance of **non-polylogarithmic integrals**. Effective approach to the calculation of such integrals is, probably, the most hot topic in multiloop calculations.

Complexity crucially depends on # of loops L and on # of scales S.

S \ L	1 loop	2 loops	3 loops	4 loops	5 loops	>5
1	✓	✓	✓	✓	a few	
2	✓	✓	some	a few		
3	✓	some	a few			
>3	✓	a few				

The following empirical "formula" describes the complexity of calculations: Complexity $= L + S + \delta_m$, where $\delta_m = 1$ ($\delta_m = 0$) for diagrams with/without massive internal lines.

- 5-loop massless propagators [Georgoudis, Gonçalves, Panzer, Pereira, Smirnov, and Smirnov, 2021].
- 4-loop $g-2$ integrals (onshell massive propagators) [Laporta, 2017]
- 4-loop form factors [RL, von Manteuffel, Schabinger, Smirnov, Smirnov, and Steinhauser, 2022]
- 3-loop massless boxes [Henn, Mistlberger, Smirnov, and Wasser, 2020]
- 2-loop 5 legs [Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, and Zoia, 2019]

- 3-loop massive form factors: numerical calculation [Fael, Lange, Schönwald, and Steinhauser, 2022].
- Partial results for 2-loop boxes with inner massive lines [Duhr, Smirnov, and Tancredi, 2021].
- Partial results for 3-loop boxes with one off-shell leg [Henn, Lim, and Torres Bobadilla, 2023].

1. Diagram generation ✓

Generate diagrams contributing to the chosen order of perturbation theory.

Tools: `qgraf` [Nogueira, 1993], `FeynArts` [Hahn, 2001], `tapir` [Gerlach et al., 2022],. . .

2. IBP reduction

Setup IBP reduction, derive differential system for master integrals.

Tools: `FIRE6` [Smirnov and Chuharev, 2020], `Kira2` [Klappert et al., 2021], `LiteRed` [RL, 2012], . . .

3. DE Solution

Reduce the system to ϵ-form, write down solution in terms of polylogarithms. Fix boundary conditions by auxiliary methods.

Tools: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], `Libra` [RL, 2021]

NB: 3rd step is not always doable.

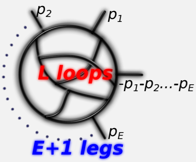
IBP reduction

IBP identities [Chetyrkin and Tkachov, 1981]

Given a Feynman diagram, consider a family

$$j(\boldsymbol{n}) = j(n_1, \ldots, n_N) = \int d\mu_L \boldsymbol{D}^{-\boldsymbol{n}} = \int \prod_{i=1}^{L} d^d l_i \prod_{k=1}^{N} D_k^{-n_k},$$

$l_1, \ldots l_L$ –loop momenta, $p_1, \ldots p_E$ — external momenta.



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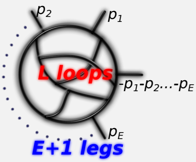


There are $N = L(L+1)/2 + L\cdot E$ scalar products involving loop momenta:

$$s_{ij} = l_i \cdot q_j, \qquad q_j = \begin{cases} l_j & j \leqslant L \\ p_{j-L} & j > L \end{cases} \qquad (1 \leqslant i \leqslant L, i \leqslant j \leqslant L+E)$$

$D_1,\ldots,D_M$ — denominators of the diagram, $D_{M+1},\ldots,D_N$ — irreducible numerators, such that $D_1,\ldots,D_N$ form a basis, i.e. any scalar product can be uniquely expressed via linear function of D_k.

IBP identities

In dim. reg. integral of divergence is zero (no surface terms):

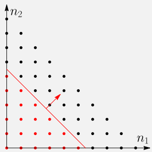
$$0 = \int d\mu_L \frac{\partial}{\partial l_i} \cdot q_j \boldsymbol{D}^{-\boldsymbol{n}} = \sum_s c_s(\boldsymbol{n}) j(\boldsymbol{n}+\delta_s).$$

Explicitly differentiating, we obtain relations between integrals.

IBP reduction

Laporta algorithm (FIRE, Kira, Reduze, . . .)

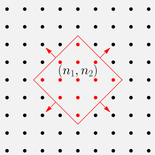
- generate identities for many numeric $\boldsymbol{n} \in \mathbb{Z}^N$.
- use Gauss elimination and collect reduction rules to database.
- twist: mapping to finite fields $\mathbb{F}_p$ + reconstruction. $\Longleftarrow$ naturally parallelizable



Heuristic search (LiteRed)

1. Generate identities for shifts around $\boldsymbol{n}$ with *symbolic* entries.
2. Use Gauss elimination until acceptable rule is found.
3. **Solve Diophantine equations to derive applicability condition.**

Demo: `Examples/LiteRed/example1.nb`



Operator representation of IBP identities

Operators A and B

$$A_k f(n_1, \ldots, n_k, \ldots, n_N) = n_k f(n_1, \ldots, n_k + 1, \ldots, n_N)\,,$$
$$B_k f(n_1, \ldots, n_k, \ldots, n_N) = f(n_1, \ldots, n_k - 1, \ldots, n_N)\,,$$

It is easy to check that $[A_k, B_m] = \delta_{km}$, i.e., these operators[a] implement (a representation of) N-th Weyl algebra $\mathbb{A}_N$.

[a] NB: these notations imply that operators act on function rather than on its value. So $A_k f = \tilde{f}$, such that $\tilde{f}(\ldots, n_k, \ldots) = n_k f(\ldots, n_k + 1, \ldots)$. Thus we will sometimes use braces, like in $(A_k f)(\boldsymbol{n})$.

Operators A and B

$$A_k f(n_1,\dots,n_k,\dots,n_N) = n_k f(n_1,\dots,n_k+1,\dots,n_N)\,,$$
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Using linearity of D_k in s_{ij} and completeness, we can write $q_j \cdot \frac{\partial}{\partial l_i} D_k = c_{km}^{(ij)} D_m + c_k^{(ij)}$, where $c_{km}^{(ij)}$ and $c_k^{(ij)}$ are some coefficients independent of loop momenta. Then IBP identities can be written as

IBP identities in terms of A and B operators

$$\int d\mu_L \underbrace{\frac{\partial}{\partial l_i}}_{\mathcal{O}^{(ij)}} \cdot q_j \boldsymbol{D}^{-\boldsymbol{n}} = -\underbrace{\left[c_{km}^{(ij)} A_k B_m + c_k^{(ij)} A_k - d\delta_{ij}\right]}_{\mathcal{P}^{(ij)}(A,B)} j(\boldsymbol{n}) = 0.$$

NB: by construction $c_{km}^{(ij)}$ and $c_k^{(ij)}$ are independent of $\boldsymbol{n}$ and d.

Feynman and Baikov representations

To derive Feynman representation for the integral

$$j(n_1 \ldots, n_M) = \int \frac{\prod_{i=1}^{L} d^d l_i}{\pi^{Ld/2} \prod_{k=1}^{M} D_k^{n_k}}$$

we use exponential parametrization $D_k^{-n_k} = \int_0^\infty \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} e^{-z_k D_k}$ to obtain

$$j(n_1 \ldots, n_M) = \int \frac{\prod_{i=1}^{L} d^d l_i}{\pi^{Ld/2}} \int\limits_{\mathbb{R}_+^M} \prod_{k=1}^{M} \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} e^{-\Sigma_{k=1}^{M} z_k D_k}$$

Since D_k are linear functions of $l_i \cdot l_j$ and $l_i \cdot p_j$, we can represent

$$\Sigma_k z_k D_k = a_{ij} l_i \cdot l_j + 2 b_i \cdot l_i + c,$$

where a, b, c are linear combinations of z_k. Taking the integrals over l_i, we obtain

$$j(n_1 \ldots, n_M) = \int\limits_{\mathbb{R}_+^M} \prod_{k=1}^{M} \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} U(z)^{-d/2} e^{-F(z)/U(z)},$$

where $U = \det a$ and $F = [c - (a^{-1})_{ij} b_i \cdot b_j] U$ are Symanzik polynomials.

Note that both U and F are homogeneous polynomials of z_k of degree L and $L+1$, respectively.

Now we insert $1 = \int_0^\infty ds \delta\left(s - \widetilde{\Sigma}_k z_k\right)$, where $\widetilde{\Sigma}_k$ denotes any nonempty partial sum (i.e., k runs over any nonempty subset of $\{1, \ldots, M\}$). After rescaling $z_k \to s z_k$ for **all** k, we pull out s from the argument of δ-function and then take the integral over s. We obtain

Feynman representation (aka alpha-representation, parametric representation)

$$j(n_1 \ldots, n_M) = \Gamma[\Sigma_k n_k - L\tfrac{d}{2}] \int\limits_{\mathbb{R}_+^M} \prod_{k=1}^{M} \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} \frac{F^{Ld/2-\Sigma_k n_k}}{U^{(L+1)d/2-\Sigma_k n_k}} \delta\left(1 - \widetilde{\Sigma}_k z_k\right)$$

A modified representation has been suggested in [RL and Pomeransky, 2013]:

Lee-Pomeransky representation

$$j(n_1 \ldots, n_M) = \frac{\Gamma(d/2)}{\Gamma[(L+1)d/2 - \Sigma_k n_k]} \int\limits_{\mathbb{R}_+^M} \prod_{k=1}^{M} \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} G^{-d/2}, \quad G = U + F$$

To prove equivalence, insert $1 = \int_0^\infty ds \delta\left(s - \widetilde{\Sigma}_k z_k\right)$, rescale $z_k \to s z_k$ and take integral $\int ds\, s^{\Sigma_k n_k - Ld/2 - 1}(U + sF)^{-d/2}$.

The integrand in the loop integral depends on loop momenta via scalar products $s_{i,j} = l_i \cdot q_j$. Therefore, we may think of "integrating out" other integration variables. Indeed, it appears to be possible [Baikov, 1997]. Let us write

$$j(n_1 \ldots, n_N) = \int \frac{\prod_{i=1}^{L} d^d l_i}{\pi^{Ld/2}} f(\boldsymbol{n}, s_{ij}), \qquad f(\boldsymbol{n}, s_{ij}) = \prod_{k=1}^{N} D_k^{-n_k}$$

We start from the integral over l_1. The integrand depends on l_1 via scalar products $s_{1,1}, \ldots, s_{1,K}$, where $K = L + E$. We write

$$\frac{d^d l_1}{\pi^{\frac{d}{2}}} = \frac{d^{K-1} l_{1\|}}{\pi^{\frac{K-1}{2}}} \frac{d^{d-K+1} l_{1\perp}}{\pi^{\frac{d-K+1}{2}}} = \frac{ds_{12} \ldots ds_{1K}}{\pi^{\frac{K-1}{2}} V^{\frac{1}{2}}(q_2, \ldots q_K)} \frac{\overbrace{ds_{11}}^{dl_1^2 = dl_{1\perp}^2} \left(\overbrace{\frac{V(q_1, \ldots q_K)}{V(q_2, \ldots q_K)}}^{l_{1\perp}^2} \right)^{\frac{d-K-1}{2}}}{\Gamma(d-K+1)/2]}$$

where $l_{1\|}$ ($l_{1\perp}$) denote the components in (the orthogonal complement of) the linear subspace spanned by $q_2 = l_2, \ldots, q_L = l_L, q_{L+1} = p_1, \ldots, q_{L+E} = p_E$.

Here $V(q_1, \ldots q_K) = \det\{q_i \cdot q_j|_{i,j=1\ldots K}\}$ is the *Gram determinant* = square of volume of the parallelepiped constructed on $q_1, \ldots q_K$. Respectively, the matrix $\widehat{V}(q_1, \ldots q_K) = \{q_i \cdot q_j|_{i,j=1\ldots K}\}$ is called the *Gram matrix*.

Repeating the same transformation for $l_2, \ldots, l_L$, we obtain

$$\frac{\pi^{(L-N)/2}}{\Gamma\left[\frac{d-K+1}{2}, \ldots, \frac{d-E}{2}\right]} \int \prod_{i=1}^{L} \prod_{j=i}^{K} ds_{ij} \frac{\left[V\left(q_1, \ldots, q_K\right)\right]^{(d-K-1)/2}}{\left[V\left(p_1, \ldots, p_E\right)\right]^{(d-E-1)/2}} f\left(\boldsymbol{n}, s_{ij}\right)$$

Since $D_1, \ldots D_N$ are linear in s_{ij} and form a basis, we have $\prod_{i=1}^{L} \prod_{j=i}^{K} ds_{ij} = J \prod_{k=1}^{N} dD_N$, where $J = \left(\det \frac{\partial D_k}{\partial s_{ij}}\right)^{-1}$ (here ij should be understood as index running over N distinct values). Also $V\left(q_1, \ldots, q_K\right) = P(D_1, \ldots, D_N)$ is polynomial in D_k (called *Baikov polynomial*). Finally we have

Baikov representation

$$j(\boldsymbol{n}) = \frac{\pi^{(L-N)/2} J}{\Gamma\left[\frac{d-K+1}{2}, \ldots, \frac{d-E}{2}\right]} \int_{\mathcal{D}} \prod_{k=1}^{N} \frac{dD_k}{D_k^{n_k}} \frac{\left[P\left(D_1, \ldots D_N\right)\right]^{(d-K-1)/2}}{\left[V\left(p_1, \ldots, p_E\right)\right]^{(d-E-1)/2}}$$

With some reservations, the integration region is

$$\mathcal{D} = \left\{\left(D_1, \ldots, D_N\right) \in \mathbb{R}^N \middle| P\left(D_1, \ldots D_N\right) > 0\right\}.$$

Dimension shifts and differentiation

Dimension shift relations [Tarasov, 1996; Derkachov et al., 1990]

Feynman representation in $d-2$ dimensions

$$j(d-2|\boldsymbol{n}) = \Gamma[\Sigma_k n_k + L - L\tfrac{d}{2}] \int_{\mathbb{R}_+^M} \prod_{k=1}^{M} \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} U \frac{F^{Ld/2-\Sigma_k n_k - L}}{U^{(L+1)d/2-\Sigma_k n_k - L}} \delta\left(1 - \widetilde{\Sigma}_k z_k\right)$$

Note an extra factor of U. Highlighted are the modifications which appeared due to the shift $d \to d-2$. Then it is easy to check that the following relation holds:

Dimension raising relation [Tarasov, 1996]

$$j(d-2|\boldsymbol{n}) = U(A_1, \ldots, A_N) j(d|\boldsymbol{n}) \qquad (U \text{ is 1st Symanzik polynomial})$$

Similarly, from Baikov representation we obtain

Dimension lowering relation [Derkachov, Honkonen, and Pis'mak, 1990]

$$j(d+2|\boldsymbol{n}) = \frac{2^L V^{-1}(p_1, \ldots, p_E)}{(d-K+1)_L} P(B_1, \ldots, B_N) j(d|\boldsymbol{n}) \quad (P \text{ is Baikov polynomial})$$

Note a remarkable correspondence:

Feynman parameters $z_k \Leftrightarrow A_k$ Baikov parameters $D_k \Leftrightarrow B_k$.

Differentiating the integral $j\left(\boldsymbol{n}\right)$ wrt to m^2 reduces to differentiating the **integrand**. Differentiating wrt some invariant $(p_i \cdot p_j)$ is trickier as the integrand depends on the scalar products of p_i, p_j with loop momenta. We have to express the derivative wrt $(p_i \cdot p_j)$ via derivatives wrt p_i and/or p_j

Differentiating wrt invariant [Remiddi, 1997]

$$\frac{\partial}{\partial\left(p_i \cdot p_j\right)} j\left(\boldsymbol{n}\right) = 2^{-\delta_{ij}} \left[\widehat{P}^{-1}\right]_{ik} p_k \cdot \partial_{p_j} j\left(\boldsymbol{n}\right) .$$

Here $\widehat{P} = \{p_i \cdot p_j | i, j = 1, \ldots, E\}$ is Gram matrix.
The derivative ∂_{p_j} can now be applied to the integrand of $j(\boldsymbol{n})$.

Alternatively, one might consider differentiation in Feynman or Baikov representations. Usually those also shift dimension, but this can be fixed as shown poreviously. E.g., using Lee-Pomeransky representation it is easy to obtain the following formula

Differentiating in Feynman representation

$$\frac{\partial}{\partial x} j\left(d-2|\boldsymbol{n}\right) = -\frac{\partial F(A_1, \ldots, A_N)}{\partial x} j(d|\boldsymbol{n}).$$

Here x is any kinematic parameter. Note the dimension shift in the lhs.

As a result of IBP reduction we express amplitudes via a finite set of master integrals $\boldsymbol{j} = (j_1, \ldots, j_K)^\intercal$. What is even more important, we can obtain closed equations for the master integrals. To obtain these equations we simply apply the dimensional shifts and/or differentiate the master integrals and then IBP-reduce the result. Then the dimension shifts and/or derivatives of the master integrals is expressed as linear combination of the same set of master integrals $\boldsymbol{j} = (j_1, \ldots, j_K)^\intercal$. We obtain

Differential equations

[Kotikov, 1991; Remiddi, 1997]

$$\partial_x \boldsymbol{j} = M(x, d)\boldsymbol{j}$$

Dimensional recurrences

[Tarasov, 1996; Derkachov et al., 1990]

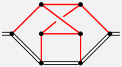
$$\boldsymbol{j}(d-2) = R(x, d)\boldsymbol{j}(d)$$

It appears that in multi-loop case it is often easier to solve these equations than to use direct methods for calculation of the master integrals.

Dimensional recurrence relations are especially useful for one-scale integrals, when the differential equations can not help. The approach is very effective when the matrix R in $\boldsymbol{j}(d-2) = R(d)\boldsymbol{j}(d)$ is triangular. Using analytical properties of integrals as functions of d to fix the arbitrary periodic functions, one can obtain the solution in the form of convergent sums. High-precision evaluation of these sums can be done with `SummerTime` package [RL and Mingulov, 2016].

Using PSLQ algorithm, one can turn the obtained numerical results into analytical expressions.

In Ref. [RL and Pikelner, 2023] the four-loop HQET propagator master integrals have been calculated using DRA method.



We will postpone more detailed discussion of the DRA method to the second lecture.

Differential equations

- Differential equations for master integrals have the form

$$\partial_x \boldsymbol{j} = M(x, \epsilon)\boldsymbol{j}$$

- One can try to simplify the equation by transformation $\boldsymbol{j} = T\tilde{\boldsymbol{j}}$, so that

$$\partial_x \tilde{\boldsymbol{j}} = \widetilde{M}\tilde{\boldsymbol{j}}, \qquad \widetilde{M} = T^{-1}\left[MT - \partial_x T\right]$$

- [Henn, 2013]: there is often a "canonical" basis $\boldsymbol{J} = T^{-1}\boldsymbol{j}$ such that

$$\partial_x \boldsymbol{J} = \epsilon S(x)\boldsymbol{J} \qquad (\epsilon\text{-form})$$

- General solution for d.e. in ϵ-form is easily expanded in ϵ:

$$U(x, x_0) = \mathrm{Pexp}\left[\epsilon \int_{x_0}^{x} dx S(x)\right] = \sum_n \epsilon^n \iiint\limits_{x>x_n>\ldots>x_0} dx_n \ldots dx_1 S(x_n)\ldots S(x_1)$$

- ϵ-form of differential system can be conjecturally obtained from an analysis of the loop integrand [Henn, 2013] or derived from the initial differential system.
- The algorithm of finding transformation to ϵ-form was presented in [RL, 2015]. It is implemented in 3 publicly available codes: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], and recently in `Libra` [RL, 2021].

General structure of reduction algorithm

Algorithm proceeds in three major stages, each involving a sequence of "elementary" transformations.

1. *Fuchsification*: Eliminating higher-order poles

Input: Rational matrix $M\left(x,\epsilon\right)$

Output: Rational matrix with only simple poles on the extended complex plane, $M\left(x,\epsilon\right)=\sum_k \frac{M_k(\epsilon)}{x-a_k}$.

2. *Normalization*: Normalizing eigenvalues

Input: Matrix from the previous step, $M\left(x,\epsilon\right)=\sum_k \frac{M_k(\epsilon)}{x-a_k}$.

Output: Matrix of the same form, but with the eigenvalues of all $M_k\left(\epsilon\right)$ being proportional to ϵ.

3. *Factorization*: Factoring out ϵ

Input: Matrix from the previous step.

Output: Matrix in ϵ-form, $M\left(x,\epsilon\right)=\epsilon S(x)=\epsilon\sum_k \frac{S_k}{x-a_k}$.

Frobenius method [RL, Smirnov, and Smirnov, 2018]

Regularized path-ordered exponent

$$U(x,\underline{0}) = \lim_{x_0\to 0} \mathrm{Pexp}\left[\int\limits_{x_0}^{x} M(x)dx\right] x_0^{M_0}, \quad M_0 = \mathrm{res}_{x=0} M(x)$$

can also be expanded in generalized power series when x is small enough.

$$U(x,\underline{0}) = \sum_{\lambda\in S} x^{\lambda} \sum_{n=0}^{\infty} \sum_{k=0}^{K_\lambda} \frac{1}{k!} C\left(n+\lambda, k\right) x^n \ln^k x \,.$$

Note that for expansion around singular point (which we usually want) non-integer powers x^{λ} and $\log x$ might appear.

The convergence radius is the distance to the nearest singularity. However, it is easy to perform analytical continuation to the whole complex plane by matching expansions at different points. Let $x = 1$ is also the singular point, then the continuation of $U(x,\underline{0})$ beyond $x = 1$ is simply

$$U(x>1,\underline{0}) = U(x,\underline{1})U^{-1}(1/2,\underline{1})U(1/2,\underline{0})$$

Recently this approach was successfully applied to the numeraical calculation of massive form factors [Fael, Lange, Schönwald, and Steinhauser, 2022].

- `Libra` is a *Mathematica* package useful for treatment of differential systems which appear in multiloop calculations.
- Tools for reduction to ϵ-form
  - Visual interface
  - Algebraic extensions
  - Birkhoff-Grothendieck factorization
- Tools for constructing solution
  - Determining boundary constants.
  - Constructing ϵ-expansion of Pexp.
  - Constructing Frobenius expansion of Pexp.

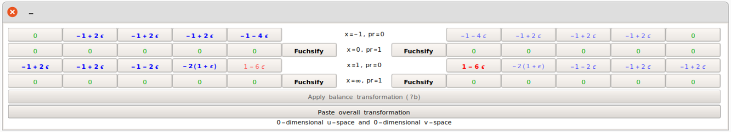
Libra tools for reduction to ϵ-form

- Fuchsification and normalization.
  - Automatic tool (useful for simple cases)

```
In[1]:  t=Rookie[M,x,ϵ];
```

  - Interactive tool (useful for most cases)

```
In[1]:  t=VisTransformation[M,x,ϵ];
```



- Factorization.

```
In[2]:  t=FactorOut[M,x,ϵ,μ];
```

- General solution

```
In[3]:  U=PexpExpansion[{M,6},x];
```

Suppose we have found a transformation $T(x) = T(x, \epsilon)$ to ϵ-form, $\boldsymbol{j} = T\boldsymbol{J}$. Then we can write

$$\boldsymbol{J}(x) = U(x, x_0)\boldsymbol{J}(x_0),$$
$$\boldsymbol{j}(x) = T(x)U(x, x_0)[T(x_0)]^{-1}\boldsymbol{j}(x_0)$$

But the point x_0 should be somewhat special to simplify the evaluation of $\boldsymbol{j}(x_0)$ as compared to $\boldsymbol{j}(x)$. As a rule, "special" boils down to "singular", i.e., we can expect simplifications for x_0 being a singular point of the differential system. Let it be $x_0 = 0$ for simplicity.

Problem

$U(x, x_0)$ diverges when x_0 tends to zero. Therefore, we have to consider not the values, but the asymptotics of $\boldsymbol{j}(x)$ at $x = 0$.

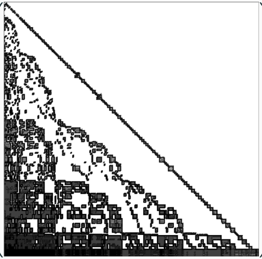
`Libra` can determine which asymptotic coefficients, $\boldsymbol{c}$, are sufficient to calculate and find the "adapter" matrix L relating those with the column of boundary constants, $\boldsymbol{C} = L\boldsymbol{c}$.

```
In[4]:  {L,cs}=GetLcs[M,T,{x,0}];
```

One of many 4-loop massless vertex topologies with two off-shell legs.

- Differential system

$\partial_x j =$



$j,$ 374 $\times$ 374 matrix

where $j =$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

- Maximum size of the diagonal blocks is "only" 11×11.
- No global rationalizing variable. Three algebraic extensions are needed for the reduction to ϵ-form:

$$x_1 = \sqrt{x}, \qquad x_2 = \sqrt{x - 1/4}, \qquad x_3 = \sqrt{1/x - 1/4}$$

End of Lecture I

Multi-loop Feynman Integrals

Lecture II: Advanced topics

Lee Roman
Monday 15th May, 2023

Budker Institute of Nuclear Physics

IBP reduction: advanced topics

Operators $\mathcal{P}^{(ij)}(A, B)$ generate a left ideal

$$\mathbb{L} = \langle \mathcal{P}^{(11)}, \ldots, \mathcal{P}^{(L,\, L+E)} \rangle_{\text{left}} = \left\{ \sum_{ij} C_{ij}(A, B)\mathcal{P}^{(ij)}(A, B) \,\middle|\, C_{ij}(A, B) \in \mathbb{A}_N \right\}.$$

Informally, $\mathbb{L}$ consists of all linear combinations of IBP identities. Any combination of IBP identities can be written as $\boxed{\mathcal{L}j(\mathbf{1}) = 0, \qquad \mathcal{L} \in \mathbb{L}}$.

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Let us write the integral $j(\boldsymbol{n})$ in the form

$$j(\boldsymbol{n}) = \boldsymbol{Y}^{\boldsymbol{n}} j(\mathbf{1}) = \prod_{k=1}^{N} Y_k^{n_k} j(\mathbf{1}), \qquad Y_k^{n_k} = \begin{cases} B_k^{1-n_k} & n_k \leqslant 0 \\ \frac{1}{(n_k-1)!} A_k^{n_k-1} & n_k > 0 \end{cases}$$

One might think of reducing $j(\boldsymbol{n})$ by finding the decomposition $\boldsymbol{Y}^{\boldsymbol{n}} = \mathcal{L}(A, B) + \mathcal{M}(A, B)$, where $\mathcal{L} \in \mathbb{L}$ and the “remainder” $\mathcal{M}$ is simplest possible[1]. Finding this decomposition is algorithmically solved via construction of Groebner basis of $\mathbb{L}$ (implemented, e.g., in `Singular`). Substituting this decomposition and using the fact that $\mathcal{L} j = 0$, we have $\boxed{j(\boldsymbol{n}) = \mathcal{M} j(\mathbf{1})}$. Assuming $\mathcal{M}$ is simple enough, we might hope for the reduction.

[1] NB: We have to fix monomial order to talk about simplicity/complexity.

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Unfortunately, a little experimenting shows that this reduction is not satisfactory, the quotient ring $\mathbb{A}_N/\mathbb{L}$ is not even finite-dimensional (the number of "master integrals" is infinite).

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What relations have we missed? We missed relations

$$(B_k A_k f)(\ldots 1_k \ldots) = 0$$

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Indeed, $(B_k A_k f)(\ldots 1_k \ldots) = (A_k f)(\ldots 0_k \ldots) = 0 \cdot f(\ldots 1_k \ldots) = 0$.

What relations have we missed? We missed relations

$$(B_k A_k f)(\ldots 1_k \ldots) = 0$$

Therefore, along with the **left** ideal $\mathbb{L} = \langle \mathcal{P}^{(11)}, \ldots, \mathcal{P}^{(L,\,L+E)} \rangle_{\text{left}}$ we have to consider also the **right** ideal
$\mathbb{R} = \langle B_1 A_1, \ldots, B_N A_N \rangle_{\text{right}} = \left\{ \sum_k B_k A_k C_k(A, B) \middle| C_k(A, B) \in \mathbb{A}_N \right\}$ and try to find the decomposition

IBP reduction as reduction wrt $\mathbb{L} + \mathbb{R}$

$$\boldsymbol{Y^n} = \mathcal{L}(A, B) + \mathcal{R}(A, B) + \mathcal{M}(A, B),$$

where $\mathcal{L} \in \mathbb{L}$, $\mathcal{R} \in \mathbb{R}$, and the "remainder" $\mathcal{M}$ is simplest possible.

Substituting this decomposition, we get the reduction $j(\boldsymbol{n}) = \mathcal{M} j(\mathbf{1})$.

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Substituting this decomposition, we get the reduction $\boxed{j(\boldsymbol{n}) = \mathcal{M} j(\boldsymbol{1})}$.
It is easy to understand that finding this $\mathcal{L} + \mathcal{R} + \mathcal{M}$ decomposition gives a full reduction. Suppose we have rule $j(\boldsymbol{n}) \to \mathcal{M} j(\boldsymbol{1})$, reducing $j(\boldsymbol{n})$ to master integrals. It means that there exists $\mathcal{L} \in \mathbb{L}$, such that $[\boldsymbol{Y}^{\boldsymbol{n}} - \mathcal{L} - \mathcal{M}] f(\boldsymbol{1}) = 0$ for **arbitrary** function f. We then claim that $\mathcal{R} = [\boldsymbol{Y}^{\boldsymbol{n}} - \mathcal{L} - \mathcal{M}]$ belongs to $\mathbb{R}$.

IBP reduction as reduction wrt sum of left and right ideals in $\mathbb{A}_N$ II

What relations have we missed? We missed relations

$$(B_k A_k f)(\ldots 1_k \ldots) = 0$$

Therefore, along with the **left** ideal $\mathbb{L} = \langle \mathcal{P}^{(11)}, \ldots, \mathcal{P}^{(L,\, L+E)} \rangle_{\text{left}}$ we have to consider also the **right** ideal
$\mathbb{R} = \langle B_1 A_1, \ldots, B_N A_N \rangle_{\text{right}} = \left\{ \sum_k B_k A_k C_k(A, B) \middle| C_k(A, B) \in \mathbb{A}_N \right\}$ and try to find the decomposition

IBP reduction as reduction wrt $\mathbb{L} + \mathbb{R}$

$$\boldsymbol{Y}^{\boldsymbol{n}} = \mathcal{L}(A, B) + \mathcal{R}(A, B) + \mathcal{M}(A, B),$$

where $\mathcal{L} \in \mathbb{L}$, $\mathcal{R} \in \mathbb{R}$, and the "remainder" $\mathcal{M}$ is simplest possible.

Substituting this decomposition, we get the reduction $j(\boldsymbol{n}) = \mathcal{M} j(\boldsymbol{1})$.

Despite an apparent similarity to $\mathcal{L} + \mathcal{M}$ decomposition, there seem to be no known effective algorithm of finding $\mathcal{L} + \mathcal{R} + \mathcal{M}$ decomposition. In particular, Groebner bases can not help. The problem looks very similar to D-modules theoretical integration problem, so there maybe such an algorithm. One should be warned though that existing (implementations of) D-modules algorithms are extremely slow.

Both Lee-Pomeransky and Baikov representations depend on one polynomial raised to the power, depending on d. It is obvious that if we act on the integrand with a random differential operator, the power of this polynomial will be shifted. Therefore, we will get relations between integrals not only with shifted indices, but also with shifted dimension. If we don't want this, we have to choose the differential operator very carefully.

Let us consider the operator $\partial_m Q_m = \frac{\partial Q_m}{\partial D_m} + Q_m \frac{\partial}{\partial D_m}$, where Q_m are some polynomials of D_k. If we act with this operator on the integrand of Baikov representation $j(\boldsymbol{n}) \propto \int \prod_k \frac{dD_k}{D_k^{n_k}} P^{\frac{d-K-1}{2}}$, we have

$$\int d\boldsymbol{D} P^{\frac{d-K-1}{2}} \left[\frac{\partial \left(\boldsymbol{D}^{-\boldsymbol{n}} Q_m \right)}{\partial D_m} + \frac{d-K-1}{2} \boldsymbol{D}^{-\boldsymbol{n}} Q_m \frac{\partial P}{P \partial D_m} \right]$$

Here we used notations $d\boldsymbol{D} = \prod_k dD_k$, $\boldsymbol{D}^{-\boldsymbol{n}} = \prod_k D_k^{-n_k}$. Extra power of P in the denominator may appear due to the term $Q_m \frac{\partial P}{P \partial D_m}$. However, if we choose Q_m s.t. $Q_m \partial_m P = QP$, where Q is also some polynomial, the P in the denominator gets cancelled. How can we do that?

Fortunately, there is a help from computational commutative algebra. Let $\boldsymbol{p} = (p_1, \dots p_n)$ be a vector of polynomials, then $\boldsymbol{Q} = (Q_1, \dots, Q_n)$ is called syzygy of $(p_1, \dots p_n)$ if the following relation holds

$$\boldsymbol{Q}\boldsymbol{p} = \sum_{m=1}^{n} Q_m p_m = 0 \,.$$

Syzygies form a module, i.e., if $\boldsymbol{Q}^{(1)}$ and $\boldsymbol{Q}^{(2)}$ are syzygies and F is any polynomial, then $\boldsymbol{Q}^{(1)} + \boldsymbol{Q}^{(2)}$ and $F\boldsymbol{Q}^{(1)}$ are also syzygies. A basis of syzygy module is a finite set $\boldsymbol{Q}^{(1)}, \dots \boldsymbol{Q}^{(n)}$, such that any syzygy is their linear combination with polynomial coefficients.

Finding a basis of syzygy module is a classical task of commutative algebra.
It is implemented in many CAS, including `Singular`, `Macaulay2`, `CoCoA`.

Thus, finding a syzygy basis of the set $\frac{\partial P}{\partial D_1}, \dots, \frac{\partial P}{\partial D_N}, -P$ we can construct IBP identities not shifting dimensions. Let $Q_m(\boldsymbol{D})\frac{\partial P}{\partial D_m} - QP = 0$ then we have

IBP identity from syzygy via A and B

$$[-Q_m(B_1, \dots, B_N)A_m + \tfrac{1}{2}(d - K - 1)Q(B_1, \dots, B_N)]j(\boldsymbol{n}) = 0, \quad K = L + E$$

Note that usual momentum-representation identities also correspond to syzygies. When the operator $\frac{\partial}{\partial l_i} \cdot q_j$ acts on a function of s_{kl}, we can represent it as follows ($K = L + E$):

$$\frac{\partial}{\partial l_i} \cdot q_j = \sum_{k=1}^{K} 2^{\delta_{ik}} s_{kj} \frac{\partial}{\partial s_{ik}} + d\delta_{ij} = \sum_{k=1}^{K} \frac{\partial}{\partial s_{ik}} 2^{\delta_{ik}} s_{kj} + (d - K - 1)\delta_{ij}$$

This representation exactly corresponds to the syzygy derived from Laplace expansion of the determinant of symmetric matrix $S = \{s_{ij} | i, j = 1, \ldots, N\}$, [Böhm et al., 2018]:

$$\sum_{k=1}^{K} 2^{\delta_{ik}} s_{kj} \frac{\partial \det S}{\partial s_{ik}} - 2\delta_{ij} \det S = 0$$

Moreover, in the same paper it was shown, that the "momentum-space" syzygies form a generating set of syzygy module. I.e., syzygy module provides exactly the same information as momentum-space IBP identities.

However, syzygy approach in Baikov representation provides a more flexible setup. In particular, they can be used to derive IBP identities for integrals without squared denominators. Also, syzygy approach gets very effective when considering the maximal cuts of the integrals (in this case, the corresponding variables D_k should be put to zero).

Note that $N = L(L+1)/2 + L \cdot E$ grows quadratically with L, while M, the # of lines in the diagram, grows only linearly. Parametric representation: only M indices. Therefore, the IBP reduction in parametric representation might be more effective for higher loops.

Let us write the parametric representation in the form

Lee-Pomeransky representation

$$j(\boldsymbol{n}) = \frac{\Gamma\left[d/2\right]\tilde{j}(\boldsymbol{n})}{\Gamma\left[(L+1)\,d/2 - \Sigma_k n_k\right]}, \qquad \tilde{j}^{(d)}(\boldsymbol{n}) = \boldsymbol{I}^{\boldsymbol{n}}\left[G^{-d/2}\right] = \prod_k I_k^{n_k}\left[G^{-d/2}\right],$$

where $G = U + F$. The functionals I_k^m are determined as

$$I_k^m[\phi(z_k)] = \begin{cases} \int_0^\infty \frac{dz_k z_k^{m-1}}{\Gamma(m)}\phi(z_k) & m > 0 \\ (-1)^m \phi^{(-m)}(0) & m \leqslant 0 \end{cases}$$

These functionals allow us to account also for negative n_k.

IBP reduction in Lee-Pomeransky representation II

It can be checked easily that these functionals satisfy relations

$$I_k^m[-\partial\phi(z_k)/\partial z_k] = I_k^{m-1}[\phi(z_k)]\,, \qquad I_k^m[z_k\phi(z_k)] = mI_k^{m+1}[\phi(z_k)]\,.$$

Suppose now that we have a syzygy $QG + Q_k\partial_k G = 0$. Then we can transform $\boldsymbol{I^n}\left[-\partial_k(Q_k/G^{d/2})\right]$ in two different ways. Using the first relation we get $\boldsymbol{I^n}\left[-\partial_k(Q_k/G^{d/2})\right] = \sum_k \boldsymbol{I^{n-1_k}}\left[Q_k/G^{d/2}\right]$. Explicitly differentiating and using the syzygy relation, we get $\boldsymbol{I^n}\left[-\partial_k(Q_k/G^{d/2})\right] = \boldsymbol{I^n}\left[(\frac{d}{2}Q - \partial_k Q_k)/G^{d/2}\right]$. Equating these two expressions and using the second relation, we get

IBP identity in LP representation

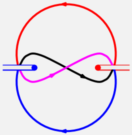
$$[Q_k(A_1,\ldots,A_N)B_k + \tfrac{d}{2}Q(A_1,\ldots,A_N)]\tilde{j}(\boldsymbol{n}) = 0$$

Note that this derivation holds both for positive and non-positive indices.

IBP reduction in Lee-Pomeransky representation is quite promising, but a fast algorithm for constructing a *minimal* (rather than Groebner) basis of syzygy module is very desirable.

- Integral in parametric representation is understood as bilinear pairing between integration cycle C and differential form ϕ.

$$\int_C G^{-\nu}\phi = \langle\phi|C] \,,$$



Simplest twisted cycle: Pochhammer contour.

- $\langle\phi|C]$ is invariant under $\phi \to \phi + \nabla_\nu\tilde{\phi}$ and/or $C \to C + \partial\tilde{C}$, where $\nabla_\nu = d - \nu G^{-1}dG$ is twisted differential and $\partial\tilde{C}$ is a boundary (contractable) cycle.
- Therefore, $\langle\cdot|\cdot]$ is defined on the elements of twisted de Rham cohomology and twisted homology. Those are finite-dimensional spaces, therefore we can use basis expansion as IBP.
- Ref. [Cho and Matsumoto, 1995] introduced pairing $\langle\phi_1|\phi_2\rangle$, correctly defined for ∇_ν and $\nabla_{-\nu}$ de Rham cohomologies.
- IBP reduction is simply a basis expansion

$$\langle\phi|C] = \sum_i \langle\phi|\phi_i\rangle\,\langle\phi_i|C] \,,$$

where $j_i = \langle\phi_i|C]$ are master integrals.
- Unfortunately, $\langle\phi_1|\phi_2\rangle$ is still very difficult to calculate in general. All examples considered so far correspond to integrals with only a few (1 or 2) indexes. Perspectives of this approach are doubtful.

DRA method

Let us briefly explain how DRA method works for triangular R. It is convenient to introduce $\nu = d/2$ and to consider all integrals as functions of ν. Then for the master integral $J = j_k$ we have the following inhomogeneous equation

$$J(\nu - 1) = C(\nu)J(\nu) + D(\nu),$$

where $C(\nu) = c\frac{\prod_i (a_i - \nu)}{\prod_i (b_i - \nu)}$ is some rational function and $D(\nu)$ is a linear combination of simpler master integrals. We assume that simpler masters are already calculated at this stage by the same method (or evaluated explicitly in terms of Γ-functions).

Using the homogeneous solution $S^{-1}(\nu) = c^{-\nu}\frac{\prod_i \Gamma(a_i - \nu)}{\prod_i \Gamma(b_i - \nu)}$, we obtain

$$Y(\nu - 1) = Y(\nu) + S(\nu - 1)D(\nu), \qquad Y(\nu) = S(\nu)J(\nu)$$

The general solution of this equation reads

$$Y(\nu) = \omega(\nu) + \Sigma_\pm S(\nu - 1)D(\nu),$$

depending on which of the two sums converges. Here $\omega(\nu) = \omega(\nu + 1)$ is arbitrary periodic function and we have introduced notations

$$\Sigma_- f(\nu) = -\sum_{k=0}^{\infty} f(\nu - k) \qquad \Sigma_+ f(\nu) = \sum_{k=1}^{\infty} f(\nu + k)$$

$$Y(\nu) = \omega(\nu) + \Sigma_{\pm} S(\nu - 1) D(\nu),$$

Two questions are in order:

1. How does one fix $\omega(\nu)$.
2. Is it possible to calculate emerging multiple sums with high precision.

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2. Is it possible to calculate emerging multiple sums with high precision.

The answer to the first question is the essence of DRA method. First, let us introduce $z = e^{2i\pi\nu}$. Then the periodic function of ν shall be understood as function of z (since z does not change upon $\nu \to \nu + 1$) Let us write

$$\omega(z) = S(\nu) J(\nu) - \Sigma_{\pm} S(\nu - 1) D(\nu).$$

We know everything about the second term in the right-hand side, but $J(\nu)$ in the first term is the goal of our calculation, so we do not know much about it. However we can extract some information about analytical properties of $J(\nu)$, e.g., from parametric representation. Suppose that we've succeeded to prove that the whole right-hand side is analytic on some stripe $\mathrm{Re}\nu \in [\nu_0, \nu_0 + 1)$ and decays when $\mathrm{Im}\nu \to \pm\infty$. Then we can claim that $\omega(z)$ has no singularities and decays when $|z| \to \infty$. These mild restrictions lead to a very concrete form: $\omega = 0$ (Note that in real analysis the same restrictions would not say much about ω).

Multiple sums with factorized summand.

Let us investigate which kind of multiple sums appear in DRA method. Suppose, for example, that $S(\nu-1)D(\nu)$ contains a term of the form $t(\nu) = F(\nu)\Sigma_+ H(\nu) = F(\nu)\sum_{m=1}^{\infty} H(\nu+m)$ Then

$$\Sigma_+ t(\nu) = \Sigma_+ F(\nu)\Sigma_+ H(\nu) = \sum_{k=1}^{\infty} F(\nu+k) \sum_{m=0}^{\infty} H(\nu+k+m)$$

Now we shift $m \to m-k$ and obtain

$$\Sigma_+ t(\nu) = \sum_{1\leqslant k\leqslant m} F(\nu+k+1)H(\nu+m+1)$$

The dependence on k and m in the summand is factorized! If the summation limits were decoupled, this would be just a product of sums. But even though this is not so, such sums can be evaluated without nested loops!

Mellin-Barnes representation (not considered here) is a powerful tool which can provide expressions for the loop integrals in the form of multiple sums.

Form of the DRA results

The DRA results are expressed in terms of the multiple sums

$$\sum_{\infty > k_1 \geqslant \ldots \geqslant k_n} f_1(k_1) \ldots f_n(k_n)$$

The summand is factorized.

Complexity scales **linearly** with n.

for $k = 0..k_{max}$ **do**
  for $i = 0..n$ **do**
    $S_i = S_i + S_{s-1} f_i(k)$
  end
end
return S_n

Form of the MB results

The MB results are expressed in terms of the multiple sums

$$\sum_{k_1} \cdots \sum_{k_n} f(k_1 \ldots k_n)$$

The summand is not factorized.

Complexity scales **exponentially**.

for $k_1 = 0..k_{max}$ **do** ...//n-fold
  for $k_n = 0..k_{max}$ **do**
    $S = S + f(k_1, \ldots)$
  end
end
return S

Demo: `Examples/SummerTime/example2.nb`

Extra topics

There is a standard approach to the simplification of the polylogarithmic expressions using symbol map. One might think of symbols as a cleaner way to represent iterated (or path-ordered) integrals with logarithmic weights (with some reservations, though):

$$I = \int\!\!\cdots\!\!\int_{1>\tau_n>\ldots>\tau_1>0} d\ln p_n(\tau_n)\ldots d\ln p_1(\tau_1) \xrightarrow{\quad\mathcal{S}\quad} p_n\otimes\ldots\otimes p_1$$

Formal symbol manipulation rules then easily follow, e.g.

$$d\ln(pq) = d\ln p + d\ln q \qquad\Longrightarrow\qquad (\ldots\otimes pq\otimes\ldots) = (\ldots\otimes p\otimes\ldots) + (\ldots\otimes q\otimes\ldots)$$

Similarly, by ordering the integration variables in the product of integrals, we get $\mathcal{S}(I_1 I_2) = \mathcal{S}(I_1) ⧢ \mathcal{S}(I_2)$, where ⧢ denotes a shuffle product, e.g.

$$(a\otimes b)⧢(c\otimes d) = a\otimes b\otimes c\otimes d + a\otimes c\otimes b\otimes d + a\otimes c\otimes d\otimes b + c\otimes a\otimes b\otimes d + c\otimes a\otimes d\otimes b + c\otimes d\otimes a\otimes b$$

We have, in particular, symbols for classical polylogarithms

$$\mathcal{S}(\mathrm{Li_n}(x)) = \underbrace{x\otimes\ldots\otimes x}_{n-1}\otimes(x-1)$$

Symbols are good for checking the identities, e.g., using $\mathcal{S}$ it is easy to establish[2]

$$7\mathrm{Li}_2\left(\frac{1+\varepsilon/z}{1-i\varepsilon}\right)-7\mathrm{Li}_2\left(\frac{1+\bar{\varepsilon}/z}{1+i\bar{\varepsilon}}\right)+7\mathrm{Li}_2\left(\frac{z+\bar{\varepsilon}}{\bar{\varepsilon}-i}\right)-7\mathrm{Li}_2\left(\frac{z+\varepsilon}{\varepsilon+i}\right)+11\mathrm{Li}_2\left(\frac{z+\varepsilon}{\varepsilon-i}\right)-11\mathrm{Li}_2\left(\frac{z+\bar{\varepsilon}}{\bar{\varepsilon}+i}\right)$$
$$+4\mathrm{Li}_2(1+z\varepsilon)-4\mathrm{Li}_2(1+z\bar{\varepsilon})+18\mathrm{Li}_2(-iz)-18\mathrm{Li}_2(iz)+11\mathrm{Li}_2\left(\frac{1+\bar{\varepsilon}/z}{1-i\bar{\varepsilon}}\right)-11\mathrm{Li}_2\left(\frac{1+\varepsilon/z}{1+i\varepsilon}\right)$$
$$=\frac{2i\pi^2}{5\sqrt{3}}-\frac{23}{3}i\pi\ln z+6i\pi\ln\left(2-\sqrt{3}\right)-\frac{i\psi'\left(\frac{1}{6}\right)}{5\sqrt{3}}-24iG,\qquad \text{where } \varepsilon=1/\bar{\varepsilon}=e^{2\pi i/3}.$$

However, strictly speaking, they are much less powerful in simplifying expressions. E.g., if we omit in the left-hand side a couple of dilogs with not so simple arguments, we could have failed to recognize in the symbol of the resulting expression that of the sum of the omitted dilogs.

Simplification algorithm idea (stay tuned)

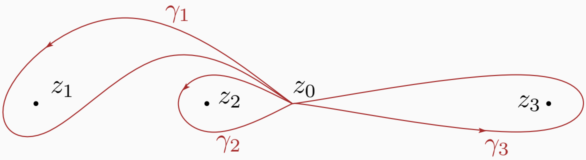
For a given expression:

1. find all possible arguments of Li_n which might enter the simplified form.
2. find equivalent form with the minimal number of polylogs.

[2] NB: This identity was used in real life (as well as some yet more complicated identities) for the simplification of the total cross section of Compton scattering @NLO [RL, Schwartz, and Zhang, 2021a].

1. “Systematic” approach.
   - Reduce the system to $(A+\epsilon B)$-form:
     $$\partial_x j = (A+\epsilon B)j.$$
   - “Integrate out” the ϵ^0 form: make substitution $j = U_0 J$, where U_0 is a fundamental matrix for the unperturbed system $\partial_x U_0 = AU_0$.
   - The system for J is in ϵ-form:
     $$\partial_x J = \epsilon \widetilde{B} J, \qquad \widetilde{B} = U_0^{-1} B U_0.$$
   - The general solution $U_1 = \mathrm{Pexp}\left[\epsilon \int dx \widetilde{B}(x)\right]$ is expanded in terms of iterated integrals with weights being the elements of $\widetilde{B}$.
     NB: irreducibility to ϵ-form means that elements of $\widetilde{B}$ are transcendental functions. In particular, the weights might be possible to represent in terms of modular forms.
   - Pros: to some extent decouples the solution of unperturbed equation and ϵ-expansion.
   - Cons: Iterated integrals with transcendental weights are poorly investigated as compared to polylogarithms. When it comes to numerical evaluation, it is often necessary to reside to some sort of Frobenius method anyway.
2. Meanwhile, the Frobenius method can be applied directly to the differential system. It seems to be the most effective approach for numerical evaluation. In particular, it works for 3-loop massive form factors [Fael, Lange, Schönwald, and Steinhauser, 2022].
3. For many cases of non-polylogarithmic integrals there exists a one-fold integral representation in terms of polylogarithms and algebraic functions.

Monodromy group



- Monodromy group $\mathcal{G}_{\circlearrowleft} \subset GL(n, \mathbb{C})$ of the differential system $\partial_z \boldsymbol{j} = M\boldsymbol{j}$ with $\boldsymbol{j} = (j_1, \ldots j_n)^\intercal$ determines how the solution space transforms under analytical continuation along nonequivalent closed paths[3]. It is generated by the monodromies around the loops encircling each singular point of the system.
- Monodromy group captures all nontrivial properties of the differential system while being blind to a specific realazation (in particular, $\mathcal{G}_{\circlearrowleft}$ is invariant wrt rational transformations of the system).
  Hilbert's 21st problem: **Proof of the existence of linear differential equations having a prescribed monodromic group**.

[3]Reminder: Let $U(z)$ is a fundamental matrix, $\partial_z U = MU$ determined in the vicinity of a regular point z_0, and let $U(z)|_\gamma$ denotes its analytical continuation along the closed path γ starting and ending in this vicinity. Then $U(z)|_\gamma = U(z)g(\gamma)$, where $g(\gamma)$ is a complex $n \times n$ matrix (i.e. $g(\gamma) \in GL(n, \mathbb{C})$). In fact, this matrix depends only on homotopy class $[\gamma]$ (they form a fundamental group $\pi_1(\overline{\mathbb{C}})$). Thus the monodromy group $\mathcal{G}_{\circlearrowleft} = \{g([\gamma])| \, [\gamma] \in \pi_1(\overline{\mathbb{C}})\}$ is a representation of the fundamental group $\pi_1(\overline{\mathbb{C}})$.

The ϵ-reducible and ϵ-irreducible systems differ intrinsically by the type of their monodromy groups at $\epsilon = 0$:

- ϵ-reducible with rational transformations: monodromy group is trivial, $\mathcal{G}_{\circlearrowleft} = \{1\}$.
- ϵ-reducible with algebraic transformations: monodromy group is finite, $|\mathcal{G}_{\circlearrowleft}| < \infty$. Monodromy group becomes trivial on the corresponding covering space.
- ϵ-irreducible: monodromy group is (isomorphic to) a subgroup of $GL(n, \mathbb{Z})$.

In particular, for elliptic cases $\mathcal{G}_{\circlearrowleft}$ is a congruence subgroup of $SL(2, \mathbb{Z})$, see [Broedel et al., 2022] for the case of 2-loop sunrise and 3-loop banana graph. This fact allows one to express the integration kernels via modular forms.

Monodromy group can be obtained numerically from Frobenius expansion, so it is not so easy to see the structure from, e.g.,

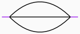
$$g_1 = \begin{pmatrix} 1 & 0. & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, g_2 = \begin{pmatrix} -2. & -5.6325 & -4.11456 \\ 0.618343 & 2.16094 & 0.84807 \\ -0.117344 & -0.220313 & 0.83906 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} -8. + 0.i & -16.8975 + 19.5116i & -12.3437 + 102.816i \\ 1.85503 - 0.296943i & 3.83906 - 4.57912i & -0.84807 - 21.5991i \\ -0.352031 + 0.406491i & 0.220313 + 1.52637i & 5.16094 + 4.57912i \end{pmatrix}.$$

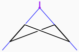
We need to find a matrix t such that $t^{-1} g_k t$ are all integer matrices. One needs some experimentation to find such a matrix. However, it appears to be possible! We find that $t = \begin{pmatrix} 1 & 0 & 3 \\ -3c - \frac{1}{32c} & \frac{i\left(1-96c^2\right)}{16\sqrt{3}c} & -3c - \frac{1}{32c} \\ c & \frac{2ic}{\sqrt{3}} & c \end{pmatrix}$ with $c = 0.11734382...$ being some unrecognized constant, renders

$$t^{-1} g_1 t = \begin{pmatrix} -2 & 0 & -3 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, t^{-1} g_2 t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, t^{-1} g_3 t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 6 \\ 0 & -4 & -5 \end{pmatrix}.$$

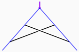
Examples of monodromy groups



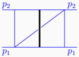
Two-loop sunrise[4]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(6) \subset SL(2,\mathbb{Z})$



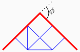
Two-loop massive vertex [von Manteuffel and Tancredi, 2017]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma(2) \subset SL(2,\mathbb{Z})$.



Two-loop EW vertex [Broedel, Duhr, Dulat, Penante, and Tancredi, 2019]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(6) \subset SL(2,\mathbb{Z})$.



3-loop forward box [Mistlberger, 2018]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(5) \subset SL(2,\mathbb{Z})$.



4-loop HQET vertex [Brüser, Dlapa, Henn, and Yan, 2020] : $\mathcal{G}_{\circlearrowleft} \cong \Gamma(3) \subset SL(2,\mathbb{Z})$.

3-loop equal-mass sunrise [Broedel, Duhr, and Matthes, 2022]:

$$\mathcal{G}_{\circlearrowleft} \cong \left\langle \begin{pmatrix} 1 & 6 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}, \begin{pmatrix} -3 & -10 & 7 \\ 12 & 31 & -21 \\ 16 & 40 & -27 \end{pmatrix} \right\rangle \subset GL(3,\mathbb{Z}).$$

3-loop HQET sunrise $\mathcal{G}_{\circlearrowleft} \cong \left\langle \begin{pmatrix} -2 & 0 & -3 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 6 \\ 0 & -4 & -5 \end{pmatrix} \right\rangle \subset GL(3,\mathbb{Z})$

[4] Here

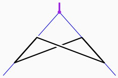
$$\Gamma_1(N) = \left\{ g \in SL(2,\mathbb{Z}) \middle| \, g = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}, \qquad \Gamma(N) = \left\{ g \in SL(2,\mathbb{Z}) \middle| \, g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

- Use ϵ-regular basis [RL and Onishchenko, 2019].
- Use Feynman parametrization to gather two denominators into one [Bezuglov and Onishchenko, 2022].
- Introduce suitable cut denominator $\delta(s - D)$ to later integrate wrt s.
- For the integrals expressible via hypergeometric functions ${}_pF_q$ use integral representation and expand in ϵ under the integral sign [Bezuglov, Kotikov, and Onishchenko, 2022].

These methods seem to be not universal, but may help in real-life calculations.

Example: maximal cut of non-planar vertex

Consider one solution of the homogeneous differential system, $J_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} + 2\epsilon, 1 + \epsilon | x\right)$. Integrating out ϵ^0 gives



$$J_1 = \sum_k \epsilon^k \sum_{i \in \{1,2\}^{k+1}} \frac{2\mathrm{K}(x_{i_0})}{\pi} \mathcal{I}(\Omega_{i_0 i_1}, \Omega_{i_1 i_2}, \ldots, \Omega_{i_{k-1} i_k}, \Omega_{i_k 1} | x)\,,$$

where $\mathcal{I}$ denotes iterated integral, $x_1 = x, x_2 = \overline{x} = 1 - x$, and

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \widetilde{B}(x)dx = \begin{pmatrix} u(\overline{x})v(x) & -u(\overline{x})v(\overline{x}) \\ u(x)v(x) & -u(x)v(\overline{x}) \end{pmatrix} \frac{dx}{\pi x \overline{x}},$$

$$u(x) = \mathrm{K}(x) - 2\mathrm{E}(x),\;\; v(x) = 2\overline{x}\mathrm{K}(x) - 2\mathrm{E}(x)\,.$$

Ω can be expressed via modular forms. Meanwhile, there are much simpler representations in terms of one-fold integrals:

$$J_1 = \frac{\Gamma(\epsilon+1)}{\sqrt{\pi}\Gamma\left(\epsilon + \frac{1}{2}\right)} \sum_k \epsilon^k \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-tx)}} \frac{\ln^k \frac{1-t}{(1-tx)^2}}{k!}$$

$$J_1 = \frac{1}{i\pi^2} \oint_{\sqrt{x}<|t|<1} \frac{dt\,\mathrm{K}(x/t^2)}{t(1-t^2)} \left[1 - 2\epsilon(1-2t)H_1 + 2\epsilon^2\left[2H_{0,1} - (1-2t)\left(3H_{1,-1} + H_{1,1}\right)\right] + \ldots\right],$$

where $H_{\boldsymbol{n}} = H_{\boldsymbol{n}}(t)$ is harmonic polylogarithm.

- Each step towards increasing the # of loops and/or # of scales requires new methods. Those involve both technological advances and new algorithms coming various fields of mathematics.
- IBP reduction still remains a bottleneck for some calculations. New ideas of IBP reduction appear, whether they will be successful is yet to find out.
- Differential equations method is already in a very good shape. However, there is still no regular approach to the computation of non-polylogarithmic integrals. From the practical point of view, there is always a Frobenius method which might be used to obtain numerical high-precision results. [5].
- New ideas and approaches to multi-loop calculations are always very welcome.

Thank you!

[5] Note that I have failed to cover here some topic related to `Libra`. I believe those require a special talk.

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